Properties of tangential and cyclic polygons: an application of circulant matrices

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In this paper, the properties of tangential and cyclic polygons discussed by Lopez-Real in [4] [5] will be proved rigorously using the theory of circulant matrices. In particular, the concepts of slippable tangential polygons and conformable cyclic polygons will be defined. It will be shown that an n-sided tangential (or cyclic) polygon P_n with n even is slippable (or conformable) and the sum of a set of non-adjacent sides (or interior angles) of P_n satisfies certain equalities. On the other hand, for a tangential (or cyclic) polygon P_n with n odd, it is rigid and the sum of a set of non-adjacent sides (or interior angles) of P_n satisfies certain inequalities. These inequalities give a definite answer to the question raised by Lopez-Real in [5] concerning the alternating sum of interior angles of a cyclic polygon.

1. Introduction

In School geometry, an important portion of the syllabus is usually occupied by the study of different type of quadrilaterals and their properties. One type of quadrilateral of particular interest is the cyclic quadrilateral. A cyclic quadrilateral is a quadrilateral that is circumscribed by a circle, i.e., the four vertices of the quadrilateral all lie on the same circle. This union between a circle and a quadrilateral imparts cyclic quadrilaterals with special properties. One such property that any student of geometry is familiar with is opposite interior angles of a cyclic quadrilateral supplement each other. In usual practice, School geometry often stops at listing, with or without formal proofs, these properties and then goes on to engage in problems applying these properties. This is a perfectly sound and fruitful practice of Mathematics. However, a curious student of Mathematics might wonder and wander in different directions. There are at least two possible paths to query about the concept of cyclic quadrilaterals. The first one is whether these well established properties for cyclic quadrilaterals will still hold, and in what sense, for cyclic n-sided polygons when n is greater than four. The second one is instead of having a circle circumscribing a quadrilateral, or in general a polygon, what about having a circle inscribed in a quadrilateral or a polygon. That is, the sides of the polygon of interest are tangential to the same circle that lies inside the polygon. In this situation, what are the properties for such a tangential polygon? Furthermore, how different or similar are these properties compared with those of cyclic polygons? One would suspect that there might exist some kind of duality in the properties between these two types of polygons since both of them are intimately related to a circle.

Lopez-Real, in a two-part paper [4] [5], studied these questions in a detailed narrative fashion describing a process of investigation into the possibility of generalizing and finding properties for tangential and cyclic polygons by looking at quadrilaterals, pentagons and hexagons. By insightful observations and heuristic arguments, interesting conclusions and questions are made at the end of the paper. Those that will form the objects of later discussions in this paper are outlined as follow.

Tangential polygons:

T1 If an even-sided polygon is tangential, the sums of alternating sides are equal (alternating sides means a sequence of every other side starting with a particular side).

T2 If an even-sided polygon is tangential, there are infinite number of polygons with the same sides that are also tangential. (This corresponds to the idea of the polygon 'slipping' around the circle. For example, the case of a quadrilateral ABCD is illustrated in figure 1).

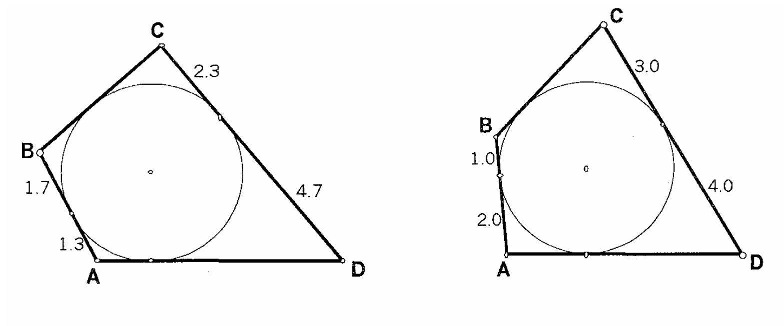

figure 1

Quadrilateral ABCD ‘slips’ around a circle

T3 If an odd-sided polygon is tangential, then it is unique (i.e. there is no other polygon with the same sides that is also tangential).

T4 If an odd-sided polygon is tangential, then the sum of any alternating set of sides is greater than the sum of the remaining sides. (Notice that the last side of any alternating sequence is in fact adjacent to the first side in the sequence).

Cyclic polygons:

C1 If an even-sided polygon is cyclic, then the sums of alternating interior angles are equal (alternating interior angles means a sequence of every other interior angle starting with a particular interior angle).

C2 Given an even-sided cyclic polygon with a particular sequence of interior angles, there are an infinite number of cyclic polygons (with the same number of sides) with the same set of angles. (In figure 2a, $AB // A'B'$, $BC // B'C'$, $CD // C'D'$, $DE // D'E'$, $EF // E'F'$ and $FA // F'A'$, hence a cyclic hexagon $A'B'C'D'E'F'$ can be constructed such that it has the same sequence of interior angles as the given cyclic hexagon ABCDEF).

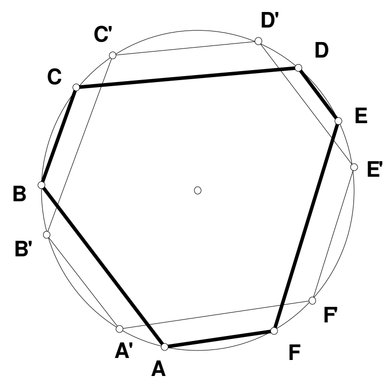

figure 2a

By constructing corresponding parallel sides, hexagon $A'B'C'D'E'F'$ has the same sequence of interior angles as the original hexagon ABCDEF

C3 If an odd-sided polygon is cyclic it is unique. (See Figure 2b for an example on the seemingly impossibility to construct another cyclic pentagon that has the same sequence of interior angles as the given cyclic pentagon ABCDE using the method suggested in Figure 2a).

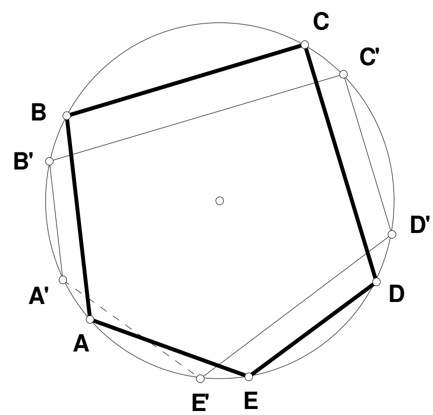

figure 2b

The construction in figure 2a for a hexagon does not work for a pentagon

C4 The sum of any alternating set of interior angles is not necessarily greater than the sum of the remaining interior angles. (In figure 3, $v+x+z<w+y$). This is quite unsatisfactory when one is trying to look for a 'dual inequality' to T4. Does there exist an inequality relating these two sets of interior angles?

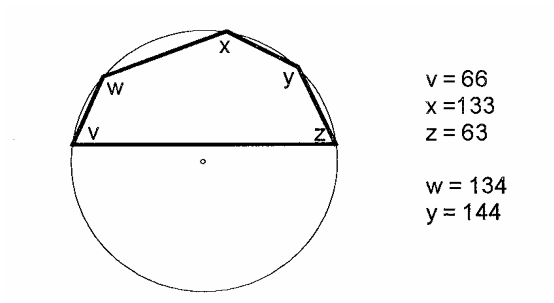

figure 3

The sum of the alternating set of interior angles v, x and z is less than the sum of the remaining interior angles w and y

Lopez-Real did not give formal proofs to the above claims and left C4 as an open question for the readers. In the process of arriving at conclusions T2 to T4, a number puzzle game inspired Lopez-Real to study systems of linear equations associated with tangential pentagons and hexagons. This 'unexpected connections' brought about fruitful insights and transformed the geometrical problems into algebraic questions. In this paper, tangential and cyclic polygons will represented by two systems of linear equations and formal proofs to T1 to T4, C1 to C3 will be provided. Furthermore, the question posed in C4 will be answered by a strict inequality.

It turns out that for each tangential (or cyclic) n-sided polygon, the coefficient matrices for the associated $n \times n$ system of linear equations take the form of *circulant* matrices. The use of circulant matrices to study problems in Euclidean geometry is not new (see Davis [2] [3]) and they have important applications in mathematical physics (see Aldrovandi [1]). The theory of circulant matrices is very rich and it is associated with the theory of Finite Fourier Transforms, however, only elementary facts are needed for the analyses in this paper. Definition, notation and some useful properties of circulant matrices will be stated in the next section. After familiarization with the needed facts, the rest of the paper goes on to explore the properties of tangential and cyclic polygons in the context of circulant matrices and give the claimed proofs.

2. Some facts on circulant matrices

An $n \times n$ matrix is said to be *circulant* if it takes the form

$$\mathbf{C} = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ c_n & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & & \vdots \\ c_2 & c_3 & \cdots & c_1 \end{pmatrix}$$

where each row of $\mathbf{C}$ is a 'shifting forward one place' of the previous row in a cyclic fashion. For convenience, a circulant matrix $\mathbf{C}$ is denoted by

$$\mathbf{C} = \text{circ}\,(c_1, c_2, \ldots, c_n) \ .$$

Notice right away that

$$(\text{circ}\,(c_1, c_2, \ldots, c_n))^T = \text{circ}\,(c_1, c_n, c_{n-1}, \ldots, c_2) \qquad (1)$$

where the superscript T stands for matrix transpose.

All circulant matrices of order n are simultaneously diagonalizable by the matrix associated with the Finite Fourier Transform (for details, see Davis [3]). This has the important consequences that circulant matrices commute with each other and if C^{-1} exits, then C^{-1} is also circulant. In fact, invertible circulants of the same order form a commutative group under matrix multiplication.

To every circulant matrix C, there is associated a complex-valued generating polynomial $P_{\mathsf{C}}(z)$ given by

$$P_{\mathsf{C}}(z) = c_1 + c_2 z + c_3 z^2 + \cdots + c_n z^{n-1} \tag{2}$$

and the eigenvalues λ_j of C are precisely

$$\lambda_j = P_{\mathsf{C}}(\omega^{j-1}), \quad j = 1, 2, \cdots, n \tag{3}$$

where $\omega = \exp(2\pi i/n)$ is the n-th root of unity. Consequently, it is not difficult to express the determinant of a circulant matrix in terms of $P_{\mathsf{C}}(z)$ and ω:

Theorem 1. (Davis [3]) Let C be a circulant matrix.

If $n = 2r+1$ is odd, then

$$\det C = \prod_{j=0}^{n-1} P_C(\omega^j) = P_C(1)\prod_{j=1}^{r}\left|P_C(\omega^j)\right|^2 .$$

If $n = 2r+2$ is even, then

$$\det C = \prod_{j=0}^{n-1} P_C(\omega^j) = P_C(1)P_C(-1)\prod_{j=1}^{r}\left|P_C(\omega^j)\right|^2 .$$

These stated facts will be suffice to furnish the analysis in the following sections. Readers are encouraged to consult [1], [2] and [3] for deeper understanding of circulant matrices.

3. Tangential Polygon

Definition 1. An n-sided polygon P_n is said to be *tangential* if the sides of P_n are tangents to an inscribed circle. A tangential P_n is said to be *slippable* if there exists another n-sided polygon P_n' that is tangential to the same circle as P_n and that preserves the sequence of lengths of the sides of P_n. Otherwise, P_n is said to be *rigid*.

Let P_n be a tangential polygon, by the tangent properties of circle, one can see that the tangential polygon P_n (see figure 4) with vertices $\mathrm{A}_1, \mathrm{A}_2, \ldots\ldots, \mathrm{A}_\mathrm{n}$ must satisfy the following $n \times n$ system of linear equations:

$$
\begin{array}{ccccccccccc}
x_1 & + & x_2 & & & & & & & = & a_1 \\
& & x_2 & + & x_3 & & & & & = & a_2 \\
\vdots & & & & \vdots & & & \vdots & & \vdots & \vdots \\
& & & & & & x_{n-1} & + & x_n & = & a_{n-1} \\
x_1 & + & & & & & & & x_n & = & a_n
\end{array}
\qquad \text{(S)}
$$

where $a_1 = \overline{\mathrm{A}_1\mathrm{A}_2}$, $a_2 = \overline{\mathrm{A}_2\mathrm{A}_3}, \cdots,$ $a_n = \overline{\mathrm{A}_\mathrm{n}\mathrm{A}_\mathrm{n\text{-}1}}$ with $x_1, x_2, \cdots, x_n, a_1, a_2, \cdots, a_n > 0$.

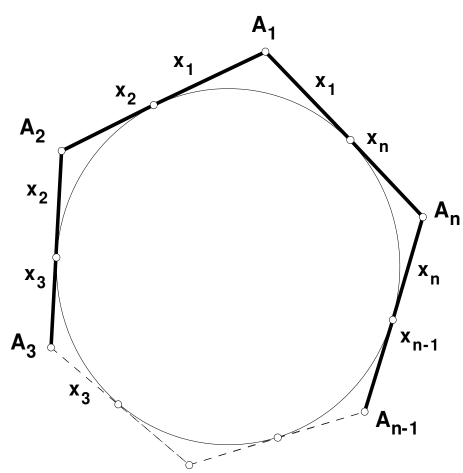

figure 4

A tangential polygon P_n

The system (S) can be written in terms of a circulant matrix as

$$\operatorname{circ}(1,1,\underbrace{0,\cdots,0}_{n-2})\cdot\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}.$$

Let $\mathsf{T}_n = \operatorname{circ}(1,1,0,\cdots,0)$ be the coefficient matrix of (S), then from (2), the generating polynomial of T_n is given by

$$P_{\mathsf{T}_n}(z) = 1 + z\,. \qquad (4)$$

By non-adjacent sides of P_n, I mean any sequence of sides of P_n in which no two sides are adjacent to each other and the sequence terminates at the side where this condition cannot be met when proceeding one step further. For example, in Figure 4, if n is even, then alternating sides are non-adjacent sides of P_n. If n is odd, then

$$A_1A_2,\ A_3A_4,\ \cdots,\ A_{n-2}A_{n-1}$$

would be a set of non-adjacent sides of P_n. The compliment of a set of non-adjacent sides will simply be called the remaining sides of P_n. Under this convention, Lopez-Real's alternating sides for odd n are the remaining sides. In this context, Properties T1 and T2, T3 and T4 for tangential polygons are stated (and proved) as Theorem Two and Theorem Three below.

Theorem 2. Suppose P_n is a tangential polygon and $n \geq 4$ is even, then P_n is slippable and the sums of non-adjacent sides of P_n are equal.

Proof. Let $a_1 = \overline{\mathrm{A}_1\mathrm{A}_2}$, $a_2 = \overline{\mathrm{A}_2\mathrm{A}_3}, \cdots,$ $a_n = \overline{\mathrm{A}_\mathrm{n}\mathrm{A}_{\mathrm{n-1}}}$ be the lengths of the sides of P_n and let each side of P_n be divided into two segments at the point of tangency with lengths x_i and x_{i+1} as shown in Figure 4. Since $n = 2r + 2, r \geq 1$ is even, using Theorem 1 and (4), the determinant of T_n is given by

$$\det T_n = \prod_{j=0}^{n-1} P_{T_n}(\omega^j) = P_{T_n}(1)P_{T_n}(-1)\prod_{j=1}^{r}\left|P_{T_n}(\omega^j)\right|^2 = 2(1-1)\prod_{j=1}^{r}\left|1+\omega^j\right|^2 = 0\,.$$

This means that T_n is singular and hence the solution set of the system (S) forms a subspace of $\mathbf{R}^{\mathrm{n}}$. By continuity argument, there must exist other positive solutions for (S). That is, there exist other sets of $x_1, x_2, \cdots, x_n > 0$ that preserve the lengths and the sequence of sides of P_n. Furthermore, each set of $x_1, x_2, \cdots, x_n > 0$ determines an *n*-sided polygon that is tangential to the same circle as P_n. Hence P_n is slippable.

Because *n* is even, one can observe from (S) (or from figure 4) that the sum of any set of non-adjacent sides must be equal to $x_1 + x_2 + x_3 + \cdots + x_{n-1} + x_n$. This concludes that the sums of non-adjacent sides of P_n are equal.

Theorem 3. Suppose P_n is a tangential polygon and $n \geq 3$ is odd, then P_n is rigid and the sum of any set of non-adjacent sides is strictly less than the sum of the remaining sides.

Proof. As in the proof of Theorem 2 above, with $n = 2r+1, r > 1$ odd, the determinant of T_n is given by:

$$\det T_n = \prod_{j=0}^{n-1} P_{T_n}(\omega^j) = P_{T_n}(1) \prod_{j=1}^{r} \left| P_{T_n}(\omega^j) \right|^2 = 2 \prod_{j=1}^{r} \left| 1 + \omega^j \right|^2 \quad . \tag{5}$$

Since n is odd, $\omega = \exp(2\pi i/n) \neq -1$. This implies that the terms $\left|1 + \omega^j\right|$ are non-zero. Hence $\det T_n > 0$ which means that T_n is non-singular and T_n^{-1} exists. Therefore the system (S) has an unique solution. This shows that P_n is rigid.

Furthermore, it can be shown that

$$T_n^{-1} = \frac{1}{2} \operatorname{circ}(1, -1, 1, -1, 1, \cdots, -1, 1) .$$

T_n is circulant implies that T_n^{-1} is also circulant. Hence, it suffices to find out the entries for the first row of T_n^{-1}. Since $T_n \cdot T_n^{-1} = I_n$ (the $n \times n$ identity matrix), we must have

$$T_n \cdot \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{6}$$

where $(u_{1,} u_2, \ldots, u_n)^T$ is the first column of T_n^{-1}. This corresponds to the linear system:

$$\begin{array}{ccccccccccc} u_1 & + & u_2 & & & & & & & = & 1 \\ & & u_2 & + & u_3 & & & & & = & 0 \\ \vdots & & & & \vdots & & & & \vdots & \vdots & \vdots \\ & & & & & & u_{n-1} & + & u_n & = & 0 \\ u_1 & + & & & & & & & u_n & = & 0 \end{array} \tag{7}$$

By backward substitution, one can check that that

$$u_n + u_1 = 0, \;\; u_{n-1} - u_1 = 0, \;\; u_{n-2} + u_1 = 0, \;\; u_{n-3} - u_1 = 0, \;\; u_{n-4} + u_1 = 0, \cdots$$

Since n is odd, when this process continues until the first two equations of the system (7), the first two equations become

$$u_2 - u_1 = 0 \quad \text{and} \quad u_1 + u_1 = 1 \ .$$

This gives $u_1 = u_2 = 1/2$. Consequently, the first column of T_n^{-1} must be

$$(1/2,\ 1/2,\ -1/2,\ 1/2,\ -1/2,\ 1/2,\ \cdots, 1/2,\ -1/2)^T \ .$$

Since T_n^{-1} is circulant, equation (1) implies that the first row of T_n^{-1} must equal

$$(1/2,\ -1/2,\ 1/2,\ -1/2,\ 1/2,\ -1/2,\ \cdots, -1/2,\ 1/2) \ .$$

Hence, $T_n^{-1} = \dfrac{1}{2}\,\text{circ}\,(1, -1, 1, -1, 1, \cdots, -1, 1)$.

From the form of T_n^{-1}, one can compute easily the solutions of the system (S):

$$\begin{aligned}
x_1 &= 1/2\,(a_1 - a_2 + a_3 - a_4 + \cdots + a_{n-2} - a_{n-1} + a_n),\\
x_2 &= 1/2\,(a_1 + a_2 - a_3 + a_4 + \cdots - a_{n-2} + a_{n-1} - a_n),\\
&\ \ \vdots \qquad\qquad\qquad\qquad \vdots\\
x_n &= 1/2\,(-a_1 + a_2 - a_3 + a_4 + \cdots - a_{n-2} + a_{n-1} + a_n).
\end{aligned}$$

$x_1, x_2, \cdots, x_n > 0$ implies that

$$
\begin{aligned}
&a_2 + a_4 + a_6 + \cdots + a_{n-1} < a_1 + a_3 + a_5 + \cdots + a_n, \\
&a_3 + a_5 + a_7 + \cdots + a_n < a_1 + a_2 + a_4 + \cdots + a_{n-1}, \\
&\qquad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&a_1 + a_3 + a_5 + \cdots + a_{n-2} < a_2 + a_4 + a_6 + \cdots + a_{n-1} + a_n.
\end{aligned}
$$

This means that when n is odd, the sum of any set of non-adjacent sides of P_n is strictly less than the sum of the remaining sides. Notice that when $n = 3$, these inequalities reduce to the triangle inequalities.

4. Cyclic Polygon

Definition 2. An n-sided polygon P_n is called *cyclic* if P_n is circumscribed by a circle. A cyclic P_n is said to be *conformable* if there exists another n-sided polygon P_n' that is circumscribed by the same circle as P_n and that preserves the sequence and the magnitudes of the interior angles of P_n. Otherwise, P_n is said to be *rigid*.

For convenience in making visual observation, the discussion in this section will be restricted to cyclic polygons with the centre of the circumscribing circle lying inside the polygons (see figure 5).

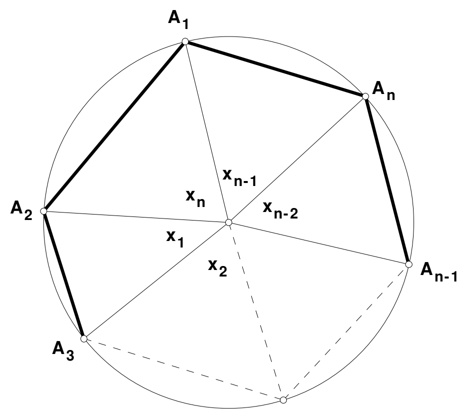

figure 5
A cyclic polygon P_n

Let P_n be a cyclic polygon. It is not difficult, using the properties of angles in a circle, to see that the cyclic polygon P_n (see figure 5) with vertices $\mathrm{A}_1, \mathrm{A}_2, \ldots\ldots, \mathrm{A}_\mathrm{n}$ must satisfy the following $n \times n$ system of linear equations:

$$\begin{array}{ccccccccccccc}
x_1 & + & x_2 & + & \cdots & + & x_{n-2} & & & & & = & \alpha_1 \\
 & & x_2 & + & x_3 & + & \cdots & + & x_{n-1} & & & = & \alpha_2 \\
 & & & & x_3 & + & x_4 & + & \cdots & + & x_n & = & \alpha_3 \\
x_1 & + & & & & & x_4 & + & \cdots & + & x_n & = & \alpha_4 \\
\vdots & & \vdots & & & & \vdots & & \vdots & & \vdots & & \\
x_1 & + & x_2 & + & \cdots & & + & x_{n-3} & & + & x_n & = & \alpha_n
\end{array} \qquad (\mathrm{S}')$$

where $x_1, x_2, \cdots, x_n$ are angles at the centre subtended by arcs $\mathrm{A}_2\mathrm{A}_3, \mathrm{A}_3\mathrm{A}_4, \cdots, \mathrm{A}_1\mathrm{A}_2$ respectively and $\alpha_1 = 2\angle \mathrm{A}_1, \cdots, \alpha_n = 2\angle \mathrm{A}_\mathrm{n}$.

The system (S') can be written in terms of a circulant matrix as

$$\operatorname{circ}(\underbrace{1,1,\cdots 1}_{n-2},0,0)\cdot\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}=\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}$$

Let $C_n = \operatorname{circ}(\underbrace{1,1,\cdots 1}_{n-2},0,0)$ be the coefficient matrix of (S'), then from (2), the generating polynomial of C_n is given by

$$P_{C_n}(z)=1+z+z^2+\cdots+z^{n-3} \qquad (8)$$

A similar 'non-adjacent' convention as in the tangential polygon case will be used here for the definition of non-adjacent interior angles for P_n. Notice in the case when n is odd, any sequence of consecutive non-adjacent interior angles starting with an arbitrary chosen interior angles must have the first member of the sequence two angles away from the last member of the sequence.

Properties C1 and C2 for cyclic polygons can be summarized as:

Theorem 4. Suppose P_n is a cyclic polygon. If $n \geq 4$ is even, then P_n is conformable and the sum of non-adjacent interior angles of P_n is equal to $(n-2)\pi/2$.

Proof. Let $\angle \mathrm{A}_1, \cdots, \angle \mathrm{A}_\mathrm{n}$ be the interior angles of P_n and $x_1, x_2, \cdots, x_n > 0$ be the angles at the centre of the circle subtended by arcs $\mathrm{A}_2\mathrm{A}_3, \mathrm{A}_3\mathrm{A}_4, \cdots, \mathrm{A}_1\mathrm{A}_2$ respectively as shown in Figure 5. Since $n = 2r+2, r \geq 1$ is even, using Theorem 1 and (8), the determinant of C_n is given by

$$\begin{aligned} \det C_n &= P_{C_n}(1) P_{C_n}(-1) \prod_{j=1}^{r} \left| P_{C_n}(\omega^j) \right|^2 \\ &= (n-2)\left(1 + (-1) + (-1)^2 + \cdots + (-1)^{n-3}\right) \prod_{j=1}^{r} \left| 1 + \omega^j + \omega^{2j} + \cdots + \omega^{(n-3)j} \right|^2 \\ &= 0 \end{aligned} .$$

Therefore C_n is singular and hence the solution set of the system (S') forms a subspace of $\mathbf{R}^\mathrm{n}$. By continuity argument, there must exist other positive solutions for (S'). That is, there exist other sets of $x_1, x_2, \cdots, x_n > 0$ such that the sequence and magnitudes of $\angle \mathrm{A}_1, \cdots, \angle \mathrm{A}_\mathrm{n}$ are preserved. Furthermore, each set of $x_1, x_2, \cdots, x_n > 0$ determines a cyclic polygon circumscribed by the same circle as P_n. Hence P_n is conformable.

Next, after some careful counting and the fact that *n* is even, one can deduce from (S') that

$$\alpha_1 + \alpha_3 + \alpha_5 + \cdots + \alpha_{n-1} = \left(\frac{n}{2} - 1\right) \cdot \left(x_1 + x_2 + x_3 + \cdots + x_n\right) = \frac{n-2}{2} \cdot 2\pi = (n-2)\pi.$$

Similarly,

$$\alpha_2 + \alpha_4 + \alpha_6 + \cdots + \alpha_n = \left(\frac{n}{2} - 1\right) \cdot \left(x_1 + x_2 + x_3 + \cdots + x_n\right) = \frac{n-2}{2} \cdot 2\pi = (n-2)\pi \ .$$

Therefore,

$$\begin{aligned} \angle \mathrm{A}_1 + \angle \mathrm{A}_3 + \cdots + \angle \mathrm{A}_{n-1} &= (1/2) \cdot \left(\alpha_1 + \alpha_3 + \alpha_5 + \cdots + \alpha_{n-1}\right) = (n-2)\pi/2\,, \\ \angle \mathrm{A}_2 + \angle \mathrm{A}_4 + \cdots + \angle \mathrm{A}_n &= (1/2) \cdot \left(\alpha_1 + \alpha_3 + \alpha_5 + \cdots + \alpha_{n-1}\right) = (n-2)\pi/2. \end{aligned}$$

This concludes that the sum of non-adjacent interior angles of P_n is equal to $(n-2)\pi/2$.

The following theorem will prove property C3 and answer the question raised in C4 with an inequality.

Theorem 5. Suppose P_n is a cyclic polygon and $n \geq 5$ is odd, then P_n is rigid. Furthermore, the sum of any set of non-adjacent interior angles of P_n is strictly greater than $(n-3)\pi/2$ and the sum of the corresponding remaining interior angles is strictly less than $(n-1)\pi/2$.

Proof. As in the proof of Theorem four, with $n = 2r+1, r > 1$ odd, the determinant of C_n is given by:

$$\begin{aligned} \det C_n &= P_{C_n}(1)\prod_{j=1}^{r} P_{C_n}(\omega^j) \\ &= (n-2)\prod_{j=1}^{r}\left|1+\omega^j+\omega^{2j}+\cdots+\omega^{(n-3)j}\right|^2 \end{aligned} \tag{9}$$

Since n is odd, $\omega = \exp(2\pi i/n) \neq -1$. This implies that the terms $\left|1+\omega^j+\omega^{2j}+\cdots+\omega^{(n-3)j}\right|$ in the last product are non-zero. Therefore, C_n is non-singular and C_n^{-1} exists. Hence the system (S') has an unique solution. Therefore, P_n is rigid.

In particular, it can be shown that

$$C_n^{-1} = \frac{1}{n-2}\operatorname{circ}\left(r, 1-r, 1-r, r, 1-r, r, 1-r, \ldots, r, 1-r\right)$$

Since C_n is circulant, C_n^{-1} is also circulant. It suffices to find out the entries for the first row of C_n^{-1}. Since $C_n \cdot C_n^{-1} = I_n$ (the $n \times n$ identity matrix), we must have

$$C_n \cdot \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where $(v_{1,} v_2, \ldots, v_n)^{\mathrm{T}}$ is the first column of C_n^{-1}. This corresponds to the linear system:

$$\begin{array}{ccccccccccccccc}
v_1 & + & v_2 & + & \cdots & + & v_{n-2} & & & & & & & = & 1 \\
 & & v_2 & + & v_3 & + & \cdots & + & v_{n-1} & & & & & = & 0 \\
 & & & & v_3 & + & v_4 & + & \cdots & + & v_n & & & = & 0 \\
v_1 & + & & & & & v_4 & + & \cdots & + & v_n & & & = & 0 \\
\vdots & & \vdots & & & & \vdots & & \vdots & & \vdots & & & & \\
v_1 & + & v_2 & + & \cdots & & + & v_{n-3} & & + & v_n & & & = & 0
\end{array} \tag{10}$$

Starting from the second equation, subtract two consecutive equations in a downward fashion until the last equation to obtain

$$v_2 = v_n,\ v_1 = v_3,\ v_2 = v_4,\ v_3 = v_5,\ v_4 = v_6, \cdots,\ v_{n-4} = v_{n-2},\ v_{n-3} = v_{n-1}\ .$$

That is,

$$v_2 = v_4 = v_6 = v_8 = \cdots = v_{n-3} = v_{n-1} = v_n \text{ and } v_1 = v_3 = v_5 = v_7 = \cdots = v_{n-4} = v_{n-2}\ .$$

Substitute these relations into equations one and two in (10) yields

$$\begin{array}{rcrcl} rv_1 & + & (r-1)v_2 & = & 1, \\ (r-1)v_1 & + & rv_2 & = & 0. \end{array}$$

The solutions to this set of equations are

$$v_1 = \frac{r}{n-2} \quad \text{and} \quad v_2 = \frac{1-r}{n-2}.$$

This gives the first column of C_n^{-1}

$$(v_{1,}v_2,\ldots,v_n)^T = \left(\frac{r}{n-2},\frac{1-r}{n-2},\frac{r}{n-2},\frac{1-r}{n-2},\frac{r}{n-2},\cdots,\frac{1-r}{n-2},\frac{1-r}{n-2}\right)^T.$$

Since C_n^{-1} is circulant, the first row of C_n^{-1} is given by

$$\left(\frac{r}{n-2},\frac{1-r}{n-2},\frac{1-r}{n-2},\frac{r}{n-2},\frac{1-r}{n-2},\frac{r}{n-2},\frac{1-r}{n-2},\ldots,\frac{r}{n-2},\frac{1-r}{n-2}\right).$$

Hence

$$C_n^{-1} = \frac{1}{n-2}\operatorname{circ}\left(r,1-r,1-r,r,1-r,r,1-r,\ldots,r,1-r\right).$$

From the form of C_n^{-1}, the solutions of the system (S') can be computed easily:

$$
\begin{aligned}
x_1 &= \frac{1}{n-2}(r\alpha_1 + (1-r)\alpha_2 + (1-r)\alpha_3 + r\alpha_4 + (1-r)\alpha_5 + \cdots + r\alpha_{n-1} + (1-r)\alpha_n) \\
x_2 &= \frac{1}{n-2}((1-r)\alpha_1 + r\alpha_2 + (1-r)\alpha_3 + (1-r)\alpha_4 + r\alpha_5 + \cdots + (1-r)\alpha_{n-1} + r\alpha_n) \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
x_n &= \frac{1}{n-2}((1-r)\alpha_1 + (1-r)\alpha_2 + r\alpha_3 + (1-r)\alpha_4 + r\alpha_5 + \cdots + (1-r)\alpha_{n-1} + r\alpha_n)
\end{aligned} \tag{11}
$$

Since $x_1, \cdots, x_n > 0$ and $\alpha_1 = 2\angle \mathrm{A}_1, \alpha_2 = 2\angle \mathrm{A}_2, \cdots, \alpha_n = 2\angle \mathrm{A}_n$, we must have

$$
\begin{aligned}
&\angle \mathrm{A}_1 + \angle \mathrm{A}_4 + \angle \mathrm{A}_6 + \cdots + \angle \mathrm{A}_{n-1} > \frac{r-1}{r}\left(\angle \mathrm{A}_2 + \angle \mathrm{A}_3 + \angle \mathrm{A}_5 + \cdots + \angle \mathrm{A}_n\right) \\
&\angle \mathrm{A}_2 + \angle \mathrm{A}_5 + \angle \mathrm{A}_7 + \cdots + \angle \mathrm{A}_n > \frac{r-1}{r}\left(\angle \mathrm{A}_1 + \angle \mathrm{A}_3 + \angle \mathrm{A}_4 + \cdots + \angle \mathrm{A}_{n-1}\right) \\
&\qquad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
&\angle \mathrm{A}_3 + \angle \mathrm{A}_5 + \angle \mathrm{A}_7 + \cdots + \angle \mathrm{A}_n > \frac{r-1}{r}\left(\angle \mathrm{A}_1 + \angle \mathrm{A}_2 + \angle \mathrm{A}_4 + \cdots + \angle \mathrm{A}_{n-1}\right)
\end{aligned} \tag{12}
$$

This means that the sum of any set of non-adjacent interior angles of P_n is strictly greater than $(r-1)/r$ times the sum of the remaining interior angles.

For completeness of argument, instead of using $x_1, \cdots, x_n > 0$ to conclude (12), one can consider the following set of inequalities (see figure 5)

$$
x_1 < 2\pi - \alpha_3, \quad x_2 < 2\pi - \alpha_4, \ \cdots, \quad x_{n-2} < 2\pi - \alpha_n, \quad x_{n-1} < 2\pi - \alpha_1, \quad x_n < 2\pi - \alpha_2.
$$

Combining these inequalities with (11) and using the fact that

$$\angle \mathrm{A}_1 + \angle \mathrm{A}_2 + \angle \mathrm{A}_3 + \cdots + \angle \mathrm{A}_n = (n-2)\pi \,,$$

the same set of inequalities (12) can be computed.

Now let Σ be the sum of a set of non-adjacent interior angles and Σ' be the sum of the set of the corresponding remaining angles, (12) can be re-written as

$$\Sigma > \frac{r-1}{r}\,\Sigma'. \tag{13}$$

Since $\Sigma + \Sigma' = (n-2)\pi$,

$$\begin{aligned} \Sigma &> \frac{r-1}{r}\,\left[(n-2)\pi - \Sigma\right], \\ \Sigma\left(1 + \frac{r-1}{r}\right) &> \frac{r-1}{r}(2r-1)\pi \,, \\ \Sigma &> (r-1)\pi \,. \end{aligned} \tag{14}$$

Substitute r with $(n-1)/2$, one arrives at

$$\Sigma > \frac{n-3}{2}\pi \,.$$

This gives the desired result. Similarly, it can be shown that

$$\Sigma' < \frac{n-1}{2}\pi .$$

The proof is complete.

5. Remarks

This has been a fruitful experience in doing Mathematics. Lopez-Real's investigation gave the inspiration, and indeed the clue, to look for generalization and mathematical rigor. In the process of developing the analyses in this paper, computer dynamic geometry environment and computer software for working with matrices were used to speculate and verify conjectures before formal theorems were stated and proofs were constructed.

School geometry is very rich in content, and it can lead to wonderful mathematical adventures if one has the openness to ask simple and obvious questions.

6. References

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