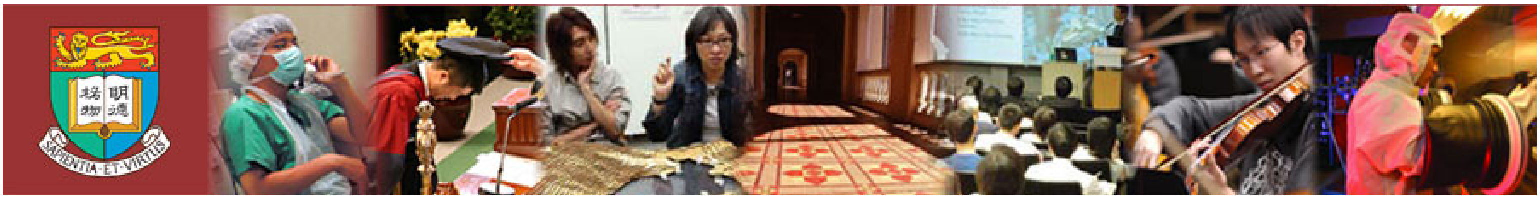

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Pole Assignment with Optimal Performance

Tingshu Hu

Department of Electrical and Computer Engineering
University of Waterloo
Waterloo, Ontario
CANADA N2L 3G1

James Lam †

Department of Mechanical Engineering
University of Hong Kong
Pokfulam Road
HONG KONG

ABSTRACT: A unified gradient-based treatment for optimizing certain performance indices under the constraint of pole assignment is provided. By introducing a free optimization parameter and solving a Sylvester matrix equation, compact gradient formulas are derived for general purpose gradient descent numerical implementation. Special problems including robust stability pole assignment and $\mathcal{H}_2$ sensitivity reduction are employed to illustrate the technique.

Keywords: Pole assignment, robustness, optimality.

1 Introduction

A classical technique in control system design for state-space systems is pole assignment. For a completely state controllable realization with two or more inputs, the feedback gain to achieve a specified set of closed-loop poles is in general nonunique. Such nonuniqueness may be exploited to optimize a variety of system performance indices. The most common application of this idea is robust pole assignment (see [1, 6, 7, 8, 11] and references therein). Relatively speaking, there has been little work on utilizing the freedom in the state feedback gain matrices to optimize other performance criteria, for instance, relating to stability radius and sensitivity reduction. The obvious reason is that pole assignment itself imposes constraints to the feedback systems and inevitably reduces the overall achievable performance. However, in certain robustness maximization problems, specific restrictions are required in order to obtain finite feedback gain solutions. Also, it is often necessary to fix or approximately fix the closed-loop poles due to practical considerations, such as transient characteristics. The tradeoff between pole assignment constraints and optimum performance is justifiable in view of control system implementation since optimal solutions may have undesirable transient behavior or unacceptably large gain. On the other hand, the specification of closed-loop poles may provide significant simplification on the solution procedures to the otherwise unconstrained optimization problems.

Motivated by the aforementioned reasons, this paper considers the optimization of a class of system performance indices under the constraints of pole assignment via state feedback. To fix the idea, the robust stability pole assignment problem and the sensitivity reduction problem will be discussed. The robust stability pole assignment for multivariable systems investigated here is to maximize certain lower bounds of the stability radius and, as such, is a problem not systematically addressed.

2 Performance Optimization with Pole Assignment Constraint

Let $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ be a set of self-conjugate complex numbers corresponding to the set of desired poles. Assume that there are n' complex conjugate pairs, $\lambda_{2i-1}, \lambda_{2i} = \alpha_i \pm \beta_i j$, $i = 1, 2, \ldots, n'$, define

$$\Lambda := \operatorname{diag}\left[\begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \ldots, \begin{bmatrix} \alpha_{n'} & \beta_{n'} \\ -\beta_{n'} & \alpha_{n'} \end{bmatrix}, \lambda_{2n'+1}, \ldots, \lambda_n\right] \tag{1}$$

It is assumed that the eigenvalues of Λ are distinct, then for a given controllable pair (A, B), $A \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times m}$, the problem of pole assignment by state feedback is to choose feedback matrix F, such that

$$V^{-1}(A + BF)V = \Lambda \tag{2}$$

for some nonsingular V. A state feedback matrix F is said to be *admissible* if the pole assignment constraint (2) is satisfied. Without loss of generality, B is assumed to have full column rank $m > 1$ and thus there are infinitely many admissible F.

We use $\operatorname{tr}(M)$, $\|M\|_F$, $\bar{\sigma}(M)$, and $\bar{\lambda}(M)$ to denote the trace, Frobenius norm, maximum singular value, and maximum eigenvalue of M respectively.

2.1 Robust Stability Pole Assignment

Consider the following uncertain system:

$$\dot{x} = (A + \Delta)x + Bu \tag{3}$$

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where $x \in \mathbf{R}^n$ is the state vector, $u \in \mathbf{R}^m$ is the input vector, and Δ is a matrix representing the uncertainties. Under static state feedback $u = Fx$, the closed-loop system is given by

$$\dot{x} = (A + BF + \Delta)x \qquad (4)$$

Suppose that $A + BF$ is stable, then from [10], the closed-loop system (4) is robustly stable if

$$\bar{\sigma}(\Delta) < \frac{1}{\bar{\sigma}(P)} \qquad (5)$$

where $P = P^T > 0$ satisfies

$$(A + BF)^T P + P(A + BF) = -2I$$

On the other hand, Hinrichsen and Pritchard [5] give a frequency domain characterization of the allowable uncertainty for robust stability as follows.

$$\bar{\sigma}(\Delta) < \frac{1}{\|(sI - A - BF)^{-1}\|_\infty} \qquad (6)$$

With $\Re(\cdot)$ denoting the real part, notice that

$$\sup_\omega \bar{\sigma}[(j\omega I - A - BF)^{-1}] \leq \bar{\sigma}(V)\bar{\sigma}(V^{-1}) \max_i |\Re(\lambda_i)|^{-1}$$

and that the eigenvalues of Λ are given *a priori*, these suggest the consideration of a simpler index, $\bar{\sigma}(V)\bar{\sigma}(V^{-1})$, which is the spectral or Euclidean condition number of V. Although V is in general not an eigenvector matrix of $A + BF$, there exists a unitary matrix L such that VL is an eigenvector matrix due to that Λ is real and normal. As a result, $\bar{\sigma}(V)\bar{\sigma}(V^{-1}) = \bar{\sigma}(VL)\bar{\sigma}(L^T V^{-1})$. Notice that the spectral condition number is traditionally used as a measure of eigenvalue robustness when a matrix is under unstructured perturbation (see [9]). Due to a few technical reasons [2, 8], the minimization of the index $\bar{\sigma}^2(V)+\bar{\sigma}^2(V^{-1})$ is preferable and that any minimizer of which serves as a minimizer of $\bar{\sigma}(V)\bar{\sigma}(V^{-1})$. Finally, as in [5], Dickman [4] has also considered the bound in (6) and suggested the minimization of the performance index $\|A + BF\|_F$. Unfortunately, Dickman did not offer any systematic procedures to minimize this index in his paper.

Consequently, robust control system design for unstructured perturbation in $A+BF$ may be considered by minimizing $J_1 := \bar{\sigma}(P)$, $J_2 := \frac{1}{2}\left(\bar{\sigma}^2(V) + \bar{\sigma}^2(V^{-1})\right)$ or $J_3 := \frac{1}{2}\|A + BF\|_F^2$, all under the constraint of (2). Now, we can formulate the Robust Stability Pole Assignment problems as:

Opt 1: $\inf J_1$
s.t. $(A + BF)^T P + P(A + BF) = -2I$
$V^{-1}(A + BF)V = \Lambda$

Opt 2: $\inf J_2$
s.t. $V^{-1}(A + BF)V = \Lambda$

Opt 3: $\inf J_3$
s.t. $V^{-1}(A + BF)V = \Lambda$

2.2 Sensitivity Reduction

Consider the following system with disturbance:

$$\dot{x} = Ax + Bu + Ew, \quad y = x, \quad z = Cx + Du \qquad (7)$$

where $w \in \mathbf{R}^r$ is the exogenous disturbance, $u \in \mathbf{R}^m$ the control, $y \in \mathbf{R}^p$ the measured output and $z \in \mathbf{R}^q$ the controlled output. The objective is to minimize certain induced norm between the controlled output z and the exogenous disturbance w. This problem is sometimes referred to as the (Almost) Disturbance Decoupling Problem. In particular, under feedback $u = Fy \equiv Fx$, one aims to minimize the $\mathcal{H}_2$ norm of the transfer function from w to z, given by $G(s) = (C+DF)(sI-A-BF)^{-1}E$. As (A, B) is controllable, this corresponds to a regular $\mathcal{H}_2$ optimal control with state feedback problem when D has full column rank and the transfer function $C(sI - A)^{-1}B + D$ has no invariant zeros on the imaginary axis. Suppose that $A + BF$ is stable, then the $\mathcal{H}_2$ norm of $G(s)$ is given by $\|G(s)\|_2$ with

$$\|G(s)\|_2^2 = \mathrm{tr}(E^T PE) =: J_4$$

where $P = P^T \geq 0$ satisfies

$$(A+BF)^T P + P(A+BF) = -(C+DF)^T(C+DF) \qquad (8)$$

In the case where the poles of $A + BF$ are not fixed *a priori*, it is possible that F may become unbounded in order to make the $\mathcal{H}_2$ norm small. To regularize the problem, the following $\mathcal{H}_2$ optimization under the constraint of pole assignment is considered.

Opt 4: $\inf J_4$
s.t. $(A + BF)^T P + P(A + BF)$
$= -(C + DF)^T(C + DF)$
$V^{-1}(A + BF)V = \Lambda$

Unlike the $\mathcal{H}_2$ optimal control problem with state feedback, the present sensitivity reduction problem always has solution as long as (A, B) is controllable.

The optimization problems discussed in this section belong to a general class of Pole Assignment Performance Index Optimization Problems:

PAPIOP : $\inf J \quad (J = J(F) \text{ or } J = J(V))$
s.t. $V^{-1}(A + BF)V = \Lambda$

3 Gradient-based Optimization

3.1 Parametric Optimization

In this paper, we will follow the idea of [3] to parametrize all the feedback matrices F and the eigenvector matrices V that satisfy (2) as the function of a free parameter $U \in \mathbf{R}^{m\times n}$. This is achieved by solving a parametric Sylvester equation in U and then recovering the feedback matrix $F = UV^{-1}$. In this way, the performance indices become functions of the free parameter U.

Given a controllable pair (A, B) and a real Λ of the form in (1) such that A and Λ have no common eigenvalues, then a function $f : U \rightarrow (F, V)$ is defined as follows. For $U \in \mathbf{R}^{m\times n}$, solve

$$AV - V\Lambda = -BU \tag{9}$$

for V and if V is nonsingular, let $F = UV^{-1}$. The function is denoted as $(F, V) = f(U)$ with domain,

$$\mathcal{D}_f := \left\{U \in \mathbf{R}^{m\times n} \mid V \text{ in (9) is nonsingular}\right\}$$

and range, $\mathcal{R}_f = f(\mathcal{D}_f)$.

Theorem 1 [1, 6]

(a) $\mathcal{D}_f$ *is a dense open set in* $\mathbf{R}^{m\times n}$.

(b) $\{(F, V) : V^{-1}(A + BF)V = \Lambda\} = \mathcal{R}_f = f(\mathcal{D}_f)$.

Since the performance indices discussed in Section 2 are uniquely determined by F and V, they are functions of the free parameter U. Consequently, they can be expressed as $J_i(U)$ for $i = 1, \ldots, 4$. As $(F, V) = f(U)$ is a rational function and $\mathcal{D}_f$ is an open set, so F and V are differentiable with respect to U for all $U \in \mathcal{D}_f$. Thus $\frac{\partial J_i}{\partial U}$ exists if J_i is differentiable with respect to F or V. It should be noted that J_1 and J_2 may be nondifferentiable at some points when the multiplicity of the largest singular value of P, V or V^{-1} is greater than one. In this case, other indices such as $\text{tr}(P)$, $\|P\|_F$, $\frac{1}{2}\left(\|V\|_F^2 + \|V^{-1}\|_F^2\right)$ may be considered.

3.2 Gradients for Optimization

3.2.1 General Case

A unified approach is taken here to treat all kinds of optimization problems under the constraint of pole assignment. To achieve this task, we assume that a given performance index J is differentiable with respect to the state feedback matrix F or the closed-loop eigenvector matrix V. This distinction is not only more convenient for the purpose of gradient computation of J which is uniquely defined by F (for example, J_1, J_3, and J_4) but is sometimes necessary. This is because there are performance indices which are not differentiable with respect to F (for example, J_2 is not well-defined for any given F). By using the function $f(U)$ defined in Section 3.1, the constraint $V^{-1}(A + BF)V = \Lambda$ may now be replaced with

$$f(U) : AV - V\Lambda = -BU, \quad F = UV^{-1}, \quad U \in \mathcal{D}_f$$

Theorem 2 *Suppose* $U \in \mathcal{D}_f$ *and*

$$AV - V\Lambda = -BU, \quad F = UV^{-1}$$

(a) For $J = J(F)$, *if* $\frac{\partial J}{\partial F}$ *is known, then*

$$\frac{\partial J}{\partial U} = \left(\frac{\partial J}{\partial F}\right) V^{-T} + B^T Y^T \tag{10}$$

where V^{-T} *denotes* $\left(V^{-1}\right)^T$ *and* Y *is the unique solution of*

$$YA - \Lambda Y = V^{-1}\left(\frac{\partial J}{\partial F}\right)^T F \tag{11}$$

(b) For $J = J(V)$, *if* $\frac{\partial J}{\partial V}$ *is known, then*

$$\frac{\partial J}{\partial U} = B^T Y^T \tag{12}$$

where Y *is the unique solution of*

$$YA - \Lambda Y = -\left(\frac{\partial J}{\partial V}\right)^T \tag{13}$$

Proof: Omitted due to page limit. □

Remark 1 *From the result in Theorem 2(a), at a differentiable minimum point of* $J(F(U))$, Y *becomes an eigenvector matrix of* $A + BF$. *In the case of* $J(V(U))$ *in Theorem 2(b),* Y *must be singular and its null space contains the range space of* B. *Since the null space of* Y *is always contained in the unobservable subspace of the pair* $((\partial J/\partial V)^T, A)$, *this subspace would also contain the range space of* B.

3.2.2 Special Cases

In the following, we give the gradient for each index discussed in Section 2. The results are summarized in Table 1. In the table, w is the normalized eigenvector of P corresponding to the eigenvalue $\bar{\sigma}(P)$ (since $P = P^T$). Also, we define $J_a = \frac{1}{2}\bar{\sigma}^2(V)$, $J_b = \frac{1}{2}\bar{\sigma}^2(V^{-1})$, then $J_2 = J_a + J_b$ and $J_2' := J_a J_b$. Furthermore, we have $\bar{\sigma}(V) = \sqrt{\bar{\lambda}(V^T V)}$ and assumed that $\bar{\lambda}\left(V^T V\right)$ is a simple eigenvalue of $V^T V$, with corresponding normalized eigenvector w_a ($w_a^T w_a = 1$) and $\bar{\lambda}((V^{-1})^T V^{-1})$ is a simple eigenvalue of $(V^{-1})^T V^{-1}$, with corresponding normalized eigenvector w_b ($w_b^T w_b = 1$).

J	$\frac{\partial J}{\partial(\cdot)}$	$\frac{\partial J}{\partial U}$	Sylvester Equations
J_1	$\frac{\partial J_1}{\partial F} = 2B^T PX$	$2B^T PXV^{-T} + B^T Y^T$	$(A+BF)X + X(A+BF)^T = -ww^T$ $YA - \Lambda Y = 2V^{-1}XPBF$
J_1'	"	"	$(A+BF)X + X(A+BF)^T = -I$
J_2	$\frac{\partial J_2}{\partial V} = Vw_a w_a^T - V^{-T}V^{-1}w_b w_b^T V^{-T}$	$B^T Y$	$YA - \Lambda Y =$ $-w_a w_a^T V^T + V^{-1}w_b w_b^T V^{-T}V^{-1}$
J_2'	$\frac{\partial J_2'}{\partial V} = J_b V w_a w_a^T - J_a V^{-T}V^{-1}w_b w_b^T V^{-T}$	"	$YA - \Lambda Y =$ $-J_b w_a w_a^T V^T + J_a V^{-1}w_b w_b^T V^{-T}V^{-1}$
J_2''	$\frac{\partial J_2''}{\partial V} = \frac{1}{2}\|V^{-1}\|_F^2 V^T - \frac{1}{2}\|V\|_F^2 V^{-T}V^{-1}V^{-T}$	"	$YA - \Lambda Y =$ $-\frac{1}{2}\|V^{-1}\|_F^2 V^T + \frac{1}{2}\|V\|_F^2 V^{-1}V^{-T}V^{-1}$
J_3	$\frac{\partial J_3}{\partial F} = B^T(A+BF)$	$B^T(A+BF)V^{-T} + B^T Y^T$	$YA - \Lambda Y = V^{-1}(A+BF)^T BF$
J_4	$\frac{\partial J_4}{\partial F} = 2MX$ $M := B^T P + D^T(C+DF)$	$2MXV^{-T} + B^T Y^T$	$(A+BF)X + X(A+BF)^T = -EE^T$ $YA - \Lambda Y = 2V^{-1}XMF$

Table 1. Gradient Formulas of Performance Indices

The gradients are obtained under the assumption that the maximum singular values concerned are simple and this is the generic situation. It may happen that the infimum of a performance index to occur at points where the maximum singular value has multiplicity greater than one. Algorithms may, however, be suitably designed to terminate at these non-differentiable critical points such as by considering the magnitude of difference between the largest two singular values less than certain prescribed value as the stopping criterion. Alternatively, smooth performance indices $J_1' := \mathrm{tr}(P)$, $J_2'' := \frac{1}{4}\|V\|_F^2\|V^{-1}\|_F^2$ may also be used.

4 Numerical Example

The matrices A, B below represents a nominal distillation model [7, 8] with 5 states and 2 inputs.

$$A = \begin{bmatrix} -0.1094 & 0.0628 & 0 & 0 & 0 \\ 1.3060 & -2.1320 & 0.9807 & 0 & 0 \\ 0 & 1.5950 & -3.1490 & 1.5470 & 0 \\ 0 & 0.0355 & 2.6320 & -4.2570 & 1.8550 \\ 0 & 0.00227 & 0 & 0.1636 & -0.1625 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0.0638 & 0 \\ 0.0838 & -0.1396 \\ 0.1004 & -0.2060 \\ 0.0063 & -0.0128 \end{bmatrix}$$

The open-loop poles are at -0.077324, -0.014232, -0.89531, -2.8408 and -5.9822.

Robust Stability Pole Assignment:

The desired closed-loop poles are $-1 \pm j, -0.2, -0.5$ and -1. The performance indices J_1, J_1', J_2, J_2', J_2'', and J_3 are minimized and the results are compared on the size of the implied robust stability bound. To facilitate comparison, we define

$$\gamma_1 := \frac{1}{\bar{\sigma}[P(F^*)]} \quad \text{and} \quad \gamma_2 := \frac{1}{\|(sI - A - BF^*)^{-1}\|_\infty}$$

where F^* denotes the optimal solution of one of the above performance optimization problems. Table 2 summarizes the results for comparison (figures in brackets correspond to the maximum values over the whole iteration processes).

From Table 2, it can be seen that by minimizing J_1 the obtained γ_1 and γ_2 are the largest. In fact, the minimizer of the smooth J_1' appears to be a good compromise in the case if a smooth performance index is insisted. The performance indices J_2, J_2' and J_2'', which all relate to the condition numbers of the closed-loop eigenvector matrix, do not give good perturbation bounds at their minimized values. The performance index J_3 does not reflect well on the allowable size of the norm-bounded perturbation Δ. Finally, the use of gradient formulas for the nonsmooth indices J_1 and J_2' derived based on the assumption of distinct eigenvalues does not present any practical difficulties in the optimization process.

Sensitivity Reduction:

The same model is now subjected to disturbance described by (7) where C, D and E are as follows

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = 0_{3\times 2}, \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

With the same prescribed set of closed-loop poles, it is found that a minimizing solution is given by

$$F_{J_4}^* = \begin{bmatrix} -51.6668 & 135.5339 & -296.9035 & 240.2631 & -30.1735 \\ -25.0055 & 62.3744 & -124.4031 & 92.0061 & 0.1746 \end{bmatrix}$$

with $J_4(F_{J_4}^*) = 4.5571$, and $\|F_{J_4}^*\|_F = 443.0413$. If the pole assignment requirement is lifted, the sensitivity reduction problem with $A + BF$ stable corresponds to a singular optimal control problem since

Performance Index	Minimized Value	γ_1	γ_2	$\|F\|_2$	$\|F\|_F$
$J_1 = \bar{\sigma}(P)$	36.07	0.0277 (0.0277)	0.1111 (0.1128)	255.71	256.61
$J_1' = \text{tr}(P)$	73.24	0.0225 (0.0225)	0.1077 (0.1100)	268.16	268.89
$J_2 = \frac{1}{2}\left(\bar{\sigma}^2(V) + \bar{\sigma}^2(V^{-1})\right)$	31.70	0.0047 (0.0063)	0.0680 (0.0768)	283.82	285.35
$J_2' = \frac{1}{4}\bar{\sigma}^2(V)\bar{\sigma}^2(V^{-1})$	248.06	0.0044 (0.0066)	0.0565 (0.0776)	286.94	289.22
$J_2'' = \frac{1}{4}\|V\|_F^2\|V^{-1}\|_F^2$	382.32	0.0052 (0.0074)	0.0701 (0.0800)	354.85	355.19
$J_3 = \frac{1}{2}\|A+BF\|_F^2$	111.34	0.0014 (0.0026)	0.0304 (0.0349)	168.98	174.89

Table 2. Comparison of Performance Indices

$D = 0$. From (8) and Table 1, a state-feedback gain matrix F which is an optimal solution must satisfy

$$\begin{aligned} (A+BF)^T P + P(A+BF) &= -C^T C \\ (A+BF)X + X(A+BF)^T &= -EE^T \\ XPB &= 0 \end{aligned}$$

These conditions may not be easily satisfied and a typical way to provide suboptimal solutions for singular problems is to regularize them via perturbation. Consider $D_\epsilon = \epsilon \begin{bmatrix} I_2 & 0_{2\times 1} \end{bmatrix}^T$ with $\epsilon > 0$ being a small number. The sequence ϵ_k, $k = 1, 2, \ldots$ gives a sequence of suboptimal state feedback controllers. When $\epsilon = 10^{-4}$, the *optimal* $\mathcal{H}_2$ sensitivity state feedback is $F^*_{\epsilon=10^{-4}}$ given by

$$\begin{bmatrix} 2.8369 & 4.7321 & -6786.6279 & -7181.5156 & -2755.7949 \\ -5.8810 & -10.5824 & -7320.0206 & 6412.6708 & 5133.3939 \end{bmatrix}$$

and the optimal $\mathcal{H}_2$ sensitivity equals 1.0967. The closed-loop system poles are -1450.6269, -221.5889, -8.6094, -0.099793, and -0.5308. It can be seen that there is a pole very near to the origin which might be undesirable due to its associated slow transient characteristics. The state feedback gain matrix has $\|F^*_{\epsilon=10^{-4}}\|_F = 1.5043 \times 10^4$. Over the range $0.0001 \leq \epsilon \leq 1$, the magnitude of the optimal state feedback matrix is $O(\epsilon^{-1})$. Finally, it should be realized that the set of closed-loop poles may be viewed as a set of design parameters not only to characterize the dynamics of the closed-loop system but also to control the gain of the optimized feedback.

5 Conclusion

Robust designs for state-space systems based on performance index optimization with state feedback pole assignment constraints are considered. By exploiting the extra degrees of freedom beyond pole assignment, a unified gradient-based treatment is offered through the introduction of a free parameter in the optimization process. The robust stability pole assignment and $\mathcal{H}_2$ sensitivity reduction are used to illustrate the technique.

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