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# A Gradient Flow Approach to Robust Pole Assignment in Second-Order Systems

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## Abstract

Robust pole assignment for second-order systems is considered. It is shown that assignment can be achieved by solving a linear matrix equation or linear system. An objective function measuring the robustness of the closed-loop spectrum is minimized via gradient flow.

## 1 Introduction

Consider a time-invariant, second-order system

$$M\ddot{q} + D\dot{q} + Kq = f \quad (1)$$

where  $q, f \in \mathbb{R}^n$  and  $M, D, K \in \mathbb{R}^{n \times n}$ . Responses of (1) can be altered by applying a feedback control force  $f = Bu$  to (1) with  $B \in \mathbb{R}^{n \times m}$  denoting the input matrix and  $u \in \mathbb{R}^m$  the control vector. The state feedback control law is defined as  $u = -F_K q - F_D \dot{q}$  where  $F_D$  and  $F_K$  are derivative and proportional feedback matrices respectively. Hence, the resulting closed-loop system is

$$M\ddot{q} + (D + BF_D)\dot{q} + (K + BF_K)q = 0. \quad (2)$$

In this note,  $M$  is assumed to be invertible and (2) can be written in the familiar first-order form

$$\dot{z} = \bar{A}z + \bar{B}u$$

where

$$\bar{A} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ M^{-1}B \end{bmatrix}.$$

For complete assignability, the pair  $(\bar{A}, \bar{B})$  is assumed to be completely controllable.

## 2 Pole Assignment

Let  $\Lambda$  be a real pseudo-diagonal matrix containing all the assigned poles, in which  $1 \times 1$  blocks represent real poles and  $2 \times 2$  blocks of the form

$$\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \quad (3)$$

represent complex conjugate pairs  $\sigma \pm i\omega$ . Note that (3) is unitary equivalent to a diagonal matrix, that is

$$U^* \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} U = \begin{bmatrix} \sigma + i\omega & 0 \\ 0 & \sigma - i\omega \end{bmatrix}$$

where  $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$  is a unitary matrix.

The idea of state feedback pole assignment is to find  $[F_K \ F_D]$  such that

$$\bar{A} + \bar{B} [F_K \ F_D] = T\Lambda T^{-1} \text{ or } \bar{A} + \bar{B}G = T\Lambda \quad (4)$$

where  $G = [F_K \ F_D]T$  and  $T$  is some nonsingular matrix. In fact, (4) corresponds to the following linear matrix equation

$$\tilde{A}T\tilde{B} - \tilde{C}T\tilde{D} = \tilde{E} \quad (5)$$

where

$$\tilde{A} = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}, \quad \tilde{B} = I, \quad \tilde{C} = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix},$$

$$\tilde{D} = \Lambda, \quad \tilde{E} = \begin{bmatrix} 0 \\ B \end{bmatrix} [F_K \ F_D]T = \begin{bmatrix} 0 \\ B \end{bmatrix} G.$$

Since the pencils  $\tilde{A} - \lambda\tilde{B}$  and  $\tilde{D} - \lambda\tilde{B}$  are regular and the spectra  $\rho(\tilde{A}, \tilde{C})$  and  $\rho(\tilde{D}, \tilde{B})$  are assumed to be disjoint, i.e. no common closed-loop and open-loop poles, (5) is uniquely solvable for a given  $G$  [1, 3]. However, by exploiting the special structures

of the coefficient matrices  $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}$ , we can express (5) into linear systems of equations which can then be solved by some effective algorithms [4]. Define  $\hat{U} = \text{diag} \left\{ \hat{U}_1, \dots, \hat{U}_{n+\frac{p}{2}} \right\}$  where  $p$ , an even number, is the number of real poles of (2) and

$$\begin{cases} \hat{U}_j = 1, & j = 1, \dots, p, \\ \hat{U}_j = U, & j = p+1, \dots, n + \frac{p}{2}. \end{cases}$$

Note that  $\hat{U} = U \otimes I_n$  if  $p = 0$  and  $\hat{U} = I_{2n}$  if  $p = 2n$ . By comparing both sides of (5), we eventually obtain the linear system

$$(\lambda_j^2 M + \lambda_j D + K) w_j = -B \hat{g}_j, \quad j = 1, \dots, 2n. \quad (6)$$

where  $\hat{g}$  is the  $j$ -th column of  $\hat{G} = G\hat{U}$ . Then for a given  $G$ , we can solve for  $W = [w_1, \dots, w_{2n}]$  from (6) and  $T = \begin{bmatrix} W\hat{U}^* \\ W\hat{U}^*\Lambda \end{bmatrix}$ . Consequently,

$$[F_K \quad F_D] = GT^{-1} \quad (7)$$

which is the augmented derivative and proportional feedback matrix such that the closed-loop system (2) have the assigned spectrum.

### 3 Robust Pole Assignment

In measuring the robustness of the closed-loop spectrum, an objective function is defined as

$$\phi(T(G)) = \|T(G)\|_F^2 + \|T(G)^{-1}\|_F^2. \quad (8)$$

It can be shown that [5] the objective function (8) has to be minimized in order to obtain a robust closed-loop system. For notational simplicity, we write  $\phi(T(G))$  as  $\phi(G)$  throughout.

For a given  $G, T$  is solved via the linear matrix equation (5) or linear system (6) at each minimization iteration for  $\phi(G)$ . When an optimum solution, say  $G^*$ , is reached such that  $\phi(G^*)$  attains its minimum, the required augmented feedback matrix  $F = [F_K \quad F_D] = G^*T^{*-1}$  is recovered from (7), where  $G^*$  and  $T^*$  satisfy (4).

With gradient flow analysis, this minimization problem comes down to solving the following system of ordinary differential equations

$$\begin{aligned} \dot{G}(t) &= 2 \left[ \text{trace} \left\{ \frac{\partial T}{\partial g_{jk}} (T^{-1}T^{-T}T^{-1} - T^T) \right\} \right]_{m \times 2n} \\ G(0) &= G_0 \in \mathcal{G}. \end{aligned} \quad (9)$$

where  $\mathcal{G} = \{G \mid T \text{ is a nonsingular solution of (4)}\}$  is open and dense in  $\mathbb{R}^{m \times 2n}$ . The solution, say  $G^*$ , to (9) will be a minimum of  $\phi(G)$  in (8). Important issues on

the existence and convergence of the solution to (9) on  $[0, \infty)$  are discussed in [5].

In the ODE (12),  $T = S\hat{U}^*$  where

$$s_i = - \begin{bmatrix} I \\ \lambda_i I \end{bmatrix} C_i^{-1} B \hat{g}_i$$

with  $C_i = \lambda_i^2 M + \lambda_i D + K$ . And also  $\frac{\partial T}{\partial g_{jk}} = \frac{\partial S}{\partial g_{jk}} \hat{U}^*$  where

$$\frac{\partial s_i}{\partial g_{jk}} = \begin{cases} -\delta_{ik} \begin{bmatrix} I \\ \lambda_i I \end{bmatrix} C_i^{-1} b_j, & k = 1, \dots, p, \\ -\frac{\delta_{ik} + \delta_{i(k+1)}}{\sqrt{2}} \begin{bmatrix} I \\ \lambda_i I \end{bmatrix} C_i^{-1} b_j, & k = p+1, p+3, \dots, 2n-1, \\ -i \frac{\delta_{i(k-1)} - \delta_{ik}}{\sqrt{2}} \begin{bmatrix} I \\ \lambda_i I \end{bmatrix} C_i^{-1} b_j, & k = p+2, p+4, \dots, 2n \end{cases}$$

where  $\delta_{jk}$  the Kronecker delta. It is important to realize that  $\frac{\partial T}{\partial g_{jk}}$  is a constant matrix and is only required to evaluate once in the computation process.

### 4 Conclusion

In this note, the problem of robust pole assignment for second-order systems by state feedback is examined. It has been shown that pole assignment can be achieved by either solving a linear matrix equation or linear system. The assigned spectrum is made optimally robust by minimizing an objective function via gradient flow which involves the solution of a system of ordinary differential equation.

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