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Robust Control of Uncertain Nonlinear Systems

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Abstract

This paper considers robust H_∞ control problems via state feedback and measurement feedback for a class of affine nonlinear systems with gain bounded uncertainties, and the solutions of the problems are derived in terms of smooth solutions of Hamilton-Jacobi inequalities. It is also shown that the results are extensions of the corresponding results for linear systems.

1 Introduction

In recent years, the H_∞ control problem for nonlinear systems has been extensively investigated by several authors [1,3-5,10,11]. In particular, in [11] it was shown that the solution of the H_∞ control problem via state feedback can be determined from the solution of a Hamilton-Jacobi equation (or inequality), which is the nonlinear version of the Riccati equation considered in [2] for the corresponding H_∞ control problem for linear systems. Sufficient conditions for the solution of the problem in the case of measurement feedback have been given in a series of papers [1,3,4], and [1] and [5] discuss the necessity of these sufficient conditions.

This paper consider the robust H_∞ control problem affine nonlinear systems with gain bounded uncertainty. Specifically, we consider an uncertain nonlinear system modeled by equations of the form

$$
\begin{aligned}
\dot{x} &= f(x) + \triangle f(x) + g_1(x)w + (g_2(x) + \triangle g_2(x))u \\
z &= h_1(x) + k_{12}(x)u \\
y &= h_2(x) + \triangle h_2(x) + k_{21}(x)w
\end{aligned}
\tag{1}
$$

where x is a state vector defined on a neighbourhood X of the origin in R^n, $u \in R^m$ denotes the control input, $w \in R^r$ the disturbance input, $z \in R^s$ the penalty variable and $y \in R^p$ the measured output. Functions $f(x)$, $g_1(x)$, $g_2(x)$, $h_1(x)$, $h_2(x)$, $k_{12}(x)$ and $k_{21}(x)$ are known smooth (i.e. C^∞) mappings defined in a neighbourhood of the origin in R^n with $f(0) = 0$, $h_1(0) = 0$ and $h_2(0) = 0$. The smooth uncertain mappings satisfy

$\triangle f(x) \in \Omega_f$, $\triangle g_2(x) \in \Omega_g$ and $\triangle h_2(x) \in \Omega_h$ with

$$
\Omega_f \triangleq \{f_{11}(x)\delta_f(x) : \delta_f^T(x)\delta_f(x) \le f_{22}^T(x)f_{22}(x), \forall x \in X\}
\tag{2}
$$

$$
\Omega_g \triangleq \{g_{11}(x)\delta_g(x) : \delta_g^T(x)\delta_g(x) \le g_{22}^T(x)g_{22}(x), \forall x \in X\}
\tag{3}
$$

$$
\Omega_h \triangleq \{h_{11}(x)\delta_h(x) : \delta_h^T(x)\delta_h(x) \le h_{22}^T(x)h_{22}(x), \forall x \in X\}
\tag{4}
$$

where $f_{11}(x)$, $g_{11}(x)$ and $h_{11}(x)$ are known smooth mappings, the smooth mappings $f_{22}(x)$, $g_{22}(x)$ and $h_{22}(x)$ are given weighting mappings with $f_{22}(0) = 0$ and $h_{22}(0) = 0$.

Definition 1.1 ([9]) : *Consider an uncertain nonlinear system Σ given by*

$$
\begin{aligned}
\dot{x} &= f(x) + \triangle f(x) + g(x)u \\
y &= h(x)
\end{aligned}
\tag{5}
$$

with $x \in R^n$, $u \in R^m$, $y \in R^p$, and $\triangle f(x) \in \Omega_f$. Let $\gamma > 0$ be given. The system Σ is said to have locally robust disturbance attenuation performance γ in $U \subset X$ (U is a neighbourhood of the origin in R^n) if for any $\triangle f(x) \in \Omega_f$, the free system with $u = 0$ is locally asymptotically stable with domain of attraction containing U, and the system Σ is with an L_2 gain less than or equal to γ, i.e.,

$$
\int_0^T \|y(t)\|^2 dt \le \gamma^2 \int_0^T \|u(t)\|^2 dt
$$

for every $T > 0$ and $u \in D$ with D being defined as

$$
D = \{u : u \in L_2(0,T)| \text{ if } x(0) = 0, x(t) \in U \text{ for } t \le T\}
$$

where $L_2(0,T)$ denotes the set of vector-valued function $u(t)$ satisfying $\int_0^T \|u(t)\|^2 dt < \infty$.

In this paper, the following two problems for the system (1) will be addressed.

Robust disturbance attenuation problem via state feedback: Given the system (1), find a state

feedback controller

$$u = \alpha(x), \qquad \alpha(0) = 0 \qquad (6)$$

such that the resulting closed-loop system has local robust disturbance attenuation performance γ.

Robust disturbance attenuation problem via measurement feedback: Given the system (1), find a controller with the following form

$$\begin{aligned} \dot{\xi} &= \eta(\xi, y) \\ u &= \theta(x) \end{aligned} \qquad (7)$$

such that the resulting closed-loop system has local robust disturbance attenuation performance γ, where ξ is defined on a neighbourhood V of the origin in R^v, and $\eta : V \times R^p \to R^v$, and $\theta :\to R^m$ are smooth functions with $\eta(0,0) = 0$ and $\theta(0) = 0$.

It should be noted that nonlinear models of the form (1) can be used to represent many real physical systems [6,7,12]. The uncertainty structure in equations (2)-(4) has been used in [9], and it was shown that the solution of the problem of robust disturbance attenuation via state feedback for the system of the form (1) with $\triangle g_2(x) = 0$ and $\triangle h_2(x) = 0$ is related to the existence of solution of a Hamilton-Jacobi inequality. The purpose of the present paper is to extend the result in [9] to the more general class of systems given by (1).

The following assumptions will be used in the sequel.

Assumption A1: The matrix $g_{22}^T(x)g_{22}(x) + k_{12}^T(x)k_{12}(x)$ is nonsingular for each $x \in X$.

Assumption A2: For each $\triangle f(x) \in \Omega_f$, $\triangle g_2(x) \in \Omega_g$, any bounded trajectory $x(t)$ of the system

$$\dot{x}(t) = f(x(t)) + \triangle f(x(t)) + [g_2(x(t)) + \triangle g_2(x(t))]u(t)$$

satisfying

$$h_1(x(t)) + k_{12}(x(t))u(t) = 0$$

for all $t \geq 0$, is such that $\lim_{t\to\infty} x(t) = 0$.

Assumption A3: The matrix $h_{11}(x)h_{11}^T(x) + k_{21}(x)k_{21}^T(x)$ is nonsingular for each $x \in X$.

2 Robust disturbance attenuation via state feedback

The following results present sufficient conditions for the solvability of the problem of robust disturbance attenuation via state feedback for the system (1).

Theorem 2.1 *Consider the system (1) and suppose Assumptions A1 and A2 hold. Suppose there exist*

scalar smooth functions $\lambda_1(x) > 0$ and $\lambda_2(x) > 0$ such that the Hamilton-Jacobi inequality

$$H_s(x, V_x^T, \lambda_1(x), \lambda_2(x))$$

$$\triangleq V_x \widetilde{f}(x) + \widetilde{h}_1^T(x)\widetilde{h}_1(x) + \frac{1}{4}V_x\widetilde{R}(x)V_x^T \leq 0 \qquad (8)$$

has a C^1 positive definite solution $V(x)$, which is defined locally defined in a neighbourhood of $x = 0$ and vanishing at $x = 0$, where $V_x = \frac{\partial V}{\partial x}$ is the Jacobian matrix of V, and

$$\widetilde{f}(x) = f(x) - g_2(x)R_2^{-1}(x)k_{12}^T(x)h_1(x) \qquad (9)$$

$$\widetilde{h}_1(x) = \begin{bmatrix} \frac{1}{\sqrt{\lambda_1(x)}}f_{22}(x) \\ \frac{1}{\sqrt{\lambda_2(x)}}g_{22}(x)R_2^{-1}(x)k_{12}^T(x)h_1(x) \\ h_1(x) - k_{12}(x)R_2^{-1}(x)k_{12}^T(x)h_1(x) \end{bmatrix} \qquad (10)$$

$$\widetilde{R}(x) = \frac{1}{\gamma^2}g_1(x)g_1^T(x) + \lambda_1(x)f_{11}(x)f_{11}^T(x)$$

$$+ \lambda_2(x)g_{11}(x)g_{11}^T(x) - g_2(x)R_2^{-1}(x)g_2^T(x) \qquad (11)$$

with

$$R_2(x) = \frac{1}{\lambda_2(x)}g_{22}^T(x)g_{22}(x) + k_{12}^T(x)k_{12}(x) \qquad (12)$$

Then, a solution of the robust disturbance attenuation problem via state feedback is given by

$$u(x) = \alpha(x) = -R_2^{-1}(x)\left(\frac{1}{2}g_2^T(x)V_x^T + k_{12}^T(x)h_1(x)\right) \qquad (13)$$

Theorem 2.2 *Consider the system (1) and suppose hypothesis A1 holds. Suppose there exist scalar smooth functions $\lambda_1(x) > 0$ and $\lambda_2(x) > 0$ such that the strict Hamilton-Jacobi inequality*

$$H_s(x, V_x^T, \lambda_1(x), \lambda_2(x)) < 0 \text{ for } x \neq 0 \qquad (14)$$

has a C^1 positive definite solution $V(x)$, which is defined locally defined in a neighbourhood of $x = 0$ and vanishing at $x = 0$. Then, a solution of the robust disturbance attenuation problem via state feedback is given by (13).

Proof of Theorem 2.1. For any $\triangle f(x) \in \Omega_f$, $\triangle g_2(x) \in \Omega_g$, $\lambda_1(x) > 0$, and $\lambda_2(x) > 0$, from (2) and (3), we have

$$V_x(f(x) + \triangle f(x) + g_1(x)w + (g_2(x) + \triangle g_2(x))u)$$

$$+ \|h_1(x) + k_{12}(x)u\|^2 - \gamma^2\|w\|^2$$

$$= V_xf(x) + V_xg_1(x)w + V_xg_2(x)u + \|h_1(x) + k_{12}(x)u\|^2$$

$$- \gamma^2\|w\|^2 + V_xf_{11}(x)\delta_f(x) + V_xg_{11}(x)\delta_g(x)u$$

$$\leq V_xf(x) + V_xg_1(x)w + V_xg_2(x)u + \|h_1(x) + k_{12}(x)u\|^2$$

$$-\gamma^2\|w\|^2+\frac{1}{4}V_x(\lambda_1(x)f_{11}(x)f_{11}^T(x)+\lambda_2(x)g_{11}(x)g_{11}^T(x))V_x^T$$

$$+\frac{1}{\lambda_1(x)}f_{22}^T(x)f_{22}(x)+\frac{1}{\lambda_2(x)}u^Tg_{22}^T(x)g_{22}(x)u$$

$$V_xf(x)+\frac{1}{4}V_x(\frac{1}{\gamma^2}g_1(x)g_1^T(x)+\lambda_1(x)f_{11}(x)f_{11}^T(x)$$

$$+\lambda_2(x)g_{11}(x)g_{11}^T(x))V_x^T-\gamma^2\left\|w-\frac{1}{2\gamma^2}g_1^T(x)V_x^T\right\|^2$$

$$+\frac{1}{\lambda_1(x)}f_{22}^T(x)f_{22}(x)+h_1^T(x)h_1(x)$$

$$+u^T\left(\frac{1}{\lambda_2(x)}g_{22}^T(x)g_{22}(x)+k_{12}^T(x)k_{12}(x)\right)u$$

$$+V_xg_2(x)u+2u^Tk_{12}^T(x)h_1(x)$$

By Assumption A1, we know that the matrix $R_2(x)$ is positive definite for $\lambda_2(x) > 0$, which further implies from (8)-(13) that

$$V_x[f(x)+\triangle f(x)+g_1(x)w+(g_2(x)+\triangle g_2(x))u]$$

$$+\|h_1(x)+k_{12}(x)u\|^2-\gamma^2\|w\|^2$$

$$\leq V_xf(x)+h_1^T(x)h_1(x)+\frac{1}{\lambda_1(x)}f_{22}^T(x)f_{22}(x)$$

$$+\frac{1}{4}V_x(\frac{1}{\gamma^2}g_1(x)g_1^T(x)+\lambda_1(x)f_{11}(x)f_{11}^T(x)$$

$$+\lambda_2(x)g_{11}(x)g_{11}^T(x))V_x^T-\gamma^2\left\|w-\frac{1}{2\gamma^2}g_1^T(x)V_x^T\right\|^2$$

$$-\alpha^T(x)R_2(x)\alpha(x)+(u-\alpha(x))^TR_2(x)(u-\alpha(x))$$

$$=V_x(f(x)-g_2(x)R_2^{-1}(x)k_{12}^T(x)h_1(x))+h_1^T(x)h_1(x)$$

$$+\frac{1}{\lambda_1(x)}f_{22}^T(x)f_{22}(x)-h_1^T(x)k_{12}(x)R_2^{-1}(x)k_{12}(x)h_1(x)$$

$$+\frac{1}{4}V_x\left(\frac{1}{\gamma^2}g_1(x)g_1^T(x)+\lambda_1(x)f_{11}(x)f_{11}^T(x)\right.$$

$$\left.+\lambda_2(x)g_{11}(x)g_{11}^T(x)-g_2^T(x)R_2^{-1}(x)g_2(x)\right)V_x^T$$

$$+(u-\alpha(x))^TR_2(x)(u-\alpha(x))-\gamma^2\left\|w-\frac{1}{2\gamma^2}g_1^T(x)V_x^T\right\|^2$$

$$=H_s(x,V_x^T,\lambda_1(x),\lambda_2(x))+(u-\alpha(x))^TR_2(x)(u-\alpha(x))$$

$$-\gamma^2\left\|w-\frac{1}{2\gamma^2}g_1^T(x)V_x^T\right\|^2\qquad(15)$$

By inequality (8) and $u=\alpha(x)$, we have

$$V_x(f(x)+\triangle f(x)+(g_2(x)+\triangle g_2(x))\alpha(x))+V_xg_1(x)w$$

$$\leq\gamma^2\|w\|^2-\|z\|^2$$

which implies from Assumption A2 and the results in [4] that the closed-loop system

$$\dot{x}\ =\ f(x)+\triangle f(x)+(g_2(x)+\triangle g_2(x))\alpha(x)+g_1(x)w$$
$$z\ =\ h_1(x)+k_{12}(x)\alpha(x)$$

is with an L_2 gain which is less than or equal to γ, and asymptotically stable for any $\triangle f(x)\in\Omega_f$ and $\triangle g_2(x)\in\Omega_g$. ◁

Proof of Theorem 2.2. By inequalities (14) and (15), we have

$$V_x(f(x)+\triangle f(x)+(g_2(x)+\triangle g_2(x))\alpha(x))+V_xg_1(x)w$$

$$<\gamma^2\|w\|^2-\|z\|^2\text{ for }x\neq 0$$

Thus, the conclusion of Theorem 2.2 is immediate from the results in [4]. ◁

Remark 2.1 When $\triangle g_2(x)=0$, Theorem 2.1 coincides with Theorem 2 in [9].

By Propositions 2.1 and 2.2 in [4], it is easy to see that the state feedback control law given by Theorem 2.1 or Theorem 2.2 also is a solution for the problem of local disturbance attenuation with internal stability for the following extended system Σ_s:

$$\dot{x}\ =\ f(x)+g_2(x)u$$
$$+\ [\ \gamma\sqrt{\lambda_1(x)}f_{11}(x)\quad\gamma\sqrt{\lambda_2(x)}g_{11}(x)\quad g_1(x)\ ]\bar{w}$$
$$\bar{z}\ =\ \begin{bmatrix}\frac{1}{\sqrt{\lambda_1(x)}}f_{22}(x)\\0\\h_1(x)\end{bmatrix}+\begin{bmatrix}0\\\frac{1}{\sqrt{\lambda_2(x)}}g_{22}(x)\\k_{12}(x)\end{bmatrix}\qquad(16)$$

Next section will discuss the relation between the uncertain system (1) and the extended system Σ_s without uncertainty. In particular, the controller given by Theorem 2.2 guarantees that the resulting closed-loop system is uniformly asymptotically stable, which corresponds to the quadratic stability for an uncertain linear system if $V(x)=x^TPx$ for some positive definite matrix P.

In linear case, consider the uncertain linear system

$$\dot{x}\ =\ (A+D_1F_1(t)E_1)x+B_1w+(B_2+D_2F_2(t)E_2)u$$
$$z\ =\ C_1x+D_{12}u\qquad(17)$$

where the uncertainties $F_1(t)$ and $F_2(t)$ satisfy $F_1^T(t)F_1(t)\leq I$ and $F_2^T(t)F_2(t)\leq I$.

Then, for the quadratic stabilizability with an H_∞-norm bound γ ([13]) of the uncertain system (17), we have the following corollary.

Corollary 2.1 *Suppose that the matrix $E_2^TE_2+D_{12}^TD_{12}$ is positive definite. Then the uncertain system (17) is quadratically stabilizable with an H_∞-norm bound $\gamma>0$ if there exist constants $\epsilon_1>0$ and $\epsilon_2>0$ such that the following Riccati inequality*

$$H_{ls}(Q,\epsilon_1,\epsilon_2)=\bar{A}^TQ+Q\bar{A}+\bar{C}_1^T\bar{C}_1+Q\bar{R}Q<0\qquad(18)$$

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has a positive definite solution Q, where

$$\bar{A} = A - B_2 R_2^{-1}(\epsilon_2) D_{12}^T C_1$$

$$\bar{C}_1 = \begin{bmatrix} \frac{1}{\sqrt{\epsilon_1}} E_1 \\ \frac{1}{\sqrt{\epsilon_2}} E_2 R_2^{-1}(\epsilon_2) D_{12}^T C_1 \\ C_1 - D_{12} R_2^{-1}(\epsilon_2) D_{12}^T C_1 \end{bmatrix}$$

$$\bar{R} = \frac{1}{\gamma^2} B_1 B_1^T + \epsilon_1 D_1 D_1^T + \epsilon_2 D_2 D_2^T - B_2 R_2^{-1}(\epsilon_2) B_2^T$$

with

$$R_2(\epsilon_2) = \frac{1}{\epsilon_2} E_2^T E_2 + D_{12}^T D_{12}$$

Moreover, a suitable feedback control law is given by

$$u = -R_2^{-1}(\epsilon_2)(B_2^T Q + D_{12} C_1)$$

Proof. It is immediate from Theorem 2.2. ◁

Remark 2.2 It should be noted that the Riccati inequality (18) involves the choices of two parameters ϵ_1 and ϵ_2. When choosing $\epsilon = \epsilon_1 = \epsilon_2$, from Theorem 3.1 in [13], the condition under which the Riccati inequality

$$H_{ls}(Q, \epsilon, \epsilon) < 0 \tag{19}$$

has a positive definite solution Q is sufficient and necessary for the quadratic stabilizability with an H_∞-norm bound γ of the following system

$$\begin{aligned} \dot{x} &= (A + \triangle A(t))x + B_1 w + (B_2 + \triangle B_2(t))u \\ z &= C_1 x + D_{12} u \end{aligned} \tag{20}$$

where

$$[\triangle A(t) \quad \triangle B_2(t)] = [D_1 \quad D_2] F(t) \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}$$

with $F^T(t)F(t) \leq I$. By the notion of overbounding in [8], the system (20) is an overbounding system of the system (17), so the condition given by (19) only is a sufficient condition for the quadratic stabilizability with an H_∞-norm bound γ of the system (17), and applying it to the system (17) may bring about some conservativeness. The condition given by (18) may be less conservative, but the necessity of the condition is a problem to be investigated.

3 Robust disturbance attenuation via measurement feedback

If the state x of the plant is not available for measurement, then consider a controller K for the system (1) of the following form

$$\begin{aligned} \dot{\xi} &= a(\xi) + b(\xi)y \\ u &= c(\xi) \end{aligned} \tag{21}$$

such that the resulting closed-loop system has locally robust disturbance attenuation performance, where $\xi \in R^v$.

Consider an auxiliary system Σ_0 given by

$$\begin{aligned} \dot{x} &= f(x) + g_2(x)u \left[\gamma\sqrt{\lambda_1(x)} f_{11}(x), \right. \\ &\quad + \left. \gamma\sqrt{\lambda_2(x)} g_{11}(x), \ 0, \ g_1(x) \right] \hat{w} \\ &= f(x) + g_{10}(x)\hat{w} + g_2(x)u \\ \hat{z} &= \begin{bmatrix} \frac{1}{\sqrt{\lambda_1(x)}} f_{22}(x) \\ 0 \\ \frac{1}{\sqrt{\lambda_3(x)}} h_{22}(x) \\ h_1(x) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\sqrt{\lambda_2(x)}} g_{22}(x) \\ 0 \\ k_{12}(x) \end{bmatrix} u \\ &= h_{10} + k_{120}(x)u \end{aligned} \tag{22}$$

$$\begin{aligned} y &= h_2(x) + \begin{bmatrix} 0 & 0 & \gamma\sqrt{\lambda_3(x)} h_{11}(x) & k_{21}(x) \end{bmatrix} \hat{w} \\ &= h_2(x) + k_{210}(x)\hat{w} \end{aligned}$$

Applying the controller K to the system (1) and the system Σ_0, respectively, we obtain the following two closed-loop systems.

$$\Sigma_{c1} : \begin{cases} \dot{x}_e = f_e(x_e) + \triangle f_e(x_e) + g_e(x_e)w \\ z = h_1(x) + k_{12}(x)c(\xi) \end{cases} \tag{23}$$

where $x_e = [x^T \ \xi^T]^T$,

$$f_e(x_e) = \begin{bmatrix} f(x) + g_2(x)c(\xi) \\ a(\xi) + b(\xi)h_2(x) \end{bmatrix} \tag{24}$$

$$\triangle f_e(x_e) = \begin{bmatrix} \triangle f(x) + \triangle g_2(x)c(\xi) \\ b(\xi) \triangle h_2(x) \end{bmatrix} \tag{25}$$

$$g_e(x_e) = \begin{bmatrix} g_1(x) \\ b(\xi)k_{21}(x) \end{bmatrix} \tag{26}$$

$$\Sigma_{c0} : \begin{cases} \dot{x}_e = f_{e0}(x_e) + g_{e0}(x_e)\hat{w} \\ \hat{z} = h_{10}(x) + k_{120}(x)c(\xi) \end{cases} \tag{27}$$

where

$$f_{e0}(x_e) = f_e(x_e) \tag{28}$$

$$g_{e0}(x_e) = \begin{bmatrix} g_{10}(x) \\ b(\xi)k_{210}(x) \end{bmatrix} \tag{29}$$

Let $U(x_e)$ be a C^1 function defined in a neighbourhood of $(0,0)$, denote

$$J_1(U, \Sigma_{c1}, \triangle f, \triangle g_2, \triangle h_2) \triangleq U_{x_e}(f_e(x_e) + \triangle f_e(x_e))$$
$$+ z^T z + \frac{1}{4\gamma^2} U_{x_e} g_e(x_e) g_e^T(x_e) U_{x_e}^T \tag{30}$$

$$J_0(U, \Sigma_{c0}, \lambda_1(x), \lambda_2(x), \lambda_3(x)) \triangleq U_{x_e}(f_{e0}(x_e) + \hat{z}^T \hat{z}$$
$$+ \frac{1}{4\gamma^2} U_{x_e} g_{e0}(x_e) g_{e0}^T(x_e) U_{x_e}^T \tag{31}$$

Then the functions J_1 and J_0 have the following relation.

Lemma 3.1 *For any* $\Delta f(x) \in \Omega_f$, $\Delta g_2(x) \in \Omega_g$, $\Delta h_2(x) \in \Omega_h$, *and* $\lambda_1(x) > 0$, $\lambda_2(x)$, $\lambda_3(x) > 0$, *the following inequality holds*

$$J_1(U, \Sigma_{c1}, \Delta f, \Delta g_2, \Delta h_2) \le J_0(U, \Sigma_{c0}, \lambda_1(x), \lambda_2(x), \lambda_3(x)) \tag{32}$$

Proof: Omitted due to space limitation. ◁

Remark 3.3 From the inequality (32) and the results in [11], we know that if there exist positive smooth functions $\lambda_1(x)$, $\lambda_2(x)$ and $\lambda_3(x)$ such that the closed-loop system Σ_{c0} of (27) is dissipative, with a C^T storage function $U(x_e)$, with respect to the supply rate $s(\hat{w}, \hat{z}) = \gamma^2 \|\hat{w}\|^2 - \|\hat{z}\|^2$, then the closed-loop system Σ_{c1} of (23) also is dissipative, with the same storage function $U(x_e)$, with respect to the supply rate $s(w, z) = \gamma^2 \|w\|^2 - \|z\|^2$ for any $\Delta f(x) \in \Omega_f$, $\Delta g_2(x) \in \Omega_g$ and $\Delta h_2(x) \in \Omega_h$.

Combining the above lemma, and Theorem 3.1 on [4], we have the following theorem.

Theorem 3.1 *Consider the system (1) and the system Σ_0 of (22), $\lambda_1(x)$, $\lambda_2(x)$ and $\lambda_3(x)$ are given positive scalar smooth functions, and suppose the following:*

(i): Assumptions A2 and A3 hold, and the matrix $k_{12}^T(x)k_{12}(x)$ is nonsingular for each x,

(ii): There exists a C^3 positive definite function $V(x)$, locally defined in a neighbourhood of $x = 0$ and vanishing at $x = 0$, which satisfies the following inequality

$$H_0(x, V_x, \lambda_1(x), \lambda_2(x), \lambda_3(x)) = V_x \hat{f}(x)$$

$$+ \hat{h}_1^T(x)\hat{h}_1(x) + \frac{1}{4} V_x \hat{R}(x) V_x^T \le 0 \tag{33}$$

where

$$\hat{f}(x) = f(x) - g_2(x) R_2^{-1}(x) k_{120}^T(x) h_{10}(x) \tag{34}$$

$$\hat{h}_1(x) = h_{10}(x) - k_{120}(x) R_2^{-1}(x) k_{120}^T(x) h_{10}(x) \tag{35}$$

$$\hat{R}(x) = \frac{1}{\gamma^2} g_{10}(x) g_{10}^T(x) - g_2(x) R_2^{-1}(x) g_2^T(x) \tag{36}$$

Denote

$$\alpha(x) = -R_2^{-1}(x)\left(\frac{1}{2} g_2^T(x) V_x^T + k_{120}^T(x) h_{10}(x)\right)$$

$$= -R_2^{-1}(x)\left(\frac{1}{2} g_2^T(x) V_x^T + k_{12}^T(x) h_1(x)\right) \tag{37}$$

$$\alpha_1(x) = \frac{1}{2\gamma^2} g_{10}^T(x) V_x^T \tag{38}$$

$$f_*(x) = f(x) + g_{10}\alpha_1(x) \tag{39}$$

$$h_{2*}(x) = h_2(x) + k_{210}(x)\alpha_1(x) \tag{40}$$

(iii): There exists a C^3 positive definite function $Q(x)$, locally defined in a neighbourhood of $x = 0$ and vanishing at $x = 0$, which satisfies the following inequality

$$F_0(x, Q_x, \lambda_1(x), \lambda_2(x), \lambda_2(x)) = Q_x \hat{f}_0 + \hat{T}(x)$$

$$+ \frac{1}{4\gamma^2} Q_x \hat{g}_1(x) \hat{g}_1^T(x) Q_x^T < 0 \text{ for } x \ne 0 \tag{41}$$

and the Hessian matrices of $F_0(x, Q_x, \lambda_1(x), \lambda_2(x)$ and $\lambda_2(x))$ are nonsingular, where

$$\hat{f}_0(x) = f_*(x) - g_{10}(x) k_{210}^T(x) R_1^{-1}(x) h_{2*}(x) \tag{42}$$

$$\hat{g}_1(x) = g_{10}(x) \left[I - k_{210}^T(x) R_1^{-1}(x) k_{210}(x)\right] \tag{43}$$

$$\hat{T}(x) = \alpha^T(x) R_2(x)\alpha(x) - \gamma^2 h_{2*}^T(x) R_1^{-1}(x) h_{2*}(x) \tag{44}$$

with

$$R_1(x) = \gamma^2 \lambda_3(x) h_{11}(x) h_{11}^T(x) + k_{21}(x) k_{21}^T(x) \tag{45}$$

(iv): There exists a matrix function $L(x)$ such that

$$Q_x L(x) = h_{2*}^T(x) \tag{46}$$

Denote

$$G(x) = (2\gamma^2 L(x) + g_{10}(x) k_{210}^T(x)) R_1^{-1}(x)$$

$$= (2\gamma^2 L(x) + g_1(x) k_{21}^T(x)) R_1^{-1}(x) \tag{47}$$

Then, the controller K given by (21) with

$$a(\xi) = f_*(\xi) + g_2(\xi)\alpha(\xi) - G(\xi) h_{2*}(\xi) \tag{48}$$

$$b(\xi) = G(\xi), \quad c(\xi) = \alpha(\xi) \tag{49}$$

is a solution of the robust disturbance attenuation problem via measurement feedback for the system (1).

Theorem 3.2 *Consider the system (1) and suppose the following:*

(i'): Assumptions A1 and A3 hold,

(ii') There exists a C^3 positive definite function $V(x)$, locally defined in a neighbourhood of $x = 0$ and vanishing at $x = 0$, which satisfies the following strict inequality

$$H_0(x, V_x, \lambda_1(x), \lambda_2(x), \lambda_3(x)) < 0 \text{ for } x \ne 0 \tag{50}$$

(iii'): (iii) and (iv) in Theorem 3.1 hold.

Then, the controller K given by (21) with (48) and (49) is a solution of the robust disturbance attenuation problem via measurement feedback for the system (1).

Proofs for Theorems 3.1 and 3.2 are omitted here due to space limitations. ◁

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Remark 3.4 From the above results, it is easy to see that the solution of the robust disturbance attenuation problem via measurement feedback for the uncertain system (1) can be conducted by the extended system Σ_0 without uncertainty, which is similar to the result for linear systems in [14]. For the case of state feedback, the similar conclusion holds for the system (1) and the extended system Σ_s of (16), i.e., the controller given in Theorem 2.1 or Theorem 2.2 can be conducted by the extended system Σ_s, but the proofs of Theorem 2.1 and Theorem 2.2 are direct from the standard "completion of the squares" argument.

Remark 3.5 By computing directly , we have

$$H_s(x, V_x^T, \lambda_1(x), \lambda_2(x)) = H_0(x, V_x^T, \lambda_1(x), \lambda_2(x), \lambda_3(x))$$

$$-\frac{1}{\lambda_3(x)} h_{22}^T(x) h_{22}(x)$$

which is different from the H_∞ control system design for affine nonlinear systems without uncertainty. This is caused by the uncertainty of the measured output.

Remark 3.6 For the linear case, the similar can be made as Corollary 2.1 and Remark 2.2, the details are omitted.

4 Conclusions

This paper addresses the robust H_∞ control problems via state feedback and measurement feedback for a class of uncertain affine nonlinear systems. The proposed solutions rely upon the existence of positive definite smooth solutions of Hamilton-Jaccobi inequalities, which can be conducted by solving the corresponding H_∞ control problem for the extended systems without uncertainty.

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