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# Robust Eigenstructure Assignment under Regional Pole Constraints

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**ABSTRACT:** This paper provides a computational procedure for a type of robust regional pole assignment problem. It allows the closed-loop poles to be settled at certain perturbation insensitive locations within some prespecified regions in the complex plane. The novelty of our approach lies in the versatility of the proposed algorithm which provides a rich set of constrained regions for the assignment of individual or subsets of closed-loop poles, in contrast to other conventional regional pole assignment methods. The algorithm is based on a gradient flow formulation on a potential function which provides a minimizing solution for the Frobenius condition number of the closed-loop state matrix.

## 1 Introduction

Robust state feedback controller design via pole assignment has been investigated quite extensively in the past ten years. In many practical design situations, "exact" pole assignment, in which poles of the closed-loop system are assigned in some fixed positions in the complex plane, is unnecessary. Instead, it would be acceptable for the closed-loop eigenvalues to be assigned within some regions prespecified by designers. If such a design freedom is considered together with the nonuniqueness associated with the state feedback gain matrix in assigning a set of self-conjugate complex eigenvalues in a general MIMO system, then potentially a much more well-conditioned closed-loop state matrix, $A_c$, can be obtained as compared to other robust pole assignment schemes which assign the set of closed-loop eigenvalues exactly [KND85,BN89,LY95].

The freedom of clustering poles within some special regions in the complex plane in order to optimize certain performance index has been most commonly used in the optimal Linear Quadratic Regulator (LQR) design [HB92]. Moreover, the technique of regional pole assignment also appears in many other aspects of control systems design. Roppenecker [Rop83] has proposed a design procedure to find a minimum norm feedback controller by assigning poles within specified eigenvalue areas. Burrows and Patton [BP91] have considered pole assignment with low eigenvalue sensitivity and small feedback gain.

In the present work, we propose a computational approach to the *Robust Regional Pole Assignment (RRPA)* problem. The idea is to minimize a potential function defined via $||V||_F$ and $||V^{-1}||_F$, where $V$ is a nonsingular eigenvector matrix of $A_c$, with the minimization problem solved based on a gradient flow formulation. In this way, the Frobenius condition number $\kappa_F(V) \triangleq ||V||_F||V^{-1}||_F$ which measures the conditioning of $A_c$ will then be minimized. While most existing regional pole assignment algorithms cluster the whole eigenspectrum within a special region or allocate individual closed-loop eigenvalues into the corresponding rectangular regions in the complex plane, the novelty of our approach lies in the versatility of the proposed algorithm which provides a rich set of constrained regions for the assignment of individual or subsets of closed-loop poles.

## 2 Problem Formulation

Consider a finite-dimensional linear time-invariant system with $q$ ( $q > 1$ ) inputs described by

$$\dot{x} = Ax + Bu \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$ is of full column rank, $x \in \mathbb{R}^{n\times 1}$, $u \in \mathbb{R}^{q\times 1}$ and the pair $(A, B)$ is assumed to be completely controllable. By applying a time-invariant state-feedback law

$$u = Kx \tag{2}$$

there results the closed-loop system given by

$$\dot{x} = (A + BK)x \tag{3}$$

As the pair $(A, B)$ is completely controllable, the spectrum of the closed-loop state matrix, $A_c \triangleq A + BK$, can be assigned to any arbitrary set of self-conjugate complex numbers of cardinality $n$ by proper choice of the feedback gain matrix $K \in \mathbb{R}^{q\times n}$.

### 2.1 Parametrization of Constrained Regions

Now suppose the set of $n$ closed-loop poles constitutes $n_1$ pairs of imaginary poles

$$\alpha_m \pm \beta_m j, \quad m = 1, \ldots, n_1$$

and $n_2$ real poles

$$\gamma_l, \quad l = 1, \ldots, n_2$$

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Here $\alpha_m$, $\beta_m$ and $\gamma_l$ are real numbers and $n = 2n_1 + n_2$. The poles are allowed to be located within the open constrained regions

$$|(\alpha_m \pm \beta_m j) - (\alpha_{m0} \pm \beta_{m0} j)| < \mathcal{R}_m(\theta_m) \qquad (4)$$

and the open intervals on the real axis

$$|\gamma_l - \gamma_{l0}| < \eta_l \qquad (5)$$

respectively with $\alpha_{m0} \pm \beta_{m0} j$ and $\gamma_{l0}$ denoting the centres of the constrained regions. With $0 < \theta_m < 2\pi$ being the polar angle measured about $\alpha_{m0} \pm \beta_{m0} j$, the polar function $\mathcal{R}_m(\theta_m)$ then describes the boundaries of some open regions in the complex plane. Suppose $A_c$ is diagonalizable (further explanation of this assumption will be given in Remark 1), then there exists a similarity transformation $T \in \mathbb{R}^{n \times n}$ such that

$$A + BK = T\Lambda T^{-1} \qquad (6)$$

or

$$AT - T\Lambda = -BG \quad ; \quad G = KT \qquad (7)$$

where

$$\Lambda = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & \cdots & \cdots & \cdots & 0 \\ -\beta_1 & \alpha_1 & 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \alpha_2 & \beta_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & -\beta_2 & \alpha_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \gamma_1 & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \ddots & \ddots \end{pmatrix} \qquad (8)$$

For any $G \in \mathbb{R}^{q \times n}$, $\Lambda \in \mathbb{R}^{n \times n}$ with the structure in (8) and satisfying (4) and (5), $T(G, \Lambda)$ is uniquely determined if we have $\mathrm{spec}\,(A) \cap \mathrm{spec}\,(\Lambda) = \emptyset$. In addition, if the regions described by (4) and (5) are non-overlapping, then the controllability of $(A, B)$ implies that $T(G, \Lambda)$ is generically non-singular for any $G$ and $\Lambda$. Now, we define $\mathcal{P}_m : \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}^{2 \times 2}$ and $\mathcal{S}_l : \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}^{1 \times 1}$ to be the projection operators on a matrix $X = (x_{ij})_{n \times n}$ resulting in submatrices such that

$$\mathcal{P}_m(X) \triangleq \begin{pmatrix} x_{2m-1,2m-1} & x_{2m-1,2m} \\ x_{2m,2m-1} & x_{2m,2m} \end{pmatrix}$$

and

$$\mathcal{S}_l(X) \triangleq (x_{2n_1+l,2n_1+l})$$

Then the (block-) diagonal elements of $\Lambda$ can be denoted as

$$\mathcal{P}_m(\Lambda) = \begin{pmatrix} \alpha_m & \beta_m \\ -\beta_m & \alpha_m \end{pmatrix} = \mathcal{L}_{m0} + \rho_m \mathcal{W}_{\theta_m} \qquad (9)$$

in which

$$\mathcal{L}_{m0} = \begin{pmatrix} \alpha_{m0} & \beta_{m0} \\ -\beta_{m0} & \alpha_{m0} \end{pmatrix}$$

and

$$\mathcal{W}_{\theta_m} = \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix}$$

Furthermore,

$$\mathcal{S}_l(\Lambda) = (\gamma_l) = (\gamma_{l0} + \eta_l \sigma(w_l)) \qquad (10)$$

Here the variables $\rho_m$ and $\theta_m$ for defining the constrained regions of the complex conjugate poles are given by

$$\rho_m = \mathcal{R}_m(\theta_m)\psi(r_m) \quad \text{and} \quad \theta_m = 2\pi\psi(s_m) \qquad (11)$$

in which $\psi : \mathbb{R} \to (0,1)$ is a binary sigmoid function

$$\psi(x) = \frac{1}{1+e^{-x}} \quad \text{with} \quad \psi'(x) = \psi(x)(1-\psi(x)) \qquad (12)$$

For the line intervals constraining the real poles, $\sigma : \mathbb{R} \to (-1,1)$ is a bipolar sigmoid function

$$\sigma(x) = \frac{1-e^{-x}}{1+e^{-x}} \quad \text{with} \quad \sigma'(x) = \frac{1}{2}(1-\sigma(x)^2) \qquad (13)$$

In other words, points within the constrained regions (4) and (5) are parametrized uniquely in terms of $(r_m, s_m)$ and $w_l$ respectively.

## 2.2 Parametric Optimization

A common measure used for the conditioning of $A_c$ in face of unstructured additive perturbation is the condition number of its eigenvector matrix. This is because by the Bauer-Fike Theorem, the spectral variation of the closed-loop state matrix $A_c$ due to an unstructured perturbation $\Delta$ in $A_c$ is bounded by $||V||_F||V^{-1}||_F||\Delta||_F$ where $||\cdot||_F$ denotes the Frobenius norm of a matrix and $V$ is a nonsingular eigenvector matrix of $A_c$. Notice that $T$ satisfying (6) is not an eigenvector matrix in general. However, there exists a unitary matrix $U$ such that $V = TU$ is an eigenvector matrix of $A_c$ and $||TU|| = ||T||$ for any unitarily invariant norm $||\cdot||$. Hence the objective function to be minimized in this paper is chosen as

$$J \triangleq \kappa_F(T) = ||T||_F||T^{-1}||_F \qquad (14)$$

Recall that the parameters $\mathbf{r} = [r_1 \ldots r_{n_1}]^T$, $\mathbf{s} = [s_1 \ldots s_{n_1}]^T$ and $\mathbf{w} = [w_1 \ldots w_{n_2}]^T$ determine $\Lambda$. For ease of notation, they are grouped with the parameters in $G$ to form

$$H = \begin{pmatrix} \mathrm{vec}(G) \\ \mathbf{r} \\ \mathbf{s} \\ \mathbf{w} \end{pmatrix} \in \mathbb{R}^{(q \times n+n) \times 1}$$

where $\mathrm{vec}(G)$ denotes the lexicographical ordering of the elements $g_{ij}$ of $G$. The set of all $H = [h_1 \ldots h_{n \times (q+1)}]^T$ resulting in the particular $G \in \mathbb{R}^{q \times n}$ and $\Lambda \in \mathbb{R}^{n \times n}$ such that $T$ is well-defined and nonsingular will hereafter be denoted by $\mathcal{H}$.

Accordingly, we now formulate the *Robust Regional Pole Assignment (RRPA)* problem as that of finding of a $H \in \mathcal{H}$ to minimize the objective function, $J$, that is,

$$\min_{H \in \mathcal{H}} J(H) : \mathbb{R}^{(q \times n+n) \times 1} \longrightarrow \mathbb{R}$$

Once $H$ is found, the corresponding $G$ and $\Lambda$ will result in the desired state feedback gain matrix, $K = GT^{-1}$ with the set of closed-loop poles,

$$\mathrm{spec}(\Lambda) = \{\alpha_1 \pm \beta_1 j, \ldots, \alpha_{n_1} \pm \beta_{n_1} j, \gamma_1, \ldots, \gamma_{n_2}\}$$

assigned within the specific constrained regions. In this way, the RRPA problem can be reduced to a pure parametric unconstrained optimization task.

## 3 Gradient Flow Based Minimization Algorithm

In this section, a gradient flow based algorithm will be proposed for minimizing the objective function $J(H)$. The algorithm consists of the gradient flow of the potential function $\phi$ defined by

$$\phi(H) = ||T(H)||_F^2 + ||T^{-1}(H)||_F^2 \tag{15}$$

in the form

$$\dot{H}(t) = -\nabla_H \phi(H) \quad ; \quad H(0) = H_0 \in \mathcal{H} \tag{16}$$

It can be shown easily that

$$n \leq J(H) \leq \frac{1}{2}\phi(H) \tag{17}$$

The equalities hold if and only if $\|T(H)\|_F = \|T^{-1}(H)\|_F$ and this is satisfied when $T(H)$ is orthogonal. It was established in Lam and Yan [LY95] that the minimization of $\phi$ will in turn minimize the Frobenius condition number $\kappa_F$ of the closed-loop eigenvector matrix. In this way, a more nearly robust closed-loop eigenstructure is obtained.

**Proposition 1** *The objective function $\phi(H)$ is strictly decreasing along the solution $H(t)$ of the flow (16) for all $t \in [0, \infty)$ if $H_0 \in \mathcal{H}$ not a stationary point of $\phi(H)$.*

**Proof:** Since the objective function $\phi(H)$ is infinitely differentiable in a neighbourhood of $H_0$ , the differential equation (16) has a unique solution defined on some interval about $t = 0$. Suppose the interval is given by $[0, t_{\max})$. Then for $0 \leq t_1 < t_2 < t_{\max}$ , we have

$$\begin{aligned} \|H(t_2) - H(t_1)\|_F &= \left\| \int_{t_1}^{t_2} \dot{H}(t)\, dt \right\|_F \\ &\leq \sqrt{\int_{t_1}^{t_2} ||\dot{H}(t)||_F^2\, dt}\, \sqrt{t_2 - t_1} \\ &= \sqrt{-\int_{t_1}^{t_2} \dot{\phi}(H(t))\, dt}\sqrt{t_2 - t_1} \\ &= \sqrt{\phi(H(t_1)) - \phi(H(t_2))}\sqrt{t_2 - t_1} \\ &\leq \sqrt{\phi(H_0)(t_2 - t_1)} \end{aligned}$$

The sequence $\{H(t_i)\}$ where $0 \leq t_i < t_{i+1}$ with $t_i$, $t_{i+1} \in [0, t_{\max})$ is Cauchy. As a consequence, we have $\lim_{t \to t_{\max}} H(t) = \bar{H}$ exists. Owing to the maximality of $t_{\max}$, $\bar{H}$ must be outside $\mathcal{H}$ and $\lim_{t \to t_{\max}} ||T(t)^{-1}||_F = \infty$ which renders the unboundedness of $\phi(H(t))$ as $t \to t_{\max}$. This contradicts the fact that $\phi(H(t))$ is nonincreasing. Hence, $t_{\max} = \infty$. Moreover,

$$\begin{aligned} \frac{d\phi(H)}{dt} &= \sum_{k=1}^{n \times (q+1)} \left(\frac{\partial \phi(H)}{\partial h_k}\right)\left(\frac{dh_k}{dt}\right) \\ &= \operatorname{tr}\left( (\nabla_H \phi(H))^T \dot{H}(t) \right) \\ &= -||\dot{H}(t)||_F^2 \\ &\leq 0 \end{aligned}$$

thus implying that $\phi(H(t_1)) \geq \phi(H(t_2))$. Since the differential equation (16) has a unique solution defined on $[0, \infty)$, $\phi(H)$ is strictly decreasing along the solution of $H(t)$ if $H_0$ is not a stationary point. ∎

**Remark 1** Suppose (16) starts with an initial $H_0 \in \mathcal{H}$. Then it corresponds to a pair of $(G_0, \Lambda_0)$ such that $T(0)$ obtained from (7) is nonsingular. If $H_0$ not a stationary point of $\phi(H)$, $\dot{\phi}(H) < 0$ and hence $T^{-1}(t)$ exits for all $t > 0$. As a result, the diagonalizability of $A_c$ can always be guaranteed along the flow in (16). ∎

The derivatives of the potential function $\phi$ with respect to $H$ can now be obtained as follows.

**Theorem 1** *Let $\phi(H)$ be defined as in (15) and*

$$P \triangleq T^{-T}\left(T^T T - T^{-1} T^{-T}\right) \tag{18}$$

*where $T$ is the unique solution to the Sylvester equation in (7). Then we have*

$$\nabla_H \phi(H) = -2 \begin{bmatrix} -\operatorname{vec}(B^T Y) \\ \left[\operatorname{tr}(Z_x^T (T^T Y))\right]_{n \times 1} \end{bmatrix}_{(q \times n + n) \times 1} \tag{19}$$

*in which*

$$Z_x = \begin{cases} \dfrac{\partial \Lambda}{\partial r_m} &, \quad x = m \\ \dfrac{\partial \Lambda}{\partial s_m} &, \quad x = n_1 + m \\ \dfrac{\partial \Lambda}{\partial w_l} &, \quad x = 2n_1 + l \end{cases} \tag{20}$$

*for $m = 1 \ldots n_1$ and $l = 1 \ldots n_2$ where $Y$ is the unique solution to the Sylvester equation*

$$A^T Y - Y \Lambda^T + P = 0 \tag{21}$$

*Block-diagonal matrices $\dfrac{\partial \Lambda}{\partial r_m}$ and $\dfrac{\partial \Lambda}{\partial s_m}$ are in $\mathbb{R}^{n \times n}$ with block-elements*

$$\mathcal{P}_m\left(\frac{\partial \Lambda}{\partial r_m}\right) = \mathcal{R}_m(\theta_m)\psi'(r_m)\mathcal{W}_{\theta_m} \tag{22}$$

*and*

$$\begin{aligned} &\mathcal{P}_m\left(\frac{\partial \Lambda}{\partial s_m}\right) \\ &= 2\pi\psi(r_m)\psi'(s_m)\begin{pmatrix} \mathcal{R}'_m(\theta_m) & \mathcal{R}_m(\theta_m) \\ -\mathcal{R}_m(\theta_m) & \mathcal{R}'_m(\theta_m) \end{pmatrix}\mathcal{W}_{\theta_m} \end{aligned} \tag{23}$$

*respectively and with zero block-elements elsewhere. Similarly, $\dfrac{\partial \Lambda}{\partial w_l}$ are matrices in $\mathbb{R}^{q \times n}$ whose elements equals zero except the $(2n_1 + l)^{th}$ diagonal element equal to $\eta_l \sigma'(w_l)$.*

**Proof:** It is noted that $\text{tr}(P^TX) = \text{tr}(C^TY)$ where $X$ and $Y$ are the solutions of the respective Sylvester equations

$$AX - X\Lambda + C = 0 \quad \text{and} \quad A^TY - Y\Lambda^T + P = 0$$

for any $C \in \mathbb{R}^{n\times n}$. Since

$$\frac{\partial\phi}{\partial h_k} = 2\text{tr}\left(P^T\frac{\partial T}{\partial h_k}\right)$$

For $k = 1 \dots q\times n$,

$$\begin{aligned}\frac{\partial\phi}{\partial h_k} &= 2\text{tr}\left(P^T\frac{\partial T}{\partial g_{ij}}\right) \text{ with } A\frac{\partial T}{\partial g_{ij}} - \frac{\partial T}{\partial g_{ij}}\Lambda = -BE_{ij} \\ &= 2\text{tr}\left((BE_{ij})^TY\right)\end{aligned}$$

or

$$\nabla_G\phi = 2B^TY$$

in which $\frac{\partial T}{\partial g_{ij}}$ is obtained by differentiating (7) with respect to $g_{ij}$ of $G$. $E_{ij} \in \mathbb{R}^{q\times n}$ is a matrix whose elements are equal to zero except the $(i,j)^{th}$ element equal to 1. Similarly, for $k = (q\times n+1)\dots(q\times n+n)$,

$$\begin{aligned}\frac{\partial\phi}{\partial h_k} &= 2\text{tr}\left(P^T\frac{\partial T}{\partial h_k}\right) \text{ with } A\frac{\partial T}{\partial h_k} - \frac{\partial T}{\partial h_k}\Lambda = T\frac{\partial\Lambda}{\partial h_k} \\ &= -2\text{tr}\left(\left(\frac{\partial\Lambda}{\partial h_k}\right)^T T^TY\right)\end{aligned}$$

in which $\frac{\partial T}{\partial h_k}$ is obtained by differentiating (7) with respect to $h_k$. Here $h_k$ may be equal to $r_m$, $s_m$ or $w_l$. The block-elements in (20) are obtained by differentiating (8) with respect to $r_m$, $s_m$ and $w_l$ accordingly. ■

**RRPA Algorithm:** Given a completely controllable pair $(A, B)$, choose $H_0 \in \mathcal{H}$ as an initial condition for $H$ such that the set of initial closed-loop poles determined by $\mathbf{r}_0$, $\mathbf{s}_0$ and $\mathbf{w}_0$ satisfying $\text{spec}(A) \cap \text{spec}(\Lambda_0) = \emptyset$.

**1.** Specify different constrained regions (4) and line intervals (5) in the complex plane for different closed-loop complex conjugate and real eigenvalues respectively to give a $\Lambda$ with the particular structure in (8).

**2.** Solve the $n\times(q+1)$ ODEs:

$$\begin{aligned}\dot{H}(t) &= 2\begin{bmatrix} -\text{vec}(B^TY) \\ \left[\text{tr}(Z_x^T(T^TY))\right]_{n\times 1}\end{bmatrix}_{(q\times n+n)\times 1} && (24) \\ H(0) &= H_0 \in \mathcal{H} && (25)\end{aligned}$$

where $T(t)$ is the unique solution of (7) while $Z_x$ and $Y$ are given by (20) and (21) respectively.

**3.** Choose $K = G(t^*)T(t^*)^{-1}$ as the feedback gain with the resulting closed-loop eigenvalues located at $\text{spec}(\Lambda(t^*))$, where $t^* > 0$ is sufficiently large.

**Remark 2** Albeit the set of constrained regions discussed so far are designed for the assignment of individual eigenvalues, they can also be used to enclose different set of closed-loop poles so that the poles can altogether settle in the most perturbation insensitive positions within the regions (see Example 3). ■

**Remark 3** Although the constrained regions specified in Step 1 should always ensure that $\text{spec}(A) \cap \text{spec}(\Lambda(t)) = \emptyset$ in order to guarantee unique solutions exist in the Sylvester equations of Step 2, the numerically singular problem when solving the Sylvester equation occurs only if some of the closed-loop eigenvalues really coincide with the open-loop ones during minimization, that is, $\text{spec}(A) \cap \text{spec}(\Lambda(t)) \neq \emptyset$ for some $t > 0$. This phenomenon rarely happens in the generic case and designers can use any regions to suit their design specifications. In case the problem mentioned above really occurs, one can repeat Step 1 and try some other initial conditions for $H$ or choose other appropriate regions which do not overlap with $\text{spec}(A)$. ■

**Remark 4** The matrix $G(t)$ is bounded and the resulting feedback gain $K(t)$ satisfies

$$||K(t)||_F \leq ||B^\dagger||_F\left[||A||_F + \frac{1}{2}||\Lambda(t)||_F\phi(H_0)\right] \qquad (26)$$

for all $t \in [0, \infty)$ where $B^\dagger = (B^TB)^{-1}B^T$ and $||\Lambda(t)||_F$ is uniformly bounded. ■

# 4 Regions in Cartesian Coordinates

For certain regions, Cartesian coordinates may admit a more convenient parametrization. Thus, for completeness, we describe some common constrained regions given in Cartesian coordinates and give the corresponding formulae for the RRPA algorithm.

The main difference occurs in the description of complex poles in $\Lambda$ while the description of real poles using (10) remained the same. Consider a $\Lambda$ satisfying the structure in (8) such that, for the $m^{th}$ complex conjugate pole pair,

$$\mathcal{P}_m(\Lambda) = \mathcal{L}_{m0} + \begin{pmatrix} x_m & y_m \\ -y_m & x_m \end{pmatrix}\mathcal{W}_{\delta_m} \qquad (27)$$

where $x_m + y_m j$ and $\delta_m$ represent respectively the translation and the rotation (measured counterclockwise) about the nominal pole position at $\alpha_{m0} + \beta_{m0} j$ (see Figure 1(b)). Notice that $x_m = x_m(r_m, s_m)$ and $y_m = y_m(r_m, s_m)$. By denoting $(x_m)_{r_m}$ and $(y_m)_{r_m}$ as the partial derivatives of $x_m$ and $y_m$ with respect to $r_m$ respectively and similar notation for $(x_m)_{s_m}$ and $(y_m)_{s_m}$, the corresponding derivative formulae (analogus to (22) and (23)) are

$$\mathcal{P}_m\left(\frac{\partial\Lambda}{\partial r_m}\right) = \begin{pmatrix} (x_m)_{r_m} & (y_m)_{r_m} \\ -(y_m)_{r_m} & (x_m)_{r_m} \end{pmatrix}\mathcal{W}_{\delta_m} \qquad (28)$$

and

$$\mathcal{P}_m\left(\frac{\partial\Lambda}{\partial s_m}\right) = \begin{pmatrix} (x_m)_{s_m} & (y_m)_{s_m} \\ -(y_m)_{s_m} & (x_m)_{s_m} \end{pmatrix}\mathcal{W}_{\delta_m} \qquad (29)$$

We summarize the expressions for $x_m$ and $y_m$ corresponding to different constrained regions in Table 1.

For the trapezoidal case described in Table 1,

$$y_m = \tau_m(x_m) = \left(\frac{c_m - b_m}{2a_m}\right)x_m + \left(\frac{c_m + b_m}{2}\right) \qquad (30)$$

is the straight line equation for the upper boundary of the region. Parameters $a_m$, $b_m$ and $c_m$ are the characteristic dimensions of the constrained regions (see Figure 1).

| | *Elliptical Region* | *Rectangular Region* | *Trapezoidal Region* |
|---|---|---|---|
| $x_m$ | $a_m\psi(r_m)\cos\theta_m$ | $a_m\sigma(r_m)$ | $a_m\sigma(r_m)$ |
| $y_m$ | $b_m\psi(r_m)\sin\theta_m$ | $b_m\sigma(s_m)$ | $\sigma(s_m)\tau(x_m)$ |
| $(x_m)_{r_m}$ | $a_m\psi'(r_m)\cos\theta_m$ | $a_m\sigma'(r_m)$ | $a_m\sigma'(r_m)$ |
| $(y_m)_{r_m}$ | $b_m\psi'(r_m)\sin\theta_m$ | 0 | $\frac{1}{2}(c_m-b_m)\sigma(s_m)\sigma'(r_m)$ |
| $(x_m)_{s_m}$ | $-2\pi a_m\psi(r_m)\psi'(s_m)\sin\theta_m$ | 0 | 0 |
| $(y_m)_{s_m}$ | $2\pi b_m\psi(r_m)\psi'(s_m)\cos\theta_m$ | $b_m\sigma'(s_m)$ | $\tau(x_m)\sigma'(s_m)$ |

Table 1. Expressions for Different Constrained Regions

## 5 Numerical Examples

Consider a 5-state, 2-input model [LY95] given by

$$A = \begin{pmatrix} -0.1094 & 0.0628 & 0 & 0 & 0 \\ 1.306 & -2.132 & 0.9807 & 0 & 0 \\ 0 & 1.595 & -3.149 & 1.547 & 0 \\ 0 & 0.0355 & 2.632 & -4.257 & 1.855 \\ 0 & 0.00227 & 0 & 0.1636 & -0.1625 \end{pmatrix}$$

$$B^T = \begin{pmatrix} 0 & 0.0638 & 0.0838 & 0.1004 & 0.0063 \\ 0 & 0 & -0.1396 & -0.2060 & -0.0128 \end{pmatrix}$$

with open-loop eigenvalues at $-0.077$, $-0.014$, $-0.895$, $-2.841$ and $-5.982$. The parameters $\mathbf{r}_0$, $\mathbf{s}_0$ and $\mathbf{w}_0$ are chosen such that the closed-loop poles are located at $\mathcal{E} = \{-1 \pm j, -0.2, -0.5, -1\}$ initially. To increase the efficiency of computation, we take the initial starting condition for $G$ as

$$G_0 = \begin{pmatrix} -58.69 & -11.84 & 38.43 & 18.77 & 27.21 \\ -18.90 & 13.18 & 22.68 & 21.55 & 25.31 \end{pmatrix}$$

which gives $\phi_0 \simeq 78.6$, $\kappa_F \simeq 39.3$, $\kappa_2 \simeq 33.6$ and $||K||_F \simeq 337.4$. $\kappa_2(T) \triangleq ||T||_2||T^{-1}||_2$ is the spectral condition number. $G_0$ is taken from the solution of the "exact" robust pole-placement algorithm proposed by Lam and Yan [LY95] with the set of closed-loop eigenvalues assigned exactly at $\mathcal{E}$. With this choice of $G_0$, any solution obtained by following the robust regional pole assignment algorithm proposed in Section 3 using (24) with $\phi(H)$ as the potential function to be minimized will be better than (or at least the same as) that of obtained from other "exact" robust pole assignment schemes [KND85,LY95].

**Example 1** Suppose the set of closed-loop poles are allowed to be relaxed by $\pm 10\%$ from their nominal positions (see Figure 2) such that

$$a_1 = 0.1 \;;\; b_1 = 0.1 \;;\; \eta_1 = 0.02 \;;\; \eta_2 = 0.05 \;;\; \eta_3 = 0.1$$

and

$$\mathbf{r}_0 = (0) \quad ; \quad \mathbf{s}_0 = (0) \quad ; \quad \mathbf{w}_0 = (\ 0 \quad 0 \quad 0\ )^T$$

where $\mathcal{L}_{10} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$, $\gamma_{10} = -0.2$, $\gamma_{20} = -0.5$ and $\gamma_{30} = -1$. At $t = 100$, the resulting feedback gain matrix is then given by

$$K_s = \begin{pmatrix} -43.88 & 106.16 & -225.39 & 191.33 & -47.70 \\ -19.38 & 35.83 & -60.34 & 45.44 & -2.21 \end{pmatrix}$$

with

$$\mathrm{spec}(\Lambda) = \{-1.10 \pm 1.10j, -0.18, -0.45, -1.09\}$$

which gives $\phi \simeq 66.62$, $\kappa_F \simeq 33.31$, $\kappa_2 \simeq 28.45$ and $||K||_F \simeq 332.03$.

**Example 2** Now assume the pair of complex conjugate poles are allowed to be assigned within a pair of elliptical regions while the set of real poles are assigned within the line intervals of equal width (see Figure 3) such that

$$a_1 = 3.6 \;;\; b_1 = 0.6 \;;\; \eta_1 = 0.1 \;;\; \eta_2 = 0.1 \;;\; \eta_3 = 0.1$$

and

$$\mathbf{r}_0 = (2.17) \quad ; \quad \mathbf{s}_0 = (-2.74) \quad ; \quad \mathbf{w}_0 = (\ 0 \quad 0 \quad 0\ )^T$$

where $\mathcal{L}_{10} = \begin{pmatrix} -4 & 0.8 \\ -0.8 & -4 \end{pmatrix}$, $\gamma_{10} = -0.2$, $\gamma_{20} = -0.5$ and $\gamma_{30} = -1$. It turns out that at $t = 80$, the resulting feedback gain matrix is given by

$$K_e = \begin{pmatrix} -43.85 & -8.17 & -57.33 & -10.38 & -19.10 \\ -17.19 & 8.98 & -19.59 & -2.86 & 4.21 \end{pmatrix}$$

with

$$\mathrm{spec}(\Lambda) = \{-5.70 \pm 1.03j, -0.13, -0.40, -1.09\}$$

In this case, $\phi \simeq 14.01$, $\kappa_F \simeq 6.85$, $\kappa_2 \simeq 3.21$ and $||K||_F \simeq 80.84$. A substantial reduction in $\phi$, $\kappa_F$, $\kappa_2$ and as well as $||K||_F$ is obtained.

**Example 3** Assume the whole set of closed-loop poles are allowed to be assigned within an trapezoidal region (see Figure 4) with the parameters

$$a_1 = 3 \;;\; b_1 = 4 \;;\; c_1 = 1 \;;\; \eta_1 = 3 \;;\; \eta_2 = 3 \;;\; \eta_3 = 3$$

and

$$\mathbf{r}_0 = (1.73) \;;\; \mathbf{s}_0 = (1.69) \;;\; \mathbf{w}_0 = (\ 4.08 \quad 2.64 \quad 1.73\ )^T$$

where $\mathcal{L}_{10} = -3.1I_2$ and $\gamma_{10} = \gamma_{20} = \gamma_{30} = -3.1$. In this case, the resulting feedback gain matrix at $t = 30$ is given by

$$K_t = \begin{pmatrix} -51.87 & -8.40 & -70.68 & 1.47 & 9.54 \\ -19.12 & 14.02 & -31.63 & 6.95 & 5.28 \end{pmatrix}$$

with

$$\mathrm{spec}(\Lambda) = \{-5.91 \pm 0.96j, -0.13, -0.24, -0.95\}$$

in which $\phi \simeq 13.75$, $\kappa_F \simeq 6.52$, $\kappa_2 \simeq 2.76$ and $||K||_F \simeq 97.40$. A huge reduction in the eigenvalue sensitivity is thus expected. Also note that even though the trapezoidal region and the open-loop eigenvalues are overlapped here, it does not cause any numerical problems.

## 6 Conclusions

We have presented a computational procedure for the RRPA problem. Poles of closed-loop systems are allowed to be settled at certain perturbation insensitive locations within some prespecified regions in the complex plane. The proposed algorithm is based on a gradient flow formulation on a potential function which provides a minimizing solution for the Frobenius condition number of the closed-loop state matrix. The effectiveness of the proposed algorithm has been illustrated by numerical examples which reveal that significant improvement on the eigenstructure robustness can be achieved.

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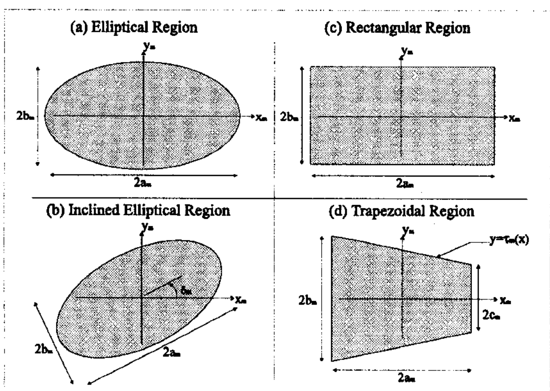


Figure 1. Special Constrained Regions for Complex Conjugate Poles (with centres corresponding to the nominal desired pole positions).

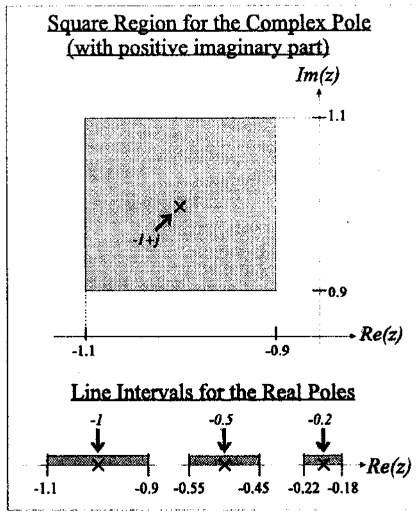


Figure 2. Different Constrained Regions for Example 1.

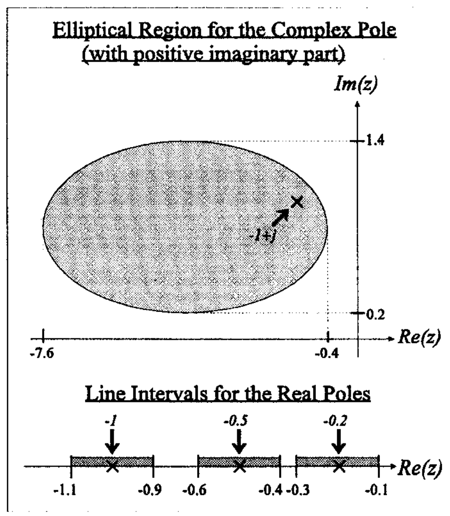


Figure 3. Different Constrained Regions for Example 2.

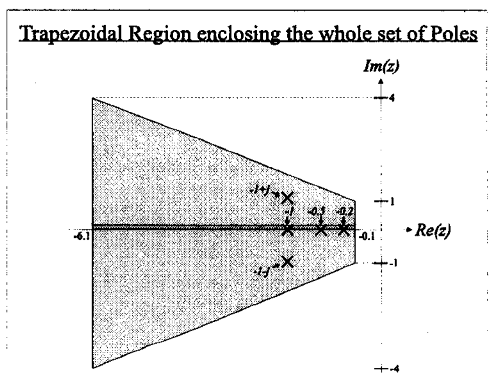


Figure 4. Constrained Region for Example 3.