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Robust Stabilization for Stochastic Time-Delay Systems with Polytopic Uncertainties

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Abstract

This paper considers the problem of robust stabilization for stochastic time-delay systems with convex polytopic uncertainties. Sufficient conditions for the solvability of the problem are obtained by using parameter-independent and parameter-dependent Lyapunov functionals, respectively. It is shown that the result derived by a parameter-dependent Lyapunov functional is less conservative. A desired state feedback controller can be designed by solving a set of linear matrix inequalities.

1 Introduction

Stochastic systems with time delays have received much attention since such systems has come to play an important role in many branches of science and engineering applications [2, 4]. The problems of robust stability analysis and stabilization for uncertain stochastic time-delay systems have been studied in the literature and a great number of results on these topics have been reported. For example, via different approaches, robust stability analysis results were obtained in [3] and [5], respectively. The robust stabilization problem was discussed in [7], where a linear matrix inequality (LMI) approach is developed and state feedback controllers were designed. The corresponding results for the discrete case can be found in [8]. It is worth mentioning that the parameter uncertainties considered in all these works are time-varying norm-bounded.

Recently, convex polytopic uncertainties have been considered. It has been shown that polytopic uncertainties can arise when the uncertain matrix in norm-bounded uncertainties provide some *a prior* known structures of uncertainties. Therefore, polytopic-type uncertainty can be regarded as an important class of parameter uncertainties. When such a kind of parameter uncertainties appears in a deterministic discrete system, the problems of robust stability and stabilization have been studied and LMI based approaches have been developed; see, e.g., [1, 6], and the references therein. To date, however, there is no available results on robust stabilization for stochastic delay systems with polytopic uncertainties. This motivates the present investigation.

In this paper, we are concerned with the problem of robust stabilization for stochastic systems with time delays and polytopic uncertainties. The problem to be addressed is the design of state feedback controllers such that the resulting closed-loop system is mean-square asymptotically stable. Firstly, a sufficient condition based on a parameter-independent Lyapunov functional is obtained. Then, in order to reduce the conservatism, we derive a sufficient condition by using a parameter-dependent Lyapunov functional. A desired state feedback controller can be constructed by solving a set of LMIs.

Notation. Throughout this paper, for symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite); I is the identity matrix with appropriate dimension; M^T represents the transpose of the matrix M; $\mathcal{E}\{\cdot\}$ denotes the expectation operator with respect to some probability measure $\mathcal{P}$; Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2 Problem Formulation

Consider the following uncertain stochastic system with time-delay:

$$
\begin{aligned}
(\Sigma): \quad dx(t) &= [A(\alpha)x(t) + A_1(\alpha)x(t-\tau) + B(\alpha)u(t)]\,dt \\
&\quad + E_1(\alpha)x(t-\tau)d\omega(t), && (1)\\
x(t) &= \phi(t),\ \forall t \in [-h, 0], && (2)
\end{aligned}
$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $\omega(t)$ is a one-dimensional Brownian motion satisfying $\mathcal{E}\{d\omega(t)\} = 0$ and $\mathcal{E}\{d\omega(t)^2\} = dt$. The vector $\alpha \in \mathbb{R}^n$ is an uncertain parameter and the scalar $\tau > 0$ is the delay of the system, which is not known. The matrices $A(\alpha)$, $A_1(\alpha)$, $B(\alpha)$ and $E_1(\alpha)$ are not precisely known, which are assumed to belong to a convex bounded (polytopic type) uncertain domain Υ given by

$$\begin{aligned}\Upsilon &= \{(A, A_1, B, E_1)(\alpha), \; : \\ & (A, A_1, B, E_1)(\alpha) = \sum_{i=1}^{N} \alpha_i (A, A_1, B, E_1)_i, \\ & \sum_{i=1}^{N} \alpha_i = 1; \; \alpha_i \geq 0\} \end{aligned} \tag{3}$$

where A_i, A_{1i}, B_i, E_{1i}, $i = 1, \ldots, N$, are known real constant matrices.

For the uncertain stochastic delay system (Σ), we consider the state feedback controller

$$u(t) = Kx(t). \tag{4}$$

Then, the resulting closed-loop system can be obtained as

$$\begin{aligned} dx(t) &= [A_c(\alpha)x(t) + A_1(\alpha)x(t-\tau)]\,dt \\ &\quad + E_1(\alpha)x(t-\tau)d\omega(t), \end{aligned} \tag{5}$$

where

$$A_c(\alpha) = A(\alpha) + B(\alpha)K. \tag{6}$$

The purpose of the robust stabilization problem to be addressed in this paper is the design of a state feedback controller (4) such that the resulting closed-loop system (5) is mean-square asymptotically stable for every $(A, A_1, B, E_1)(\alpha) \in \Upsilon$.

3 Main Results

We first provide a sufficient condition for the solvability of the robust stabilization problem by using a parameter-independent Lyapunov functional candidate.

Theorem 1 *Consider the uncertain stochastic time-delay system (Σ). Then, there exists a state feedback controller (4) such that the resulting closed-loop system (5) is mean-square asymptotically stable if there exist matrices Y, $P > 0$ and $Q > 0$ such that the following LMIs hold for $i = 1, \ldots, N$,*

$$\begin{bmatrix} A_iP + PA_i^T + B_iY + Y^TB_i^T + Q & A_{1i}P & 0 \\ PA_{1i}^T & -Q & PE_{1i}^T \\ 0 & E_{1i}P & -P \end{bmatrix} < 0. \tag{7}$$

In this case, a desired feedback gain can be chosen as

$$K = YP^{-1}. \tag{8}$$

Proof. By (7), it is easy to show that

$$\sum_{i=1}^{N} \alpha_i \Delta_i < 0,$$

where

$$\Delta_i = \begin{bmatrix} A_iP + PA_i^T + B_iY + Y^TB_i^T + Q & A_{1i}P & 0 \\ PA_{1i}^T & -Q & PE_{1i}^T \\ 0 & E_{1i}P & -P \end{bmatrix}.$$

with (8), we have

$$\begin{bmatrix} A_c(\alpha)P + A_c(\alpha)^TP + Q & A_1(\alpha)P & 0 \\ PA_1(\alpha)^T & -Q & PE_1(\alpha)^T \\ 0 & E_1(\alpha)P & -P \end{bmatrix} < 0 \tag{9}$$

where

$$\begin{aligned} A_{ci} &= A_i + B_iYP^{-1}, \quad i = 1, \ldots, N, \\ A_c(\alpha) &= \sum_{i=1}^{N} \alpha_i A_{ci}. \end{aligned}$$

Let

$$\hat{P} = P^{-1}, \quad \hat{Q} = \hat{P}Q\hat{P}.$$

Then, pre- and post-multiplying (9) by $\mathrm{diag}\left(\hat{P}, \hat{P}, \hat{P}\right)$, we obtain

$$\begin{bmatrix} \hat{P}A_c(\alpha) + A_c(\alpha)^T\hat{P} + \hat{Q} & \hat{P}A_1(\alpha) & 0 \\ A_1(\alpha)^T\hat{P} & -\hat{Q} & E_1(\alpha)^T\hat{P} \\ 0 & \hat{P}E_1(\alpha) & -\hat{P} \end{bmatrix} < 0.$$

By applying the Schur complement formula to the above inequality, we have

$$J(\alpha) < 0, \tag{10}$$

where

$$J(\alpha) = \begin{bmatrix} \hat{P}A_c(\alpha) + A_c(\alpha)^T\hat{P} + \hat{Q} & \hat{P}A_1(\alpha) \\ A_1(\alpha)^T\hat{P} & E_1(\alpha)^T\hat{P}E_1(\alpha) - \hat{Q} \end{bmatrix}$$

Now, define the following Lyapunov functional candidate for the closed-loop system in (5):

$$V(x(t), t) = x(t)^T\hat{P}x(t) + \int_{t-\tau}^{t} x(s)^T\hat{Q}x(s)ds. \tag{11}$$

Then, by Itô's formula, we obtain the stochastic differential as

$$dV(x(t), t) = \mathcal{L}V(x(t), t)dt + 2x(t)^T\hat{P}E_1(\alpha)x(t-\tau)d\omega(t)$$

where

$$\begin{aligned} \mathcal{L}V(x(t), t) &= 2x(t)^T\hat{P}\left[A_c(\alpha)x(t) + A_1(\alpha)x(t-\tau)\right] \\ &\quad + x(t)^T\hat{Q}x(t) - x(t-\tau)^T\hat{Q}x(t-\tau) \\ &\quad + x(t-\tau)^TE_1(\alpha)^T\hat{P}E_1(\alpha)x(t-\tau) \\ &= \begin{bmatrix} x(t)^T & x(t-\tau)^T \end{bmatrix} J(\alpha) \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}. \end{aligned}$$

Noting (10), we have

$$\mathcal{L}V(x(t), t) < 0.$$

By this and [2], we have that the resulting closed-loop system (5) is mean-square stable. This completes the proof. □

Theorem 1 provides a sufficient condition for the solvability of the robust stabilization stochastic time-delay systems with polytopic uncertainties and a parameter-independent Lyapunov functional candidate is used. Usually the result obtained by such a method is conservative. To reduce the conservatism, we provide a solvability condition in the following theorem by using a parameter-dependent Lyapunov functional candidate.

Theorem 2 *There exists a state feedback controller (4) such that the resulting closed-loop system (5) is mean-square asymptotically stable if there exist matrices F, G, S, Y, Ω, H_i, Φ_i, $i = 1, \ldots, 4$ and $P_j > 0$, $Q_j > 0$, $j = 1, \ldots, N$, such that the following LMIs hold for $i = 1, \ldots, N$,*

$$\begin{bmatrix} \Gamma_{1i} & A_{1i}H_2^T & -A_{1i}H_3^T - \Phi_1 E_{1i} & \Theta_i & H_1 + A_{1i}S^T & \Phi_1 \\ H_2 A_{1i}^T & -Q_i & FE_{1i} & 0 & P_i - H_2 & P_i - F \\ -H_3 A_{1i}^T - E_{1i}^T \Phi_1^T & E_{1i}^T F^T & -P_i - \Phi_2 E_{1i} - E_{1i}^T \Phi_2^T & -E_{1i}^T \Phi_3^T & H_3 - E_{1i}^T \Phi_4^T & \Phi_2 + E_{1i}^T \Omega^T \\ \Theta_i^T & 0 & -\Phi_3 E_{1i} & -G - G^T & H_4 & \Phi_3 \\ H_1^T + S A_{1i}^T & P_i - H_2^T & H_3^T - \Phi_4 E_{1i} & H_4^T & -S - S^T & \Phi_4 \\ \Phi_1^T & P_i - F & \Phi_2^T + \Omega^T E_{1i} & \Phi_3^T & \Phi_4^T & -\Omega - \Omega^T \end{bmatrix} < 0, \qquad (12)$$

where

$$\begin{aligned} \Gamma_i &= Q_i + A_i G^T + G A_i^T + B_i Y \\ &\quad + Y^T B_i^T - H_1 A_{1i}^T - A_{1i} H_1^T, \\ \Theta_i &= P_i - G + A_i G^T - A_{1i} H_4^T. \end{aligned}$$

In this case, a desired feedback gain can be chosen as

$$K = Y G^{-T}. \qquad (13)$$

Proof. First, from the LMI in (12), it can be seen that

$$-G - G^T < 0,$$

which implies that G is invertible. Therefore, (13) is well-defined when the LMIs in (12) are feasible. Now, by (12), it is easy to see that

$$\sum_{i=1}^{N} \alpha_i \begin{bmatrix} \Gamma_{1i} & A_{1i}H_2^T & -A_{1i}H_3^T - \Phi_1 E_{1i} & \Theta_i & H_1 + A_{1i}S^T & \Phi_1 \\ H_2 A_{1i}^T & -Q_i & FE_{1i} & 0 & P_i - H_2 & P_i - F \\ -H_3 A_{1i}^T - E_{1i}^T \Phi_1^T & E_{1i}^T F^T & -P_i - \Phi_2 E_{1i} - E_{1i}^T \Phi_2^T & -E_{1i}^T \Phi_3^T & H_3 - E_{1i}^T \Phi_4^T & \Phi_2 + E_{1i}^T \Omega^T \\ \Theta_i^T & 0 & -\Phi_3 E_{1i} & -G - G^T & H_4 & \Phi_3 \\ H_1^T + S A_{1i}^T & P_i - H_2^T & H_3^T - \Phi_4 E_{1i} & H_4^T & -S - S^T & \Phi_4 \\ \Phi_1^T & P_i - F & \Phi_2^T + \Omega^T E_{1i} & \Phi_3^T & \Phi_4^T & -\Omega - \Omega^T \end{bmatrix} < 0 \qquad (14)$$

Let

$$\begin{aligned} A_{ci} &= A_i + B_i Y G^{-T}, \quad i = 1, \ldots, N, \\ A_c(\alpha) &= \sum_{i=1}^{N} \alpha_i A_{ci}, \;\; P(\alpha) = \sum_{i=1}^{N} \alpha_i P_i, \\ Q(\alpha) &= \sum_{i=1}^{N} \alpha_i Q_i. \end{aligned}$$

Then, the inequality in (14) can be re-written as

$$\begin{bmatrix} Q(\alpha) + G A_c(\alpha)^T + A_c(\alpha) G^T - H_1 A_1(\alpha)^T - A_1(\alpha) H_1^T & A_1(\alpha) H_2^T & -A_1(\alpha) H_3^T - \Phi_1 E_1(\alpha) & P(\alpha) - G + A_c(\alpha) G^T - A_1(\alpha) H_4^T & H_1 + A_1(\alpha) S^T & \Phi_1 \\ H_2 A_1^T(\alpha) & -Q(\alpha) & F E_1(\alpha) & 0 & P(\alpha) - H_2 & P(\alpha) - F \\ -H_3 A_1^T(\alpha) - E_1(\alpha)^T \Phi_1^T & E_1(\alpha)^T F^T & -P(\alpha) - \Phi_2 E_1(\alpha) - E_1(\alpha)^T \Phi_2^T & -E_1(\alpha)^T \Phi_3^T & H_3 - E_1(\alpha)^T \Phi_4^T & \Phi_2 + E_1(\alpha)^T \Omega^T \\ P(\alpha) - G^T + G A_c(\alpha)^T - H_4 A_1(\alpha)^T & 0 & -\Phi_3 E_1(\alpha) & -G - G^T & H_4 & \Phi_3 \\ H_1^T + S A_1^T(\alpha) & P(\alpha) - H_2^T & H_3^T - \Phi_4 E_1(\alpha) & H_4^T & -S - S^T & \Phi_4 \\ \Phi_1^T & P(\alpha) - F & \Phi_2^T + \Omega^T E_1(\alpha) & \Phi_3^T & \Phi_4^T & -\Omega - \Omega^T \end{bmatrix} < 0. \qquad (15)$$

Note that

$$\Lambda(\alpha) = \begin{bmatrix} I & 0 & 0 & A_c(\alpha) & A_1(\alpha) & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & E_1(\alpha)^T \end{bmatrix}$$

is of full row rank. Then, pre- and post-multiplying (15) by $\Lambda(\alpha)$ and $\Lambda(\alpha)^T$, respectively, we obtain

$$\begin{bmatrix} A_c(\alpha) P(\alpha) + P(\alpha) A_c(\alpha)^T + Q(\alpha) & A_1(\alpha) P(\alpha) & 0 \\ P(\alpha) A_1(\alpha)^T & -Q(\alpha) & P(\alpha) E_1(\alpha)^T \\ 0 & E_1(\alpha) P(\alpha) & -P(\alpha) \end{bmatrix} < 0 \qquad (16)$$

Considering $P(\alpha) > 0$, we can set

$$\tilde{P}(\alpha) = P(\alpha)^{-1}.$$

Pre- and post-multiplying (16) by $\operatorname{diag}\left(\tilde{P}(\alpha), \tilde{P}(\alpha), \tilde{P}(\alpha)\right)$ give

$$\left[\begin{array}{cc} \tilde{P}(\alpha) A_c(\alpha) + A_c(\alpha)^T \tilde{P}(\alpha) + \tilde{Q}(\alpha) & \tilde{P}(\alpha) A_1(\alpha) \\ A_1(\alpha)^T \tilde{P}(\alpha) & -\tilde{Q}(\alpha) \\ 0 & \tilde{P}(\alpha) E_1(\alpha) \end{array}\right.$$

$$\begin{bmatrix} \cdots & 0 \\ \cdots & E_1(\alpha)^T\tilde{P}(\alpha) \\ \cdots & -\tilde{P}(\alpha) \end{bmatrix} < 0 \quad (17)$$

where

$$\tilde{Q}(\alpha) = \tilde{P}(\alpha)Q(\alpha)\tilde{P}(\alpha).$$

By applying the Schur complement formula to (17), we have

$$\begin{bmatrix} \tilde{P}(\alpha)A_c(\alpha) + A_c(\alpha)^T\tilde{P}(\alpha) + \tilde{Q}(\alpha) & \tilde{P}(\alpha)A_1(\alpha) \\ A_1(\alpha)^T\tilde{P}(\alpha) & E_1(\alpha)^T\tilde{P}(\alpha)E_1(\alpha) - \tilde{Q}(\alpha) \end{bmatrix} < 0 \quad (18)$$

Now, define the following Lyapunov functional candidate for the closed-loop system in (5):

$$V(x(t),t) = x(t)^T\tilde{P}(\alpha)x(t) + \int_{t-\tau}^{t} x(s)^T\tilde{Q}(\alpha)x(s)ds. \quad (19)$$

Then, by using (18) and following similar line as in the proof of Theorem 1 we have that the resulting closed-loop system (5) is mean-square stable. This completes the proof. □

Remark 1 *In the proof of Theorem 2, it can be seen that the solvability condition is obtained by using a parameter-dependent Lyapunov functional candidate. Furthermore, it can be shown that Theorem 1 is a special case of Theorem 2. Therefore, Theorem 2 is less conservative than Theorem 1.*

4 Conclusion

This paper has studied the problem of robust stabilization for stochastic time-delay systems with polytopic uncertainties. Solvability conditions based on parameter-independent and parameter-dependent Lyapunov functionals have been proposed, respectively. It can be shown that the parameter-dependent result is less conservative than the parameter-independent one.

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