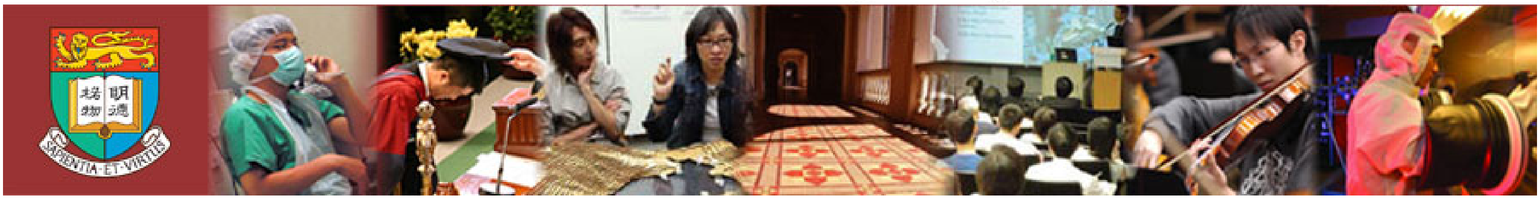

HALF-QUADRATIC REGULARIZATION FOR MRI IMAGE RESTORATION

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ABSTRACT

We consider the reconstruction of MRI images by minimizing regularized cost-functions. To accelerate the computation of the estimate, two forms of half-quadratic regularization, multiplicative and additive, are often used. In [13], we have compared both theoretically and experimentally the efficiency of these two forms using one-dimensional signals. The goal of this paper is to compare experimentally the efficiency of these two forms using MRI image reconstruction. We find that using the additive form is more computationally effective than using the multiplicative form.

1. INTRODUCTION

We address image reconstruction where a sought image $\hat{x} \in R^p$ is estimated from degraded data $y \in \mathrm{R}^q$ by minimizing a cost function $J : \mathrm{R}^p \to \mathrm{R}$ which combines a quadratic data-fidelity term and a regularization term Φ via a parameter $\beta > 0$:

$$\hat{x} = \min_{x \in \mathrm{R}^p} J(x), \quad \text{where} \quad J(x) = \|Ax - y\|^2 + \beta \Phi(x). \quad (1)$$

We shall assume that the observation operator $A \in \mathrm{R}^{q \times p}$ is known. We focus on regularization term Φ of the form

$$\Phi(x) = \sum_{i=1}^{r} \phi(g_i^T x), \quad (2)$$

where $\phi : \mathrm{R} \to \mathrm{R}$ is a potential function and g_i^T, for $i = 1, \ldots, r$, are linear operators. Typically, $\{g_i^T x\}$ are first or second-order differences between neighboring pixels. If G is the $r \times p$ matrix whose ith row is g_i^T, for $i = 1, \ldots, r$, a basic requirement is $\ker(A^T A) \cap \ker(G^T G) = \{0\}$. We suppose that ϕ is smooth and convex, and *edge-preserving*, i.e. $\phi(t) < t^2$ for $|t| \to \infty$. Such functions can be found in [2, 3, 1, 5], e.g., Huber potential function:

$$\phi(t) = \begin{cases} t^2/2 & \text{if} \quad |t| \leq \alpha, \\ \alpha|t| - \alpha^2/2 & \text{if} \quad |t| > \alpha. \end{cases} \quad (3)$$

Cost-functions of this form are popular in various inverse problems such as denoising, deblurring, seismic imaging, tomography.

However, the resultant minimizers $\hat{x}$ are non-linear with respect to data y their computation is costly, especially when A has many non-zero entries. In order to cope with numerical slowness, *half-quadratic* (HQ) reformulation of J has been pioneered, using two different ways, in [7] and [8]. The idea is to construct an *augmented cost function* $\mathcal{J} : \mathrm{R}^p \times \mathrm{R}^r \to \mathrm{R}$ which involves an auxiliary variable $s \in \mathrm{R}^r$, and two new functions, $Q : \mathrm{R} \times \mathrm{R} \to \mathrm{R}$, where $Q(., s_i)$ is quadratic $\forall s_i \in \mathrm{R}$, and $\psi : \mathrm{R} \to \mathrm{R}$,

$$\mathcal{J}(x, s) = \|Ax - y\|^2 + \beta \sum_{i=1}^{r} Q(g_i^T x, s_i) + \beta \sum_{i=1}^{r} \psi(s_i), \quad (4)$$

$$\text{so that} \quad \phi(t) = \min_{s \in \mathrm{R}} \{Q(t, s) + \psi(s)\}, \quad \forall t \in \mathrm{R}. \quad (5)$$

By (5), the global minimizer $(\hat{x}, \hat{s})$ of $\mathcal{J}$ yields the solution initially defined in (1), since $J(x) = \min_{s \in \mathrm{R}^r} \mathcal{J}(x, s)$, $\forall x \in \mathrm{R}^p$. In [7], Geman & Reynolds first considered a quadratic term Q of the *multiplicative form*,

$$Q(t, s) = t^2 s, \quad \text{for } t \in \mathrm{R}, \ s \in \mathrm{R}_+. \quad (6)$$

Later, Geman & Yang [8] proposed an *additive form* for Q:

$$Q(t, s) = (t - s)^2, \quad \text{for } t \in \mathrm{R}, \ s \in \mathrm{R}. \quad (7)$$

In both cases (6) and (7), the dual function ψ, which ensures (5), is obtained using the theory of convex conjugacy [9].

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The augmented cost-function $\mathcal{J}$ is minimized using an *alternating minimization scheme*. Let the solution obtained at iteration $(k-1)$ read $(x^{(k-1)}, s^{(k-1)})$. At the next iteration k we calculate

$s^{(k)}$ such that $\mathcal{J}(x^{(k-1)}, s^{(k)}) \leq \mathcal{J}(x^{(k-1)}, s), \quad \forall s \in \mathrm{R}^r,$

$x^{(k)}$ such that $\mathcal{J}(x^{(k)}, s^{(k)}) \leq \mathcal{J}(x, s^{(k)}), \quad \forall x \in \mathrm{R}^p.$

These minimizations give rise to two *minimizer mappings*, $x \to \left[\sigma(g_1^T x), \ldots, \sigma(g_r^T x)\right]^T$ with $\sigma : \mathrm{R} \to \mathrm{R}$, and $s \to \chi(s)$ with $\chi : \mathrm{R}^r \to \mathrm{R}^p$. The alternate minimization thus reads

$$s_i^{(k)} = \sigma(g_i^T x^{(k-1)}), \ \forall i = 1, \ldots, r, \qquad (8)$$

$$x^{(k)} = \chi(s^{(k)}). \qquad (9)$$

These ideas has been pursued and deepened by many authors [3, 6, 1, 5, 11, 10]. Although the intuition that HQ regularization does indeed increase the speed of the minimization of regularized cost-functions of the form (1), this critical question has never been considered in a theoretical way. In [13], the performance of both formulations (6) and (7) has been compared using one-dimensional signals. The goal of this paper is to compare experimentally the efficiency of these two forms using MRI image reconstruction. We find that using the additive form is more computationally effective than using the multiplicative form.

2. SOME FACTS ABOUT HQ REGULARIZATION

2.1. Multiplicative form

We consider potential functions ϕ such that

$$\begin{array}{ll} t \to \phi(\sqrt{t}) \text{ is concave on } \mathrm{R}_+, & \lim_{t \searrow 0} \phi'(t)/t < \infty, \\ t \to \phi(t) \text{ is convex on } \mathrm{R}, & \lim_{t \to \infty} \phi(t)/t^2 = 0, \\ \phi \text{ is twice differentiable on } \mathrm{R}, & \phi(t) = \phi(-t), \ \forall t \in \mathrm{R}. \end{array} \qquad (10)$$

Then the expressions below are equivalent [7, 5, 10]:

$$\begin{array}{lcl} \phi(t) & = & \inf_{s \in \mathrm{R}} \left\{ st^2 + \psi(s) \right\}, \\ \psi(s) & = & \sup_{t \in \mathrm{R}} \left\{ \phi(t) - st^2 \right\}. \end{array} \qquad (11)$$

Notice that ψ is convex and $\psi(s) = +\infty$ for $s < 0$; hence the infimum in (11) can be considered only for $s \geq 0$. The resultant augmented cost-function $\mathcal{J}$ is defined on $\mathrm{R}^p \times \mathrm{R}_+^r$ and reads

$$\mathcal{J}(x, s) = \|Ax - y\|^2 + \beta (Gx)^T \mathrm{diag}(s)\, Gx + \beta \sum_{i=1}^{r} \psi(s_i), \qquad (12)$$

where $\mathrm{diag}(s)$ is a diagonal matrix whose diagonal elements are s_i, for $i = 1, \ldots, r$. The function σ, as given in (8), reads [5, 10]

$$\sigma(t) = \begin{cases} \dfrac{\phi'(t)}{2\,t} & \text{if} \quad t \neq 0 \\ \zeta & \text{if} \quad t = 0 \end{cases} \quad \text{where } \zeta := \lim_{\tau \searrow 0} \frac{\phi'(\tau)}{2\,\tau}, \qquad (13)$$

where clearly $\sigma(t) \geq 0$, $\forall t \in \mathrm{R}$. The minimizer mapping χ, introduced in (9), satisfies $D_1 \mathcal{J}(\chi(s), s) = 0$, $\forall s \in \mathrm{R}^r$, and reads:

$$\begin{array}{lcl} \chi(s) & = & (H(s))^{-1} A^T y, \\ \text{where } H(s) & = & A^T A + \beta G^T \mathrm{diag}(s) G. \end{array} \qquad (14)$$

2.2. Additive form

This form is considered under the condition that the function

$$t \to t^2/2 - \phi(t) \qquad (15)$$

is convex, continuous and finite for every $t \in \mathrm{R}$. Then the following expressions are equivalent:

$$\begin{array}{lcl} \phi(t) & = & \inf_{s \in R} \left\{ \psi(s) + (t - s)^2/2 \right\}, \\ \psi(s) & = & \sup_{t \in R} \left\{ \phi(t) - (t - s)^2/2 \right\}. \end{array} \qquad (16)$$

The condition (15) implies that $\phi'(t^-) \geq \phi'(t^+)$, for any $t \in \mathrm{R}$. Whenever ϕ is convex, it implies that ϕ is differentiable. The augmented cost-function now reads

$$\mathcal{J}(x, s) = \|Ax - y\|^2 + \frac{\beta}{2} \|Gx - s\|^2 + \beta \sum_{i=1}^{r} \psi(s_i). \qquad (17)$$

The minimizer function σ reads [4, 1]:

$$\sigma(t) = t - \phi'(t). \qquad (18)$$

The minimizer function χ relevant to $\mathcal{J}(., s)$ reads

$$\begin{array}{lcl} \chi(s) & = & H^{-1} \left(2A^T y + \beta G^T s \right), \\ \text{where } H & = & 2A^T A + \beta\, G^T G. \end{array} \qquad (19)$$

3. EXPERIMENTAL RESULTS

In this section, we first consider restoring a two-dimensional image (128×128) and use this example to compare the performance of the additive and the multiplicative forms of the HQ regularization. In Figures 1 and 2, we display the original signal and the observed blurred and noisy version. We consider a spatial-invariant blurring process, and therefore the corresponding blurring matrix A is a Toeplitz-like matrix [14]. Since A is Toeplitz-like and G is the discretization matrix of the first-order differentiation operator, the coefficient matrix $A^T A + \beta G^T G$ in the additive form can be diagonalized by the cosine transform matrix. It follows that the computational complexity required for solving (19) at each HQ iteration is $O(n^2 \log n)$ operations for an n-by-n image. However, for the multiplicative form, the coefficient matrix is $A^T A + \beta G^T \mathrm{diag}(s) G$ and it cannot be diagonalized by the fast transform matrix even A and G have Toeplitz structures. We employ conjugate gradient methods (inner iterations) to solve such linear systems. The computational complexity required at each inner iterations is $O(n^2)$ operations.

The restored images using a cost-function of the form (1)-(2) where ϕ is a Huber function (3) are displayed in Figures 3 and 4 for the additive and the multiplicative forms respectively ($\alpha = \beta = 1$). The stopping criterion of the HQ iterations is $\|f^{(k)} - f^{(k-1)}\|_2 / \|g\|_2 < 1 \times 10^{-3}$. All the computations are done using MATLAB. Visually, two restored images using the additive and the multiplicative form are almost the same. In Table 1, we compare the performance of the HQ iterations using the additive and the multiplicative forms for different parameters α and β. We see that

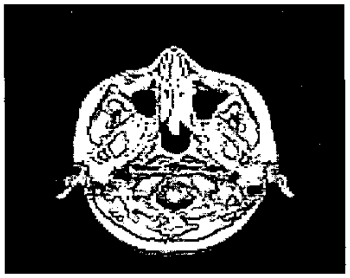

Fig. 1. The original image.

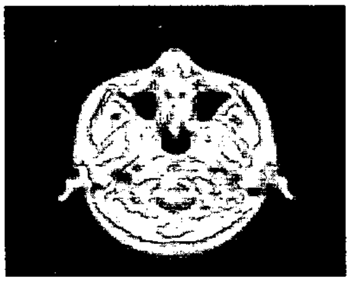

Fig. 2. The blurred and noisy image. (relative error = 14.5)

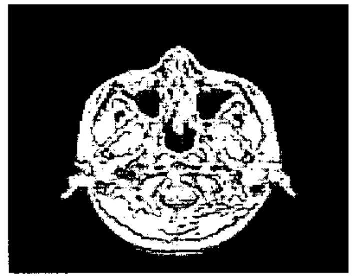

Fig. 3. The restored image using the additive form. (relative error = 0.1240)

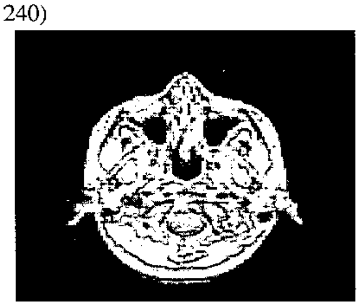

Fig. 4. The restored image using the multiplicative form. (relative error = 0.1239)

the multiplicative form of the HQ regularization is more effective than the additive form in terms of the objective function values and the relative errors. However, the differences are not significant.

On the other hand, the computational time required by the additive form of the HQ regularization is significantly less than that by the multiplicative form. As we have mentioned that fast cosine transform can be used to solve the linear system in the additive form, but inner iterations are required to solve the linear system in the multiplicative form. Thus the additive form is more efficient.

Next we consider the problem of increasing the spatial resolution of three-dimensional fMRI images [12]. A slice of a real image is given in Figures 5–8 using the additive and the multiplicative forms of HQ regularization ($\alpha = 0.01$ and $\beta = 1$). Both forms restore the images quite well. Again the fast cosine transform can be applied to solve the corresponding linear system in the additive form. The inner iterations are required in the multiplicative form. Our numerical results show that the additive form takes 103 seconds for the restoration, but the multiplicative form takes almost an hour for the restoration.

4. CONCLUDING REMARKS

We performed a numerical comparison of the two forms of HQ regularization, multiplicative and additive. The obtained results clearly stipulate that the additive form is more

$\alpha = 1$ and $\beta = 1$	additive	multiplicative
Number of iterations required	4	4
Objective function value	259.3545	259.1436
Relative error of the restored image	0.12408	0.12399
CPU time required (seconds)	3.9	158.9
$\alpha = 0.5$ and $\beta = 1$	additive	multiplicative
Number of iterations required	4	4
Objective function value	258.1902	258.4929
Relative error of the restored image	0.12372	0.12226
CPU time required (seconds)	3.4	156.1
$\alpha = 0.25$ and $\beta = 1$	additive	multiplicative
Number of iterations required	5	4
Objective function value	235.1825	235.4126
Relative error of the restored image	0.12293	0.11727
CPU time required (seconds)	4.0	158.0
$\alpha = 1$ and $\beta = 0.5$	additive	multiplicative
Number of iterations required	4	4
Objective function value	159.3879	145.2088
Relative error of the restored image	0.13388	0.12751
CPU time required (seconds)	3.1	156.8
$\alpha = 1$ and $\beta = 2$	additive	multiplicative
Number of iterations required	4	4
Objective function value	437.9721	468.3296
Relative error of the restored image	0.12243	0.12215
CPU time required (seconds)	2.8	154.2

Table 1. Comparisons between the additive and the multiplicative forms for Huber potential function.

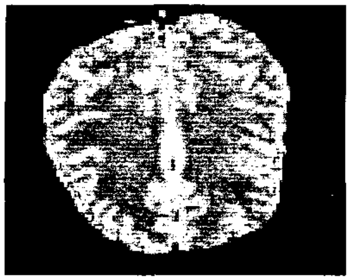

Fig. 5. The original image.

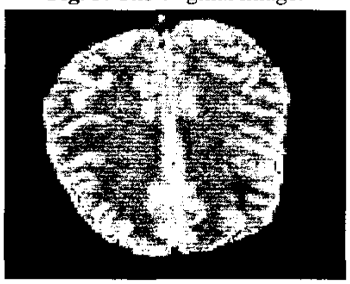

Fig. 6. The blurred and noisy image. (relative error = 5.521)

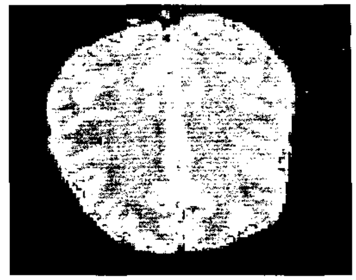

Fig. 7. The restored image using the additive form. (relative error = 0.04848)

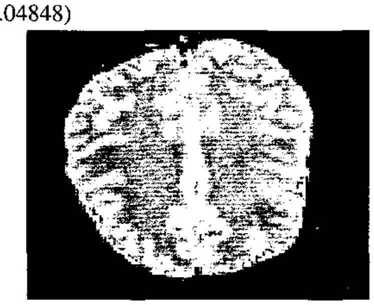

Fig. 8. The restored image using the multiplicative form. (relative error = 0.04756)

attractive in terms of computational cost.

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