M-channel Cosine-Modulated Wavelet Bases

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Abstract

In this paper, we propose a new *M*-channel wavelet bases called the cosine-modulated wavelets. We first generalize the theory of two-channel biorthogonal compactly supported wavelet bases to the *M*-channel case. A sufficient condition for the *M*-channel perfect reconstruction filter banks to construct *M*-channel compactly supported wavelet bases is given. By using this condition, a family of orthogonal and biorthogonal *M*-channel cosine-modulated wavelet bases is constructed by iterations of cosine-modulated filter banks (CMFB). The advantages of the approach are their simple design procedure, efficient implementation and good filter quality. A method for imposing the regularity on the cosine-modulated filter banks is also introduced and design example is given.

1. Introduction

Wavelets are functions generated from the dilations and translations of one basic function called the wavelet function [1]. More recently, wavelet functions have been constructed and studied extensively both in the mathematical and signal processing communities [1]-[4]. In signal analysis, wavelet transform (WT), which is a representation of a signal in terms of a set of wavelet basis functions, allows the signal to be analyzed in different resolutions or scales. WT makes a different resolution trade-off in the time-frequency plane as compared with the short-time Fourier transform. It has better time resolution in high frequency and better frequency resolution in low frequency. This property is useful to detect discontinuity in non-stationary signals which usually have slowly varying components accompanied with transient high frequency spikes.

The theory of wavelets is closely related to perfect reconstruction (PR) multirate filter banks. Daubechies [1] constructed compactly supported orthonormal wavelets from iterations of two-channel discrete filter banks with certain regularity condition. Since two-channel paraunitary PR filter banks cannot have non-trivial linear phase solution, more general biorthogonal filter banks were studied. In [2], more general biorthogonal compactly supported wavelet bases were introduced with similar regularity condition. The idea of constructing orthonormal wavelets by multirate filter banks has also been extended to the more general case of *M*-channel orthonormal wavelets [3], [4]. Like the dyadic case, it is also possible to obtain *M*-channel wavelets bases from *M*-channel PR orthogonal filter banks with added regularity condition.

In this paper, we first derive a sufficient condition for a biorthogonal *M*-channel PR filter banks to construct a biorthogonal *M*-channel compactly supported wavelet bases. It is shown that the lowpass filter in the PR filter bank has to satisfy similar regularity condition and the bandpass and highpass filters have to satisfy the admissible condition. Then, we propose to use the cosine-modulated filter banks (CMFB) [6]-[9], [11] to construct such wavelet bases. The design of *M*-channel wavelet bases is more difficult than the two-channel case due to the large number of design parameters and difficulties in meeting the regularity condition. The advantages of the CMFB are their low design and implementation complexities, good filter quality, and ease in imposing the regularity conditions. Both orthogonal and biorthogonal *M*-channel wavelets can be constructed by CMFB.

We shall use the following notations. $h_i[n]$ and $g_i[n]$, $i = 0,1,\ldots, M-1$, respectively, represent analysis and synthesis filters of *M*-channel filter banks [5]. $\tilde{g}_i[n]$ represents the mirror image of $g_i[n]$, namely, $\tilde{g}_i[n] = g_i[-n]$. $\phi(x)$ and $\psi_i(x)$, $i = 1,2,\ldots, M-1$, represent, respectively, the scaling and wavelet functions. While $\tilde{\phi}(x)$ and $\tilde{\psi}_i(x)$ are the corresponding dual functions. Caption letter represents discrete-time Fourier transform or Fourier transform.

In Section 2, we shall briefly review the theory of the *M*-channel wavelets. Section 3 is devoted to an overview of *M*-channel CMFB. Design procedure and design example of the *M*-channel cosine-modulated wavelet bases are given in Section 4. Finally, we summarize our results in the conclusion.

2. Theory of *M*-channel Wavelets

In [3] and [4], the *M*-channel orthonormal wavelets are constructed by iteration of *M*-channel orthogonal filter banks. The analysis and synthesis filters are time-inverse of each other. In the biorthogonal cases, there is not such restriction.

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Here, we have two dual bases, each generated from a set of wavelet functions. First of all, we start with the discrete-time Fourier transforms (scaled by $M^{-1/2}$) of $h_0[n]$ and $\tilde{g}_0[n]$,

$$H_0(\omega) = M^{-1/2}\sum_n h_0[n]e^{-jn\omega}, \qquad (2\text{-}1)$$

$$G_0(\omega) = M^{-1/2}\sum_n \tilde{g}_0[n]e^{-jn\omega}. \qquad (2\text{-}2)$$

By iterating these discrete-time filters, it is possible to define the Fourier transform $\Phi(\xi)$ and $\tilde{\Phi}(\xi)$ of the scaling function $\phi(x)$ and its dual $\tilde{\phi}(x)$ using the following infinite products:

$$\Phi(\xi) = (2\pi)^{-1/2}\prod_{j=1}^{\infty} H_0(M^{-j}\xi), \qquad (2\text{-}3)$$

$$\tilde{\Phi}(\xi) = (2\pi)^{-1/2}\prod_{j=1}^{\infty} G_0(M^{-j}\xi). \qquad (2\text{-}4)$$

These infinite products can only converge if,

$$H_0(0) = G_0(0) = 1. \qquad (2\text{-}5)$$

(2-3) and (2-4) will then converge uniformly and absolutely on compact sets to $\Phi(\xi)$ and $\tilde{\Phi}(\xi)$ which are well-defined C^∞ functions. From (2-3) and (2-4), we also have:

$$\Phi(\xi) = H_0(\frac{\xi}{M})\cdot\Phi(\frac{\xi}{M}), \qquad (2\text{-}6)$$

$$\tilde{\Phi}(\xi) = G_0(\frac{\xi}{M})\cdot\tilde{\Phi}(\frac{\xi}{M}). \qquad (2\text{-}7)$$

Taking the inverse Fourier Transform leads to the well known two-scale difference equations of $\phi(x)$ and its dual $\tilde{\phi}(x)$ as follows:

$$\phi(x) = \sqrt{M}\sum_n h_0[n]\phi(Mx-n), \qquad (2\text{-}8)$$

$$\tilde{\phi}(x) = \sqrt{M}\sum_n \tilde{g}_0[n]\tilde{\phi}(Mx-n). \qquad (2\text{-}9)$$

(2-8) and (2-9) tell us that $\phi(x)$ and $\tilde{\phi}(x)$ can be written as a linear combination of their contracted (by M) and shifted versions. Therefore the space spanned by:

$$\phi^{(j,k)}(x) = M^{-j/2}\phi(M^{-j}x-k),$$
$$\tilde{\phi}^{(j,k)}(x) = M^{-j/2}\tilde{\phi}(M^{-j}x-k),\ k\in Z \qquad (2\text{-}10)$$

at a given resolution j can be viewed as a multiscale approximation of a signal $f(x)$. To show that $\phi(x)$ and $\tilde{\phi}(x)$ can be used to generate a basis, we need M-1 wavelet functions and their duals to describe the remaining "details" in the approximation.

$$\psi_i(x) = \sqrt{M}\sum_n h_i[n]\phi(Mx-n), \qquad (2\text{-}11)$$

$$\tilde{\psi}_i(x) = \sqrt{M}\sum_n \tilde{g}_i[n]\tilde{\phi}(Mx-n), \qquad (2\text{-}12)$$

where $i = 1,2,\ldots, M-1$.

We can also define $\psi^{(j,k)}(x)$ and $\tilde{\psi}^{(j,k)}(x)$ as:

$$\psi_i^{(j,k)}(x) = M^{-j/2}\psi_i(M^{-j}x-k), \qquad (2\text{-}13)$$

$$\tilde{\psi}_i^{(j,k)}(x) = M^{-j/2}\tilde{\psi}_i(M^{-j}x-k), \qquad (2\text{-}14)$$

$k\in Z, i = 1,2,\ldots, M-1$.

We give an important theorem without proof. For more details on the proof, interested readers can refer to [10].

Theorem 2.1

The functions $\psi_i^{(j,k)}(x)$ and $\tilde{\psi}_i^{(j,k)}(x)$, constructed as (2-13) and (2-14), generate a biorthogonal M-channel wavelet bases in $L^2(\Re)$ if they satisfy the following three conditions:

(C1) $h_i[n]$ and $g_i[n]$, $i = 0,1,\ldots, M-1$. constitute a PR M-channel filter bank;

(C2) $\frac{1}{\sqrt{M}}\sum_n h_i[n] = 0$ and $\frac{1}{\sqrt{M}}\sum_n \tilde{g}_i[n] = 0$, $i = 1,2,\ldots, M-1$;

(C3) Both $H_0(\omega)$ and $G_0(\omega)$ have K order zeros at $\omega_\ell = \frac{2\pi\ell}{M}, \ell = 1,2,\ldots, M-1, K\ge 1$.

(C2) ensures that the wavelets will satisfy the admissible conditions[1]:

$$\int\frac{|\Psi_i(\xi)|^2}{|\xi|}d\xi < \infty \text{ and } \int\frac{|\tilde{\Psi}_i(\xi)|^2}{|\xi|}d\xi < \infty \qquad (2\text{-}15)$$

(C3) is the familiar regularity condition. (C3) together with (C1) will ensure that $\psi_i^{(j,k)}(x)$ and $\tilde{\psi}_i^{(j,k)}(x)$ constitute a Riesz bases in $L^2(\Re)$. Therefore, we have, for any $f(x)\in L^2(\Re)$, $f(x) = \sum_{j,k,i} c_i^{(j,k)}\psi_i^{(j,k)}$, $j,k\in Z$ with $c_i^{(j,k)} = \langle f(x), \tilde{\psi}_i^{(j,k)}\rangle$. These two equations are called the wavelet series and wavelet coefficients, respectively.

3. Theory of CMFB

In this section, we shall introduce the theory of CMFB and its design procedure. More details can be found in [6]-[9], [11]. In CMFB, the analysis filter bank $f_k(n)$ and synthesis filter bank $g_k(n)$ are obtained respectively by modulating the prototype filters $h(n)$ and $s(n)$,

$$f_k(n) = h(n)c_{k,n},$$
$$g_k(n) = s(n)\bar{c}_{k,n},$$
$$k = 0,1,\ldots M-1,\ n = 0,1,\ldots,N-1, \qquad (3\text{-}1)$$

where M is the number of channels and N is the length of the filters. Two possible modulations can be used:

$$c_{k,n} = 2\cos\left((2k+1)\frac{\pi}{2M}(n-\frac{N-1}{2})+(-1)^k\frac{\pi}{4}\right), \qquad (3\text{-}2a)$$

$$c_{k,n} = \sqrt{\frac{2}{M}}\cos\left((2k+1)\frac{\pi}{2M}(n+\frac{M+1}{2})\right). \qquad (3\text{-}2b)$$

(3-2b) is used in Malvar's ELT[8]. Without losing generality we use the (3-2a) as the modulation. For simplicity, we shall consider the case $N=2mM$ in this paper. The arbitrary length case is considered in [11]. It can be seen that analysis and synthesis modulations are time reverse of each other. Therefore that, if $h(n)$ is equal to $s(n)$ and is a linear phase filter, then $h_k(n)$ and $g_k(n)$ will be time-reverse of each other and we obtain the orthogonal CMFB. On the other hand, if $h(n)$ is a non-linear phase filter, then we obtain a biorthogonal CMFB. $h_k(n)$ and $g_k(n)$ will not be time-reverse of each other any more.

Let $H(z)=\sum_{q=0}^{2M-1} z^{-q} G_q(z^{2M})$ be the type-I polyphase decomposition [5] of the prototype filter, it can be shown [11] that the PR conditions are given by:

$$G_k(z)G_{2M-k-1}(z)+G_{M+k}(z)G_{M-k-1}(z)=(-1)^s z^{-\alpha}. \quad (3\text{-}3)$$

For orthogonal CMFB, the PR conditions are further simplified to:

$$\tilde{G}_k(z)G_k(z)+\tilde{G}_{M+k}(z)G_{M+k}(z)=1. \quad (3\text{-}4)$$

It can be seen that the PR conditions in (3-3) and (3-4) depend only on the prototype filter $h(n)$. This is the reason why the design complexity has been greatly reduced. In the orthogonal case, the number of free parameters is further reduced by half due to the linear-phase property of the prototype filter.

For the CMFB, the analysis filters will be frequency shifted version of the prototype filter. Therefore, the optimization objective function is reduced to:

$$\Phi=\int_{\omega_s}^{\pi}\left|H(e^{i\omega})\right|^2 d\omega, \quad (3\text{-}5)$$

where ω_s is the stop-band cutoff frequency whose value should be between $\frac{\pi}{2M}$ and $\frac{\pi}{M}$. Larger ω_s leads to larger stop-band attenuation but the overlap between adjacent analysis, or synthesis filters will also increase. It is also possible to replace the integral in (3-5) by a summation. This has the advantage of being able to put different weighting to different parts of the stop-band and provides more control over the stop-band attenuation.

The design problem is then formulated as the following constrained optimization problem:

$$\min_{\boldsymbol{h}} \int_{\omega_s}^{\pi}\left|H(e^{j\omega})\right|^2 d\omega, \quad (3\text{-}6)$$

subjected to the PR conditions in (3-3) or (3-4) for orthogonal and biorthogonal CMFB, where $\boldsymbol{h}$ is the vector containing the impulse response of the prototype filter.

In the case of biorthogonal CMFB, the linear phase requirement of the prototype filter is relaxed. Therefore we have more freedom in choosing its coefficients. At the same time, the number of the PR conditions in (3-3) also increases accordingly. Interested readers can refer to [9], [11] for more details. We shall see in next section that the frequency shifted property of the CMFB is very useful to satisfy (C2) and (C3) in Theorem 2.1 for constructing orthogonal and biorthogonal cosine modulated wavelet bases.

4. Design of M-channel Cosine-Modulated Wavelets

To generate wavelet filter bank, the lowpass analysis and synthesis filters should be of form [4][9]:

$$F_0(z)=C(1+z^{-1}+....+z^{-(M-1)})^K B(z), \quad (4\text{-}1)$$

where $B(z)$ is a polynomial of z and C is a constant.

In [9], we have proposed a method to design the M-channel orthogonal cosine-modulated wavelets that can structurally impose the regularity condition on $F_0(z)$. This can also be used in the biorthogonal case. Our approach is to decompose the prototype filter $H(z)$ into two parts:

$$H(z)=Q(z)P(z), \quad (4\text{-}2)$$

and determine the polynomial $P(z)$ such that after modulation $F_0(z)$ will have the required zeros at $\omega_\ell=\frac{2\pi\ell}{M}$, $\ell=1,2,...,M-1$. From (4-2), we notice that $f_0(n)$ is derived from $h(n)$ using the following cosine modulation:

$$f_0(n)=2h(n)\cos\left(\frac{\pi}{2M}(n-\frac{N-1}{2})+\frac{\pi}{4}\right)$$

$$=2h(n)\cos(\frac{\pi}{2M}n+\varphi), \quad \text{where,} \quad \varphi=\frac{(M-N+1)\pi}{4M},$$

$n=0,1,...,N-1$. Therefore the frequency response of $f_0(n)$, $F_0(e^{j\omega})$, is given by:

$$F_0(e^{j\omega})=e^{j\varphi}H(e^{j(\omega-\frac{\pi}{2M})})+e^{-j\varphi}H(e^{j(\omega+\frac{\pi}{2M})}). \quad (4\text{-}3)$$

It means that $H(e^{j\omega})$ is shifted along the frequency axis by $\frac{\pi}{2M}$ and $-\frac{\pi}{2M}$. If ω_ℓ are zeros of $F_0(e^{j\omega})$, then the right hand side of (4-3) should also be zero. This will be the case if $H(e^{j\omega})$ have zeros at $\omega_\ell\pm\frac{\pi}{2M}$. Therefore, for the M-channel CMFB to have K-vanishing moment, $H(e^{j\omega})$ should have zeros of order K at $\omega_n=\frac{(4n\pm1)\pi}{2M}$, $n=1,2,...,M-1$. The polynomial $P(z)$ is then given by:

$$P(z)=\left[\prod_{n=1}^{M-1}(z^{-1}-e^{j\frac{4n+1}{2M}\pi})(z^{-1}-e^{j\frac{4(M-n)-1}{2M}\pi})\right]^K. \quad (4\text{-}4)$$

Hence, by multiplying $P(z)$ with $Q(z)$, which contains the free parameters, the prototype filter $H(z)$ will always satisfy (C3) of Theorem 2.1. It is interesting to note that due to the frequency shifted properties, all the highpass and bandpass filters will have the same

number of zeros at $\omega = 0$ and satisfy (C2) automatically.

This design procedure for the orthogonal and biorthogonal cases is similar except for the different filter banks (orthogonal or biorthogonal). Here we only give an example of biorthogonal cosine-modulated wavelets. In this example, the parameters of the CMFB are M=4, K=1 and N=40. Fig.1 are the scaling function, the three wavelet functions and their duals. Fig.2 are the impulse and frequency response of prototype filter. In our example, these basis functions are obtained from iterating the corresponding two-level tree-structured CMFB.

5. Conclusion

In this paper we have presented a sufficient condition for constructing M-channel wavelets and proposed a family of M-channel cosine-modulated wavelet bases. The advantages of the approach are their simple design procedure, efficient implementation and good filter quality. Design example is given to demonstrate the usefulness of the method.

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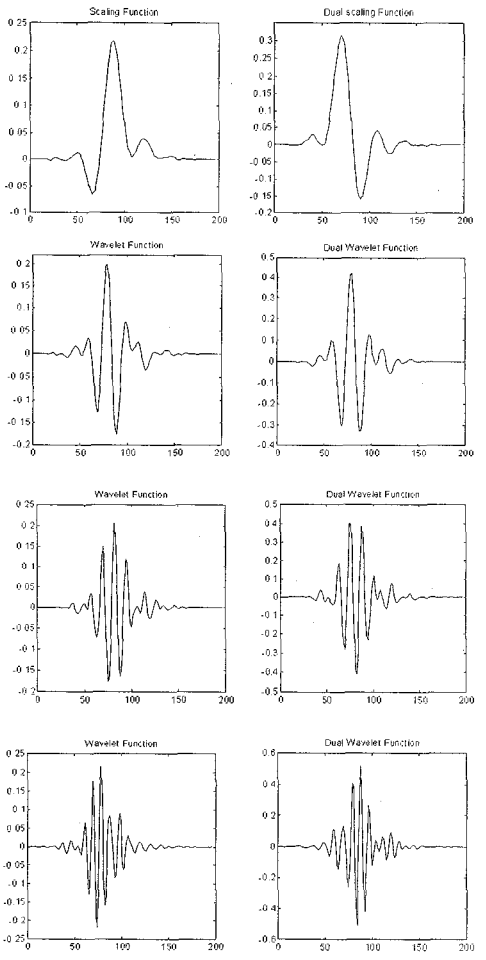

Fig.1 Scaling and wavelet functions and their dual functions

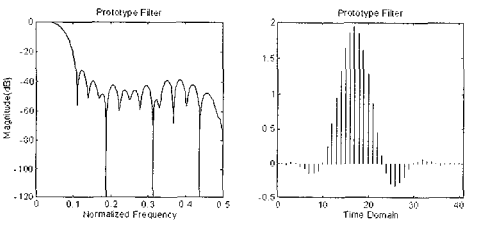

Fig.2 Prototype filter of 4-channel CMFB
(a) Frequency response (b) Impulse response