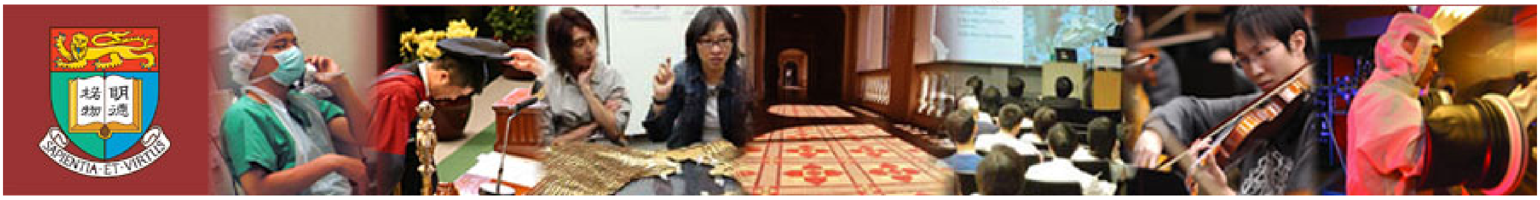

Iterative CZT-Based Frequency Offset Estimation for Frequency-Selective Channels

Feng-Xiang Ge, Dongxu Shen, Aifen Sui and Victor O.K. Li
Department of Electrical & Electronic Engineering
The University of Hong Kong, Hong Kong, China
Email: {fxge, dxshen, vli}@eee.hku.hk

***Abstract*— In this paper, we present an accurate frequency offset estimation method for frequency-selective channels. Through the iterative use of the chirp z-transform (CZT) algorithm, an accurate frequency offset estimator is proposed, approaching the Cramer-Rao Bound (CRB) even at low signal-to-noise ratio (SNR). Further, the estimation can be achieved within one block of training sequence, thus avoiding the transmission of repetitive known blocks as is usually required in many conventional methods. Meanwhile, the overall complexity is acceptable. More importantly, the CZT operation can utilize the fast Fourier transform (FFT) structure that is favourable for digital signal processor (DSP) implementation. Simulation results show that two or at most three iterations of the CZT computation are sufficient for an accurate frequency offset estimation in the SNR range from 0 dB to 30 dB.**

I. Introduction

Wireless communication systems are subject to channel impairments such as multipath propagation and fading in addition to additive noise [?], [?]. Frequency offset due to either limited oscillator precision or the Doppler shift can cause a significant performance loss in the coherent detection [?], [?] if the nontrivial frequency offset is not properly compensated. Frequency offset compensation in such systems will help improve their performance. Many wireless communication systems operating over frequency-selective fading channels employ training sequences (TS) to estimate the frequency offset and the channel impulse response [?]-[?].

The objective of this paper is to develop a frequency offset estimator for frequency-selective channels. In [?], TS is utilized for frequency offset estimation and a maximum likelihood (ML) estimator is derived. The frequency offset estimation requires a two-step procedure. First is a coarse search as in [?]. Then various fine-frequency estimators based on interpolation techniques have been used for the fine search ([?]-[?] and references therein) to improve the estimate accuracy. However, the performance of the interpolation-based techniques are frequency dependent. Furthermore, they involve operations of division and square root, which are undesirable for digital signal processing (DSP) implementation. The pulse-pair based (PP-based) methods [?]-[?] always perform a correlation across the TSs first, then obtain the angle of the cross-correlation result ([?] and references therein). The PP-based methods are quite simple, thus amenable for implementation. However, its performance is only satisfactory at sufficiently high signal-to-noise ratio (SNR); in other words, its performance deteriorates significantly in the low SNR region [?]. The performance of the PP-based methods can be improved with more TSs. However, this improvement is achieved at the price of increased redundancy, and the overhead of the TSs can be significant when the data burst is short.

In this paper, we propose an accurate frequency offset estimation algorithm based on the iterative use of the chirp z-transform (CZT). The algorithm works in the following steps: 1) determine the approximate range of the true frequency offset, based on the fast Fourier transform (FFT); 2) over the approximate range, CZT is carried out to obtain a better frequency offset estimate; 3) based on the previous frequency offset estimation result, a finer region for the next CZT operation can be identified; 4) repeat steps 2 and 3 until a satisfactory frequency offset estimate is obtained.

Our algorithm has a number of desirable properties. First, it is an accurate frequency offset estimation algorithm. Since the search range of CZT is much narrowed after each iteration, performance enhancement is obtained with each additional application of the CZT algorithm, until the desired accuracy has been achieved. Second, our algorithm does not require the transmission of multiple TSs, and the whole operation can be finished within one TS. Thus, the above two contributions make our proposed method much better than the PP-based methods [?]-[?]. Third, the computational complexity is also acceptable, and is feasible for practical implementation. An iterative approach is also adopted in [?], which proposes a method called Luise and Reggiannini estimator with iterative filtering (LRIF). However, its complexity is higher than our approach. Further, the CZT algorithm can be realized by using the FFT structure that is favourable for DSP implementation, thus avoiding extra hardware cost. Through simulations, we show that two or at most three iterations of the CZT algorithm are sufficient to approach the Cramer-Rao Bound (CRB).

This paper is organized as follows. In Section II, we present the signal model for frequency-selective channels. In Section III, we propose the frequency offset estimation method based on the iterative use of the CZT, and analyze the performance and computational load. In Section IV, simulation results are presented. Conclusion is given in Section V.

This work is supported in part by the Research Grant Council of Hong Kong, under grant No. HKU 7047/00E.

II. Signal Model

Under the assumption of a linear modulation (e. g. PSK and QAM) and a frequency-selective channel with a slow variation in time compared with the signaling interval, the received signal sampled at the symbol rate are expressed by

$$y(n)=e^{j2\pi n v_0}x(n)+w(n), \quad n=0,1,\cdots,N-1, \tag{1}$$

where v_0 is the normalized frequency offset which results from oscillator frequency mismatch and Doppler shift, $w(n)$ is the sample of zero-mean complex Gaussian noise with variance σ_w^2, and $x(n)$ is given by

$$x(n)=\sum_{l=0}^{L-1}h(l)s(n-l), \quad n=0,1,\cdots,N-1, \tag{2}$$

where $\mathbf{h}=\left[\begin{array}{cccc} h(0) & h(1) & \cdots & h(L-1)\end{array}\right]^T$ is the channel impulse response that has a length of L, $s(n)$, $-L+1\leq n\leq N-1$, are the training symbols, and T denotes transpose. The received signal can be written in vector form as

$$\mathbf{y}=\Gamma(v_0)Sh+w, \tag{3}$$

where $\mathbf{y}=\left[\begin{array}{cccc} y(0) & y(1) & \cdots & y(N-1)\end{array}\right]^T$, $\mathbf{\Gamma}(v_0)=\text{diag}\left\{\begin{array}{cccc} 1 & e^{j2\pi v_0} & \cdots & e^{j2\pi(N-1)v_0}\end{array}\right\}$, $[\mathbf{S}]_{i,j}=s_{i-j}$, $\mathbf{w}=\left[\begin{array}{cccc} w(0) & w(1) & \cdots & w(N-1)\end{array}\right]^T$ and $\mathbf{R_w}=E(\mathbf{w}\mathbf{w}^H)=\sigma_w^2\mathbf{I}_N$, $\mathbf{I}_N$ is the $N\times N$ identity matrix, $E(\cdot)$ is the expectation operator, and H denotes Hermitian transpose.

III. Frequency Offset Estimation

In this section, we present our iterative CZT-based algorithm for frequency offset estimation, and analyze its performance and computational complexity.

A. ML Frequency Offset Estimation

From (**??**), the likelihood function for the parameter v_0 is

$$\Lambda(v_0)=\frac{1}{(\pi\sigma_w^2)^N}\exp\left\{-\frac{[\mathbf{y}-\Gamma(v_0)Sh]^H[\mathbf{y}-\Gamma(v_0)Sh]}{\sigma_w^2}\right\}. \tag{4}$$

Then the maximum likelihood (ML) estimate of v_0 is given by

$$\hat{v}_0=\arg\max_v\left\{\sum_{n=0}^{N}y(n)s^*(n)e^{-j2\pi nv}\right\}. \tag{5}$$

The parameter estimation in (**??**) can be efficiently computed through discrete/fast Fourier transform (DFT/FFT). In frequency offset estimation and other applications ([**?**] and references therein), however, we are only interested in obtaining frequency samples equally spaced over a portion of the unit circle, rather than the whole circle. Unfortunately, the DFT/FFT results are over the whole frequency range, including the points outside the region of interest. To improve the frequency resolution of the DFT/FFT method, one possible scheme is to augment the original sequence with zero-padding, which increases the computational load. A more efficient method to achieve the above purpose is to adopt the chirp z-transform (CZT) [**?**], [**?**].

B. Chirp Z-Transform (CZT)

The z-transform of a sequence $d(n)$, $n=0,1,\cdots,N-1$, is given by

$$D(z)=\sum_{n=0}^{N-1}d(n)z^{-n}. \tag{6}$$

When the above operation is computed at equal-spaced points around the unit circle, i.e., $z=z_k=e^{j2\pi k/N}$, this sampled z-transform in (**??**) is equivalent to the familiar DFT [**?**].

Using CZT [**?**], [**?**], $D(z_k)$ can be computed at the points z_k given by

$$z_k=AW^{-k}, \qquad k=0,1,\cdots,K-1, \tag{7}$$

where $W=W_0e^{-j2\pi\zeta_0}$, $A=A_0e^{j2\pi\rho_0}$, with W_0 and A_0 being positive real numbers. The results from CZT can be viewed as the contour of a spiral in the z-plane along which the samples are obtained. The parameter W_0 controls the rate at which the contour spirals. The parameters A_0 and $2\pi\rho_0$ are the location in radius and angle, respectively, of the first sample, i.e., for $k=0$. The remaining samples are located along the spiral contour with an angular spacing of $2\pi\zeta_0$. Consequently, if $W_0=1$, the spiral is, in fact, a circular arc, and if $A_0=1$, this circular arc is part of the unit circle. In this paper, we set $W_0=1$ and $A_0=1$, and let $K=N$. Then the CZT starts from $2\pi\rho_0$ to $2\pi\rho_0-(N-1)\cdot\zeta_0$, covering a range of $2\pi\zeta_0 N$. Thus, the performance of the CZT algorithm can only be the same as that of the DFT algorithm when the sequence $y(n)$ is zero-padded to length $1/\zeta_0\geq N$ when $N\zeta_0\leq 1$. Obviously, the complexity of DFT will always be much higher than that of CZT, depending on the frequency range of interest. From the Bluestein identity [**?**], [**?**], CZT can also be efficiently computed by FFT.

C. Iterative CZT-Based Estimator

In our approach, the high accuracy frequency offset estimation algorithm is achieved by the iterative use of CZT. The procedure is given as follows.

First, the optimization involved in (**??**) can be computed by the FFT. Thus, we can obtain a coarse frequency offset estimate $\hat{\varepsilon}_0$ and determine the approximate range of the true frequency offset. $\Delta v_0=1/N$ is the frequency sampling interval.

Then an fine-frequency offset estimator based on iterative CZT can be performed, and a finer frequency offset estimate can be obtained based on the previous frequency offset estimation result and the finer search range.

Let $\hat{\varepsilon}_i$ and Δv_i, $i=1,2,3,\cdots$, be the estimated frequency offset and the normalized frequency sampling interval of the ith CZT iteration. Then the range for the ith CZT iteration can be determined as

$$-\left(\Delta v_{i-1}/2+p\sigma_v\right)+\hat{\varepsilon}_{i-1}<v<\left(\Delta v_{i-1}/2+p\sigma_v\right)+\hat{\varepsilon}_{i-1}, \quad i=1,2,3,\cdots, \tag{8}$$

where σ_v^2 is the CRB of frequency offset estimate given by [**?**], [**?**] when there is no error from the finite frequency sampling interval (which has been accounted by Δv_{i-1}). Thus, $\Delta v_i=$

$(\Delta v_{i-1} + 2p\sigma_v)/N$, $i = 0, 1, 2, \cdots$. The error boundary terms in (**??**) are further explained as follows.

The DFT/FFT is an unbiased maximum likelihood estimator (MLE) at moderately high SNR [**?**], and the asymptotic probability density function (PDF) of the maximum likelihood estimate for a scalar parameter is Gaussian distributed [**?**]. Furthermore, since CZT can be viewed as a refined implementation of the DFT [**?**], [**?**], the frequency estimate $\hat{\varepsilon}_{i-1}$, $i = 1, 2, 3, \cdots$, should be a Gaussian random variable with zero-mean and variance σ_v^2 when there is no error caused by the finite sampling interval (i.e., assuming $N \to \infty$, the sampling interval is infinitely small). Therefore, we choose $\pm p\sigma_v$ as the possible range for v as in (**??**), where p should be a positive value that is high enough so that the probability for v to fall outside of $(-p\sigma_v + \hat{\varepsilon}_{i-1},\ p\sigma_v + \hat{\varepsilon}_{i-1})$, $i = 1, 2, 3, \cdots$, (occurrence of out-lier) is very small. For example, when $p = 4$, the probability for having an outlier is smaller than 0.0001.

Since the errors Δv_{i-1} and σ_v are independent, we have the range in (**??**) for the iterative CZT computation.

After determining the finer search region, another CZT operation can be performed to produce

$$\hat{\varepsilon}_i = \arg \max_{(k\Delta\varepsilon_i)} \left\{ \left| \sum_{n=0}^{N-1} y(n) s^*(n) e^{-j2\pi(\varepsilon_i + k\Delta\varepsilon_i)n} \right|^2 \right\}, \quad i = 1, 2, 3, \cdots, \tag{9}$$

where $\varepsilon_i = -(\Delta v_{i-1}/2 + p\sigma_v) + \hat{\varepsilon}_{i-1}$ is the location in normalized frequency of the first sample, and $\Delta\varepsilon_i = 2 \times (\Delta v_{i-1}/2 + p\sigma_v)/N$ is the finer step size.

Here we specify the stopping rule for the CZT iteration. If Δv_i, $i = 1, 2, \cdots$, is smaller than the required frequency offset precision, or if Δv_i is on the order of σ_v, CZT is terminated; otherwise the CZT loop computation continues until the above conditions are met. Finally, we obtain the frequency offset estimate $\hat{\varepsilon}_M$, where M is the total number of the CZT iterations. Considering the CRB of the frequency offset estimate, 2 or 3 times of CZT iterations are always sufficient for achieving highly accurate frequency offset estimate, which is also confirmed by simulation results. In practice, the term σ_v^2 cannot be known exactly due to the unknown SNR, and has to be approximated. For this purpose, we always assume SNR takes a fixed and small SNR value. In this way, we can have a larger region for the CZT search, reducing the outlier probability.

As a summary, the framework of the proposed method is shown in Fig. 1. The control module (CM) in Fig. 1 is responsible for determining the number of CZT iterations.

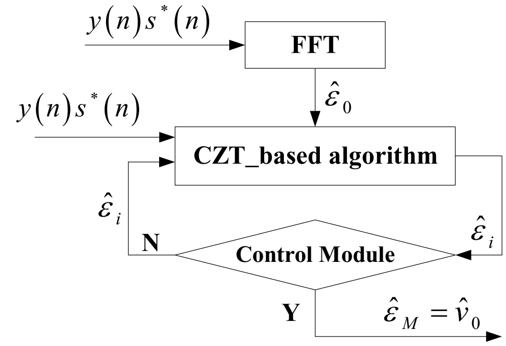

Fig. 1. The framework of the iterative CZT-based frequency offset estimation.

D. Analysis of the Iterative CZT-Based Method

In our algorithm, the final estimation error of the frequency estimate consists of two parts. One is the error resulting from the frequency quantization. The frequency quantization can be assumed to be uniformly distributed in the range of $(\Delta v_M)^2$, where Δv_M is the normalized frequency sampling interval of the final CZT iteration. Obviously, the mean squared error (MSE) contributed by this part is $(\Delta v_M)^2/12$. The other is from the frequency estimator itself, which is σ_v^2. The above two errors are independent. When the CRB of the frequency offset estimate is approached at the last iteration, the final MSE of the frequency offset estimate ($\hat{v}_0 = \hat{\varepsilon}_M$) is on the order of

$$\text{MSE} \sim \frac{(\Delta v_M)^2}{12} + \sigma_v^2. \tag{10}$$

Since the CZT iteration is not terminated until Δv_i, $i = 1, 2, 3, \cdots$, is on the order of σ_v, $(\Delta v_M)^2$ is on the order of σ_v^2 and $(\Delta v_M)^2/12 \ll \sigma_v^2$, i.e., the MSE of the frequency offset estimate approaches σ_v^2.

E. Comparison of Computational Load

Since the computational load of complex multiplications is much higher than that of complex additions, we use the number of complex multiplications as an indicator for the complexity of different frequency offset estimation methods.

Based on the Bluestein identity, the computational load (CL) of CZT can be reduced significantly. The number of complex multiplications required for the whole estimation algorithm is on the order of [**?**], [**?**]

$$\begin{aligned} \text{CL}_{CZT} &= T_{\text{CZT}} \cdot (3N \log_2 N + 11N) \\ &= 3T_{\text{CZT}} N \log_2 N + 11 T_{\text{CZT}} N, \end{aligned} \tag{11}$$

We then compare the complexity of our iterative CZT-based algorithm with other popular methods. For an FFT estimator to achieve the same estimation accuracy, data should be zero-padded, which means a large number of frequency samples will be computed. The number of complex multiplications for FFT with zero-padding is on the order of

$$\text{CL}_{FFTzp} = \frac{N_{zp}}{2} \cdot \log_2 N_{zp}, \tag{12}$$

where $N_{zp} = 2\pi/\phi_0$, and ϕ_0 is the bin-width for the last CZT operation, since the sequence should be zero-padded to length N_{zp}. For example, if the accuracy of the normalized frequency estimate is less than 10^{-4}, then N_{zp} should be larger than 10^4, which makes $\text{CL}_{FFT} \gg \text{CL}_{CZT}$.

The number of complex multiplications required for the traditional PP method with one block of training sequence is on the order of

$$\text{CL}_{PP} > N. \tag{13}$$

This computational load is less than that of the CZT-based method. However, the traditional PP method only yields good performance at sufficiently high SNR (SNR $\gg 1$) [?], [?].

The number of complex multiplications required for the computation of the Luise and Reggiannini estimator with iterative filtering (LRIF) [?] is on the order of

$$\mathrm{CL}_{LRIF} = 4N(\chi+1) + T_{\mathrm{IF}}(16N-10), \quad (14)$$

where χ denotes the parameter for the LRIF method and the optimal value is $\chi = N/2$, T_{IF} is the number of iterations and T_{IF} always varies between 2 and 4. We show in the next section that LRIF is more complex than our method.

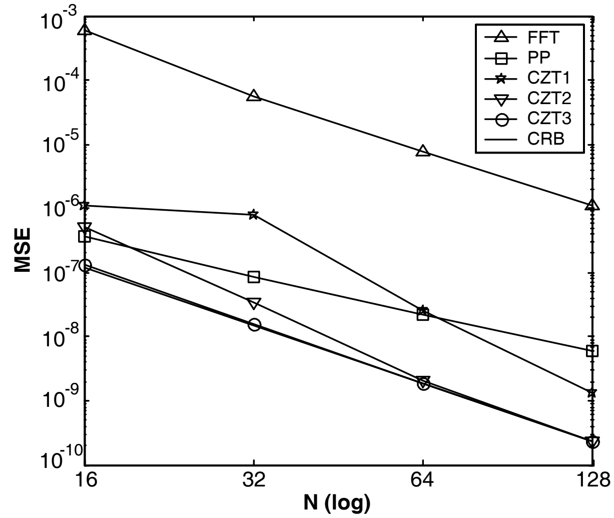

Fig. 3. CRB and MSE of frequency offset estimation versus N.

IV. SIMULATION RESULTS

To demonstrate the performance of the proposed method, simulations have been performed for frequency-selective channels. The training sequence is a QPSK format with only one block. The channel impulse response is given by

$$h(k) = \sum_{i=0}^{5} A_i g(kT_s - \tau_n - t_0), \quad (15)$$

where A_n and τ_n are the random attenuation and delay of each path, T_s is the sampling period, $t_0 = 3T_s$, and $g(t)$ is a raised cosine rolloff filter with a rolloff of 0.5. Simulation results are obtained through 3000 independent Monte Carlo trials.

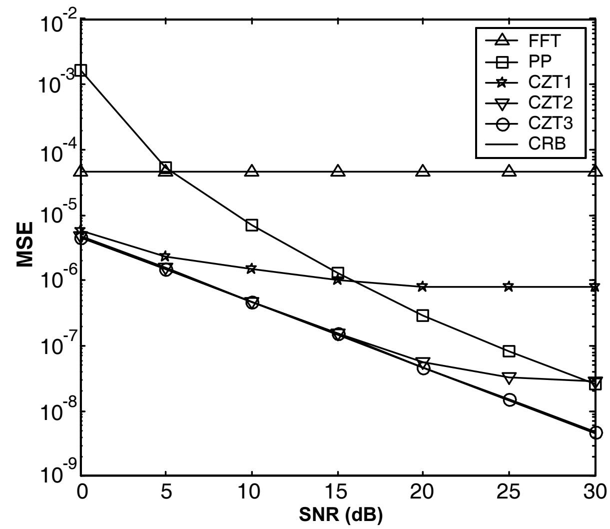

Fig. 2. CRB and MSE of frequency offset estimation versus SNR for different estimators. FFT: FFT method without zero-padding; PP: traditional PP method; CZT1, CZT2, CZT3: CZT method with 1, 2, and 3 iterations.

A. MSE Performance

Fig. 2 shows the MSEs of the frequency offset estimation for different SNRs, which are also compared with the CRB. The SNR is defined as $\mathrm{SNR} = \frac{1}{N}\sum_{n=0}^{N-1} x^2(n)/\sigma_w^2$. In the experiment, sequence length $N = 32$. In the SNR range from 0 dB to 30 dB, it is shown that CZT2 has better performance than that of the traditional PP method, and CZT3 approaches the corresponding CRB. Due to the error resulting from frequency quantization, the FFT, CZT1 and CZT2 algorithms all exhibit MSE floors at different SNR. However, we can reduce the error with more CZT iterations. Thus, it is always possible to approach CRB of the frequency offset estimate for our algorithm. In Fig. 2, it is shown that our proposed method can still achieve highly accuracy frequency offset estimate even at low SNR.

Fig. 3 shows the MSEs of the frequency offset estimation for different value N, which are also compared with the CRB at a fixed SNR of 25 dB. For N from 16 to 128, it is shown that CZT2 has better performance than that of the traditional PP method and CZT3 approaches the corresponding CRB. We should note that even under an SNR as high as 25 dB, the traditional PP and FFT still cannot approach CRB, while CZT3 does.

B. Complexity Comparison

In Fig. 4 and Fig. 5, the computational complexity comparison of four typical frequency offset algorithms is illustrated: FFT with zero-padding ($\mathrm{CL}_{FFT_{ZP}}$), LRIF (CL_{LRIF}), CZT with 1, 2 and 3 iterations(CL_{CZT1}, CL_{CZT2}, CL_{CZT3}), and the traditional PP algorithm (CL_{PP}). It is obvious PP has

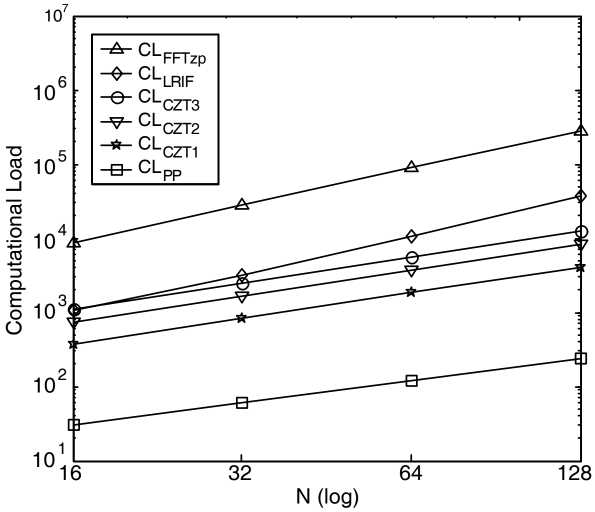

Fig. 5. Computational load of frequency offset estimation versus N. $\mathrm{SNR} = 25\mathrm{dB}$ and $T_{\mathrm{IF}} = 2$.

the least complexity, and CZT is more complex than PP by almost an order of magnitude. However, it should be noted that the performance of PP is much inferior to that of CZT2, and CZT2 is still less complex than the other two methods. Furthermore, the CZT operation can utilize the FFT structure that is favourable for DSP implementation.

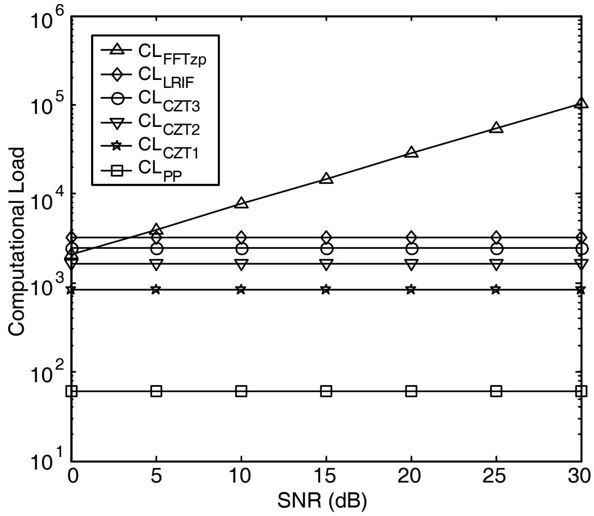

Fig. 4. Computational load of frequency offset estimation versus SNR. $N = 32$ and $T_{\mathrm{IF}} = 2$.

V. CONCLUSIONS

In this paper, we propose a novel frequency offset estimation algorithm for frequency-selective channels. Through the iterative use of the CZT algorithm, highly accurate frequency offset estimation can be achieved, approaching the CRB, even at low SNR. Further, the estimation can be finished within one block of training sequence, thus avoiding the transmission of repetitive known blocks as is usually required in many conventional methods. Meanwhile, the complexity is only higher than the PP method, but is still lower than other approaches that deliver the same level of accuracy. More importantly, the CZT operation can utilize the FFT structure that is favourable for DSP implementation. Simulation results show that two or at most three iterations of CZT is sufficient for an accurate estimation in the SNR range from 0 dB to 30 dB. In all, our approach is an accurate, flexible and affordable method for frequency offset estimation.

REFERENCES

[1] T. S. Rappaport, "*Wireless communications: principles and practice*," 2nd ed., N.J.: Prentice Hall, 2002.

[2] J. Proakis, *Digital Communications*, 3rd ed., New York: McGraw-Hill,1995.

[3] M. G. Hebley, and D.P. Taylor, "The effect of diversity on a burst-mode carrier-frequency estimator in the frequency-selective multipath channel," *IEEE Trans. Commun.*, vol. 46, no. 4, pp. 553 - 560, 1998.

[4] E. R. Jeong, G. Choi, Y. H. Lee, "Data-aided frequency estimation for PSK signaling in frequency-selective fading," *IEEE J. Select. Areas Commun.*, vol. 19, no. 7, pp. 1408 - 1419, 2001

[5] M. Morelli, and U. Mengali, "Carrier-frequency estimation for transmissions over selective channels," *IEEE Trans. Commun.*, vol. 48, no. 9, pp. 1580-1589, 2000.

[6] H. Viswanathan, and R. Krishnamoorthy, "A frequency offset estimation technique for frequency-selective fading channels," *IEEE Commun. Lett.*, vol. 5, no. 4, pp. 166-168, 2001.

[7] P. Stoica, and O. Besson, "Training sequence design for frequency offset and frequency-selective channel estimation," *IEEE Trans. Commun.*, vol. 51, no. 11, pp. 1910-1917, 2003.

[8] D. Rife, R. Boorstyn, "Single tone parameter estimation from discrete-time observations," *IEEE Trans. Info. Theory*, vol. 20, no. 5, pp. 591 - 598, 1974.

[9] M. D. Macleod, "Fast nearly ML estimation of the parameters of real or complex single tones or resolved multiple tones," *IEEE Trans. Signal Processing*, vol. 46, no. 1, pp. 141-148, 1998.

[10] E. Aboutanios, "A modified dichotomous search frequency estimator," *IEEE Signal Processing Lett.*, vol. 11, no. 2, pp. 186-188, 2004.

[11] S. M. Kay, *Modern spectral estimation: Theory and Application,* N. J.: Prentice Hall, 1988.

[12] M. Ghogho, A. Swami, and T. S. Durrani, "Frequency estimation in the presence of Doppler spread: performance analysis," *IEEE Trans. Signal Processing*, vol. 49, no. 4, pp. 771-789, 2001.

[13] S. Kay, "Statistically/computationally efficient frequency estimation," *Proc. ICASSP*, Apr. 1988, pp. 2292-2295.

[14] G. Olmo, L. Presti, D. Dosetto, "Progressive refinement approach to MLE: An application to carrier frequency recovery," *IEEE Trans. Commun.*, vol. 49, no. 7, pp. 1148-1153, Jul. 2001.

[15] L. Rabiner, R. Schafer, and C. Rader, "The chirp z-transform algorithm," *IEEE Trans. Audio and Electroacoustics*, vol. 17, no. 2, pp. 86 - 92, 1969.

[16] A. V. Oppenheim, and R. W. Schafer, *Digital signal processing*, N. J.: Prentice-Hall, 1975.

[17] S. M. Kay, *Fundamentals of statistical signal processing: estimation theory,* N. J.: Prentice Hall, 1993.