Approximation for Minimum Triangulation of Convex Polyhedra

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Abstract

The minimum triangulation of a convex polyhedron is a triangulation that contains the minimum number of tetrahedra over all its possible triangulations. Since finding the minimum triangulation of convex polyhedra was recently shown to be NP-hard, it becomes significant to find algorithms that give good approximation. In this paper, we give a new triangulation algorithm with an improved approximation ratio $2 - \Omega(\frac{1}{\sqrt{n}})$. We also show that this is best possible for algorithms that only consider the combinatorial structure of the polyhedra.

1 Introduction

Triangulation is the subdivision of d-dimensional polyhedron into simplices. In this paper we are concerned with triangulations of 3-dimensional convex polyhedra with vertices in general position. Triangulation has important applications in computer-aided design, computer graphics, finite element analysis, etc.

Triangulation in 3-D has many interesting properties. Convex polyhedra can always be triangulated, but non-convex polyhedra may not: the Schönhardt polyhedron [8] is such an example. It is even NP-complete to determine whether a given non-convex polyhedron can be triangulated [7]. Different triangulations of convex polyhedra may result in different numbers of tetrahedra, and finding the minimum triangulation was recently shown to be NP-hard [2], [3]. Thus designing good approximation algorithms for this problem becomes significant. We modified a well-known triangulation heuristic to obtain a better bound of $2 - \Omega(\frac{1}{\sqrt{n}})$ on the approximation ratio in Section 3. On the other hand, it is shown in [1] that the minimum triangulation of polyhedra is not an invariant of the face lattice. In Section 4 we extend this to show that any algorithm that only considers the combinatorial structure of polyhedra cannot give an approximation ratio better than $2 - O(\frac{1}{\sqrt{n}})$. Thus our algorithm is best possible in this sense.

We begin with a few definitions. A *dome* of a vertex v in a polyhedron P is the region between P and the convex hull (CH) of $P - v$. An edge in a triangulation is called an *interior edge* if it does not lie on the surface of the polyhedron. A *3-cycle* is a closed path of three edges on the surface graph of a polyhedron such that each side of the cycle contains at least one vertex not on the cycle. Throughout this paper, let n denote the number of vertices of a given polyhedron, e_i the minimum number of interior edges required in any triangulation of the polyhedron, and Δ the maximum degree on the surface graph. For any polyhedron, the number of interior edges e in a triangulation is directly related to the number of tetrahedra t by the formula $t = e + n - 3$ [5]. This implies a lower bound of $n - 3$ tetrahedra for any triangulation.

One way to triangulate a convex polyhedron P is to remove the dome of a vertex v

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(which can easily be triangulated) to get a smaller polyhedron $CH(P - v)$, and then to iterate this process for $CH(P - v)$. This is known as 'peeling' [5]. Peeling a vertex of degree d gives $d - 2$ tetrahedra. It can be shown that peeling yields an approximation ratio of $3 - \Omega(\frac{1}{n})$. 'Fanning' (also called 'coning', 'starring' or 'emission') is another well-known approach. Fanning picks a vertex of the polyhedron and uses it to form tetrahedra with every non-adjacent triangular facets, producing $2n - 4 - \Delta \leq 2n - 7$ tetrahedra, which, considering the $n - 3$ lower bound, gives an approximation ratio of $2 - \Omega(\frac{1}{n})$.

As finding the minimum triangulation of convex polyhedra is shown to be NP-hard, we wish to find better approximation algorithms for this problem. However, it is shown in [9] that there exist polyhedra requiring as many as $2n - 10$ tetrahedra for any triangulation. For $n > 12$, $\Delta \geq 6$ and the fanning heuristic gives no more than $2n - 10$ tetrahedra. Thus the fanning heuristic is worst-case optimal, in terms of the absolute number of tetrahedra produced. This motivates us to analyse the approximation ratio of the fanning heuristic.

2 Vertex-Edge Chain Structure

A key structure that appeared in [1] will be used extensively in this paper. This so-called '*vertex-edge chain structure*' (VECS) consists of $2m + 2$ faces (a, q_i, q_{i+1}) and (b, q_i, q_{i+1}) for $i = 0, ..., m$, with the additional restriction that the line segment $q_0 q_{m+1}$ goes through the interior of the polyhedron formed by tetrahedra abq_iq_{i+1} for all i. This polyhedron is called a *wedge* (Fig. 1). All the $2m+2$ faces lie on its convex hull. We say this VECS has *size* m, and call edge ab the *main diagonal.* A tetrahedron in a triangulation is said to be *incident* to a VECS if at least 3 out of its 4 vertices belong to the VECS.

We now extend a lemma in [1] concerning VECS.

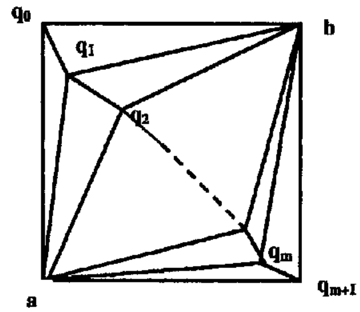

Figure 1: A VECS of size m.

LEMMA 2.1. *(i) In a polyhedron containing a VECS of size m as a substructure, if the main diagonal is not present in a triangulation, at least $2m$ tetrahedra must be incident to the VECS.*
(ii) If a polyhedron has k mutually disjoint VECSs each of size m as substructures, all without the main diagonal in a triangulation, at least $2mk$ tetrahedra are incident to these VECS.

Proof. (i) There are $2m+2$ faces in the VECS. If each face is associated with a separate tetrahedron, then there exist at least $2m$ tetrahedra. Faces aq_iq_{i+1} and bq_iq_{i+1} cannot be in the same tetrahedron because this will induce edge ab. If $aq_{i-1}q_i$ and aq_iq_{i+1} are in the same tetrahedron, (similarly for $bq_{i-1}q_i$ and bq_iq_{i+1}), $aq_{i-1}q_{i+1}$ will be a new face incident to this VECS, which determines another incident tetrahedron. If this new face is in the same tetrahedron with other face triangle, this will in turn determine another incident tetrahedron. This process can continue until the two faces aq_0q_{m+1} and bq_0q_{m+1} appear. A simple induction argument on this idea can show that at least $2m$ tetrahedra are incident to this VECS.
(ii) Consider k mutually disjoint VECSs. A tetrahedron incident to one VECS has at least 3 vertices in that VECS, leaving at most one vertex not in that VECS, thus this tetrahedron cannot be incident to other VECSs.

Therefore, the k VECS must have $2mk$ distinct incident tetrahedra. □

Two VECSs $(a, b, q_0, ..., q_{m+1})$ and $(a', b', q'_0, ..., q'_{m'+1})$ are said to be *interlocked* if ab intersects faces $a'b'q'_0$ and $a'b'q'_{m'+1}$, and $a'b'$ intersects faces abq_0 and abq_{m+1}.

LEMMA 2.2. *Given k VECSs in an interlocked position, at most one of the main diagonals can be used to triangulate its corresponding VECS, so as to reduce the number of tetrahedra incident to that VECS.*

Proof. To use a main diagonal to reduce the number of tetrahedra incident to a particular VECS, two triangle faces of that VECS have to form a tetrahedra with the main diagonal (see proof of Lemma 2.1). Thus any other interlocked VECS's main diagonal, if used, will penetrate this tetrahedron, thus cannot be used for triangulation. □

3 The Triangulation Algorithm

3.1 A restricted case. We first consider the case when the given polyhedron has no 3-cycles. This restriction will be removed in the next subsection. We analyse the approximation ratio of the fanning heuristic in this case.

LEMMA 3.1. *Let P be a convex polyhedron that has no 3-cycles, $n > 4$. Then*
(i) At least one interior edge must be incident to the dome of any vertex;
(ii) $e_i + \Delta \geq \sqrt{2n} - 1$, and this is tight within a constant factor.

Proof. (i) Consider a particular vertex v_0 and its neighbors v_1, v_2, ..., v_k where $4 \leq k \leq \Delta$. Note that if $k = 3$, P contains a 3-cycle. The triangle $v_0v_1v_2$ must belong to a tetrahedron. If the fourth vertex v of this tetrahedron is one from v_3, v_4, ..., v_k, then either v_1v or v_2v (or both) is an interior edge (otherwise there would be a 3-cycle). If v is not one from $v_3, v_4, ..., v_k$, then v is not a neighbor of v_0 and hence v_0v is an interior edge. In either case, an interior edge is incident to the dome of v_0.
(ii) Suppose we count the total number of interior edges by counting the number of interior edges incident to all the n possible domes. By (i) above, this number is greater than n. However, each interior edge is counted more than once; it is counted by the domes of their two endpoints as well as the endpoints' neighbors. The total number of these domes does not exceed $2(\Delta + 1)$. Therefore each edge is counted at most $2(\Delta + 1)$ times, giving $e_i \times 2(\Delta + 1) \geq n$. Applying the Arithmetic-Geometric-Mean inequality [10], $\frac{e_i+(\Delta+1)}{2} \geq \sqrt{e_i(\Delta + 1)}$, thus $e_i + \Delta \geq \sqrt{2n} - 1$.

We construct a convex polyhedron (Fig. 2) to show this inequality is tight up to constant factor. The polyhedron consists of a prism with top and bottom faces being a convex m-gon. Then on each rectangular side faces we attach a VECS of size m. The structures are placed in such a way that their main diagonals are compatible with a triangulation of the prism. They are also made flat enough so that the resulting polyhedron remains convex. This polyhedron has $\Theta(m^2)$ vertices; the maximum hull degree Δ is $\Theta(m)$; and the interior edges are those needed to triangulate the prism, which is $\Theta(m)$ since there are only $\Theta(m)$ vertices in the prism. These edges will automatically triangulate the attached VECSs. It can also be easily seen that this is the minimum triangulation. Thus $e_i + \Delta = \Theta(m) = \Theta(\sqrt{n})$. □

The approximation ratio of the fanning heuristic under the no-3-cycle restriction is shown in the lemma below.

LEMMA 3.2. *Using the fanning heuristic, the approximation ratio r is bounded above by $\frac{2n-\sqrt{2n}-2}{n-2}$.*

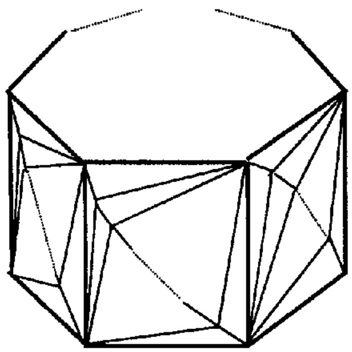

Figure 2: Tight example for Lemma 3.1(ii).

Proof. The approximation ratio is bounded by $\frac{2n-4-\Delta}{n-3+e_i}$. Hence by Lemma 3.1(ii)

$$r \leq \frac{2n-4-(\sqrt{2n}-1-e_i)}{n-3+e_i}$$
$$= \frac{2n-\sqrt{2n}-2+(e_i-1)}{n-2+(e_i-1)}$$
$$\leq \frac{2n-\sqrt{2n}-2}{n-2},$$

since $e_i \geq 1$ by Lemma 3.1(i). □

3.2 The general case. We now deal with the general case for polyhedra having 3-cycles. We first find all 3-cycles by using algorithms such as those in [6] and [4]. Then we cut along all 3-cycles to produce subpolyhedra, each is free of 3-cycles. Finally we apply the fanning heuristic to each subpolyhedra. The algorithm is shown below.

ALGORITHM 3.1. CutFan(P)
Input: A convex polyhedron P with n vertices.
Output: T, a set of tetrahedra that triangulates P.

/* partition the polyhedron into 3-cycle-free components */
$\mathcal{P} \leftarrow \emptyset$
$C \leftarrow$ Enumerate-triangles(P) /* finds all 3-cycles, but also include faces of P */
$\mathcal{P} \leftarrow$ polyhedra obtained by cutting P through all 3-cycles in C
/* apply fanning to each subpolyhedron */
$T \leftarrow \emptyset$
for each polyhedron Q in $\mathcal{P}$
  pick a vertex v of highest degree in Q
  $T \leftarrow T \cup$ { the set of tetrahedra of Q fanned from v }
End.

LEMMA 3.3. *Algorithm 3.1 takes $O(n)$ time.*

Proof. We need to describe the algorithm in more detail in order to prove the time complexity. Finding all 3-cycles takes linear time [6], [4]. The process of cutting is as follows. Each cycle divides the surface graph into an interior part and an exterior part. We select an edge and use breadth first search to find all vertices inside the cycle, without crossing other cycles. If the edge selected is on some other cycles nested inside, we process them recursively first. The set of vertices reached determines a sub-polyhedron. We remove the edges traversed. Since the search and removal takes time proportional to the size of the components, and each edge is processed a constant number of times only, they take $O(n)$ time in total. Finally, picking maximum degree vertices and fanning also take $O(n)$ time for all subpolyhedra. □

LEMMA 3.4. *Let P be a convex polyhedron with maximum degree Δ and having a 3-cycle. Suppose we cut through the 3-cycle to produce two polyhedra P_1 and P_2, having maximum degrees Δ_1 and Δ_2 respectively. Then $\Delta \leq \Delta_1 + \Delta_2 - 2$.*

Proof. Clearly, $3 \leq \Delta_1 \leq \Delta, 3 \leq \Delta_2 \leq \Delta$. We apply a case-by-case analysis.
Case 1. Δ is not on the 3-cycle being cut. Then it is at one side (or both) of P_1 and P_2, say $\Delta_1 = \Delta$. Then $\Delta_1 + \Delta_2 - 2 \geq \Delta + 3 - 2 > \Delta$.
Case 2. Δ is on the 3-cycle being cut. Let d_1,

d_2 be the new degrees in P_1, P_2 respectively at the original maximum-degree vertex. Then $\Delta = d_1 + d_2 - 2$. There are two subcases to consider:
(i) $\Delta_1 = d_1, \Delta_2 = d_2$. Then $\Delta = \Delta_1 + \Delta_2 - 2$.
(ii) $\Delta_1 \geq d_1, \Delta_2 \geq d_2$, and the two equalities do not hold at the same time. Then $\Delta < \Delta_1 + \Delta_2 - 2$.
In all cases the inequality holds. Equality only holds for case 2(i). □

LEMMA 3.5. *Let P, P_1 and P_2 be the same polyhedra as defined in Lemma 3.4, with n, n_1, n_2 vertices respectively. Let F, F_1 and F_2 be the number of tetrahedra produced by the fanning heuristic applied to P, P_1 and P_2 respectively. Then $F \geq F_1 + F_2$.*

Proof. Note that $F = 2n - 4 - \Delta, F_1 = 2n_1 - 4 - \Delta_1$, and $F_2 = 2n_2 - 4 - \Delta_2$. Thus

$$\begin{aligned} F_1 + F_2 &= 2(n_1 + n_2) - 8 - (\Delta_1 + \Delta_2) \\ &= 2(n + 3) - 8 - (\Delta_1 + \Delta_2) \\ &= 2n - 4 - (\Delta_1 + \Delta_2 - 2) \\ &\leq 2n - 4 - \Delta = F. \end{aligned}$$

The last inequality follows from Lemma 3.4. □

The above lemma shows that cutting along a 3-cycle will not increase the number of tetrahedra using the fanning heuristic. Note also that any cut will not create new 3-cycles since no new surface edge is created, and two 3-cycles will never 'cross' each other (i.e. vertices of a cycle C_1 will not be on different sides of another cycle C_2). Therefore the lemma is also true for multiple cuts. The following theorem gives the analysis of the approximation ratio of our algorithm.

THEOREM 3.1. *Algorithm 3.1 always gives an approximation ratio bounded above by $2 - \Omega(\frac{1}{\sqrt{n}})$.*

Proof. Suppose there are k 3-cycles in P, and we apply k cuts to partition P into $k+1$ parts, each is free of 3-cycles. Consider the following two cases:
(i) $k = o(n)$. Since each 3-cycle contains exactly 3 vertices, there are at least $n - 3k = \Theta(n)$ vertices that do not lie on any 3-cycle. Since they do not lie on any 3-cycle, by the same argument as in Lemma 3.1(i), each of these $\Theta(n)$ vertices has an interior edge incident to their corresponding domes in any triangulation, unless they are of degree 3. However, the number of such degree-3 vertices must be sub-linear. It is because each of them must determine a cycle, so its number must not be more than k, by our assumption. Thus there is still a linear number of vertices each has interior edges incident to their corresponding domes. Similar to the proof of Lemma 3.1(ii), we have $2e_i(\Delta + 1) \geq \Theta(n)$, giving $e_i + \Delta \geq \Theta(\sqrt{n})$. Using a proof similar to that of Lemma 3.2, we have

$$r \leq \frac{2n - 4 - \Delta}{e_i + n - 3} \leq 2 - \Omega(\frac{1}{\sqrt{n}})$$

This means even if we apply the fanning heuristic directly to the original polyhedron (without cutting), we still have a bound of $2 - \Omega(\frac{1}{\sqrt{n}})$. By Lemma 3.5, dividing the polyhedron along all 3-cycles before fanning will not increase the number of tetrahedra. Thus our algorithm gives a bound of $2 - \Omega(\frac{1}{\sqrt{n}})$.
(ii) $k = \Omega(n)$. By Lemmas 3.4 and 3.5, apart from Case 2(i), each cut will reduce the number of tetrahedra by at least 1. If Case 2(i) occurs only sub-linear number of times, the other cases will occur a linear number of times, thus reducing a linear number of tetrahedra. Therefore the approximation ratio

$$r \leq \frac{2n - 4 - \Delta - cn}{e_i + n - 3} \leq 2 - c \leq 2 - \Omega(\frac{1}{\sqrt{n}})$$

for some constant $c > 0$. If Case 2(i) occurs a linear number of times, that means the maximum degree is cut a linear number of times. Each cut reduce Δ by at least 1. Thus Δ must be linear. Therefore

$$r \leq \frac{2n-4-c'n}{e_i+n-3} \leq 2-c' \leq 2-\Omega(\frac{1}{\sqrt{n}})$$

for constant $c' > 0$. □

Note that although cutting 3-cycles is an important step in our algorithm, it does not always produce minimum triangulations. A counterexample can be constructed using a 'cupola' [3], that is 3 VECSs of size m attached to a Schönhardt polyhedron. We add a vertex joining to the triangular bottom face of the cupola, making sure that it is inside the visibility cone of the top face, and the whole polyhedron remains convex (Fig. 3). Now the original bottom face becomes a 3-cycle. The polyhedron has $3m + 7$ vertices. If we cut through the 3-cycle, the 3 main diagonals cannot be used together, because then the Schönhardt polyhedron cannot be triangulated. Hence at least 1 main diagonal cannot be used, and therefore the cupola needs at least $4m$ tetrahedra to triangulate. On the other hand, since the new vertex can see all non-convex facets of the Schönhardt polyhedron from inside, we can triangulate the whole polyhedron by 'fanning' from the added vertex, while the 3 VECSs are triangulated using their own main diagonals. This gives only $3m+10$ tetrahedra. Therefore cutting 3-cycles does not always produce minimum triangulations.

Note also that finding 3-cycles, cutting them, and fanning the resulting polyhedra, are all combinatorially invariant. In other words, they only require the surface graph of the polyhedron to be given. The next section shows that this restriction implies a lower bound of approximability.

4 Lower Bound of Approximation

The combinatorial structure of a polyhedron is completely determined by its surface graph,

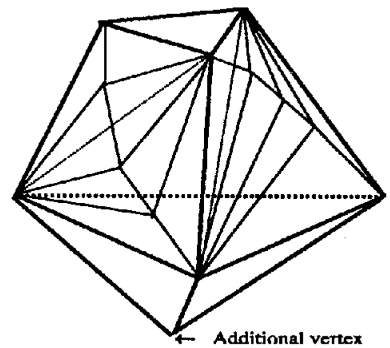

Figure 3: A cupola with a 3-cycle.

or equivalently by its *face lattice* [11]. Two combinatorially equivalent polyhedra can still have different geometric properties, e.g. having different vertex coordinates. It is shown in [1] that the minimum triangulation of convex polyhedra is not an invariant of the face lattice. We shall extend this idea to show that the difference in the minimum numbers of tetrahedra for two different convex polyhedra with the same face lattice can be linear in the number of vertices. As a major result, we show that, given only the face lattice but without the vertex coordinates, no approximation algorithms for finding minimum triangulation of convex polyhedra can have an approximation ratio better than $2 - O(\frac{1}{\sqrt{n}})$.

The idea for our proof is to construct two polyhedra, P1 and P2, such that they have the same surface graph, but their minimum triangulations differ by a linear number of tetrahedra. Both polyhedra consist of m thin wedges, each wedge being a VECS of size m. Both polyhedra have the same number of vertices n where $n = m^2 + 4m$. In P1 the m VECSs interlock each other, thus only one of the m main diagonals can be present in a triangulation, resulting in a greater number of tetrahedra. P2 is constructed similarly but the wedges do not interlock each other, and thus all m the main diagonals can be present in a triangulation, resulting in much smaller number of tetrahedra.

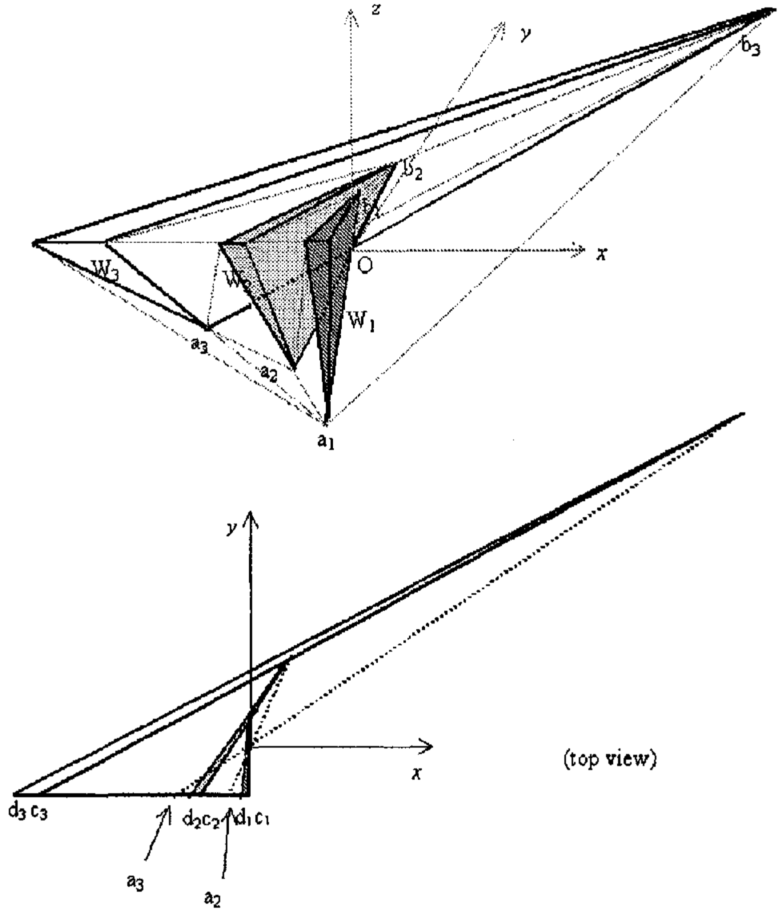

Figure 4: The placement of 3 wedges. q^i's are not shown.

4.1 The detailed construction. Fig.4 shows a rough idea of the construction of P1. We first arrange m wedges in an xyz-coordinate system. Wedge W_k $(1 \leq k \leq m)$ has vertices a_k, b_k, c_k, d_k where $a_k b_k$ is the main diagonal. All faces $a_k c_k d_k$ lie on the vertical plane $y = -1$ and all faces $b_k c_k d_k$ lie on the horizontal plane $z = 1$. W_1 is at $a_1(0, -1, -1)$, $b_1(0, 1, 1)$, $c_1(0, -1, 1)$, $d_1(-0.1, -1, 1)$. Assuming wedges W_1 to W_k have been constructed, W_{k+1} is constructed as follows (Fig.4 also shows a top view from the positive z-axis): pick a_{k+1} so that its x-coordinate is smaller than that of d_k and z-coordinate larger than a_k, and such that $a_1, a_2, ..., a_k, a_{k+1}$ form a convex chain w.r.t. the point $(0, -1, -\infty)$. b_{k+1} is uniquely determined on the $z = 1$ plane such that $a_{k+1}b_{k+1}$ passes through the origin $(0, 0, 0)$. Find a point c_{k+1} such that $b_{k+1}c_{k+1}$ does not intersect all previous wedge faces on the horizontal plane. d_{k+1} is placed slightly to the 'left' (negative x direction) of c_{k+1}.

The following two facts are not difficult to be established:

1. b_{k+1} has larger y-coordinate than b_k.
2. Vertices $b_1, b_2, ..., b_k, b_{k+1}$ form a convex chain w.r.t. the point $(\infty, 0, 1)$.

Moreover, m points $q_k^1, q_k^2, ..., q_k^m$ are created between each interval (c_k, d_k) and edges $a_k q_k^i, b_k q_k^i$ are added so that they form a VECS of size m.

This construction puts all the vertices on two planes. It is not difficult to see that this degeneracy can be removed by slightly bending the horizontal and vertical planes so that the polyhedron is convex, while keeping the orientations of all the main diagonals unchanged.

Finally, we move the wedges slightly along the positive x-axis: the k-th wedge W_k is moved by a distance of $k\delta$ where $\delta > 0$ is a small constant, so that any two wedges are in an interlocked position. It is easy to see that small enough perturbations will not change the surface graph of the polyhedron. P1 is formed by taking the convex hull of all the wedges, i.e. having the edges $a_kc_k, a_kd_k, b_kc_k, b_kd_k (1 \leq k \leq m)$; $d_ka_{k+1}, d_{k+1}c_k, b_kc_{k+1}, a_ka_{k+1}, b_kb_{k+1} (1 \leq k \leq m-1)$; $a_1a_k, b_1b_k (3 \leq k \leq m)$; c_1b_m, a_1b_m, a_1d_m; $q_k^iq_k^{i+1}, q_k^1c_k, q_k^md_k (1 \leq i \leq m-1, 1 \leq k \leq m)$; $q_k^ia_k, q_k^ib_k (1 \leq i \leq m, 1 \leq k \leq m)$.

The construction of P2 is almost identical to that of P1, except that the wedges do not move but shrink. We draw a vertical line passing through each vertex c_k and intersect edge d_ka_k at a'_k. We shrink the wedge W_k from $a_kb_kc_kd_k$ to $a'_kb_kc_kd_k$. Then the chain $a'_1(=a_1), a'_2, ..., a'_m$ is still convex. Clearly, the surface graph will not be changed by this transformation, and all wedges do not intersect. Thus we have the following.

CLAIM 4.1. *P1 and P2 have the following properties.*
(i) Both P1 and P2 are convex.
(ii) P1 and P2 have the same surface graph.
(iii) All VECS faces $a_kq_k^iq_k^{i+1}$ and $b_kq_k^iq_k^{i+1}$ are on the surface of P1 and P2.
(iv) For P1, all wedges interlock one another. For P2, all wedges do not intersect one another.

4.2 Size of the triangulations. We next show that the minimum triangulation of P1 contains a much greater number of tetrahedra than that of P2.

LEMMA 4.1. *The minimum triangulation of P1 contains at least $2m^2 - m + 1$ tetrahedra.*

Proof. Among the m interlocked main diagonals, only one can be present in any triangulation of P1 (Lemma 2.2). Note that all m VECSs are disjoint in our construction, then by Lemma 2.1, at least $(m-1)(2m)$ tetrahedra must incident to these $m-1$ VECSs not using their main diagonals. The VECS with the main diagonal creates $m+1$ tetrahedra. Thus, a total of at least $(m-1)(2m)+(m+1) = 2m^2 - m + 1$ tetrahedra must appear in any triangulation. □

LEMMA 4.2. *The minimum triangulation of P2 contains at most $m^2 + 8m - 8$ tetrahedra.*

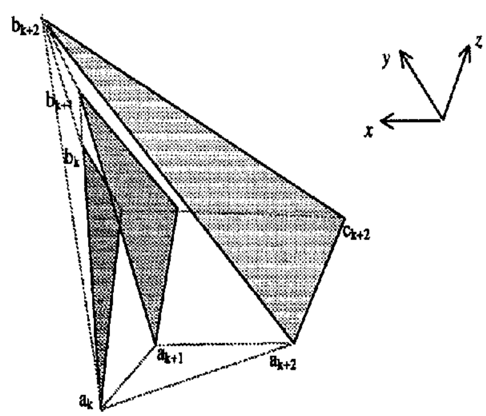

Figure 5: The space between 3 wedges. Thickness of each wedge is not shown.

Proof. We triangulate each wedge by using the main diagonal of its corresponding VECS, producing $m+1$ tetrahedra each. Now we have to triangulate the remaining space. Consider any three consecutive wedges W_k, W_{k+1}, W_{k+2} $(1 \leq k \leq m-2)$. We add the interior edges a_kb_{k+1} and $a_{k+1}b_{k+2}$ so that the two regions $a_k b_k d_k a_{k+1} b_{k+1} c_{k+1}$ and $a_{k+1} b_{k+1} d_{k+1} a_{k+2} b_{k+2} c_{k+2}$ are convex, and they can be triangulated using 4 tetrahedra each. Finally, the remaining space can be triangulated with 3 tetrahedra $a_k a_{k+1} a_{k+2} b_{k+2}$, $b_k b_{k+1} b_{k+2} a_k$, and $a_{k+1} b_{k+1} a_k b_{k+2}$ (Fig.5). Using the above procedure, wedge W_{k+1} is effectively 'shielded' and the triangulation can proceed with the remaining $m-1$ wedges as if there is no W_{k+1}. We use this to triangulate P2 starting from $W_1 W_2 W_3$, then $W_1 W_3 W_4$ and so on until only two wedges W_1 and W_m remain. The space beyond W_1 and W_m can be triangulated by adding the tetrahedra $a_1 b_1 c_1 b_m$ and $a_m b_m d_m a_1$ for the two outer faces.

There are m VECSs, each giving $m+1$ tetrahedra, each of the $m-1$ convex regions between W_k and W_{k+1} gives 4 tetrahedra, each of the $m-2$ inductive steps gives 3 more tetrahedra, and the final two wedges gives 2 more. Thus the total number of tetrahedra is $m(m+1)+4(m-1)+3(m-2)+2 = m^2+8m-8$. □

THEOREM 4.1. *There exist convex polyhedra with the same face lattice, such that the minimum number of tetrahedra required for their respective triangulations can vary with respect to their vertex coordinates. The difference can be linear in the number of vertices.*

Proof. This follows directly from Lemmas 4.1 and 4.2. P1 has at least $2m^2-m+1 = 2n-\Theta(\sqrt{n})$ tetrahedra while P2 has at most $m^2+8m-8 = n+\Theta(\sqrt{n})$ tetrahedra. □

THEOREM 4.2. *Any approximation algorithm for finding the minimum triangulation of convex polyhedra, provided with only the face lattice, cannot have approximation ratio better than $2-O(\frac{1}{\sqrt{n}})$.*

Proof. Since the minimum triangulation of polyhedra with the same face lattice can have either at least $2n-\Theta(\sqrt{n})$ or at most $n+\Theta(\sqrt{n})$ tetrahedra, any approximation algorithm that considers the face lattice only must produce a triangulation with the worst case number, i.e. $2n-\Theta(\sqrt{n})$. Otherwise, this algorithm would produce a triangulation with less than optimal number of tetrahedra in some cases, a contradiction. Note that there exists polyhedron whose minimum triangulation contains at most $n+\Theta(\sqrt{n})$ tetrahedra while any approximation algorithm can only give a triangulation with at least $2n-\Theta(\sqrt{n})$ tetrahedra. Thus the approximation ratio is bounded by $r \geq \frac{2n-\Theta(\sqrt{n})}{n+\Theta(\sqrt{n})} \geq 2-O(\frac{1}{\sqrt{n}})$. □

5 Conclusion

In this paper, we give an algorithm for finding the minimum triangulation of convex polyhedra with approximation ratio $2-\Omega(\frac{1}{\sqrt{n}})$. We also show a $2-O(\frac{1}{\sqrt{n}})$ bound on the approximability of the minimum triangulation problem. Thus there is no better constant-ratio approximation algorithm without considering the actual coordinates of the vertices. If vertex coordinates are taken into account, whether better approximation ratio can be achieved, or similar non-approximability results exist, is still an open problem. When the maximum degree is bounded by a constant, it is also unclear whether these bounds can be improved.

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