Periodic solutions for systems of coupled nonlinear Schrödinger equations with three and four components

K. W. Chow* and D. W. C. Lai
Department of Mechanical Engineering, University of Hong Kong, Pokfulam Road, Hong Kong
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Periodic solutions for systems of coupled nonlinear Schrödinger equations (CNLS) are established by the Hirota bilinear method and elliptic functions. The interesting feature is the choice of theta functions in the formulation. The sum of moduli of the components or the total intensity of the beam in physical terms, will now be a rational function, instead of a polynomial, of elliptic functions. Each component of the CNLS may have multiple peaks within one period.

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Systems of coupled nonlinear Schrödinger equations (CNLS) arise in various fields of applications, e.g., hydrodynamics and optics [1]. In the present work, we consider CNLS of the form

$$i\frac{\partial \psi_n}{\partial t}+\varepsilon_n\frac{\partial^2 \psi_n}{\partial x^2}+\left(\sum_{m=1}^{N}\beta_{mn}\psi_m\psi_m^*\right)\psi_n=0,\qquad n=1,2,\ldots,N. \quad (1)$$

The focus here will be on periodic waves. Periodic waves for the single-component regime, or the $N=1$ case of Eq. (1) have, in fact, been studied intensively earlier. By employing special transformations, elegant, exact solutions of the ordinary, uncoupled nonlinear Schrödinger equation have been derived [2–4]. These expressions can describe periodic wave trains, as well as wave patterns generated by modulational instability, periodic evolution of bright solitons on a continuous wave background, and in a special regime, collision of dark solitary waves.

Let us now turn our attention to the fully coupled system, Eq. (1) (N greater than 1). Analytical solutions of Eq. (1) as single or products of Jacobi elliptic functions have been derived earlier in the literature for both the regimes of anomalous and normal dispersion, $\varepsilon_n=+1$ or -1, respectively, and all interaction coefficients β_{mn} being $+1$. These explicit exact solutions in terms of products of Jacobi elliptic functions are available for CNLS systems ranging from two to six components [5–9]. In fact, a general algorithm based on an ansatz of Lamè functions has been formulated. Physical properties of the waves, e.g., the frequency and the wave number, can be solved as a system of algebraic equations. Reductions to the known solutions can be established directly for a small number of components [10,11]. Recently, the case of β_{mn} being allowed to be both positive and negative has been considered [12,13]. The Lamè and elliptic functions are shown to be applicable as well. The eigenvalue spectrum and Bäcklund transformation of Eq. (1) have been derived [14], and thus such CNLS systems are of fundamental theoretical interest.

*FAX: (852) 2858 5415; email address: kwchow@hkusua.hku.hk

In hydrodynamics, CNLS systems arise in the propagation of surface and internal wave packets [15]. The induced mean flow must be computed before the coefficient of the cubic nonlinearity for each waveguide is established. This mechanism will provide the coupling. From the perspective of optics CNLS models are relevant in the propagation of multimode, incoherent, spatial solitons in noninstantaneous Kerr media, where diffraction and light-induced waveguiding effects are in balance. In one theoretical model, the total intensity is the superposition of the contributions from all the relevant modes in the nonlinear waveguide. Analytically, the self- and cross-phase nonlinearities are replaced by a linear term. When this term is the square of a hyperbolic secant or square of a Jacobi elliptic dn function, calculations have been performed [16,17].

The Hirota form has been proven to be effective in obtaining solitary and periodic waves for evolution equations, and will be employed here. The goal of the present work is to derive some further solutions of CNLS systems using a different choice of theta functions in the bilinear formulation. Details of the calculations are similar to earlier works [9,18] and will be omitted. Basically, for each component, a single trilinear equation is considered instead of two uncoupled bilinear equations. A proper ansatz of theta functions is assumed and the Hirota derivatives are handled by identities of theta functions. An interesting feature here is the choice of theta functions in the bilinear formulation. This results in solutions in terms of rational functions, instead of just polynomials, of Jacobi elliptic functions.

CNLS of three components. Expressions and definitions of elliptic and theta functions can be found in the literature [19,20,9]. The Hirota forms of the solutions in terms of theta functions are

$$g_1=A_1[c_1\theta_4^2(\alpha x)-\theta_3^2(\alpha x)]\theta_1(\alpha x),$$
$$g_2=A_2[c_2\theta_4^2(\alpha x)-\theta_3^2(\alpha x)]\theta_2(\alpha x), \quad (2)$$

$$g_3=A_3\theta_3^3(\alpha x),\quad f=\theta_3(\alpha x)\theta_4^2(\alpha x),$$
$$\psi_m=\frac{g_m\exp(-i\Omega_m t)}{f},\quad m=1,2,3. \quad (3)$$

The choice here is to allow for different theta functions in f rather than θ_4 alone. These solutions of the CNLS system are more compact in terms of the Jacobi elliptic functions:

$$\psi_1=A_1\sqrt{k}(1-k^2)^{1/4}\left[c_1-\frac{\mathrm{dn}^2(rx)}{(1-k^2)^{1/2}}\right]\frac{\mathrm{sn}(rx)\exp(-i\Omega_1 t)}{\mathrm{dn}(rx)}, \tag{4}$$

$$\psi_2=A_2\sqrt{k}\left[c_2-\frac{\mathrm{dn}^2(rx)}{(1-k^2)^{1/2}}\right]\frac{\mathrm{cn}(rx)\exp(-i\Omega_2 t)}{\mathrm{dn}(rx)}, \tag{5}$$

$$\psi_3=\frac{A_3\,\mathrm{dn}^2(rx)\exp(-i\Omega_3 t)}{(1-k^2)^{1/2}}. \tag{6}$$

c_1 is a root of

$$3c^2-2c\sqrt{1-k^2}-1=0$$

and c_2 is a root of

$$3c^2-\frac{2c}{\sqrt{1-k^2}}-1=0.$$

In contrast to the previous studies [18,9], the present choice of theta functions for f [Eq. (3)] allows a one-parameter family of solutions in the amplitudes. More precisely, there are only *two* equations constraining the three unknowns A_1, A_2, A_3,

$$c_1^2A_1^2-c_2^2\sqrt{1-k^2}A_2^2=2r^2k\sqrt{1-k^2},$$

$$\sqrt{1-k^2}A_1^2-A_2^2=kA_3^2.$$

The sum of intensities, which is related to the refractive index, is

$$\sum_{m=1}^{3}\psi_m\psi_m^*=\frac{2r^2(1-k^2)}{\mathrm{dn}^2(rx)}+B_1+6r^2\,\mathrm{dn}^2(rx), \tag{7}$$

$$B_1=\frac{c_2(2\sqrt{1-k^2}+c_2)A_2^2}{k}-\frac{c_1(2+c_1\sqrt{1-k^2})A_1^2}{k}. \tag{8}$$

The angular frequencies are

$$\Omega_1=6c_1r^2\sqrt{1-k^2}-r^2(5-4k^2)-B_1,$$

$$\Omega_2=6c_2r^2\sqrt{1-k^2}-r^2(5-k^2)-B_1,$$

$$\Omega_3=-4r^2(2-k^2)-B_1.$$

The long wave limit ($k\to1$) produces, not surprisingly, solitary waves

$$\psi_1=\beta_1\tan rx\,\mathrm{sech}\,rx\exp[i(B_{10}+r^2)t], \tag{9}$$

$$\psi_2=\beta_2\left(\frac{2}{3}-\mathrm{sech}^2\,rx\right)\exp(iB_{10}t), \tag{10}$$

$$\psi_3=\beta_3\,\mathrm{sech}^2\,rx\exp[i(B_{10}+4r^2)t], \tag{11}$$

$$\beta_1^2=\frac{4}{3}\beta_2^2+6r^2,\quad \beta_3^2=\frac{\beta_2^2}{3}+6r^2,\quad \beta_2\ \text{arbitrary},$$

$$B_{10}=\frac{4\beta_2^2}{9}.$$

(B_{10} is the long wave limit of B_1.) The remarkable point is that one of the components [Eq. (10)] is a dark solitary wave, even though Eq. (1) is in the anomalous dispersion regime. This solution is consistent with that found in Ref. [13]. The reasoning is that in the long wave limit ($k\to1$), Eq. (7) reduces to a polynomial again ($\mathrm{dn}\,z\to\mathrm{sech}\,z, k\to1$).

CNLS of four components. Similarly a solution for the case of four components is

$$g_1=A_1[c_1\theta_4^4(\alpha x)-\delta_1\theta_3^2(\alpha x)\theta_4^2(\alpha x)-\theta_3^4(\alpha x)],$$

$$g_2=A_2[c_2\theta_4^4(\alpha x)-\delta_2\theta_3^2(\alpha x)\theta_4^2(\alpha x)-\theta_3^4(\alpha x)],$$

$$g_3=A_3[c_3\theta_4^2(\alpha x)-\theta_3^2(\alpha x)]\theta_1(\alpha x)\theta_2(\alpha x),$$

$$g_4=A_4[c_4\theta_4^2(\alpha x)-\theta_3^2(\alpha x)]\theta_1(\alpha x)\theta_2(\alpha x),$$

$$f=\theta_3(\alpha x)\theta_4^3(\alpha x),\quad \psi_m=\frac{g_m\exp(-i\Omega_m t)}{f},$$

$$m=1,2,3,4.$$

Similarly to the previous case, the four amplitude parameters are determined from the *three* constraints

$$A_1^2+A_2^2-\frac{\sqrt{1-k^2}(A_3^2+A_4^2)}{k^2}=0,$$

$$2\delta_1A_1^2+2\delta_2A_2^2+\left[\frac{2(1+c_3\sqrt{1-k^2})}{k^2}-1\right]A_3^2+\left[\frac{2(1+c_4\sqrt{1-k^2})}{k^2}-1\right]A_4^2=0,$$

$$c_1^2A_1^2+c_2^2A_2^2-\frac{c_3^2\sqrt{1-k^2}A_3^2}{k^2}-\frac{c_4^2\sqrt{1-k^2}A_4^2}{k^2}=2r^2\sqrt{1-k^2},$$

with auxiliary constants defined below. This is thus a one-parameter family of solutions. The constant B_2 given by

$$B_2=-2c_1\delta_1A_1^2-2c_2\delta_2A_2^2+[c_3^2(2-k^2)+2c_3\sqrt{1-k^2}]\frac{A_3^2}{k^2}+[c_4^2(2-k^2)+2c_4\sqrt{1-k^2}]\frac{A_4^2}{k^2} \tag{12}$$

will be related to the total intensity or "refractive index" through

$$\sum_{m=1}^{4}\psi_m\psi_m^*=\frac{2r^2(1-k^2)}{\mathrm{dn}^2(rx)}+B_2+12r^2\,\mathrm{dn}^2(rx). \tag{13}$$

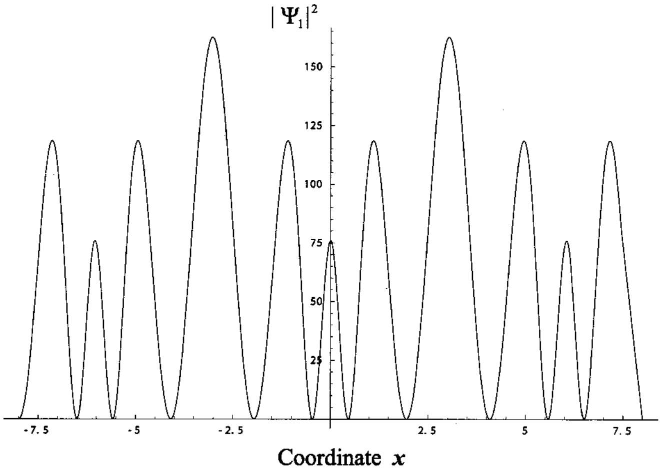

FIG. 1. Square of the modulus of the amplitude $|\psi_1|^2$ versus the coordinate x, Eq. (14), $k=0.98$, $\delta_1=-4.63$, $c_1=-2.06$, $r=1$.

This form of the intensity is different from the cases studied earlier in the literature [16,17]. In terms of the Jacobi elliptic functions the components are, for $n=1,2$,

$$\psi_n=A_n\left[c_n-\frac{\delta_n\,\mathrm{dn}^2(rx)}{(1-k^2)^{1/2}}-\frac{\mathrm{dn}^4(rx)}{1-k^2}\right]\frac{(1-k^2)^{1/4}\exp(-i\Omega_n t)}{\mathrm{dn}(rx)}. \tag{14}$$

c_n and δ_n, $n=1,2$ are related by

$$c_n=-\frac{\delta_n}{5\,\delta_n+4(\sqrt{1-k^2}+1/\sqrt{1-k^2})},$$

where δ_n is selected from the roots of

$$25\delta^3+40\left(\sqrt{1-k^2}+\frac{1}{\sqrt{1-k^2}}\right)\delta^2+\frac{4(17-17k^2+4k^4)\,\delta}{1-k^2}+8\left(\sqrt{1-k^2}+\frac{1}{\sqrt{1-k^2}}\right)=0.$$

The angular frequencies Ω_n are

$$\Omega_n=-B_2-9r^2(2-k^2)-10\delta_n r^2\sqrt{1-k^2},\quad n=1,2.$$

The other components, $n=3,4$, are

$$\psi_n=A_n k\left(c_n-\frac{\mathrm{dn}^2(rx)}{(1-k^2)^{1/2}}\right)\frac{\mathrm{sn}(rx)\mathrm{cn}(rx)\exp(-i\Omega_n t)}{\mathrm{dn}(rx)}, \tag{15}$$

where c_n are roots of

$$5c^2-2\left(\sqrt{1-k^2}+\frac{1}{\sqrt{1-k^2}}\right)c-1=0,$$

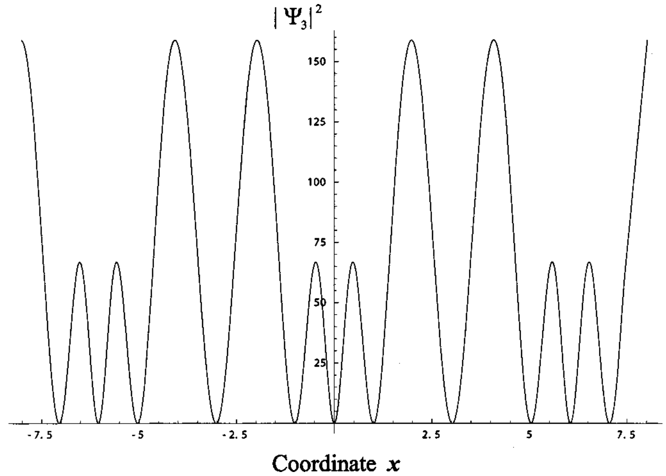

FIG. 3. Square of the modulus of the amplitude $|\psi_3|^2$ versus the coordinate x, Eq. (15), $k=0.98$, $c_3=2.18$, $r=1$.

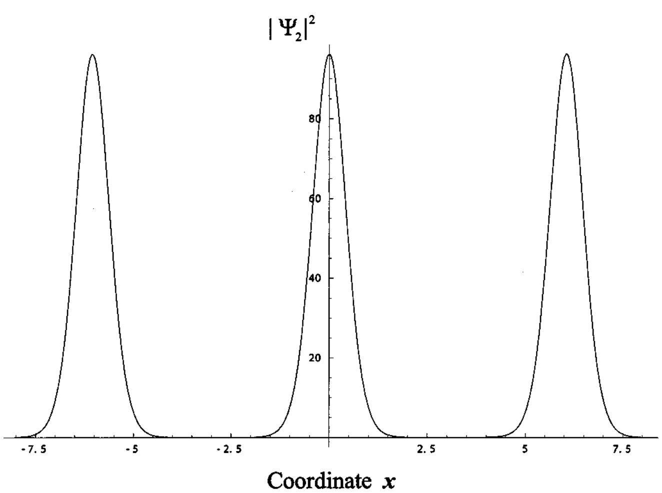

FIG. 2. Square of the modulus of the amplitude $|\psi_2|^2$ versus the coordinate x, Eq. (14), $k=0.98$, $\delta_2=-0.099$, $c_2=0.004\,88$, $r=1$.

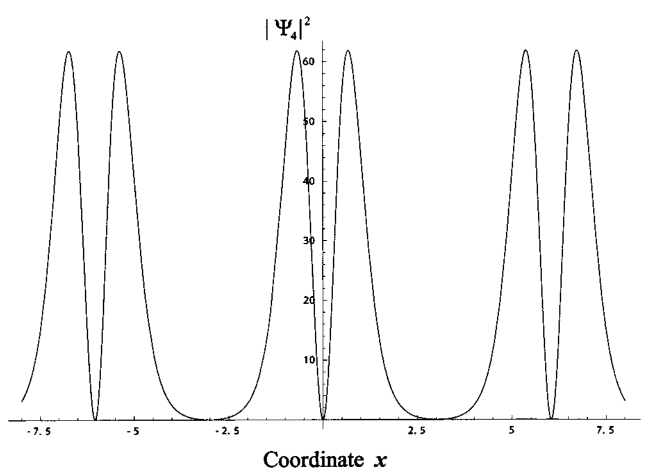

FIG. 4. Square of the modulus of the amplitude $|\psi_4|^2$ versus the coordinate x, Eq. (15), $k=0.98$, $c_4=-0.092$, $r=1$.

and the angular frequencies Ω_n are

$$\Omega_n = -B_2 + 10c_n r^2 \sqrt{1-k^2} - 4r^2(2-k^2), \quad n=3,4.$$

Graphs for the square of the moduli $|\psi_n|^2$, $n=1,2,3,4$ are shown in Figs. 1–4, where, in general, multiple peaks occur within one period. The long wave limits can be taken, and solitary waves will result. Details will be similar to the case of three components and earlier works in the literature [18,9], and hence will be omitted.

In conclusions, periodic waves for CNLS systems with three or four components are established by a different choice of theta functions in the bilinear formulation. Besides the theoretical importance in the context of evolution equations, such CNLS waves are relevant in fluids with multiple layers. The physical significance in optics hinges on the fact that the total intensity of the beam is a more complicated function than cases treated earlier in the literature [16,17,21]. In consideration of self-trapping of mutually incoherent wave packets in photorefractive media, the total intensity is a critical factor in the response of the system [16,17]. This type of total intensity for the solutions here, Eqs. (7) and (13), permits two different local maxima per period, and thus enhances our capability in modeling wave dynamics in physics.

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