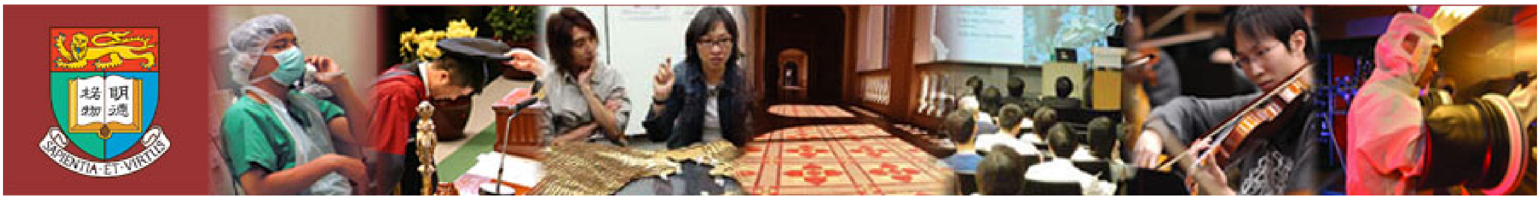

Title	A linear matrix inequality (LMI) approach to robust H2 sampled-data control for linear uncertain systems
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Case 2: $\|\hat{W}\| = w_m$ and $\hat{W}^T S(z)h(x)e_s > 0$

In this case, it can be seen from (34) that

$$\left|S^T(z_\sigma)\dot{\hat{W}}\right| = \gamma \left| \left[S^T(z)S(z_\sigma) - \frac{S^T(z)\hat{W}\hat{W}^T S(z_\sigma)}{\|\hat{W}\|^2} \right] h(x)e_s \right| \le \gamma h(x)|e_s| \left[S^T(z)S(z_\sigma) + \left| \frac{S^T(z)\hat{W}\hat{W}^T S(z_\sigma)}{\|\hat{W}\|^2} \right| \right].$$

Since $S^T(z)\hat{W}$ and $\hat{W}^T S(z_\sigma)$ are scalar functions and bounded by $|\hat{W}|_1$, we obtain

$$\left|S^T(z_\sigma)\dot{\hat{W}}\right| \le \gamma h(x)|e_s| \left(l + \frac{|\hat{W}|_1^2}{\|\hat{W}\|^2} \right).$$

By the relationship $|\hat{W}|_1^2 \le l\|\hat{W}\|^2$, it is shown that $\left|S^T(z_\sigma)\dot{\hat{W}}\right| \le 2\gamma l h(x)|e_s|$.

Combining the above two cases, we conclude that $\left|S^T(z_\sigma)\dot{\hat{W}}\right| \le 2\gamma l h(x)|e_s|$.

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A Linear Matrix Inequality (LMI) Approach to Robust H_2 Sampled-Data Control for Linear Uncertain Systems

Li-Sheng Hu, James Lam, Yong-Yan Cao, and Hui-He Shao

***Abstract*—In this paper, we consider the H_2 sampled-data control for uncertain linear systems by the impulse response interpretation of the H_2 norm. Two H_2 measures for sampled-data systems are considered. The robust optimal control procedures subject to these two H_2 criteria are proposed. The development is primarily concerned with a multirate treatment in which a periodic time-varying robust optimal control for uncertain linear systems is presented. To facilitate multirate control design, a new result of stability of hybrid system is established. Moreover, the single-rate case is also obtained as a special case. The sampling period is explicitly involved in the result which is superior to traditional methods. The solution procedures proposed in this paper are formulated as an optimization problem subject to linear matrix inequalities. Finally, we present a numerical example to demonstrate the proposed techniques.**

***Index Terms*—H_2 performance, multirate, optimal control, robust control, sampled-data system, uncertain system.**

I. Introduction

The H_2 performance of optimal control has a significant physical background in practice. The robust H_2 problem, rooted from the efforts to provide stability margin to the H_2-optimal (LQG) regulator, has been a focal point of research since the 1970s. The main difficulty encountered in this problem is the tradeoff between robustness and H_2 performance. A wealth of literature on the simultaneous considerations of robustness and H_2 performance, such as H_2 and H_∞ mixed performances in various forms (see [6] and [13]) has been reported. In

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[6], a review of the methods, robust H_2 performance analysis in the state-space, and operator version of the robust H_2 performance (which views the H_2 norm as the worst case response under a suitable class of signals) are provided. Nowadays, digital control, digital signal processing, and digital communication in networks are widely used. Traditionally, the synthesis and analysis methods for these applications are all based on discretized and continuous-based formulations. However, these are approximate formulations [7]. During the last few years, the development of sampled-data system control (controllers are composed of a sampler-and-hold device and a digital controller), which takes into account of the effects of the intersampling behavior of the system and has no degradation of the closed-loop performance, has received wide attention [4], [7]. Moreover, past research had also revealed that sampled-data control exhibits some additional favorable properties. For instance, the effects of the sampler and the holder are taken into account in the control design. A complex digital controller, such as time-varying or periodically time-varying, can be easily constructed by the piecewise static controllers. The multirate output feedback control is equivalent to state feedback control. In particular, it is very suitable to be used to control nonlinear systems, and can asymptotically stabilize almost all classes of nonlinear systems in which some of them cannot be asymptotically stabilized, even locally, using smooth feedback. However, a sampled-data system is a hybrid system which involves both continuous-time and discrete-time signals. This makes traditional synthesis and analysis methodologies using purely discrete-time and continuous-time formulations difficult to apply. Many research studies on H_2 and H_∞ control for the sampled-data systems reported are based on the frequency response [1], [16], the lifting technique [3], and the L_2-induced norm [15]. Moreover, the results of the robustness and stability of the uncertain sampled-data systems are studied [5], [9], [14]. In [4], the authors discussed H_2 performance measures of sampled-data systems in detail and provided H_2 sampled-data control procedures for linear systems using the lifting technique. However, the H_2 sampled-data optimal control for uncertain systems is still open. In this paper, we will show that these H_2 performance criteria can be extended to the case of uncertain linear systems. Robust multirate H_2 sampled-data optimal control (a time-varying periodic control) procedure for such systems is provided. Moreover, we also present a result on the robust single-rate sampled-data control for the uncertain linear systems to minimize these H_2 measures as a special case. The solution procedures proposed in this paper are formulated as an optimization problem subject to linear matrix inequalities.

This paper is organized as follows. In Section II, we present a formulation of problems considered in this paper and some preliminary materials. Furthermore, a new result on the stability of hybrid systems is established to facilitate a multirate control design. Section III contains results on robust sampled-data optimal control of uncertain linear systems for two H_2 measures. A multirate robust sampled-data control (a piecewise constant periodic control) procedure for uncertain linear systems is proposed. Moreover, the single-rate solution is provided as a special case. Numerical examples are provided in Section IV. Finally, concluding remarks are given in Section V.

II. Problem Formulation

In the following, T_s is used to denote the sampling period, with which all the states of the system are sampled simultaneously by ideal samplers. The discrete-time signals will be represented by $[\cdot]$ and with the associated variable (for example $\tilde{u}[t_k]$. For a matrix $M, \bar{\sigma}(M)(\underline{\sigma}(M))$ denotes the maximum (minimum) singular value of M. If M is real symmetric, $M < 0 (M > 0)$ denotes M is negative (positive) definite.

Consider the following uncertain linear system:

$$\dot{x}(t) = (A + \Delta A)x(t) + B_1 w(t) + (B_2 + \Delta B_2)u(t) \tag{1}$$
$$z(t) = C_1 x(t) + D_{12} u(t) \tag{2}$$

where
- $x(t) \in \mathbb{R}^n$ state vector of the plant;
- $u(t) \in \mathbb{R}^m$ control vector;
- $z(t) \in \mathbb{R}^q$ output of the plant to be controlled;
- $w(t) \in \mathbb{R}^p$ impulsive disturbance vector.

ΔA and ΔB_2 are uncertainties satisfying

$$\Delta A = H_1 \Delta E_1, \quad \Delta B_2 = H_2 \Delta E_2$$

where A, B_1, B_2, C_1, D_{12} and H_1, H_2, E_1, E_2 are the real matrices with compatible dimensions. The uncertain matrix Δ satisfies $\Delta^T \Delta \leq I$. In this paper, we assume that the uncertainty is linear time-invariant (LTI).

For the system in (1) and (2), the H_2 sampled-data optimal control problem is to design a single-rate digital control and a multirate digital control to minimize an upper bound of the H_2 measure of the system. The traditional H_2 measure is defined as

$$J = \sup_{\Delta:\Delta^T\Delta \leq I} \left(\sum_{i=1}^{p} \|T_{zw}(\Delta)\delta(t)e_i\|_2^2 \right)^{1/2} \tag{3}$$

where $\{e_i\}_{i=1,2,\ldots,p}$ are the standard basis in $\mathbb{R}^p, \delta(t)$ the impulse applied at $t = 0, T_{zw}(\Delta)$ is a closed-loop system from w to z involving a sampled-data controller ($T_{zw}(\Delta)$ is periodic with period T_s). Thus, there is no transfer function in the usual sense such that its H_2 norm could be minimized. Measure (3) will be used to circumvent this problem [4]. Since $T_{zw}(\Delta)$ is time-varying, in [4] the authors proposed an alternative specification, a generalized H_2 measure defined as

$$J_g = \left(\int_0^{T_s} J_\tau^2 d\tau \right)^{1/2} \tag{4}$$

was considered where

$$J_\tau = \sup_{\Delta:\Delta^T\Delta \leq I} \left(\sum_{i=1}^{p} \|T_{zw}(\Delta)\delta_\tau(t)e_i\|_2^2 \right)^{1/2}$$

and $\delta_\tau(t) = \delta(t - \tau), 0 \leq \tau \leq T_s$. Clearly, if $\tau = 0$, then $J_\tau \equiv J$. If T_{zw} is LTI, then $J_\tau = J$ for all τ and $J_g = \sqrt{T_s}J$ which implies that J_g is a generalization of the traditional H_2 measure. The H_2 measures in (3) and (4) are considered in [2], [4], and [10] for sampled-data control systems using the so-called lifting technique. In this paper, they will be extended to uncertain sampled-data control systems.

An important sampled-data control problem is concerned with designing a control

$$u(t) = \tilde{u}[t_k], \quad \text{for } t \in (t_k, t_{k+1}], \quad t_0 = 0 \tag{5}$$
$$\tilde{u}[t_k] = Fx(t_k), \quad k = 0, 1, 2, \ldots, \tag{6}$$

to stabilize the system in (1) and (2) and minimize upper bounds of the H_2 measures in (3) and (4). Equation (6) denotes a digital controller with static gain F and (5) indicates that the digital control is fed into the continuous system by means of an ideal zeroth-order hold. In this situation, all the states are sampled by ideal samplers, and the control action is synchronously switched with a common sampling period T_s. This is called a *single-rate digital control.*

In this paper, we will explore the procedure of computing a robust periodic time-varying sampled-data optimal control $F(t)$ with period

T_s for the uncertain linear systems of (1) and (2). The control actions are switched with a shorter period T, and $N_T = (T_s/T)$. Here, T_s is referred to as the frame period and N_T the input multiplicity. Moreover, the time-varying digital control signals are fed into the plant with ideal zeroth-order holds. At every instant $t_k + iT$, its mechanism is described as

$$u(t) = \tilde{u}[i \mid t_k], \quad \text{for } t \in (t_k + iT, t_k + (i+1)T] \tag{7}$$

for $i = 0, 1, 2, \ldots, N_T - 1$, where $\tilde{u}[i \mid t_k]$ is defined as

$$\tilde{u}[i \mid t_k] := F(i)x(t_k), \quad \text{for } i = 0, 1, 2, \ldots, N_T - 1 \tag{8}$$

where, with a slight abuse of notation, we have $F(i) := F(iT)$ time-varying and periodic $F(i) = F(i+N_T)$ (that is, $F(iT+T_s) = F(iT)$). In the frame period, the control gain is switched at $t = t_k + iT$. Consequently, the problem is cast into the design of a periodic time-varying controller (7) and (8) to optimally stabilize plant (1) and (2) and minimize certain upper bounds of the H_2 criteria in (3) and (4). In contrast with the single-rate case, such a control problem is a *multirate* one.

To facilitate later developments, we consider the following nonlinear hybrid system:

$$\dot{x}(t) = f(t, x(t)), \quad t \in (t_j, t_{j+1}) \tag{9}$$

$$x(t_j^+) = I_j(t_j, x(t_j)), \quad j = 0, 1, 2, \ldots, x(0) = x_0 \tag{10}$$

where $I_j : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $f(t, x(t))$ is uniformly norm-bounded in $x(t)$ (that is, $\|f(t,x(t))\| \le \alpha \|x(t)\|$ for some constant $\alpha > 0$) and Lipschitz on the interval $(t_j, t_{j+1}]$ with right limit at t_j, $f(0,0) = 0$. $\{t_j\}_{j=0,1,2,\ldots}$ is the sequence of jumps with $t_{j+1} - t_j$ equals a constant. Assume the system (9) and (10) satisfies the conditions for the existence and uniqueness of the solution [11].

Lemma 1: Let $a > 0, b > 0$ and $c > 0$ be real scalars. Consider system (9) and (10), if there exists a Lyapunov function $V(t, x(t))$ where $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is continuous on $(t_j, t_{j+1}]$ with right limit at t_j and locally Lipschitzian in x, such that

$$a\|x\|^2 \le V(t, x) \le b\|x\|^2 \tag{11}$$

and

$$D^+ V(t, x) < -c\|x\|^2 \tag{12}$$

for $t \in (t_j, t_{j+1})$ and

$$\sum_{j=kN_T}^{kN_T+l-1} \{V(t_j^+, x(t_j^+)) - V(t_j, x(t_j))\} \le 0 \tag{13}$$

for $k = 0, 1, \ldots, l = 1, 2, \ldots, N_T$, some integer $N_T \ge 0$, then system (9) and (10) is asymptotically stable.

The detailed proof of Lemma 1 is given in the Appendix .

Remark 1: Lemma 1 is an adaptation of [11, Th. 3.7.3]. In Theorem 3.7.3, at the all sampling instants, the Lyapunov function is decreasing, which is relaxed in Lemma 1. While the decrease the of Lyapunov function is accumulatively evaluated over every frame period in this lemma. In the frame period, jumping up of Lyapunov function values at the sampling instants is admitted. At a certain instant, the accumulation of jumping must be negative from the beginning of the frame period, as shown in (13).

Let $\tilde{x}(t) = (x^T(t), u^T(t))^T$, the closed-loop system composed of the system in (1) and (2), and the controller in (7) and (8) is rewritten as

$$\dot{\tilde{x}}(t) = (\bar{A} + \tilde{H}\tilde{\Delta}\tilde{E})\tilde{x}(t) + \bar{B}_1 w(t) \tag{14}$$

$$\tilde{x}(t_k^+ + iT) = \tilde{A}\tilde{x}(t_k + iT) + \tilde{B}F(i)\bar{C}_2\tilde{x}(t_k) \tag{15}$$

$$z(t) = \bar{C}_1\tilde{x}(t) \tag{16}$$

for $t \in (t_k + iT, t_k + (i+1)T)$, $i = 0, 1, 2, \ldots, N_T - 1$, where

$$\bar{A} = \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \bar{C}_1 = [C_1 \quad D_{12}], \quad \tilde{H} = \begin{bmatrix} H_1 & H_2 \\ 0 & 0 \end{bmatrix}$$

$$\tilde{E} = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \quad \tilde{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}$$

$$\bar{C}_2 = [I \quad 0], \quad \mathbb{I} = [I \quad 0]^T.$$

Clearly, if $N_T \equiv 1$, the multirate sampled-data control problem is reduced to the single-rate one.

III. Main Results

First, we consider the multirate sampled-data control problem of minimizing an upper bound of the H_2 measure J_g in (4) for the system in (1) and (2).

Theorem 1: For the system in (1) and (2), if there exists a time-varying periodic matrix function $X(t) = X(t + T_s) > 0$ for $t \in (iT, (i+1)T], t \ge -T_s$, defined as

$$X(t) = X(i^+) + \frac{t - iT}{T}(X(i+1) - X(i^+)) \tag{17}$$

with the notations $X(i+1) := X((i+1)T), X(i^+) := X((iT)^+)$, for $i = 0, 1, \ldots, N_T - 1$, matrices $U > 0, \Lambda = \mathrm{diag}(\lambda_1 I, \lambda_2 I) > 0$ and $Z(i)$, such that the linear matrix inequalities

$$\begin{bmatrix} U & B_1^T \mathbb{I}^T \\ \mathbb{I}B_1 & X(i+1) \end{bmatrix} > 0 \tag{18}$$

$$\begin{bmatrix} U & B_1^T \mathbb{I}^T \\ \mathbb{I}B_1 & X(i^+) \end{bmatrix} > 0 \tag{19}$$

$$\begin{bmatrix} \bar{X}(i^+) & X(i^+)\bar{C}_1^T & X(i^+)\tilde{E}^T \\ \bar{C}_1 X(i^+) & -I & 0 \\ \tilde{E}X(i^+) & 0 & -\Lambda \end{bmatrix} < 0 \tag{20}$$

$$\begin{bmatrix} \bar{X}(i+1) & X(i+1)\bar{C}_1^T & X(i+1)\tilde{E}^T \\ \bar{C}_1 X(i+1) & -I & 0 \\ \tilde{E}X(i+1) & 0 & -\Lambda \end{bmatrix} < 0 \tag{21}$$

$$\begin{bmatrix} \tilde{X} & \bar{C}_2^T \mathbb{Z}^T \mathbb{B}^T & \tilde{B}Z(0) \\ \mathbb{B}\mathbb{Z}\bar{C}_2 & -\mathbb{X}^+ + \mathbb{A}\mathbb{X}\mathbb{A}^T & \mathbb{B}\mathbb{Z} \\ Z^T(0)\tilde{B}^T & \mathbb{Z}^T\mathbb{B}^T & -\mathbb{I}^T X(N_T)\mathbb{I} \end{bmatrix} < 0 \tag{22}$$

where $\bar{X}(i^+) = (1/T)(X(i^+) - X(i+1)) + \bar{A}X(i^+) + X(i^+)\bar{A}^T + \tilde{H}\Lambda\tilde{H}^T$, $\bar{X}(i+1) = (1/T)(X(i^+) - X(i+1)) + \bar{A}X(i+1) + X(i+1)\bar{A}^T + \tilde{H}\Lambda\tilde{H}^T$, $\tilde{X} = -X(0^+) + \tilde{A}X(N_T)\tilde{A}^T + \tilde{B}Z(0)\bar{C}_2 + \bar{C}_2^T Z^T(0)\tilde{B}^T$, $\mathbb{X} = \mathrm{diag}(X(1), X(2), \ldots, X(N_T - 1))$, $\mathbb{X}^+ = \mathrm{diag}(X(1^+), X(2^+), \ldots, X((N_T - 1)^+)$, $\mathbb{Z} = (Z^T(1), Z^T(2), \ldots, Z^T(N_T - 1))^T$, $Z(i) = F(i)\mathbb{I}^T X(N_T)\mathbb{I}$, $\mathbb{A} = \mathrm{diag}(\tilde{A}, \tilde{A}, \ldots, \tilde{A})$, $\mathbb{B} = \mathrm{diag}(\tilde{B}, \tilde{B}, \ldots, \tilde{B})$, hold, then $F(i) = Z(i)(\mathbb{I}^T X(N_T)\mathbb{I})^{-1}$ for $i = 0, 1, \ldots, N_T - 1$, are stabilizing periodic time-varying feedback gains with which the H_2 measure J_g in (4) of the closed-loop system satisfies

$$J_g < \sqrt{T_s \mathrm{trace}(U)}.$$

Furthermore, a set of optimal feedback gains can be obtained by solving the following optimization:

$$\inf_{\Lambda, U Z(i), X(i+1), X(i^+), i=0,1,\ldots,N_T-1} \mathrm{trace}(U) \tag{23}$$

subject to linear matrix inequalities (18)–(22).

Proof: Let $\tilde{x}(t) = (x^T(t), u^T(t))^T$, choose a Lyapunov function candidate $V(t,\tilde{x}) = \tilde{x}^T(t)P(t)\tilde{x}(t)$, where $P(t) = P(t+T_s) > 0$ is a piecewise continuous function on $t \geq -T_s$. Here, take $P(t) = X^{-1}(t)$, where $X(t)$ defined in (17) is a solution of the linear matrix inequalities (18)–(22). Let

$$\mathbb{C}_2 = \begin{bmatrix} \bar{C}_2^T & \bar{C}_2^T & \cdots & \bar{C}_2^T \end{bmatrix}^T$$
$$\mathbb{P} = \mathrm{diag}(P(1), P(2), \ldots, P(N_T-1))$$
$$\mathbb{P}^+ = \mathrm{diag}(P(1^+), P(2^+), \ldots, P((N_T-1)^+)$$
$$\mathbb{F} = \mathrm{diag}(F(1), F(2), \ldots, F(N_T-1)).$$

For $t \in (t_k + iT, t_k + (i+1)T)$, noting that (14), for any $\Lambda = \mathrm{diag}(\lambda_1 I, \boldsymbol{\lambda}_2 I) > 0$, we have

$$D^+V(t,\tilde{x}(t)) \leq \tilde{x}^T(t)(D^+P(t) + P(t)\bar{A} + \bar{A}^T P(t) + P(t)\tilde{H}\Lambda\tilde{H}^T P(t) + \tilde{E}^T\Lambda^{-1}\tilde{E})\tilde{x}(t).$$

From the construction in (17), the periodic matrix function $X(t)$ is convex on the interval $(iT, (i+1)T]$ and is a convex combination of $X(i^+)$ and $X(i+1)$. As the convex combination of (20) and (21) is negative definite, we have

$$\frac{X(i^+) - X(i+1)}{T} + \bar{A}X(t) + X(t)\bar{A}^T + \tilde{H}\Lambda\tilde{H}^T + X(t)\tilde{E}^T\Lambda^{-1}\tilde{E}X(t) + X(t)\bar{C}_1^T\bar{C}_1X(t) < 0. \quad (24)$$

Pre- and post-multiplying (24) by $P(t)$ and noting $P(t)X(t) = I$, we obtain

$$D^+V(t,\tilde{x}(t)) < -\tilde{x}^T(t)\bar{C}_1^T\bar{C}_1\tilde{x}(t) \leq 0 \quad (25)$$

for $t \in (t_k + iT, t_k + (i+1)T)$. The computational problem of the H_2 measure J_g in (4) is equivalent to an LQ problem of the system with an initial condition $x(t_0) = x_0$ excited by an impulse $\delta_\tau(t) = \delta(t-\tau), 0 \leq \tau \leq T_s$, in all the input channels and the system responded autonomously [4]. Consider the LQ problem for the system with a fixed initial condition at $t = -\tau$ given by $\tilde{x}_{-\tau} := \tilde{x}(-\tau) = \bar{B}_1 e_i$ and no external input thereafter

$$J_\tau(\tilde{x}_{-\tau}) := \sup_{\Delta:\Delta^T\Delta\leq I} \|z\|_2^2.$$

Let $\bar{k}$ be the largest integer such that $\bar{k}T \leq \tau$. The effect of the impulse $\delta_\tau(t)$ in the ith channel is given by

$$\begin{aligned} J_\tau(\tilde{x}_{-\tau}) &= \sup_\Delta \left(\int_{-\tau}^{\infty} \left(\tilde{x}^T(t)\bar{C}_1^T\bar{C}_1\tilde{x}(t) + D^+V(t,\tilde{x}(t)) - D^+V(t,\tilde{x}(t)) \right) dt \right) \\ &< \inf_{\Lambda,P} \left(-\int_{-\tau}^{\infty} D^+V(t,\tilde{x}(t))\, dt \right) \\ &= \inf_{\Lambda,P} \left(V(-\tau,\tilde{x}) + \lim_{N\to\infty} \sum_{k=0}^{N} \left[\sum_{i=-\bar{k}}^{N_T-1-\bar{k}} (V(t_k^+ + iT,\tilde{x}) - V(t_k + iT,\tilde{x})) - V(t_k + (N_T - \bar{k})T,\tilde{x}) \right] \right). \end{aligned}$$

Let

$$\Theta(t_k) := (\tilde{x}^T(t_k), \tilde{x}^T(t_k+T), \ldots, \tilde{x}^T(t_k + (N_T-1)T))^T$$

we have

$$\sum_{i=0}^{N_T-1} [V(t_k^+ + iT,\tilde{x}) - V(t_k + iT,\tilde{x})] = \Theta^T(t_k)\Xi\Theta(t_k)$$

where

$$\Xi = \begin{bmatrix} M & * & \cdots & * \\ \tilde{A}^T P(1^+)\tilde{B}F(1)\bar{C}_2 & S_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{A}^T P((N_T-1)^+)\tilde{B}F(N_T-1)\bar{C}_2 & 0 & \cdots & S_{N_T-1} \end{bmatrix}$$

and

$$\begin{aligned} M &= -P(N_T) + (\tilde{A} + \tilde{B}F(0)\bar{C}_2)^T P(0^+)(\tilde{A} + \tilde{B}F(0)\bar{C}_2) + \sum_{i=1}^{N_T-1} \bar{C}_2^T F^T(i)\tilde{B}^T P(i^+)\tilde{B}F(i)\bar{C}_2 \\ S_i &= \tilde{A}^T P(i^+)\tilde{A} - P(i). \end{aligned}$$

For $\Theta(t_k) \neq 0$, if

$$\Xi < 0 \quad (26)$$

then

$$\sum_{i=1}^{N_T-1} [V(t_k^+ + iT,\tilde{x}) - V(t_k + iT,\tilde{x})] + V(t_k^+,\tilde{x}) < V(t_k,\tilde{x}). \quad (27)$$

From (25) and (27) and the construction of $X(t)$, if there exist solutions to (20) and (21), then (17) shows that the piecewise linear matrix $X(t)$ satisfying

$$X(t) \geq \min\{\underline{\sigma}(X(i^+)), \underline{\sigma}(X(i+1))\}I$$
$$X(t) \leq \max\{\bar{\sigma}(X(i^+)), \bar{\sigma}(X(i+1))\}I$$

for $t \in (t_k + iT, t_k + (i+1)T]$, which implies that $P(t)$ is bounded and the Lyapunov function satisfies bounds of the type in (11) of Lemma 1. It is easy to obtain that of Ξ are negative definite from (26), which implies $\sum_{i=0}^{l}[V(t_k^+ + iT,\tilde{x}) - V(t_k + iT,\tilde{x})] < 0$, for $l = 0, 1, \ldots, N_T - 1$. Then, by Lemma 1 with (25) and (27), the closed-loop system is asymptotically stable. As $P(t)$ and $F(i)$ are periodic with period T_s, then we have

$$\sum_{i=-\bar{k}.}^{N_T-1-\bar{k}} [V(t_k^+ + iT,\tilde{x}) - V(t_k + iT,\tilde{x})] < 0$$

which implies

$$J_\tau(\tilde{x}_{-\tau}) < \inf_{F,\Lambda,P} V(-\tau,\tilde{x}) = \inf_{F,\Lambda,P} \tilde{x}_{-\tau}^T P(-\tau)\tilde{x}_{-\tau}.$$

Summing the effect of the impulses over the channels, we have

$$\begin{aligned} \sup_{\Delta:\Delta^T\Delta\leq I} \sum_{i=1}^{p} \|T_{zw}(\Delta)\delta_\tau(t)e_i\|_2^2 &\leq \sum_{i=1}^{p} \sup_{\Delta:\Delta^T\Delta\leq I} \|T_{zw}(\Delta)\delta_\tau(t)e_i\|_2^2 \\ &= \sum_{i=1}^{p} J_\tau(\bar{B}_1 e_i) \\ &< \sum_{i=1}^{p} \inf_{F,\Lambda,P} e_i^T \bar{B}_1^T P(-\tau)\bar{B}_1 e_i \\ &\leq \inf_{F,\Lambda,P} \mathrm{trace}\left(B_1^T \mathbb{I}^T P(-\tau)\mathbb{I} B_1 \right). \end{aligned}$$

Then, the generalized H_2 measure has the following upper bound:

$$J_g^2 < \inf_{F,\Lambda,P} \int_0^{T_s} \mathrm{trace}\left(B_1^T \mathbb{I}^T P(-\tau)\mathbb{I} B_1 \right) d\tau.$$

Let U be such that

$$U > B_1^T \mathbb{I}^T P(-\tau)\mathbb{I}B_1 = B_1^T \mathbb{I}^T \left(\frac{\tau - iT}{T}X(i^+) + \frac{(i+1)T-\tau}{T}X(i+1)\right)^{-1} \mathbb{I}B_1$$

for $\tau \in [iT, (i+1)T)$, then

$$J_g^2 < T_s \inf_{F,\Lambda,P,U} \mathrm{trace}(U).$$

These directly lead to (23) and (18) and (19). Condition (26) is equivalent to

$$\begin{bmatrix} -P(N_T) & 0 & (\tilde{A}+\tilde{B}F(0)\bar{C}_2)^T & \mathbb{C}_2^T\mathbb{F}^T\mathbb{B}^T \\ 0 & -\mathbb{P} & 0 & \mathbb{A}^T \\ \tilde{A}+\tilde{B}F(0)\bar{C}_2 & 0 & -P^{-1}(0^+) & 0 \\ \mathbb{B}\mathbb{F}\mathbb{C}_2 & \mathbb{A} & 0 & -(\mathbb{P}^+)^{-1} \end{bmatrix} < 0$$

which is again equivalent to

$$\begin{bmatrix} -X(0^+) & 0 \\ 0 & -\mathbb{X}^+ \end{bmatrix} + \begin{bmatrix} \tilde{A}+\tilde{B}F(0)\bar{C}_2 & 0 \\ \mathbb{B}\mathbb{F}\mathbb{C}_2 & \mathbb{A} \end{bmatrix}\begin{bmatrix} X(N_T) & 0 \\ 0 & \mathbb{X} \end{bmatrix} \times \begin{bmatrix} (\tilde{A}+\tilde{B}F(0)\bar{C}_2)^T & \mathbb{C}_2^T\mathbb{F}^T\mathbb{B}^T \\ 0 & \mathbb{A}^T \end{bmatrix} < 0.$$

Expand the second group on the left side of the above inequality

$$\begin{aligned} &\begin{bmatrix} \tilde{A}+\tilde{B}F(0)\bar{C}_2 & 0 \\ \mathbb{B}\mathbb{F}\mathbb{C}_2 & \mathbb{A} \end{bmatrix}\begin{bmatrix} X(N_T) & 0 \\ 0 & \mathbb{X} \end{bmatrix} \times \begin{bmatrix} (\tilde{A}+\tilde{B}F(0)\bar{C}_2)^T & \mathbb{C}_2^T\mathbb{F}^T\mathbb{B}^T \\ 0 & \mathbb{A}^T \end{bmatrix} \\ &= \begin{bmatrix} \tilde{A}X(N_T)\tilde{A}^T & 0 \\ 0 & \mathbb{A}\mathbb{X}\mathbb{A}^T \end{bmatrix} + \tilde{M} + \tilde{M}^T + \begin{bmatrix} \tilde{B}F(0)\bar{C}_2 & 0 \\ \mathbb{B}\mathbb{F}\mathbb{C}_2 & 0 \end{bmatrix} \\ &\quad \times \begin{bmatrix} X(N_T) & 0 \\ 0 & \mathbb{X} \end{bmatrix}\begin{bmatrix} \bar{C}_2^T F^T(0)\tilde{B}^T & \mathbb{C}_2^T\mathbb{F}^T\mathbb{B}^T \\ 0 & 0 \end{bmatrix} \end{aligned}$$

where

$$\tilde{M} = \begin{bmatrix} \tilde{A}X(N_T)\bar{C}_2^T F^T(0)\tilde{B}^T & \tilde{A}X(N_T)\mathbb{C}_2^T\mathbb{F}^T\mathbb{B}^T \\ 0 & 0 \end{bmatrix}.$$

Let $F(i)\mathbb{I}^T X(N_T)\mathbb{I} := Z(i)$, we directly have the inequality (22). We then obtain the desired result. ■

As explained in Section II, the H_2 measure J is a special case of the generalized H_2 measure J_g. The following is a direct result of Theorem 1 by setting $\tau = 0$.

Corollary 1: For the system in (1) and (2), if there exists a matrix function $X(t) = X(t+T_s) > 0$ defined as (17), matrices $U > 0, \Lambda = \mathrm{diag}(\lambda_1 I, \lambda_2 I) > 0$ and $Z(i)$, such that the linear matrix inequalities (20)–(22), and

$$\begin{bmatrix} U & B_1^T\mathbb{I}^T \\ \mathbb{I}B_1 & X(N_T) \end{bmatrix} > 0 \tag{28}$$

hold, then $F(i) = Z(i)(\mathbb{I}^T X(N_T)\mathbb{I})^{-1}$ for $i = 0, 1, \ldots, N_T - 1$, are stabilizing periodic feedback gains with which the H_2 measure J in (3) of the closed-loop system satisfies

$$J < \sqrt{\mathrm{trace}(U)}.$$

Furthermore, a set of optimal feedback gains can be obtained by solving the following optimization:

$$\inf_{\Lambda,U,Z(i),X(i+1),X(i^+),i=0,1,\ldots,N_T-1} \mathrm{trace}(U)$$

subject to linear matrix inequalities (20)–(22) and (28).

Remark 2: The piecewise continuous matrix function $X(t)$ in (17) is linear and its positive definiteness convex in t, which leads to a linear matrix inequality (LMI) formulation of the multirate H_2 sampled-data control. In fact, similar results can be obtained if we choose $X(t) = X(t+T_s) > 0$, for $t \geq 0 (t \geq -T_s)$ with the structure

$$X(t) = X(i^+) + \varphi(t)(X(i+1) - X(i^+)), t \in (iT, (i+1)T]$$

where $\varphi((iT)^+) = 0$, and $\varphi((i+1)T) = 1$, for the multirate case, where $\varphi(t)$ is a real monotonic increasing function. Of course, it is possible to choose $X(t)$ as a constant, but it is clearly less flexible to obtain a solution.

As discussed earlier, the single-rate control problem is a special case of the multirate one with $N_T = 1$. From Theorem 1 and Corollary 1, we obtain the following results for the single-rate case.

Corollary 2: For the system in (1) and (2), if there exists a matrix function $X(t) = X(t+T_s) > 0$ for $t \geq -T_s$, defined as

$$X(t) = X(0^+) + \frac{t}{T_s}(X(1) - X(0^+)), \quad \text{for } t \in (0, T_s] \tag{29}$$

matrices $Z, U > 0$ and $\Lambda = \mathrm{diag}(\lambda_1 I, \lambda_2 I) > 0$, such that the linear matrix inequalities

$$\begin{bmatrix} U & B_1^T\mathbb{I}^T \\ \mathbb{I}B_1 & X(1) \end{bmatrix} > 0 \tag{30}$$

$$\begin{bmatrix} U & B_1^T\mathbb{I}^T \\ \mathbb{I}B_1 & X(0^+) \end{bmatrix} > 0 \tag{31}$$

$$\begin{bmatrix} \bar{X}(0^+) & X(0^+)\bar{C}_1^T & X(0^+)\tilde{E}^T \\ \bar{C}_1 X(0^+) & -I & 0 \\ \tilde{E}X(0^+) & 0 & -\Lambda \end{bmatrix} < 0 \tag{32}$$

$$\begin{bmatrix} \bar{X}(1) & X(1)\bar{C}_1^T & X(1)\tilde{E}^T \\ \bar{C}_1 X(1) & -I & 0 \\ \tilde{E}X(1) & 0 & -\Lambda \end{bmatrix} < 0 \tag{33}$$

$$\begin{bmatrix} -X(0^+) + \tilde{A}X(1)\tilde{A}^T + \bar{C}_2^T Z^T\tilde{B}^T + \tilde{B}Z\bar{C}_2 & \tilde{B}Z \\ Z^T\tilde{B}^T & -\mathbb{I}^T X(1)\mathbb{I} \end{bmatrix} < 0 \tag{34}$$

hold, then $F = Z(\mathbb{I}^T X(1)\mathbb{I})^{-1}$ is a stabilizing feedback gain with which the H_2 measure J_g in (4) of the closed-loop system satisfies

$$J_g < \sqrt{T_s \mathrm{trace}(U)}.$$

Furthermore, an optimal control gain can be obtained by solving the following optimization:

$$\inf_{Z,\Lambda,U,X(0^+),X(1)} \mathrm{trace}(U)$$

subject to linear matrix inequalities (32)–(34).

Corollary 3: For the system in (1) and (2), if there exists a periodic matrix function $X(t) = X(t+T_s) > 0$ for $t \geq 0$, defined as (29), matrices $U > 0, \Lambda = \mathrm{diag}(\lambda_1 I, \lambda_2 I) > 0$, and Z, such that linear matrix inequalities (32)–(34) and

$$\begin{bmatrix} U & B_1^T\mathbb{I}^T \\ \mathbb{I}B_1 & X(1) \end{bmatrix} > 0 \tag{35}$$

hold, then $F = Z(\mathbb{I}^T X(1)\mathbb{I})^{-1}$ is a stabilizing feedback gain with which the H_2 measure J in (3) of the closed-loop system satisfies

$$J < \sqrt{\mathrm{trace}(U)}.$$

Furthermore, an optimal feedback gain can be obtained by solving the following optimization:

$$\inf_{Z,\Lambda,U,X(0^+),X(1)} \mathrm{trace}(U)$$

subject to linear matrix inequalities (32)–(34) and (35).

TABLE I
VALUES OF PARAMETERS FOR DIFFERENT FLOW RATES

k	1	2	3	4	5
L_k	40.8	35.8	35.8	45.8	45.8
G_k	66.7	61.7	71.7	71.7	61.7

TABLE II
SUMMARY OF RESULTS

Methods	$(J_g <)\sqrt{T_s \min \operatorname{trace}(U)}$	$(J <)\sqrt{\min \operatorname{trace} U}$
Theorem 1	3.2364	/
Corollary 1	/	2.8587
Corollary 2	3.1228	/
Corollary 3	/	2.8120

IV. NUMERICAL EXAMPLE

Consider a robust optimal control problem of a 2-input gas absorber in [8] and [12]. The plant model is

$$\dot{x}(t) = \begin{bmatrix} b & c & 0 & 0 & 0 & 0 \\ a & b & c & 0 & 0 & 0 \\ 0 & a & b & c & 0 & 0 \\ 0 & 0 & a & b & c & 0 \\ 0 & 0 & 0 & a & b & c \\ 0 & 0 & 0 & 0 & a & b \end{bmatrix} x(t) + \begin{bmatrix} b & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & c \end{bmatrix} u(t) \tag{36}$$

where $a = L_k/(H\alpha + h), b = (L_k + G_k\alpha)/(H\alpha + h), c = G_k\alpha/(H\alpha + h)$. The nominal parameter values are $\alpha = 0.72, H = 1, h = 75$. Five different pairs of flow rates, L_k and $G_k, k = 1, 2, \ldots, 5$, are given in Table I.

We try to design digital control procedures with single-rate and multirate to minimize the following index:

$$J = \int_0^\infty (x^T Qx + u^T Ru)\, dt$$

where $Q = I, R = I$. After performing a Cholesky factorization, we have

$$[C_1 \quad D_{12}]^T [C_1 \quad D_{12}] = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$$

and define

$$z(t) = C_1 x(t) + D_{12} u(t) \tag{37}$$

and assume the disturbance $w(t)$ is injected to the system through all the state channels. That is, $B_1 = I$. In the example, the above system is converted into the form of systems in (1) and (2). We choose the nominal parameters to be: $L_k^0 = (1/5)\sum_{k=1}^{k=5} L_k, G_k^0 = (1/5)\sum_{k=1}^{k=5} G_k$, and assume $L_k = L_k^0 + (\max(L_k) - L_k^0)\delta_1, G_k = G_k^0 + (\max(G_k) - G_k^0)\delta_2$, where δ_1 and δ_2 are uncertain real scalars with $|\delta_1| < 1, |\delta_2| < 1$. Then, (36) and (37) can be rewritten as (1) and (2), where the matrices A, B_2, H, E, H_1, and E_1 can be easily obtained. The above LQ problem is now cast into the H_2 problem. Using the proposed procedures and choosing $T_s = 1$ and $N_T = 2$, the result is summarized in Table II.

The result shows the upper bounds of the generalized H_2 measure (J_g) are larger than that of the traditional H_2 measure (J). In fact, what we computed for the generalized H_2 measure case is an upper bound of the accumulated effect excited by impulses in the interval $(0, T_s]$.

V. CONCLUSION

In this paper, two H_2 measures are used to design uncertain linear systems. Under the framework of hybrid systems, the multirate and single-rate H_2 sampled-data control procedures are proposed for the uncertain linear systems. The sampling period is explicitly involved in the result which is superior to traditional methods. The solution procedures proposed in this paper are formulated as an optimization problem subject to linear matrix inequalities. A numerical example is also presented to demonstrate the proposed techniques.

APPENDIX
PROOF OF LEMMA 1

Proof: For a given scalar $\delta > 0$, consider the system in (9) and (10) with an initial state $x(t_0) = x_0$, satisfying $\|x_0\| < \delta$. Let $\sigma > 0$ be a scalar such that $a\sigma^2 > b\delta^2$. For any $t > t_0$, let $\bar{k}$ be the largest nonnegative integer such that the jumping instant $t_{\bar{k}}$ satisfying $t_{\bar{k}} \leq t - t_0$, we have

$$\begin{aligned} \int_{t_0}^t D^+V(\tau, x)d\tau &= V(t, x) - V(t_{\bar{k}}^+, x) \\ &\quad + \sum_{i=0}^{\bar{k}-1} (V(t_{i+1}, x) - V(t_i^+, x)) \\ &= V(t, x) - V(t_0, x) \\ &\quad - \sum_{k=0}^{[\frac{\bar{k}}{N_T}]-1} \sum_{i=kN_T}^{(k+1)N_T-1} (V(t_i^+, x) - V(t_i, x)) \\ &\quad - \sum_{i=[\frac{\bar{k}}{N_T}]N_T}^{\bar{k}} (V(t_i^+, x) - V(t_i, x)). \end{aligned}$$

That is

$$\begin{aligned} a\|x\|^2 \leq V(t, x) &= \int_{t_0}^t D^+V(\tau, x)d\tau + V(t_0, x) \\ &\quad + \sum_{k=0}^{[\frac{\bar{k}}{N_T}]-1} \sum_{i=kN_T}^{(k+1)N_T-1} \left(V\left(t_i^+, x\right) - V(t_i, x)\right) \\ &\quad + \sum_{i=[\frac{\bar{k}}{N_T}]N_T}^{\bar{k}} \left(V\left(t_i^+, x\right) - V(t_i, x)\right) \\ &\leq V(t_0, x) \leq b\|x_0\|^2 < b\delta^2 < a\sigma^2 \end{aligned}$$

from which we obtain $\|x\| < \sigma$ for $t \geq t_0$. Then, we claim that

$$\lim_{t\to\infty} \inf \|x(t)\| = 0. \tag{38}$$

If this is not true, then there exists a scalar $\gamma > 0$ such that

$$\|x(t)\| \geq \gamma, \quad t \geq \bar{T} + t_0 \tag{39}$$

for some $\bar{T} > 0$. From the conditions in (12), (13), and (39), we obtain

$$\begin{aligned} \lim_{t\to\infty} V(t, x) &= \lim_{t\to\infty} \int_{t_0}^t D^+V(\tau, x)d\tau + V(t_0, x_0) \\ &\quad + \lim_{\bar{k}\to\infty} \left\{ \sum_{k=0}^{[\frac{\bar{k}}{N_T}]-1} \sum_{j=kN_T}^{(k+1)N_T-1} \left(V\left(t_j^+, x\right) - V(t_j, x)\right) \right. \\ &\quad \left. + \sum_{j=[\frac{\bar{k}}{N_T}]N_T}^{\bar{k}} \left(V\left(t_j^+, x\right) - V(t_j, x)\right) \right\} \\ &\leq V(t_0, x_0) - \lim_{t\to\infty} \int_{t_0}^t c\|x(\tau)\|^2 d\tau \\ &\leq V(t_0, x_0) - c\gamma^2 \lim_{t\to\infty} \int_{t_0+\bar{T}}^t ds = -\infty \end{aligned}$$

which is a contradiction.

We now claim that $\lim_{t\to\infty} \sup \|x(t)\| = 0$, if this is not true, then we can choose a scalar $\varepsilon > 0$ such that $\varepsilon < \lim_{t\to\infty} \sup \|x(t)\|$. Since

(38) holds, we can find sequences $\{\tilde{t}_i\}_{i=1,2,\ldots}$ and $\{\bar{t}_i\}_{i=1,2,\ldots}, t_0 < \tilde{t}_i < \bar{t}_i \le \tilde{t}_{i+1} < \bar{t}_{i+1}$, such that

$$\|x(\tilde{t}_i)\| = \frac{\varepsilon}{2}, \|x(\bar{t}_i)\| = \varepsilon$$

and

$$\frac{\varepsilon}{2} < \|x(t)\| < \varepsilon, t \in (\tilde{t}_i, \bar{t}_i), \quad i = 1, 2, \ldots. \tag{40}$$

Of course, we could have, instead of the relationship in (40)

$$\|x(\tilde{t}_i)\| = \varepsilon, \|x(\bar{t}_i)\| = \frac{\varepsilon}{2}$$

and

$$\frac{\varepsilon}{2} < \|x(t)\| < \varepsilon, t \in (\tilde{t}_i, \bar{t}_i), \quad i = 1, 2, \ldots. \tag{41}$$

The value of ε can be chosen such that at least an infinite sequence of intervals satisfies either (40) or (41) with no sampling instants in $(\tilde{t}_i, \bar{t}_i)$. As $f(t, x)$ is uniformly norm-bounded in x and $\|x\| < \sigma$ for $t \ge t_0$, then we can deduce that $\|D^+ x\|$ is bounded; that is, there exists a scalar $M > 0$ such that

$$\|D^+ x\| = \|f(t, x)\| \le M, \quad \text{for } t \in (t_k, t_{k+1}).$$

This implies that for any $x(t_1)$ and $x(t_2), t_2 > t_1, t_1, t_2 \in (t_k, t_{k+1})$, we have

$$|\|x(t_2)\| - \|x(t_1)\|| \le \|x(t_2) - x(t_1)\| \le M(t_2 - t_1)$$

and hence

$$\left|D^+\|x\|\right| = \left|\lim_{t_2 \to t_1^+} \frac{\|x(t_2)\| - \|x(t_1)\|}{t_2 - t_1}\right| \le M.$$

Since $D^+\|x\| \le M$, from (40), we obtain the relation $\bar{t}_i - \tilde{t}_i \ge (\varepsilon/2M)$. In view of (12) and (13), we have

$$\begin{aligned}
&\lim_{n\to\infty} V(\bar{t}_n, x(\bar{t}_n)) \\
&= V(t_0, x(t_0)) + \lim_{n\to\infty} \int_{t_0}^{\bar{t}_n} D^+ V(\tau, x) d\tau \\
&\quad + \lim_{n\to\infty} \left\{ \sum_{i=[\frac{n}{N_T}]N_T}^{n} \left(V\left(t_i^+, x\right) - V(t_i, x)\right) \right. \\
&\quad \left. + \sum_{k=0}^{[\frac{n}{N_T}]-1} \sum_{i=kN_T}^{(k+1)N_T-1} \left(V\left(t_i^+, x\right) - V(t_i, x)\right) \right\} \\
&\le V(t_0, x(t_0)) + \lim_{n\to\infty} \sum_{0 \le i \le n} \int_{\tilde{t}_i}^{\bar{t}_i} D^+ V(s, x(s)) ds \\
&\le V(t_0, x(t_0)) - \lim_{n\to\infty} \sum_{i=0}^{n} c\left(\frac{\varepsilon}{2}\right)^2 \frac{\varepsilon}{2M} \\
&= -\infty
\end{aligned}$$

which is a contradiction. Thus, $\|x(t)\| \to 0$ as $t \to \infty$ and hence $x(t) \to 0$ as $t \to \infty$. The case when $D^+\|x(t)\|$ is bounded from below can be proved using (41) with similar arguments. All of these showed the desired results, which complete the proof. ■

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