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A_{11} is assumed to be stable so that $(-1)^p \det(\overline{M}_0) > 0$. The critical stability for the above inequality is

$$\det(I_p + \varepsilon_{2c}\overline{M}_1\overline{M}_0^{-1} + \varepsilon_{2c}^2\overline{M}_2\overline{M}_0^{-1})$$
$$= 0 \qquad \text{for } \varepsilon < \varepsilon_{2c} \tag{15}$$

and (15) is equivalent to the following equation [25]:

$$\det(I_{2p} + \varepsilon_{2c}\overline{A}_M) = 0.$$

Then the upper bound ε, to guarantee the condition 3) of Lemma 3 to be satisfied, can be given by $\varepsilon < \varepsilon_{2c}$.

APPENDIX B
PROOF OF THEOREM 2

The proof of conditions 1) and 2) of Lemma 3 are similar to those of Theorem 1, so we concentrate on searching the bound ε_2. Connecting Lemma 2 with (8c) and (9), we can assert that matrix E is Hurwitz. Then E is invertible, and the third condition of Lemma 3 is now equivalent to computing the minimum real eigenvalue of FE^{-1}. Hence, the upper bound, to guarantee the third condition of Lemma 3 to be satisfied, is given by ε_2.

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Improvement of Parametric Stability Margin Under Pole Assignment

Tingshu Hu and James Lam

Abstract—In this paper, the improvement of the parametric stability margin of state-space uncertain systems via a maximization formulation under the constraints of pole assignment is investigated. The class of systems considered is where the uncertainty may be modeled as the, possibly nonlinear, variation of a parameter appearing in the entries of the system and input matrices. The continuity and differentiability properties of the stability margin are discussed. A gradient-based approach is presented for the improvement of the stability margin and a compact formula to compute the gradient is provided. Numerical examples are used to demonstrate the effectiveness of the approach.

Index Terms— Gradient, optimization, pole assignment, robustness, stability margin.

I. INTRODUCTION

Over the last decade, a vast amount of research has been devoted to robust stability analysis for systems with parametric uncertainties or perturbations; see, e.g., [6], [10], [12], [14], and [15]. In these papers, the perturbations in the system matrix are assumed to be affine, multilinear, or polynomial functions of the uncertain parameters. Some bounds on the parameters to ensure robust stability were provided. However, less attention has been paid to designing a controller to enhance robust stability.

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From the designer's point of view, it would be important to construct a closed-loop system so that it is maximally tolerable toward uncertainties. A classical technique in control system design for state-space systems is pole assignment. For a completely state controllable realization, it is well known that the closed-loop poles can be arbitrarily assigned. For systems with two or more inputs, the feedback gain to achieve a given pole assignment specification is in general nonunique. Such nonuniqueness may be exploited to optimize a variety of system performance indexes. The most common application of this idea is robust pole assignment. Research in this area may be found in [1], [2], [4], [5], [8], and references therein. There is little work on utilizing the freedom in the feedback matrices to improve stability margin. The obvious reason is that the pole assignment itself imposes constraints to the feedback systems and inevitably reduces the overall achievable stability margin if only closed-loop stability is concerned. However, it is often necessary to fix or approximately fix the closed-loop poles due to practical considerations, such as transient characteristics. The tradeoff between pole assignment constraints and optimum performance is justifiable in view of control system implementation since optimal solutions may have undesirable transient behavior or unacceptably large gain.

Motivated by the aforementioned reasons, this paper considers the improvement of a parametric stability margin under the constraints of pole assignment via state feedback. As a first step toward a more general computation procedure, it is assumed that the (nonlinear) perturbation is parameterized by a single parameter. In contrast to previous works, the development is given in terms of state-space matrices with a gradient-based optimization treatment.

II. STABILITY MARGIN

Consider the following parametric uncertain system:

$$\dot{x} = M(F, p)x \tag{1}$$

where $x \in \mathbf{R}^n$ is the state, F is a real matrix containing all the design parameters, $p \in \mathbf{R}$ is the uncertain parameter, and $M(F, p) \in \mathbf{R}^{n \times n}$ is a matrix function that is continuously differentiable with respect to F and p.

For a given F, suppose $M(F, 0)$ is stable, then there exists a real number $r > 0$ such that $M(F, p)$ is stable for all $p \in (-r, r)$. A practical problem is to select an F such that this r is maximized. To formulate the problem, we define the function of stability margin as follows.

Definition 1: Let $\mathcal{F}$ be the set of F such that $M(F, 0)$ is stable. For $F \in \mathcal{F}$, define

$$\rho_M(F) := \begin{cases} \min\{|p|: M(F, p) \text{ is unstable}\} \\ \infty, \quad \text{if } M(F, p) \text{ is stable for all } p. \end{cases} \tag{2}$$

In this paper, we are particularly interested in the following closed-loop system:

$$\dot{x} = M(F, p)x = [A(p) + B(p)F]x \tag{3}$$

where $F \in \mathbf{R}^{m \times n}$ is the state feedback matrix and $A(p) \in \mathbf{R}^{n \times n}$, $B(p) \in \mathbf{R}^{n \times m}$ are matrix functions that are continuously differentiable with respect to the uncertain parameter $p \in \mathbf{R}$. Our objective is to select an F such that $\rho_M(F)$ is maximized under the constraint of pole assignment. For simplicity, denote $A(0) = A_0$, $B(0) = B_0$.

Since p is a scalar, for a given F, $\rho_M(F)$ can be computed by the bisection method. It is clear that functions of this kind are very complicated and can possess discontinuities. To maximize $\rho_M(F)$ based on gradient information, one must have knowledge about under what conditions $\rho_M(F)$ is continuous and differentiable.

Assume that $M(F, 0)$ is stable. Denote

$$Q(F, p) = \begin{bmatrix} M(F, p) & 0 \\ 0 & M(F, -p) \end{bmatrix}$$

then

$$\rho_M(F) = \begin{cases} \min\{p > 0 : \operatorname{Re}\lambda_i[Q(F, p)] = 0, \quad \text{for some } i\} \\ \infty, \quad \text{if } \operatorname{Re}\lambda_i[Q(F, p)] \neq 0, \quad \text{for all } p > 0 \text{ and } i \end{cases} \tag{4}$$

where $\operatorname{Re}\lambda_i[\cdot]$ denotes the real part of the ith eigenvalue of a matrix. For a given F, each locus $\lambda_i[Q(F, p)]$ is continuous and piecewise smooth and $\rho_M(F)$ equals the smallest p at which one of the loci hits the imaginary axis.

Theorem 1: For a given F_0, let $p_0 = \rho_M(F_0)$. Suppose $Q(F_0, p_0)$ has ℓ distinct eigenvalues $\lambda_i[Q(F_0, p_0)], 1 \leq i \leq \ell$ on the imaginary axis, then we have the following.

1) $\rho_M(F)$ is continuous in a neighborhood of F_0 if there is one $i, 1 \leq i \leq \ell$ such that

$$\frac{\partial \operatorname{Re}\lambda_i[Q(F_0, p_0)]}{\partial p} \neq 0.$$

2) $\rho_M(F)$ is differentiable at F_0 if

$$\frac{\partial \operatorname{Re}\lambda_i[Q(F_0, p_0)]}{\partial p} \neq 0, \qquad \text{for all } 1 \leq i \leq \ell$$

and the following ℓ items are equal:

$$\frac{\dfrac{\partial \operatorname{Re}\lambda_i[Q(F_0, p_0)]}{\partial F}}{\dfrac{\partial \operatorname{Re}\lambda_i[Q(F_0, p_0)]}{\partial p}}, \qquad 1 \leq i \leq \ell.$$

In this case, the partial derivative of $\rho_M(F)$ at F_0 is given as

$$\frac{\partial \rho_M(F_0)}{\partial F} = -\frac{\dfrac{\partial \operatorname{Re}\lambda_1[Q(F_0, p_0)]}{\partial F}}{\dfrac{\partial \operatorname{Re}\lambda_1[Q(F_0, p_0)]}{\partial p}}. \tag{5}$$

To prove the above theorem, define

$$\rho_M^i(F) := \begin{cases} \min\{p > 0 : \operatorname{Re}\lambda_i[Q(F, p)] = 0\} \\ \infty, \quad \text{if } \operatorname{Re}\lambda_i[Q(F, p)] \neq 0 \text{ for all } p > 0. \end{cases}$$
$$i = 1, 2, \cdots, 2n.$$

It is easy to see that

$$\rho_M(F) = \min\{\rho_M^i(F), i = 1, 2, \cdots, 2n\}. \tag{6}$$

For each $\rho_M^i(F)$, we have the following result.

Lemma 1: For a given F_0, assume $\rho_M^i(F_0) < \infty$. Let $p_0 = \rho_M^i(F_0)$, then by definition $\operatorname{Re}\lambda_i[Q(F_0, p_0)] = 0$. Suppose that the following conditions are satisfied:

1) $\lambda_i[Q(F_0, p_0)]$ is a simple eigenvalue of $Q(F_0, p_0)$;
2) $(\partial \operatorname{Re}\lambda_i[Q(F_0, p_0)])/\partial p \neq 0$;

then $\rho_M^i(F)$ is continuously differentiable in a neighborhood of F_0 with

$$\frac{\partial \rho_M^i(F_0)}{\partial F} = -\frac{\dfrac{\partial \operatorname{Re}\lambda_i[Q(F_0, p_0)]}{\partial F}}{\dfrac{\partial \operatorname{Re}\lambda_i[Q(F_0, p_0)]}{\partial p}}. \tag{7}$$

Proof: From Condition 2 and the definition of $\rho_M^i(F)$ we have

$$\frac{\partial \operatorname{Re}\lambda_i[Q(F_0, p_0)]}{\partial p} > 0. \tag{8}$$

From Condition 1, there exist $\delta_1, \xi_1 > 0$ such that when $\|F - F_0\| < \delta_1, |p - p_0| < \xi_1, \lambda_i[Q(F, p)]$ is continuously differentiable. So there exists $\xi \in (0, \xi_1)$ such that

$$\frac{\partial \operatorname{Re}\lambda_i[Q(F_0, p)]}{\partial p} > 0, \qquad \text{for all } p \in (p_0 - \xi, p_0 + \xi). \tag{9}$$

By the implicit function theorem, there exists $\delta_2 \in (0, \delta_1]$, such that when $\|F - F_0\| < \delta_2$, there is a unique $p \in (p_0 - \xi, p_0 + \xi)$ satisfying $\operatorname{Re}\lambda_i[Q(F, p)] = 0$. By the definition of $\rho_M^i(F)$, we also have $\max\{\operatorname{Re}\lambda_i[Q(F_0, p)]: p \in [0, p_0 - \xi]\} < 0$. Hence there exists $\delta \in (0, \delta_2]$ such that $\max\{\operatorname{Re}\lambda_i[Q(F, p)]: p \in [0, p_0 - \xi], \|F - F_0\| \le \delta\} < 0$. Thus for any F such that $\|F - F_0\| \le \delta$, there is a unique $p' \in (0, p_0 + \xi)$ satisfying $\operatorname{Re}\lambda_i[Q(F, p')] = 0$. This implies that $\rho_M^i(F) = p'$, where $p' \in (0, p_0 + \xi)$ is uniquely determined from $\operatorname{Re}\lambda_i[Q(F, p)] = 0$. By using the implicit function theorem again, we know that $\rho_M^i(F)$ is continuously differentiable in a neighborhood of F_0 with the partial derivative given by (7). $\square$

The following result is similar to Proposition 2.1 in Hinrichsen and Pritchard [3].

Lemma 2: $\rho_M^i(F)$ is semicontinuous from above. That is, given F_0, if $\rho_M^i(F_0) > \alpha$, then there exists $\delta > 0$, such that $\rho_M^i(F) > \alpha$ whenever $\|F - F_0\| < \delta$.

Proof: Since $\rho_M^i(F_0) > \alpha$, thus $\max\{\operatorname{Re}\lambda_i[Q(F_0, p)]: p \in [0, \alpha]\} < 0$ and there exists $\delta > 0$ such that

$$\max\{\operatorname{Re}\lambda_i[Q(F, p)]: p \in [0, \alpha], \|F - F_0\| \le \delta\} < 0$$

and the result follows. $\square$

Proof of Theorem 1: Notice that $\rho_M^i(F_0) = \rho_M(F_0)$ for $i \le \ell$ and $\rho_M^i(F_0) > \rho_M(F_0)$ for $i > \ell$.

1) Without loss of generality, assume that $(\partial \operatorname{Re}\lambda_1[Q(F_0, p_0)])/\partial p \ne 0$, and it follows from Lemma 1 that $\rho_M^1(F)$ is continuous at F_0. Thus for any $\xi > 0$, there exists $\delta_1 > 0$ such that when $\|F - F_0\| < \delta_1, |\rho_M^1(F) - \rho_M^1(F_0)| < \xi$. Since $\rho_M^i(F_0) > \rho_M^1(F_0) - \xi$, by Lemma 2, there exists $\delta \in (0, \delta_1]$ such that when $\|F - F_0\| < \delta, \rho_M^i(F) > \rho_M^1(F_0) - \xi$, $i = 2, \cdots, 2n$. This implies $\rho_M(F) > \rho_M(F_0) - \xi$. On the other hand, since $\rho_M^1(F) < \rho_M(F_0) + \xi$, we have $\rho_M(F) < \rho_M(F_0) + \xi$ by (6). It follows that $|\rho_M(F) - \rho_M(F_0)| < \xi$ for all F satisfying $\|F - F_0\| < \delta$. This shows $\rho_M(F)$ is continuous at F_0.

2) When the conditions are satisfied, $\rho_M^i(F), i \le \ell$ are continuous at F_0. Let $\xi = 1/2 \min_{i>\ell}(\rho_M^i(F_0) - \rho_M(F_0))$, there exists $\delta > 0$ such that when $\|F - F_0\| < \delta, |\rho_M^i(F) - \rho_M(F_0)| < \xi$ for $i \le \ell$ and $\rho_M^i(F) > \rho_M(F_0) + \xi$ for $i > \ell$ (by Lemma 2).

This shows $\rho_M(F) = \min\{\rho_M^i(F), i = 1, 2, \cdots, \ell\}$ when $\|F - F_0\| < \delta$. Thus, together with the conditions, we know $\rho_M(F)$ is continuously differentiable and the partial derivative formula (5) follows. $\square$

Here, we provide a formula to compute $(\partial \rho_M(F_0)/\partial F)$. Denote the left eigenvector and the right eigenvector of $Q(F_0, p_0)$ corresponding to λ_1 as t^T and v, $t^T v = 1$. Furthermore, t, v are partitioned as $t^T = [t_1^T \quad t_2^T], v^T = [v_1^T \quad v_2^T], t_1, t_2, v_1, v_2 \in \mathbf{C}^n$. It can be shown (see (10) at the bottom of the page) that $A'(p_0), B'(p_0)$ denote the derivatives of $A(p), B(p)$ at p_0, respectively.

With the above formula, a gradient-based algorithm can be devised to increase $\rho_M(F)$. The constraint that $A_0 + B_0 F$ is stable will be guaranteed in each step since $\rho_M(F)$ is increased after each iteration. In the following section, we present a method to increase $\rho_M(F)$ under the pole assignment constraint.

III. OPTIMIZING STABILITY MARGIN UNDER POLE ASSIGNMENT

Let $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ be a set of self-conjugate complex numbers corresponding to the set of desired poles. Assume that there are n' complex conjugate pairs, $\lambda_{2i-1}, \lambda_{2i} = \alpha_i \pm j\beta_i, i = 1, 2, \cdots, n'$, then one can define the real block diagonal matrix shown in (11), at the bottom of the page. It is assumed that the eigenvalues of Λ are distinct, then for a given controllable pair $(A, B), A \in \mathbf{R}^{n \times n}$ and $B \in \mathbf{R}^{n \times m}$, the problem of pole assignment by state feedback is to choose feedback matrix F, such that

$$V^{-1}(A + BF)V = \Lambda \tag{12}$$

for some nonsingular V.

Now we turn back to (3). At the nominal working point $p = 0$, the closed-loop state matrix is $A_0 + B_0 F$. It is required that the eigenvalues of $A_0 + B_0 F$ be the set $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. Our objective is to choose an F such that the stability margin $\rho_M(F)$ is maximized. This problem can be formulated as

$$\sup \rho_M(F) \quad \text{s.t.} \quad V^{-1}(A_0 + B_0 F)V = \Lambda. \tag{13}$$

In the following, we will follow the idea of [1] and [2] to parameterize all the feedback matrices F that satisfy (12) as the function of a free parameter $U \in \mathbf{R}^{m \times n}$. In this way, $\rho_M(F)$ becomes a function of the free parameter U. Explicit formulas to compute the gradient can be derived.

Given a controllable pair (A_0, B_0) and a real block diagonal matrix Λ with the form in (11) such that A_0 and Λ have no common eigenvalues, then a function $f: U \to F$ is defined as follows. For $U \in \mathbf{R}^{m \times n}$, solve

$$A_0 V - V\Lambda = -B_0 U \tag{14}$$

for V and if V is nonsingular, let $F = UV^{-1}$. The function is denoted as $F = f(U)$. The domain of f is

$$\mathcal{D}_f := \{U \in \mathbf{R}^{m \times n} | V \text{ in (14) is nonsingular}\}$$

and the range of f is $\mathcal{R}_f = f(\mathcal{D}_f)$.

$$\frac{\partial \rho_M(F_0)}{\partial F} = -\frac{[\operatorname{Re}(v_1 t_1^T)B(p_0) + \operatorname{Re}(v_2 t_2^T)B(-p_0)]^T}{\operatorname{Re}\{t_1^T[A'(p_0) + B'(p_0)F]v_1 - t_2^T[A'(-p_0) + B'(-p_0)F]v_2\}} \tag{10}$$

$$\Lambda := \operatorname{diag}\left[\begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \cdots, \begin{bmatrix} \alpha_{n'} & \beta_{n'} \\ -\beta_{n'} & \alpha_{n'} \end{bmatrix}, \lambda_{2n'+1}, \cdots, \lambda_n\right] \tag{11}$$

The condition that A_0 and Λ have no common eigenvalues ensures that (14) always has a unique solution for each U.

The following result justifies the use of the parameter U as a means to optimize the stability margin under pole assignment constraints.

Theorem 2 [1], [4]:

1) $\mathcal{D}_f$ is a dense open set in $\mathbf{R}^{m \times n}$.
2) $\{F: V^{-1}(A_0 + B_0 F)V = \Lambda\} = \mathcal{R}_f = f(\mathcal{D}_f)$.

This shows that all the F's satisfying the constraint in (13) can be parameterized as the function of a free parameter U. Since $\rho_M(F)$ is a function of F which is in turn uniquely determined by U, consequently, it can be expressed as $J(U) := \rho_M(F(U))$. By Theorem 2, the constraint in (13) can be relaxed and we get an equivalent optimization problem

$$\sup_{U \in \mathcal{D}_f} J(U). \tag{15}$$

As $F = f(U)$ is a rational function and $\mathcal{D}_f$ is an open set, then F is differentiable with respect to U for all $U \in \mathcal{D}_f$. Thus $(\partial J/\partial U)$ exists if $\rho_M(F)$ is differentiable with respect to F. To facilitate the derivation of the gradient formula, we first state, with the proof omitted, the following lemma.

Lemma 3: For $M, N, Q, R, X, Y \in \mathbf{R}^{n \times n}$ satisfying

$$MX + XN = Q, \qquad YM + NY = R$$

$\text{tr}(RX) = \text{tr}(QY)$.

Theorem 3: Suppose $U \in \mathcal{D}_f$ and

$$A_0 V - V\Lambda = -B_0 U, \qquad F = UV^{-1}.$$

If $(\partial \rho_M(F)/\partial F)$ exists, then the gradient of $J(U) = \rho_M(F(U))$ is given by

$$\frac{\partial J}{\partial U} = \left(\frac{\partial \rho_M}{\partial F}\right)V^{-T} + B_0^T Y^T \tag{16}$$

where V^{-T} denotes $(V^{-1})^T$ and Y is the unique solution of

$$Y A_0 - \Lambda Y = V^{-1}\left(\frac{\partial \rho_M}{\partial F}\right)^T F. \tag{17}$$

Proof: Consider $U = [u_{ij}]_{m \times n}$, and we have

$$\frac{\partial U}{\partial u_{ij}} = e_i e_j^T$$

where e_i and e_j are the ith and the jth basis vectors of $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively. Also

$$\frac{\partial F}{\partial u_{ij}} = \frac{\partial UV^{-1}}{\partial u_{ij}} = e_i e_j^T V^{-1} - F\frac{\partial V}{\partial u_{ij}}V^{-1}$$

where $(\partial V/\partial u_{ij})$ satisfies

$$A_0 \frac{\partial V}{\partial u_{ij}} - \frac{\partial V}{\partial u_{ij}}\Lambda = -B_0 e_i e_j^T. \tag{18}$$

Write F as $F = [f_{ij}]_{m \times n}$, and we have

$$\frac{\partial J}{\partial u_{ij}} = \sum_{p=1}^{m}\sum_{q=1}^{n} \frac{\partial \rho_M}{\partial f_{pq}} \frac{\partial f_{pq}}{\partial u_{ij}}$$

$$= \text{tr}\left[\left(\frac{\partial \rho_M}{\partial F}\right)^T \frac{\partial F}{\partial u_{ij}}\right]$$

$$= e_j^T V^{-1}\left(\frac{\partial \rho_M}{\partial F}\right)^T e_i - \text{tr}\left[V^{-1}\left(\frac{\partial \rho_M}{\partial F}\right)^T F\frac{\partial V}{\partial u_{ij}}\right].$$

By Lemma 3

$$-\text{tr}\left[V^{-1}\left(\frac{\partial \rho_M}{\partial F}\right)^T F\frac{\partial V}{\partial u_{ij}}\right] = \text{tr}[(B_0 e_i e_j^T)Y] = e_j^T Y B_0 e_i$$

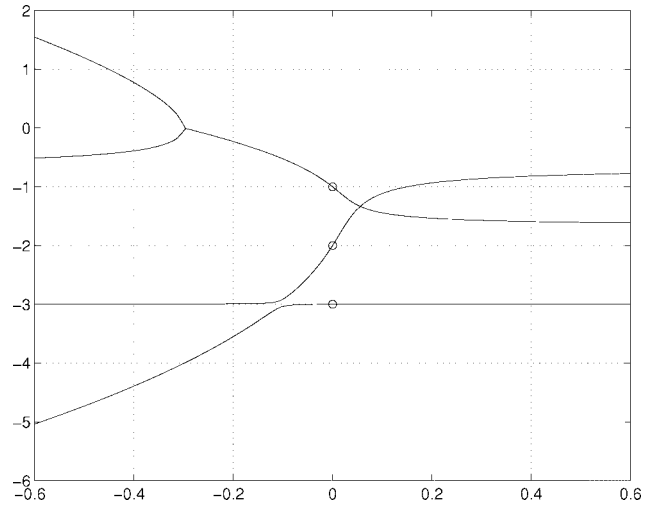Fig. 1. Variation of real parts as functions of p in Example 1.

where Y is the unique solution of (17). Consequently

$$\frac{\partial J}{\partial u_{ij}} = e_j^T\left(V^{-1}\left(\frac{\partial \rho_M}{\partial F}\right)^T + YB_0\right)e_i$$

$$= e_i^T\left(V^{-1}\left(\frac{\partial \rho_M}{\partial F}\right)^T + YB_0\right)^T e_j$$

and the result follows. $\qquad \blacksquare$

With $(\partial J/\partial U)$, the stability margin $J(F(U))$ can be increased with a gradient algorithm.

IV. NUMERICAL EXAMPLES

Consider the system of two identical penduli coupled by a spring [11]

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{g}{l} - \frac{k\alpha^2}{ml^2} & 0 & \frac{k\alpha^2}{ml^2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k\alpha^2}{ml^2} & 0 & \frac{g}{l} - \frac{k\alpha^2}{ml^2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ \frac{1}{ml^2} & 0 \\ 0 & 0 \\ 0 & \frac{1}{ml^2} \end{bmatrix} u.$$

Example 1: Suppose α is the uncertain parameter and other parameters are constants: $l = 1$, $k = 2$, $m = 0.2$. The nominal value of α is $\sqrt{0.5}$. Let $p = \alpha^2 - 0.5$, then

$$A(p) = A_0 + pA_1, \qquad B(p) = B_0$$

where

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4.8 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & 0 & 4.8 & 0 \end{bmatrix}, \qquad B_0 = \begin{bmatrix} 0 & 0 \\ 5 & 0 \\ 0 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \\ 10 & 0 & -10 & 0 \end{bmatrix}. \tag{19}$$

The open-loop system is unstable for all p and the nominal system matrix A_0 has eigenvalues ± 3.1305 and $\pm j0.4472$. The desired closed-loop eigenvalues of $A_0 + B_0 F$ are $-1 \pm j$, -2, -3.

Let

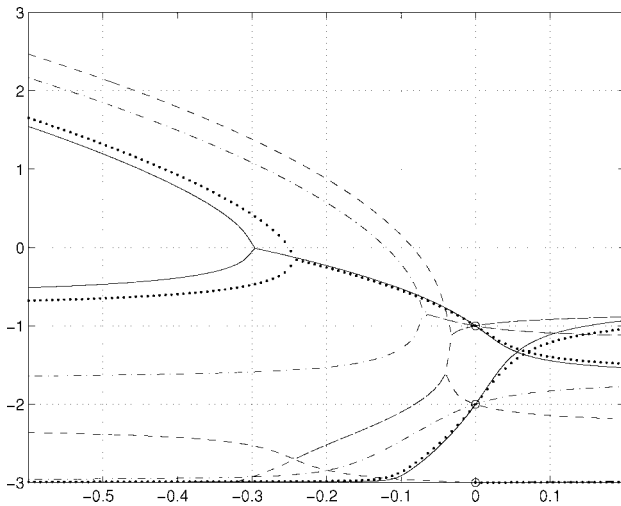$$U_0 = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}.$$

Fig. 2.   Intersections defining stability margin for different iterates (F_0- -, F_1- $\cdot$, $F_2$$\cdots$, F_4—).

We obtain the initial feedback matrix F_0 with stability margin given by $J(U_0) = \rho_M(F(U_0)) = 0.0844$. After four iterations, the gradient algorithm terminates at a local minimum U_4, with

$$F_4 = \begin{bmatrix} -2.0849 & -0.9109 & -1.6376 & -0.2889 \\ -0.1118 & 0.2245 & -0.8832 & -0.4891 \end{bmatrix}$$

and $\rho_M(F_4) = 0.3004$ which represents a significant improvement of the stability margin. For the closed-loop system, the variation of the real parts of the closed-loop poles as functions of p is depicted in Fig. 1 (the three circles correspond to the position of the real parts of the three closed-loop poles when $p = 0$). It can be seen that when $p \approx -0.3$, the system has a pair of complex conjugate poles which coalesce at the origin that destabilizes the system. Correspondingly, this first intersection of one of the curves with the abscissa equals the stability margin. To appreciate the improvement of the stability margin, the first intersections corresponding to the iterates F_0, F_1, F_2, and F_4 are shown in Fig. 2.

Example 2: Now suppose l is the uncertain parameter. The other parameters are constants, $k = 2$, $\alpha = \sqrt{0.5}$, $m = 0.2$. The nominal value of l is one. Let $p = (1/l) - 1$, then

$$A(p) = A_0 + pA_1 + p^2 A_2, \qquad B(p) = B_0 + pB_1 + p^2 B_2$$

where A_0, B_0 are the same as those in Example 1 and

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.2 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \\ 10 & 0 & -0.2 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 & 0 \\ 10 & 0 \\ 0 & 0 \\ 0 & 10 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -5 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 5 & 0 & -5 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 \\ 5 & 0 \\ 0 & 0 \\ 0 & 5 \end{bmatrix}.$$

The desired closed-loop eigenvalues are the same as Example 1.

By using the gradient algorithm, different local minima are detected. It is very interesting to note that the value of $\rho_M(F)$ at these minima are exactly the same, as far as the computation results showed. The optimal stability margin is $\rho_M(F^*) = 0.2428$. A particular optimal feedback that achieves this stability margin is F^*

$$F^* = \begin{bmatrix} -1.2510 & -0.5367 & -0.7251 & -0.0740 \\ -1.5804 & -0.2938 & -2.0610 & -0.8633 \end{bmatrix}.$$

V. CONCLUSION

We have studied the improvement of the parametric stability margin of state-space uncertain systems via a maximization process under the constraints of pole assignment. The uncertainty is modeled as a one-parameter, generally nonlinear, variation in the system and input matrices. The conditions on continuity and differentiability of the stability margin as functions of the feedback matrix is analyzed. A gradient-based approach is derived to improve the stability margin. The effectiveness of the approach is demonstrated by numerical examples.

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