Reliable H_∞ Control for Affine Nonlinear Systems

Guang-Hong Yang, James Lam, and Jianliang Wang

***Abstract*— This paper addresses the reliable H_∞-control problems for affine nonlinear systems. Based on the Hamilton–Jacobi inequality approach developed in the H_∞-control problems for affine nonlinear systems, a method for the design of reliable nonlinear control systems is presented. The resulting nonlinear control systems are reliable in that they provide guaranteed local asymptotic stability and H_∞ performance not only when all control components are operational, but also in the case of some component outages within a prespecified subset of control components. A numerical example is also given.**

***Index Terms*— H_∞ control, Hamilton–Jacobi inequalities, nonlinear systems, reliable control.**

I. Introduction

In recent years, considerable attention has been paid to the design problems of reliable linear control systems achieving various reliability goals, and some design methods have been given by several authors (see [3], [9], [12]–[14], and the references therein). In particular, Veillette *et al.* [12] present a methodology for the design of reliable linear control systems by means of the algebraic Riccati equation approach from linear H_∞-control theory, such that the resulting designs guaranteed closed-loop stability and H_∞ performance not only when all control components are operating, but also in the case of some admissible control component outages.

In the area of nonlinear H_∞ control, some important advances have been made by several authors (see [1], [4]–[6], and [8]–[11]). In particular, in [11] it was shown that the solution of the H_∞-control problem via state feedback can be determined from the solution of a Hamilton–Jacobi equation (or inequality), which is the nonlinear version of the Riccati equation for the corresponding linear H_∞-control problem in [2]. The solution to the problem in the case of measurement feedback has also been given in terms of the solutions of a pair of Hamilton–Jacobi inequalities in [1], [5], and [8]. For the computational method to find Taylor series approximations to the solutions of the Hamilton–Jacobi inequalities, the reader is referred to [7] and [11].

In this paper, we investigate the reliable H_∞-control problem for affine nonlinear systems by using the Hamilton–Jacobi inequality approach. A new method is given to design controllers that are reliable in the sense that closed-loop internal stability and H_∞-disturbance attenuation performance are guaranteed not only when all sensors and actuators are functional, but also when some sensor and/or actuator outages occur. This extends some of the reliable control results for linear systems to nonlinear systems. An example is given to illustrate the design procedure and to show (by computer simulation) the effectiveness of the proposed design method.

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G.-H. Yang and J. Wang are with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798 (e-mail: egyang@ntuvax.ntu.ac.sg).

J. Lam is with the Department of Mechanical Engineering, The University of Hong Kong, Hong Kong.

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II. Problem Formulation

Consider an affine nonlinear system Σ described by equations of the form

$$\dot{x} = f(x) + g_1(x)w_0 + \sum_{j=1}^{m} g_{2j}(x)u_j \tag{1}$$

$$y_i = h_{2i}(x) + w_i, \qquad i = 1, 2, \cdots, q \tag{2}$$

$$z = \begin{bmatrix} h_1(x) \\ u_1 \\ \vdots \\ u_m \end{bmatrix} \tag{3}$$

where x is a state vector defined on a neighborhood X of the origin in R^n, $u = [u_1 \quad u_2 \quad \cdots \quad u_m]^T \in R^m$ denotes the control input, $w_r = [w_0^T \quad w_1 \quad \cdots \quad w_q]^T \in R^r$ the disturbance input, $z \in R^s$ the output to be regulated, $y = [y_1 \quad y_2 \quad \cdots \quad y_q]^T \in R^q$ the measured output, $f(x), g_1(x), h_1(x), g_{2j}(x)$ $(j = 1, \cdots, m)$ and $h_{2i}(x)$ $(i = 1, \cdots, q)$ are known smooth mappings defined in a neighborhood of the origin in R^n, and $f(0) = 0, h_1(0) = 0$, and $h_{2i}(x) = 0$ $(i = 1, \cdots, q)$.

Denote

$$g_2(x) = [g_{21}(x) \quad g_{22}(x) \quad \cdots \quad g_{2m}(x)] \tag{4}$$

$$h_2(x) = [h_{21}(x) \quad h_{22}(x) \quad \cdots \quad h_{2q}(x)]^T. \tag{5}$$

Let $\Omega_a \subset \{1, 2, \cdots, m\}$ and $\Omega_s \subset \{1, 2, \cdots, q\}$ correspond to a selected subset of actuators susceptible to outages and a selected subset of sensors susceptible to outages, respectively. Then, the problem considered in this paper is as follows.

Given the system Σ described by (1)–(3) and a positive constant γ, find a controller K of the following form:

$$\begin{aligned} \dot{\xi} &= a(\xi) + b(\xi)y \\ u(\xi) &= c(\xi) \end{aligned} \tag{6}$$

where $\xi \in R^v$, such that for actuator outages corresponding to any $\omega_a \subset \Omega_a$ and sensor outages corresponding to any $\omega_s \subset \Omega_s$, the resulting closed-loop system is locally asymptotically stable and has a local L_2 gain which is less than or equal to γ.

For $\omega_a \subset \Omega_a$ and $\omega_s \subset \Omega_s$, introduce the decomposition

$$\begin{aligned} g_2(x) &= g_{2\omega_a}(x) + g_{2\bar{\omega}_a}(x) \\ u &= u_{\omega_a} + u_{\bar{\omega}_a} \\ h_2(x) &= h_{2\omega_s}(x) + h_{2\bar{\omega}_s}(x) \\ y &= y_{\omega_s} + y_{\bar{\omega}_s} \\ w &= [w_1 \quad \cdots \quad w_q]^T = w_{\omega_s} + w_{\bar{\omega}_s} \\ b(x) &= [b_1(x) \quad b_2(x) \quad \cdots \quad b_q(x)] = b_{\omega_s}(x) + b_{\bar{\omega}_s}(x) \end{aligned}$$

where

$$\begin{aligned} g_{2\omega_a}(x) &= \left[\delta_{\omega_a}(1)g_{21}(x) \quad \delta_{\omega_a}(2)g_{22}(x) \quad \cdots \quad \delta_{\omega_a}(m)g_{2m}(x)\right] \\ u_{\omega_a} &= \left[\delta_{\omega_a}(1)u_1 \quad \delta_{\omega_a}(2)u_2 \quad \cdots \quad \delta_{\omega_a}(m)u_m\right]^T \\ h_{2\omega_s}(x) &= \left[\delta_{\omega_s}(1)h_{21}(x) \quad \delta_{\omega_s}(2)h_{22}(x) \quad \cdots \quad \delta_{\omega_s}(m)h_{2q}(x)\right]^T \\ y_{\omega_s} &= \left[\delta_{\omega_s}(1)y_1 \quad \delta_{\omega_s}(2)y_2 \quad \cdots \quad \delta_{\omega_s}(q)y_q\right]^T \\ w_{\omega_s}(x) &= \left[\delta_{\omega_s}(1)w_1 \quad \delta_{\omega_s}(2)w_2 \quad \cdots \quad \delta_{\omega_s}(q)w_q\right]^T \\ b_{\omega_s}(x) &= \left[\delta_{\omega_s}(1)b_1(x) \quad \delta_{\omega_s}(2)b_2(x) \quad \cdots \quad \delta_{\omega_s}(q)b_q(x)\right] \end{aligned}$$

with δ_{ω_a} and δ_{ω_s} defined as follows:

$$\delta_{\omega_a}(i) = \begin{cases} 1, & \text{if } i \in \omega_a \\ 0, & \text{if } i \notin \omega_a \end{cases}$$
$$\delta_{\omega_s}(i) = \begin{cases} 1, & \text{if } i \in \omega_s \\ 0, & \text{if } i \notin \omega_s. \end{cases}$$

Applying the controller K of (6) to the system Σ, when actuator and sensor outages corresponding to $\omega_a \subset \Omega_a$ and $\omega_s \subset \Omega_s$ occur, the resulting closed-loop system $\Sigma_{\omega_a,\omega_s}$ is given by

$$\dot{x} = f(x) + g_{2\bar{\omega}_a}(x)c_{\bar{\omega}_a}(\xi) + g_1(x)w_0 \tag{7}$$
$$\begin{aligned}\dot{\xi} &= a(\xi) + b_{\bar{\omega}_s}(\xi)y_{\bar{\omega}_s} \\ &= a(\xi) + b_{\bar{\omega}_s}(\xi)h_{2\bar{\omega}_s}(x) + b_{\bar{\omega}_s}(\xi)w_{\bar{\omega}_s}\end{aligned} \tag{8}$$
$$z_{\bar{\omega}_a} = \begin{bmatrix} h_1(x) \\ c_{\bar{\omega}_a}(\xi) \end{bmatrix}. \tag{9}$$

The goal is to select the functions $a(\xi)$, $b(\xi)$, and $c(\xi)$ such that for any $\omega_a \subset \Omega_a$ and $\omega_s \subset \Omega_s$, the system $\Sigma_{\omega_a,\omega_s}$ is locally asymptotically stable and is locally dissipative with respect to the supply rate $s(w_{r\bar{\omega}_s}, z_{\bar{\omega}_a}) = \gamma^2\|w_{r\bar{\omega}_s}\|^2 - \|z_{\bar{\omega}_a}\|^2$, where

$$w_{r\bar{\omega}_s} = \begin{bmatrix} w_0^T & w_{\bar{\omega}_s}^T \end{bmatrix}^T. \tag{10}$$

The next section will present a design procedure for the reliable controller design problem.

The following two inequalities are obvious and will be used in the sequel:

$$g_{2\omega_a}(x)g_{2\omega_a}^T(x) \le g_{2\Omega_a}(x)g_{2\Omega_a}^T(x), \qquad \text{for } \omega_a \subset \Omega a \tag{11}$$
$$h_{2\omega_s}(x)h_{2\omega_s}^T(x) \le h_{2\Omega_s}(x)h_{2\Omega_s}^T(x), \qquad \text{for } \omega_s \subset \Omega s. \tag{12}$$

III. Main Results

In order to describe the main result of the section, we first recall a notion of detectability.

Definition 3.1 [4]: Suppose $f(0) = 0$ and $h(0) = 0$. The pair $\{f, h\}$ is said to be locally detectable if there exists a neighborhood U of the point $x = 0$ such that if $x(t)$ is any integral curve of $\dot{x} = f(x)$ satisfying $x(0) \in U$, then $h(x(t))$ is defined for all $t \ge 0$ and $h(x(t)) = 0$ for all $t \ge 0$ implies $\lim_{t\to\infty} x(t) = 0$.

Define the Hamiltonians $H_s(x, p)$ and $H_0(x, p)$ as follows:

$$\begin{aligned} H_s(x,p) &= p^T f(x) + h_1^T(x)h_1(x) + \gamma^2 h_{2\Omega_s}^T(x)h_{2\Omega_s}(x) \\ &\quad + \frac{1}{4}p^T\left(\frac{1}{\gamma^2}g_1(x)g_1^T(x) - g_{2\bar{\Omega}_a}(x)g_{2\bar{\Omega}_a}^T(x)\right)p \end{aligned} \tag{13}$$
$$\begin{aligned} H_0(x,p) &= p^T f(x) + \frac{1}{4\gamma^2}p^T g_1(x)g_1^T(x)p \\ &\quad + \frac{1}{4}p^T g_{2\Omega_a}(x)g_{2\Omega_a}^T(x)p + h_1^T(x)h_1(x) \\ &\quad - \gamma^2 h_{2\bar{\Omega}_s}^T(x)h_{2\bar{\Omega}_s}(x). \end{aligned} \tag{14}$$

Then the following theorem presents a sufficient condition for the solvability of the reliable controller design problem.

Theorem 3.2: Consider the system Σ described by (1)–(3) and suppose the following:

1) the pair $\{f, h_1\}$ is locally detectable;
2) there exists some C^2 function $\psi(x) \ge 0$ with $\psi(0) = 0$ such that:
   a) there exists a C^3 positive definite function $V(x)$, locally defined in a neighborhood of $x = 0$ and vanishing at $x = 0$, which satisfies the Hamilton–Jacobi equation
   $$H_s\left(x, V_x^T\right) + \psi(x) = 0 \tag{15}$$
   b) there exists a C^3 positive definite function $U(x)$, locally defined in a neighborhood of $x = 0$ and vanishing at $x = 0$, which satisfies the Hamilton–Jacobi inequality
   $$H_0\left(x, U_x^T\right) + \psi(x) \le 0 \tag{16}$$
   and such that $H_0(x, U_x^T) + \psi(x)$ has nonsingular Hessian matrix at $x = 0$;
   c) $U(x) - V(x)$ is positive definite, and
   $$(U_x - V_x)L(x) = 2\gamma^2 h_2^T(x) \tag{17}$$
   has a solution $L(x)$;

where V_x and U_x are the Jacobian matrices of V and U, respectively.

Then, the controller K of (6) with

$$\begin{aligned} a(\xi) &= f(\xi) + \frac{1}{2\gamma^2}g_1(\xi)g_1^T(\xi)V_x^T(\xi) \\ &\quad - \frac{1}{2}g_{2\bar{\Omega}_a}(\xi)g_{2\bar{\Omega}_a}^T(\xi)V_x^T(\xi) - L(\xi)h_2(\xi) \end{aligned} \tag{18}$$
$$b(\xi) = L(\xi) \tag{19}$$
$$c(\xi) = -\frac{1}{2}g_2^T(\xi)V_x^T(\xi) \tag{20}$$

is a solution of the reliable controller design problem for the system Σ of (1)–(3).

The following preliminaries are required in the proof of Theorem 3.2.

For the system Σ described by (1)–(3), consider an extended system Σ_e given by

$$\dot{x} = f(x) + \begin{bmatrix} g_1(x) & \gamma g_{2\Omega_a}(x) \end{bmatrix}\bar{w}_0 + g_2(x)u \tag{21}$$
$$y = h_2(x) + w \tag{22}$$
$$\bar{z} = \begin{bmatrix} h_1(x) \\ \gamma h_{2\Omega_s}(x) \\ u \end{bmatrix}. \tag{23}$$

Applying the controller K of (6) to the system Σ_e, the resulting closed-loop system Σ_{ce} is as follows:

$$\dot{x}_e = \bar{f}_e(x_e) + \bar{g}_e(x_e)\bar{w} \tag{24}$$
$$\bar{z} = \begin{bmatrix} h_1(x) \\ \gamma h_{2\Omega_s}(x) \\ c(\xi) \end{bmatrix} \tag{25}$$

where $x_e = [x^T \ \ \xi^T]^T$ $\bar{w} = [\bar{w}_0^T \ \ w^T]^T$

$$\bar{f}_e(x_e) = \begin{bmatrix} f(x) + g_2(x)c(\xi) \\ a(\xi) + b(\xi)h_2(x) \end{bmatrix}$$
$$\bar{g}_e(x_e) = \begin{bmatrix} [g_1(x) \quad \gamma g_{2\Omega_a}(x)] & 0 \\ 0 & b(\xi) \end{bmatrix}.$$

The closed-loop system $\Sigma_{\omega_a,\omega_s}$ of (7)–(9) can be written as

$$\dot{x}_e = f_{as}(x_e) + g_{as}(x_e)w_{r\bar{\omega}_s} \tag{26}$$
$$z_{\bar{\omega}_a} = \begin{bmatrix} h_1(x) \\ c_{\bar{\omega}_a}(\xi) \end{bmatrix} \tag{27}$$

where $w_{r\bar{\omega}_s}$ is given by (10)

$$f_{as}(x_e) = \begin{bmatrix} f(x) + g_{2\bar{w}_a}(x)c_{\bar{\omega}_a}(\xi) \\ a(\xi) + b_{\bar{\omega}_s}(\xi)h_{2\bar{w}_s}(x) \end{bmatrix}$$
$$g_{as}(x_e) = \begin{bmatrix} g_1(x) & 0 \\ 0 & b_{\bar{\omega}_s}(\xi) \end{bmatrix}.$$

Let $X(x_e)$ be a C^1 function defined in a neighborhood of $(0,0)$, and denote

$$J_{ce}(X,\Sigma_{ce}) = X_{x_e}\bar{f}_e(x_e) + \bar{z}^T\bar{z} + \frac{1}{4\gamma^2}X_{x_e}\bar{g}_e(x_e)\bar{g}_e^T(x_e)X_{x_e}^T \tag{28}$$

$$J_{as}\big(X,\Sigma_{\omega_a,\omega_s}\big) = X_{x_e}f_{as}(x_e) + z_{\bar{\omega}_a}^T z_{\bar{\omega}_a} + \frac{1}{4\gamma^2}X_{x_e}g_{as}(x_e)g_{as}^T(x_e)X_{x_e}^T. \tag{29}$$

Then, we have the following lemmas.

Lemma 3.3: For any $\omega_a \subset \Omega_a$ and $\omega_s \subset \Omega_s$, the following inequality holds:

$$J_{as}\big(X,\Sigma_{\omega_a,\omega_s}\big) \le J_{ce}(X,\Sigma_{ce}). \tag{30}$$

Proof: By (24)–(27), we have

$$\begin{aligned} X_{x_e}f_{as}(x_e) &= X_{x_e}\bar{f}_e(x_e) - X_{x_e}\begin{bmatrix} g_{2\omega_a}(x)c_{\omega_a}(\xi) \\ b_{\omega_s}(\xi)h_{2\omega_s}(x)\end{bmatrix} \\ &= X_{x_e}\bar{f}_e(x_e) - X_{x_e}\begin{bmatrix} g_{2\omega_a}(x)c_{\omega_a}(\xi) \\ 0\end{bmatrix} \\ &\quad - X_{x_e}\begin{bmatrix} 0 \\ b_{\omega_s}(\xi)h_{2\omega_s}(x)\end{bmatrix} \\ &\le X_{x_e}\bar{f}_e(x_e) + \frac{1}{4}X_{x_e}\begin{bmatrix} g_{2\omega_a}(x)g_{2\omega_a}^T(x) & 0 \\ 0 & 0\end{bmatrix} \\ &\quad \times X_{x_e}^T + c_{\omega_a}^T(\xi)c_{\omega_a}(\xi) + \frac{1}{4\gamma^2}X_{x_e} \\ &\quad \times \begin{bmatrix} 0 & 0 \\ 0 & b_{\omega_s}(\xi)b_{\omega_s}^T(\xi)\end{bmatrix}X_{x_e}^T + \gamma^2 h_{2\omega_s}^T(x)h_{2\omega_s}(x) \end{aligned} \tag{31}$$

$$\begin{aligned} z_{\bar{\omega}_a}^T z_{\bar{\omega}_a} &= h_1^T(x)h_1(x) + c_{\bar{\omega}_a}^T(\xi)c_{\bar{\omega}_a}(\xi) \\ &= h_1^T(x)h_1(x) + c^T(\xi)c(\xi) - c_{w_a}^T(\xi)c_{w_a}(\xi) \end{aligned} \tag{32}$$

$$\begin{aligned} X_{x_e}g_{as}(x_e)g_{as}^T(x_e)X_{x_e}^T &= X_{x_e}\begin{bmatrix} g_1(x)g_1^T(x) & 0 \\ 0 & b_{\bar{\omega}_s}(\xi)b_{\bar{\omega}_s}^T(\xi)\end{bmatrix}X_{x_e}^T \\ &= X_{x_e}\begin{bmatrix} g_1(x)g_1^T(x) & 0 \\ 0 & b(\xi)b^T(\xi)\end{bmatrix}X_{x_e}^T \\ &\quad - X_{x_e}\begin{bmatrix} 0 & 0 \\ 0 & b_{\omega_s}(\xi)b_{\omega_s}^T(\xi)\end{bmatrix}X_{x_e}^T. \end{aligned} \tag{33}$$

Combining equations (31)–(33), (25), and inequalities (11) and (12), it follows that

$$\begin{aligned} &J_{as}\big(X,\Sigma_{\omega_a,\omega_s}\big) \\ &\le X_{x_e}\bar{f}_e(x_e) + h_1^T(x)h_1(x) \\ &\quad + c^T(\xi)c(\xi) + \gamma^2 h_{2\omega_s}^T(x)h_{2\omega_s}(x) + \frac{1}{4\gamma^2}X_{x_e} \\ &\quad \times \begin{bmatrix} g_1(x)g_1^T(x) + \gamma^2 g_{2\omega_a}(x)g_{2\omega_a}^T(x) & 0 \\ 0 & b(\xi)b^T(\xi)\end{bmatrix}X_{x_e}^T \\ &\le X_{x_e}\bar{f}_e(x_e) + \bar{z}^T\bar{z} + \frac{1}{4\gamma^2}X_{x_e} \\ &\quad \times \begin{bmatrix} g_1(x)g_1^T(x) + \gamma^2 g_{2\Omega_a}(x)g_{2\Omega_a}^T(x) & 0 \\ 0 & b(\xi)b^T(\xi)\end{bmatrix}X_{x_e}^T \\ &= J_{ce}(X,\Sigma_{ce}). \end{aligned}$$

□

Lemma 3.4: Under the assumptions of Theorem 3.2, the equilibrium $x=0$ of the system

$$\dot{x} = f(x) + \frac{1}{2\gamma^2}g_1(x)g_1^T(x)V_x^T(x) - L(x)h_2(x) \tag{34}$$

is locally asymptotically stable.

Proof: Let $Q(x) = U(x) - V(x)$ and

$$\begin{aligned} H_w\big(x,Q_x^T\big) &= Q_x\bigg(f(x) + \frac{1}{2\gamma^2}g_1(x)g_1^T(x)V_x^T \\ &\qquad + \frac{1}{2}g_{2\Omega_a}(x)g_{2\Omega_a}^T(x)V_x^T\bigg) - Q_xL(x)h_2(x) \\ &\quad + \frac{1}{4\gamma^2}Q_x\big[g_1(x)g_1^T(x) + \gamma^2 g_{2\Omega_a}(x)g_{2\Omega_a}^T(x)\big]Q_x^T \\ &\quad + c^T(x)c(x) + \frac{1}{4\gamma^2}Q_xL(x)L^T(x)Q_x^T. \end{aligned} \tag{35}$$

Then, by (13)–(17) and (20), it follows:

$$\begin{aligned} H_w\big(x,Q_x^T\big) &= H_0\big(x,U_x^T\big) - H_s\big(x,V_x^T\big) \\ &= H_0\big(x,U_x^T\big) + \psi(x) \le 0. \end{aligned} \tag{36}$$

By computing directly, we have

$$\begin{aligned} H_w\big(x,Q_x^T\big) &\ge Q_x\bigg(f(x) + \frac{1}{2\gamma^2}g_1(x)g_1^T(x)V_x^T - L(x)h_2(x)\bigg) \\ &\quad + c^T(x)c(x) - \frac{1}{4}V_xg_{2\Omega_a}(x)g_{2\Omega_a}^T(x)V_x^T \\ &\quad + \frac{1}{4\gamma^2}Q_xg_1(x)g_1^T(x)Q_x^T + \frac{1}{4\gamma^2}Q_xL(x)L^T(x)Q_x^T \\ &\ge Q_x\bigg(f(x) + \frac{1}{2\gamma^2}g_1(x)g_1^T(x)V_x^T - L(x)h_2(x)\bigg) \end{aligned}$$

which further implies from (36) and the condition under which $H_0(x,U_x^T) + \psi(x)$ has a nonsingular Hessian matrix at $x=0$ that the system (34) is locally asymptotically stable. □

Lemma 3.5: Under the assumptions of Theorem 3.2, let $Q(x) = U(x) - V(x)$, $X(x_e) = V(x) + Q(x-\xi)$, then there exists a neighborhood of $(x,\xi) = (0,0)$ in which the following inequality holds:

$$J_{ce}(X,\Sigma_{ce}) \le 0. \tag{37}$$

Proof: In the extended system Σ_e described by (21)–(23), let $\bar{g}_1(x) = [g_1(x) \;\; \gamma g_{2\Omega_a}(x)]$ and $\bar{h}_1(x) = \left[\begin{smallmatrix} h_1(x) \\ \gamma h_{2\Omega_s}(x)\end{smallmatrix}\right]$. Then, from (13), we have

$$\begin{aligned} &V_xf(x) + \bar{h}_1^T(x)\bar{h}_1(x) + \frac{1}{4}V_x\bigg(\frac{1}{\gamma^2}\bar{g}_1(x)\bar{g}_1^T(x) - g_2(x)g_2^T(x)\bigg)V_x^T \\ &= H_s\big(x,V_x^T\big). \end{aligned} \tag{38}$$

Denote $c_1(x) = \frac{1}{2\gamma^2}\bar{g}_1^T(x)V_x^T$ and $\bar{f}(x) = f(x) + \bar{g}_1(x)c_1(x)$. By (17), (35), and (36), it follows:

$$\begin{aligned} &Q_x\bar{f}(x) + c_1^T(x)c_1(x) - \gamma^2 h_2^T(x)h_2(x) + \frac{1}{4\gamma^2}Q_x\bar{g}_1(x)\bar{g}_1^T(x)Q_x^T \\ &= H_w\big(x,Q_x^T\big) \\ &= H_0\big(x,U_x^T\big) + \psi(x). \end{aligned} \tag{39}$$

Then, from the assumptions of Theorem 3.2, (38), (39), and the proof of [5, Th. 3.1], it follows that (37) holds in a neighborhood of $(x,\xi) = (0,0)$. □

Proof of Theorem 3.2: By Lemma 3.3, Lemma 3.5, and [11, Th. 2], it follows that for any $\omega_a \subset \Omega_a$ and $\omega_s \subset \Omega_s$, the system $\Sigma_{\omega_a,\omega_s}$ of (7)–(9) or (26) and (27) is locally dissipative with respect to the supply rate $s(w_{r\bar{\omega}_s}, z_{\bar{\omega}_a}) = \gamma^2\|w_{r\bar{\omega}_s}\|^2 - \|z_{\bar{\omega}_a}\|^2$.

In the following, we show that the system $\Sigma_{\omega_a,\omega_s}$ is locally asymptotically stable.

From $J_{as}(X,\Sigma_{\omega_a,\omega_s}) \le 0$ and $w_{r\bar{\omega}_s} = 0$, it follows:

$$\begin{aligned} \frac{dX(x_e(t))}{dt} &= X_{x_e}f_{as}(x_e(t)) \le -z_{\bar{\omega}_a}^T z_{\bar{\omega}_a} \\ &= -\|h_1(x(t))\|^2 - \big\|c_{\bar{\omega}_a}(\xi)\big\|^2. \end{aligned}$$

This proves that the system $\Sigma_{\omega_a,\omega_s}$ is stable at the equilibrium $(x,\xi)=(0,0)$, and any trajectory satisfying

$$\frac{dX(x_e(t))}{dt}=0$$

is necessarily a trajectory of

$$\dot{x}=f(x)+g_{2\bar{\omega}_a}(x)c_{\bar{\omega}_a}(\xi)$$

such that $x(t)$ is bounded and $h_1(x(t))=0, c_{\bar{\omega}_a}(\xi(t))=0$, which further follows from assumption 1) that $\lim_{t\to\infty} x(t)=0$. Thus, the ω-limit set of such a trajectory is a subset of

$$M=\{(x,\xi)\colon x=0,\ c_{\bar{\omega}_a}(\xi(t))=0\}.$$

By (18), and $\bar{\Omega}_a\subset\bar{\omega}_a$, any initial condition on this ω-limit set yields a trajectory in which $x(t)=0$ for all $t\geq 0$, while $\xi(t)$ is a trajectory of

$$\begin{aligned}\dot{\xi}&=a(\xi)+b_{\bar{\omega}_s}(\xi)h_{2\bar{\omega}_s}(x)=a(\xi)\\&=f(\xi)+\frac{1}{2\gamma^2}g_1(\xi)g_1^T(\xi)V_x^T(\xi)\\&\quad-\frac{1}{2}g_{2\bar{\Omega}_a}(\xi)g_{2\bar{\Omega}_a}^T(\xi)V_x^T(\xi)-L(\xi)h_2(\xi)\\&=f(\xi)+\frac{1}{2\gamma^2}g_1(\xi)g_1^T(\xi)V_x^T(\xi)\\&\quad-L(\xi)h_2(\xi)+g_{2\bar{\Omega}_a}(\xi)c_{\bar{\Omega}_a}^T(\xi)\\&=f(\xi)+\frac{1}{2\gamma^2}g_1(\xi)g_1^T(\xi)V_x^T(\xi)-L(\xi)h_2(\xi).\end{aligned}$$

By Lemma 3.4, it follows that $\lim_{t\to\infty}\xi(t)=0$. Thus, by the invariance principle, the system $\Sigma_{\omega_a,\omega_s}$ is locally asymptotically stable. □

In the case of a linear system

$$\dot{x}=Ax+Gw_0+Bu \tag{40}$$

$$y=Cx+w \tag{41}$$

$$z=\begin{bmatrix}Hx\\u\end{bmatrix} \tag{42}$$

a solution of the corresponding reliable controller design problem is given by the following corollary.

Corollary 3.6: Consider the linear system described by (40)–(42) and suppose the following.

1) The pair (A,H) is detectable.
2) The following algebraic Riccati equation and inequality

$$\begin{aligned}&A^TX+XA-XB_{\bar{\Omega}_a}B_{\bar{\Omega}_a}^TX+\frac{1}{\gamma^2}XGG^TX\\&\quad+H^TH+\gamma^2C_{\Omega_s}^TC_{\Omega_s}=0\end{aligned} \tag{43}$$

$$\begin{aligned}&A^TY+YA+YB_{\Omega_a}B_{\Omega_a}^TY+\frac{1}{\gamma^2}YGG^TY\\&\quad+H^TH-\gamma^2C_{\bar{\Omega}_s}^TC_{\bar{\Omega}_s}<0\end{aligned} \tag{44}$$

have positive definite solutions X and Y, respectively, and $Y>X$, where the matrices B_{Ω_a}, $B_{\bar{\Omega}_a}$, C_{Ω_s}, and $C_{\bar{\Omega}_a}$ have meanings similar to those of $g_{2\Omega_a}(x)$, $g_{2\bar{\Omega}_a}(x)$, $h_{2\Omega_a}(x)$, and $h_{2\bar{\Omega}_a}(x)$ in (13) and (14).

Denote

$$G_+=[G\quad \gamma B_{\Omega_a}],\qquad K_{d+}=\frac{1}{\gamma^2}G_+^TX$$

$$K=-B^TX$$

$$L=\gamma^2(Y-X)^{-1}C^T.$$

Then the controller

$$\dot{\xi}=(A+BK+G_+K_{d+}-LC)\xi+Ly \tag{45}$$

$$u=K\xi \tag{46}$$

is a control law such that for actuator outages corresponding to any $\omega_a\subset\Omega_a$ and sensor outages corresponding to any $\omega_s\subset\Omega_s$, the resulting closed-loop system is asymptotically stable and has an H_∞-norm bound γ.

Remark 3.7: Note that in Corollary 3.6, (44) is a strict inequality. This is stronger than the corresponding condition in [12, Th. 4.2] where only an equality is needed. But with this strict inequality, the controller in this paper need not be assumed asymptotically stable, as is the case in [12, Th. 4.2].

Remark 3.8: The result presented in this paper provides a reliable controller design methodology for the so-called *primary contingency* problem where the set of sensors and actuators that are susceptible to outages is known *a priori*. This is very different from the so-called *single contingency* problem where any single sensor or any single actuator may have an outage. The single contingency problem is addressed in another paper [15].

IV. An Example

In this section, we present an example to illustrate the design procedure and the effectiveness of our reliable controller.

Consider the following second-order system:

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}&=\begin{bmatrix}-2x_1+x_1x_2^2\\x_2^3\end{bmatrix}+\begin{bmatrix}1\\x_1\end{bmatrix}w_0+\begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}\\y&=2x_1+2x_2+w_1\\z&=[x_1\quad x_2^4\quad u_1\quad u_2]^T\end{aligned}$$

with $\Omega_a=\{2\}$ and $\Omega_s=\emptyset$. That is, only actuator u_2 is susceptible to outage and the sensor y is 100% reliable. It is easy to check that $\{f,h_1\}$ is locally detectable. By selecting $\gamma=0.81$ and

$$\phi(x)=\frac{1}{2}x_1^2+\frac{8\gamma^2}{\gamma^2-x_1^2}x_2^6$$

approximate solutions to the Hamilton–Jacobi equality (15) and inequality (16) are given by, respectively

$$\begin{aligned}V(x)&=0.4533x_1^2+0.3463x_1^2x_2^2+\frac{2\gamma^2}{\gamma^2-x_1^2}x_2^4\\U(x)&=1.5938x_1^2+0.3496x_1x_2+1.2667x_2^2.\end{aligned}$$

By solving (17) we have the following approximate solution:

$$L(x)=[1.0132\quad 0.8961]^T.$$

Then the final controller K is given by (6) with $a(x)$, $b(x)$, and $c(x)$ as given in (18)–(20).

Computer simulations are performed to check the actual system response. The actual achieved disturbance attenuation γ is approximated by

$$\begin{aligned}\gamma_{\max}&\triangleq\max_{0\leq t\leq T}\gamma(t)\\&\triangleq\max_{0\leq t\leq T}\sqrt{\int_0^T\frac{z^T(t)z(t)}{w_0^2(t)+w_1^2(t)}\,dt}\end{aligned} \tag{47}$$

where $T=50$ s and the disturbances w_0 and w_1 are taken as

$w_0(t)=0.3\sin(4\pi t)$

$w_1(t)=$ square wave of amplitude 0.05 and period 0.1 s.

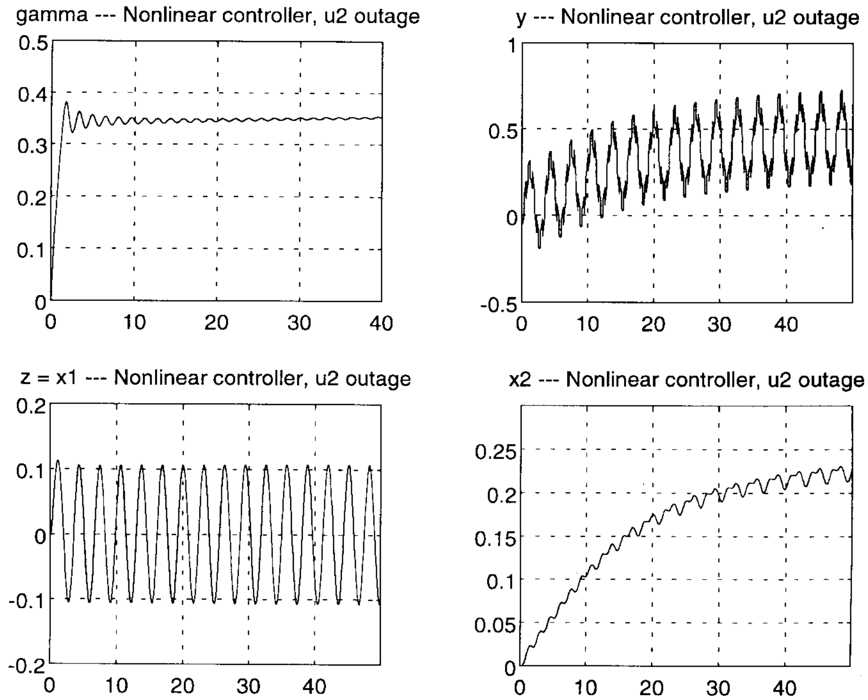

Fig. 1. Response for u_2 outage.

TABLE I
RELIABLE CONTROLLER DESIGN RESULTS

	No Outage	u_2 Outage	u_1 Outage
γ_{max}	0.3678	0.3805	0.3678

The system response in the u_2 outage case is given in Fig. 1. Although the controller is designed for u_2 outage only, computer simulation shows that it also provides very good closed-loop performance in the case of u_1 outage. Responses for the "no-outage" and the "u_1-outage" cases are similar to those of Fig. 1 and are omitted here. The achieved H_∞-disturbance attenuation performance, as specified in (47), is tabulated in Table I. It is noted that this actually achieved γ is less than the designed value of $\gamma = 0.81$, as guaranteed by Theorem 2.

Finally, it is worth noting that if the linearization of the designed nonlinear controller is used to control the nonlinear system, the closed-loop system is actually unstable in the no-outage case as well as in the u_1-outage and the u_2-outage cases.

V. CONCLUSION

This paper presents a solution to the reliable controller design problem for affine nonlinear systems. The solution is shown to be related to the existence of solutions of a Hamilton–Jacobi equation and a Hamilton–Jacobi inequality. The resulting closed-loop nonlinear control system is reliable in the sense that it achieves asymptotic stability and H_∞-disturbance attenuation performance not only when all sensors and actuators of the system are operating properly, but also when control components (sensors and/or actuators) that are within a prespecified subset of control components become faulty (outage). This result is also verified, via computer simulation, by using a nontrivial example.

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Approximate $\mathcal{H}_\infty$ Identification Using Partial Sum Operators in a Disc Algebra Basis

László Gianone, József Bokor, and Ferenc Schipp

***Abstract*—$\mathcal{H}_\infty$-system identification using a basis in the disc algebra is presented. The approximate model is represented by a partial sum with respect to this basis. The identification problem is to estimate the expansion coefficients of this partial sum. Since the constructed basis functions cannot be represented analytically, they are approximated in order to arrive at a model in a suitable form. An algorithm is presented which calculates the model parameters from the frequency domain data set.**

***Index Terms*—Disc algebra basis, H_∞ identification, parameter estimation, system identification, summation technique.**

I. Introduction

System identification with $\mathcal{H}_\infty$ criteria has received a growing interest since the appearance of $\mathcal{H}_\infty$ formulation of robust control. The recent state of the art is characterized in most recent survey papers like [13] and [15]. Information-based complexity (IBC) theory offers a common framework for the setup of the different identification algorithms and for examining their properties, such as optimality and convergence [14]. Until now several linear and nonlinear algorithms have been developed for solving the identification problem. A large class of nonlinear methods is built by the interpolatory-type algorithms [5], [6]. Another type is based on $\mathcal{H}_\infty$ approximation of systems in $\mathcal{L}_\infty$. Examples for these so-called two-stage algorithms can be found, e.g., in [12], [17], and [11]. Nonlinear algorithms obviously need much higher computational effort compared to the linear ones. Meanwhile, it is proved that no linear algorithm exists under unknown but bounded noise condition which is convergent in worst case and is not tuned to the prior information on the system. However, linear algorithms do not show such dramatic—if any—divergence in practice. This is due to the (possibly) slow speed of divergence under worst case conditions and that the worst case situations happen rarely.

Among the existing approaches, our problem setup is related to linear identification given in [16] and [11] and also to the terminology in [17] and [12]. In [16] the approximate model was the truncated Taylor series of a rational transfer function (FIR model). The parameters were estimated under the criterion that the frequency response of the model should exactly match the frequency response at prescribed finite number of frequencies. This approach leads to the solution of a Lagrange interpolation problem. In [11] the authors apply an FIR-model represented by Cesaro summation, i.e., by arithmetic means of partial sums of the Taylor series or impulse response function.

The basic disadvantage of the FIR modeling is that the system $\{z^k, z \in C, k = 0, 1, \cdots\}$ is not a basis in the disc algebra, and the Fourier partial sum operators are not uniformly bounded, i.e., the $\mathcal{H}_\infty$ norm of identification error will grow by $\log n$ even in the noiseless case, where n is the order of the FIR model. The approach based on Cesaro summation overcomes this problem in the noiseless case since the Cesaro means converge in the l_∞ norm when approximating the system on the unit circle. For l_∞-norm bounded deterministic noise, however, like any other linear algorithm, the identification error will grow proportionally with $\varepsilon \log n$ where ε is the l_∞ norm of the noise.

The approach offered in this paper is based on construction of a basis in the disc algebra. The approximate model will be represented by a partial sum in this basis. The partial sum operator in this basis will be uniformly bounded in the $\mathcal{H}_\infty$ norm, and the convergence rate is proportional to the modulus of continuity of the system transfer function. Note that one difference between the previous two approaches is that the system $\{z^k, k = 0, 1, \cdots\}$ is not a basis in the disc algebra. What we expect from our approach (in addition to the uniform boundedness of the partial sum operator) is a faster convergence to zero of the approximation error with respect to the model order when compared to the algorithms based on some types of summation procedures. This rather heuristic expectation is supported by the fact that the proposed algorithm is optimal in the chosen basis (from an approximation point of view), which is not true for the other summation algorithms.

The use of specific basis functions appeared recently in l_2 identification; see [20] and [19]. This idea is extended to approximate $\mathcal{H}_\infty$ identification of systems with transfer function in the disc algebra. This basis is derived from the Faber–Schauder system and using orthogonalization from the Franklin system defined on the unit circle. The order of the model, i.e., the number of basis functions, can be much less than the number of measurement data. In the case of equality the identification algorithm reduces to a Faber–Schauder interpolation of the frequency response function.

The outline of the paper is as follows. This introduction is followed by the problem formulation of the identification in the $\mathcal{H}_\infty$ setting. Section III considers the $\mathcal{H}_\infty$ identification using a disc algebra basis. The basis will be constructed first; the model is a partial sum of the biorthogonal expansion of the transfer function of the system with respect to this basis. Then the estimation of the model parameters is considered on the basis of frequency domain measurements. Finally, an estimate for the identification error is given. The results are demonstrated by simulation examples in Section IV. (The conference version of this paper has appeared in [3].)

II. Problem Formulation of Identification in $\mathcal{H}_\infty$

The usual frequency-domain problem formulation of the identification in the $\mathcal{H}_\infty$ space will be summarized briefly. The basic property of this approach is that the maximum distance between the system and the model is minimized, and it is assumed that the noise on the output of the system is considered as an unknown-but-bounded (UBB)-type deterministic disturbance.

Denote by R and C the field of real and complex numbers, respectively, and by N the set of nonnegative integers. Denote the

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L. Gianone and J. Bokor are with the Computer and Automation Research Institute, Hungarian Academy of Sciences, 1518 Budapest, P.O.B. 63, Hungary (e-mail: gianone@decst.scl.sztaki.hu).

F. Schipp is with Eötvös Loränd University, Budapest, 1068 Budapest.

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