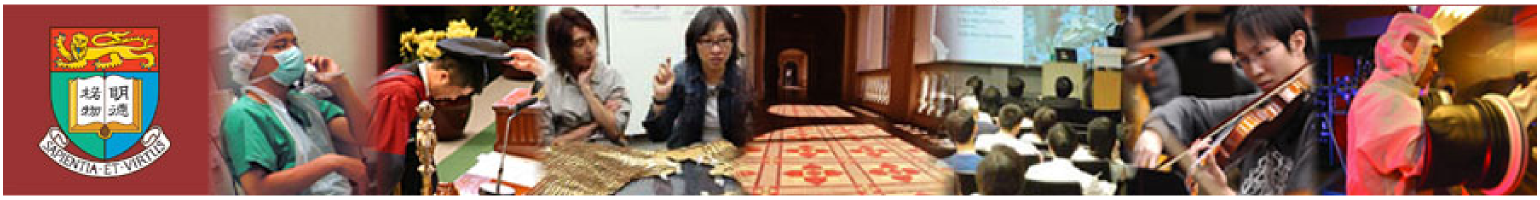

A CLASS OF UNITARILY INVARIANT NORMS ON $B(H)$

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Abstract. Let H be a complex Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on H. For $c = (c_1, \dots, c_k)$, where $c_1 \geq \cdots \geq c_k > 0$, and $p \geq 1$, define the (c,p)-norm of $A \in B(H)$ by

$$\|A\|_{c,p} = \left(\sum_{i=1}^{k} c_i s_i(A)^p \right)^{\frac{1}{p}},$$

where $s_i(A)$ denotes the ith s-numbers of A. In this paper we study some basic properties of this norm and give a characterization of the extreme points of its closed unit ball. Using these results, we obtain a description of the corresponding isometric isomorphisms on $B(H)$.

1. Introduction

Let H be a Hilbert space over $\mathbb{C}$ and let $B(H)$ be the algebra of all bounded linear operators on H. When H is of finite dimension n, we shall identify $B(H)$ with M_n, the algebra of all $n \times n$ complex matrices. For a compact operator $A \in B(H)$, the ith s-number (or singular value) of A is the i-th largest eigenvalue of $|A| = (A^*A)^{\frac{1}{2}}$, where each eigenvalue repeats according to its multiplicity. If necessary, the numbers will be appended by 0's to form an infinite sequence. The ith s-number of A will be denoted by $s_i(A)$. Let Φ be a symmetric gauge function on $\mathbb{R}^{\dim H}$. (We refer readers to [4] for the basic definitions and properties.) Then Φ determines a symmetric norm ideal C_Φ of compact operators by decreeing $A \in C_\Phi$ if $\Phi(\{s_i(A)\}) < \infty$. Norm ideals of this type include the Schatten class and the Hilbert-Schmidt class. Moreover $\|A\|_\Phi = \Phi(\{s_i(A)\})$ is a complete norm on C_Φ, which is unitarily invariant in the sense that $\|UAV\|_\Phi = \|A\|_\Phi$ for any unitary operators U and V. In particular, when $\dim H = n < \infty$, every symmetric gauge function defines a unitarily invariant norm on M_n. The algebras C_Φ and M_n under these norms have been studied extensively. For example, see [1], [5], [8] and [10]. An important topic is to describe isometric isomorphisms under these norms. In most cases, as in the papers cited, they are multiplication by unitary operators, possibly followed by transposition.

Now the definition of s-number can be extended to a bounded (non-compact) operator as follows. For $A \in B(H)$, let $s_\infty(A)$ denote the essential norm of A. Then $s_\infty(A)$ is either an accumulation point of $\sigma(|A|)$ or an eigenvalue of $|A|$ of infinite multiplicity. The operator A is compact if and only if $s_\infty(A) = 0$. Every element of $\sigma(|A|)$ exceeding $s_\infty(A)$ is an eigenvalue of $|A|$ of finite multiplicity. The s-numbers of A is defined to be the eigenvalues $s_1(A) \geq s_2(A) \geq \cdots$ of $|A|$, where each of them repeats according to its multiplicity. If there are only finitely many of them, we put $s_i(A) = s_\infty(A)$ for the remaining i's.

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Alternatively, one has $s_i(A) = \inf\{\|A - X\| : X \in B(H) \text{ has rank} < i\}$. We refer readers to Gohberg and Krein [4] for a detailed account. One observation is that s-numbers of bounded operators enjoy many nice properties as their compact operator counterparts.

In this paper we study the following class of norms on $B(H)$: let $c = (c_1, \ldots, c_k) \in \mathbb{R}^k$, where $c_1 \geq \cdots \geq c_k > 0$, and let $1 \leq p < \infty$. Define the (c, p)-norm of an operator $A \in B(H)$ by

$$\|A\|_{c,p} = \left(c_1 s_1(A)^p + \cdots + c_k s_k(A)^p\right)^{\frac{1}{p}}.$$

When $p = 1$, the norm is simply called the c-norm and denoted $\|\cdot\|_c$. Further, if $c = (1, \ldots, 1) \in \mathbb{R}^k$, the above definition reduces to the Ky-Fan k-norm. In particular, $\|A\|_1$ is the operator norm and will be denoted by the usual symbol $\|A\|$.

In the next section we prove some facts about the (c, p)-norm. Then in section 3, we study the extreme points of the closed unit ball for $\|\cdot\|_{c,p}$. The cases $p = 1$ and $p > 1$ will be discussed separately. Using the properties obtained, we are able to describe isometric isomorphisms on $B(H)$ under these norms. They are of the form depicted earlier. This is given in the last section.

2. Some properties of $\|\cdot\|_{c,p}$

Let H be an infinite-dimensional Hilbert space over $\mathbb{C}$, $c = (c_1, \ldots, c_k)$ where $k > 1$ and $c_1 \geq \cdots \geq c_k > 0$, and $p \geq 1$. These assumptions prevail unless otherwise stated. It is known that the (c, p)-norm given by

$$\|A\|_{c,p} = \left(c_1 s_1(A)^p + \cdots + c_k s_k(A)^p\right)^{\frac{1}{p}}$$

is actually a norm on $B(H)$. In fact, all (c, p)-norms are equivalent. For example they are equivalent to the operator norm:

$$c_1\|A\| \leq \|A\|_c \leq \left(\sum_{i=1}^{k} c_i\right)\|A\|.$$

More generally, we have

Theorem 2.1. *Let $1 \leq p, q < \infty$, $c = (c_1, \ldots, c_k)$, $d = (d_1, \ldots, d_k)$, where $c_1 \geq \cdots \geq c_k \geq 0$, $d_1 \geq \cdots \geq d_k \geq 0$, and not both c_k and d_k are zero. Then*

$$M = \max\left\{\left(d_1 z_1^q + \cdots + d_k z_k^q\right)^{\frac{1}{q}} : z_1 \geq \cdots \geq z_k \geq 0,\ c_1 z_1^p + \cdots + c_k z_k^p \leq 1\right\}$$

is the smallest positive number satisfying

$$\|A\|_d \leq M\|A\|_c \text{ for all } A \in B(H).$$

In particular, if $p = q = 1$, we have $M = \max\{(d_1 + \cdots + d_i)/(c_1 + \cdots + c_i) : 1 \leq i \leq k\}$.

Proof. The general assertion is clear. To verify the optimality, one only needs to consider a finite rank operator A with s-numbers $z_1, \ldots, z_k, 0, \ldots$ that yield the maxium M in the optimization problem.

For the particular case, let $z_k = s_k$ and $z_i = s_i - s_{i+1}$ for $i = 1, \ldots, k-1$. Then

$$\begin{aligned}
\sum_{i=1}^{k} d_i s_i &= \sum_{i=1}^{k} d_i \sum_{j=i}^{k} z_j = \sum_{i=1}^{k} \left(\sum_{j=1}^{i} d_j \right) z_i \\
&\leq \sum_{i=1}^{k} \left(M \sum_{j=1}^{i} c_j \right) z_i = M \sum_{j=1}^{k} c_j \sum_{i=j}^{k} z_i \\
&= M \sum_{j=1}^{k} c_j s_j \, ,
\end{aligned}$$

by the definition of M. One easily checks that M is optimal. $\square$

Suppose $A \in B(H)$ is compact. Then A admits a Schmidt expansion

$$A = \sum_{i=1}^{\infty} s_i(A) \langle \cdot, x_i \rangle y_i \, ,$$

where $\{x_i\}$ and $\{y_i\}$ are orthonormal sequences. It follows that $s_i(A) = \langle A x_i, y_i \rangle$. In general we have

Lemma 2.2. (C.f. [3, Lemma 4.4]) *For every $A \in B(H)$ and $\varepsilon > 0$, there exist orthonormal sequences $\{x_i\}$ and $\{y_i\}$ such that $|\langle A x_i, y_i \rangle - s_i(A)| < \varepsilon$ for all i.*

Proof. If the operator A is compact, there are orthonormal sequences $\{x_i\}$ and $\{y_i\}$ such that $\langle A x_i, y_i \rangle = s_i(A)$, as shown. In general let E be the spectral measure for $|A|$. If the projection $E\big((s_\infty(A), \|A\|]\big)$ is of infinite rank, there are infinity many eigenvalues for $|A|$ and the same argument as in the compact case applies. Otherwise there are finitely many orthonormal vectors $x_1, \ldots, x_n$ such that $|A| x_i = s_i(A) x_i$ for $i = 1, \ldots, n$. Choose a δ, $0 < \delta < \varepsilon$ and $s_\infty(A) - \delta > 0$. The projection $E\big((s_\infty(A) - \delta, s_\infty(A)]\big)$ is necessarily of infinite rank. Take any orthonormal sequence $\{x_{n+i}\}$ in $\operatorname{Im} E\big((s_\infty(A) - \delta, s_\infty(A)]\big)$. We have $\| |A| x_{n+i} - s_\infty(A) x_{n+i} \| \leq \delta$. As $\operatorname{Im} E\big((s_\infty(A) - \delta, s_\infty(A)]\big) \subseteq \operatorname{Im} |A|$ (Conway [2, p.274]), which is contained in the initial space of U, the sequence $\{U x_{n+i}\}$ is orthonormal. We have

$$\begin{aligned}
|\langle A x_{n+i}, U x_{n+i} \rangle - s_\infty(A)| &= |\langle U^* A x_{n+i}, x_{n+i} \rangle - s_\infty(A)| \\
&= |\langle |A| x_{n+i} - s_\infty(A) x_{n+i}, x_{n+i} \rangle| \\
&\leq \delta < \varepsilon \, .
\end{aligned}$$

The proof is complete. $\square$

We have the following description of the (c,p)-norm, which is an extension of [4, Lemma II.4.1].

Theorem 2.3.

$$\|A\|_{c,p} = \sup \left\{ \left(\sum_{i=1}^{k} c_i |\langle A x_i, y_i \rangle|^p \right)^{\frac{1}{p}} : \{x_i\}_{i=1}^{k}, \{y_i\}_{i=1}^{k} \textit{ are orthonormal sets in } H \right\}$$

Proof. Let $\{x_i\}_{j=1}^k$ and $\{y_i\}_{j=1}^k$ be orthonormal sets in H. If H is finite-dimensional, then

$$\left(\sum_{i=1}^k c_i|\langle Ay_i, x_i\rangle|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^k c_i s_i(A)^p\right)^{\frac{1}{p}}.$$

In general let P be the projection onto span $\{x_i, Ay_i, y_i : i = 1, \ldots, k\}$. Then by the corresponding finite-dimensional result,

$$\begin{aligned}\left(\sum_{i=1}^k c_i|\langle Ay_i, x_i\rangle|^p\right)^{\frac{1}{p}} &= \left(\sum_{i=1}^k c_i|\langle PAy_i, x_i\rangle|^p\right)^{\frac{1}{p}} \\ &\le \left(\sum_{i=1}^k c_i s_i(PA)^p\right)^{\frac{1}{p}} \\ &\le \left(\sum_{i=1}^k c_i s_i(A)^p\right)^{\frac{1}{p}}.\end{aligned}$$

The reverse inequality follows from Lemma 2.2. □

Recall that a norm N on $B(H)$ is submultiplicative if $N(AB) \le N(A)N(B)$ for all $A, B \in B(H)$. If, in addition, $N(I) = 1$, it is an algebra norm. We have the following observations.

Theorem 2.4. *A (c,p)-norm on $B(H)$ is submultiplicative if and only if $c_1 \ge 1$; and it is an algebra norm if and only if it is the operator norm.*

Proof. Suppose $c_1 \ge 1$. Then for any $A, B \in B(H)$,

$$\|AB\|_{c,p} \le \|A\|\|B\|_{c,p} \le \|A\|_{c,p}\|B\|_{c,p}.$$

Conversely, if $c_1 < 1$, let A be a rank one operator with s-numbers $1, 0, \ldots$. Then $\|A\|_{c,p} = c_1 > c_1^2 = \|A\|_{c,p}^2$. Therefore, $\|\cdot\|_{c,p}$ is not submultiplicative.

The second assertion is clear. □

3. Extreme operator for $\|\cdot\|_{c,p}$

Let $S_{c,p}$ denote the closed unit ball for $B(H)$ under $\|\cdot\|_{c,p}$ and let $\operatorname{ext} S_{c,p}$ denote the set of extreme points of $S_{c,p}$. When $p = 1$, then as usual, we suppress the index p. To describe $\operatorname{ext} S_c$, let $r_j = \sum_{i=1}^j c_i$ and let R_j be the set of all rank j partial isometries. A maximal partial isometry is either an isometry or a co-isometry, i.e. its adjoint is an isometry. The set of all maximal partial isometries will be denoted by $R_{\max}$. Note that when H is finite-dimensional, a complete description of $\operatorname{ext} S_c$ is given in [8, Theorem 2]. We include their part (c) and (d) below for quick reference.

Lemma 3.1. *Suppose $c_1 = \cdots = c_h > \cdots > c_{n-l+1} = \cdots = c_n \ge 0$. Then in M_n,*

$$\operatorname{ext} S_c = r_1^{-1}R_1 \cup \left(\cup_{h<j<n-l}\ r_j^{-1}R_j\right) \cup r_n^{-1}R_n.$$

If $n = h + l + 1$, the middle summand is empty.

When H is infinite-dimensional, we have to replace R_n by $R_{\max}$.

Theorem 3.2. *Suppose* $c_1 = \cdots = c_h > \cdots \geq c_k > 0$*. Then*

$$\operatorname{ext} S_c = r_1^{-1} R_1 \cup \big(\cup_{h<j<k}\ r_j^{-1} R_j \big) \cup r_k^{-1} R_{\max}$$

If $k = h + 1$*, the middle summand is empty.*

We divide the proof of Theorem 3.1 into Lemma 3.3 to Lemma 3.7.

Lemma 3.3.

$$r_1^{-1} R_1 \cup \big(\cup_{h<j<k}\ r_j^{-1} R_j \big) \subseteq \operatorname{ext} S_c$$

Proof. Suppose $A = \langle \cdot, x \rangle y$ for $\|x\| = \|y\| = 1$ and $r_1^{-1} A = \frac{1}{2}(B + C)$ for $B, C \in S_c$. For every rank n ($n > k$) projections P and Q with $x \in \operatorname{Im} P$ and $y \in \operatorname{Im} Q$, we have

$$r_1^{-1} A = r_1^{-1} QAP = \frac{1}{2}(QBP + QCP)\,.$$

Since $\|QBP\|_c, \|QCP\|_c \leq 1$, we conclude from Lemma 3.1 that $QBP = QCP = r_1^{-1} A$. As P and Q are arbitrary, $B = C = r_1^{-1} A$.

Similarly, if $A \in \cup_{h<j<k} r_j^{-1} R_j$, then $r_j^{-1} A \in \operatorname{ext} S_c$. □

Again with a similar argument we get

Lemma 3.4.

$$r_k^{-1} I \in \operatorname{ext} S_c$$

Lemma 3.5.

$$r_k^{-1} R_{\max} \subseteq \operatorname{ext} S_c$$

Proof. Suppose U is an isometry and $r_k^{-1} U = \frac{1}{2}(C + D)$ for C, D in S_c. Multiplying both sides by U^*, we get $r_k^{-1} I = \frac{1}{2}(U^* C + U^* D)$. By Lemma 3.3, $r_k^{-1} I = U^* C$. If $C \neq r_k^{-1} U$, there is a unit vector $x \in H$ such that $(C - r_k^{-1} U)x = y \neq 0$. As $U^* C = r_k^{-1} I$, $(C - r_k^{-1} U)x \perp r_k^{-1} Ux$ and hence

$$\begin{aligned}
\|Cx\|^2 &= \|(C - r_k^{-1} U)x + r_k^{-1} Ux\|^2 \\
&= \|(C - r_k^{-1} U)x\|^2 + r_k^{-2} \|Ux\|^2 \\
&= \|y\|^2 + r_k^{-2} \\
&> r_k^{-2}\,.
\end{aligned}$$

Take an orthonormal set $\{x_1, \ldots, x_k\}$ in H with $x_1 = x$ and let $y_1 = \frac{Cx}{\|Cx\|}$, $y_i = Ux_i$ for $i = 2, \ldots, k$. Then $\{y_1, \ldots, y_k\}$ is also an orthonormal set. In fact for $i = 2, \ldots, k$,

$$\begin{aligned}
\langle y_1, y_i \rangle &= \frac{1}{\|Cx\|} \langle Cx, Ux_i \rangle \\
&= \frac{1}{\|Cx\|} \langle U^* Cx, x_i \rangle \\
&= \frac{1}{r_k \|Cx\|} \langle x, x_i \rangle \\
&= 0\,.
\end{aligned}$$

But then

$$\begin{aligned}\|C\|_c &\geq \sum_{i=1}^{k} c_i |\langle Cx_i, y_i\rangle| \\ &= c_1|\langle Cx_1, y_1\rangle| + \sum_{j=2}^{k} c_i|\langle Cx_i, x_i\rangle| \\ &= c_1\|Cx\| + \sum_{j=2}^{k} c_i|\langle U^*Cx_i, x_i\rangle| \\ &> c_1 r_k^{-1} + \left(\sum_{i=2}^{k} c_i\right) r_k^{-1} \\ &= 1\,,\end{aligned}$$

contradicting $C \in S_c$. Hence $r_k^{-1}U = C = D$.

If U is a co-isometry, then U^* is an isometry. The same argument yields $r_k^{-1}U \in \operatorname{ext} S_c$. $\square$

Lemma 3.6. *If $A \in \operatorname{ext} S_c$ is a scalar multiple of a partial isometry, then*

$$A \in r_1^{-1}R_1 \cup \big(\cup_{h<j<k}\; r_j^{-1}R_j\big) \cup r_k^{-1}R_{\max}\,.$$

Proof. First of all, it follows from Lemma 3.1 that if A is of finite rank, then $A \in r_1^{-1}R_1 \cup \big(\cup_{h<j<k}\; r_j^{-1}R_j\big)$. If A is of infinite rank, then $A = r_k^{-1}U$ for a partial isometry U. We claim that U is maximal. Otherwise both subspaces $\ker U$ and $(\operatorname{Im} U)^{\perp}$ are non-zero. Take unit vectors $x \in \ker U$, $y \in (\operatorname{Im} U)^{\perp}$ and consider the operator $B = \langle \cdot, x\rangle y$. For small enough $\varepsilon > 0$, we have $\|A \pm \varepsilon B\|_c = 1$. This contradicts $A \in \operatorname{ext} S_c$. $\square$

Lemma 3.7. *Every $A \in \operatorname{ext} S_c$ is a scalar multiple of a partial isometry.*

Proof. Suppose $A \in \operatorname{ext} S_c$. We shall prove that $\sigma(|A|) \subseteq \{0, \|A\|\}$ and hence A is a scalar multiple of a partial isometry. If every non-zero element in $\sigma(|A|)$ is an eigenvalue of $|A|$, then by [7, Lemma 1], the assertion is true. Otherwise $s_\infty(A) \neq 0$ is an accumulation point of $\sigma(|A|)$. There is an $\varepsilon > 0$ such that $\sigma(|A|) \cap \big(0, s_\infty(A) - \varepsilon\big) \neq \emptyset$. Let

$$f(t) = \begin{cases} 2t, & 0 \leq t < \frac{s_\infty - \varepsilon}{2} \\ s_\infty - \epsilon, & \frac{s_\infty - \varepsilon}{2} \leq t < s_\infty - \varepsilon \\ t, & s_\infty - \varepsilon \leq t \leq \|A\| \end{cases}$$

and $g(x) = 2t - f(t)$. Then $f(|A|) \neq g(|A|)$. By the spectral mapping theorem,

$$\begin{aligned}\sigma\big(f(|A|)\big) \cap [s_\infty(A) - \varepsilon, \|A\|] &= \sigma\big(g(|A|)\big) \cap [s_\infty(A) - \varepsilon, \|A\|] \\ &= \sigma(|A|) \cap [s_\infty(A) - \varepsilon, \|A\|]\,,\end{aligned}$$

and hence $s_\infty(A)$ is an accumulation point of both $\sigma\big(f(|A|)\big)$ and $\sigma\big(g(|A|)\big)$. We conclude that $\|f(|A|)\|_c = \|g(|A|)\|_c = 1$. Now $|A| = \frac{1}{2}\big(f(|A|) + g(|A|)\big)$. If $A = U|A|$ is the polar decomposition of A, then $A = \frac{1}{2}\big(Uf(|A|) + Ug(|A|)\big)$. As $\operatorname{Im} f(|A|), \operatorname{Im} g(|A|) \subseteq \overline{\operatorname{Im} |A|}$, which is the initial space of U, we have $Uf(|A|) \neq Ug(|A|)$. This contradicts the fact that $A \in \operatorname{ext} S_c$. $\square$

A refinement of the notion of an extreme point is the following. Let Q be a convex subset of a normed linear space X. A point $q \in Q$ is called an exposed point of Q if there is a bounded $\mathbb{R}$-linear functional $f : X \to \mathbb{R}$ such that $f(q) > f(p)$ for every $q \in Q \setminus \{q\}$. An exposed point q is said to be strongly exposed if for every sequence $\{q_n\}$ in Q such that $f(q_n) \to f(q)$, we have $q_n \to q$. Clearly an exposed point is an extreme point of Q.

Grzaślewicz [6, Theorem 2] showed that under the operator norm, the closed unit ball for $B(H)$ does not have any strongly exposed point. We shall show that in the $\|\cdot\|_c$ case, an extreme point of S_c is strongly exposed if and only if it is of finite rank.

Theorem 3.8. *Let $A \in \operatorname{ext} S_c$. Then A is a strongly exposed point of S_c if and only if $A \in r_1^{-1}R_1 \cup \big(\cup_{h<j<k}\ r_j^{-1}R_j\big)$.*

Proof. Suppose $A \in r_k^{-1}U$ is a strongly exposed point of S_c. Then it is easy to see that U is a strongly exposed point of the closed unit ball for the operator norm. This contradicts [6, Theorem 2] mentioned above.

For finite rank extreme points, we shall show that every $A \in \cup_{h<j<k} r_j^{-1}R_j$ is a strongly exposed point of S_c. The proof for $A \in r_1^{-1}R_1$ is similar.

Suppose $A = r_j^{-1}\sum_{i=1}^{j}\langle\cdot, x_i\rangle y_i$ for orthonormal sets $\{x_i\}$ and $\{y_i\}$. Define $f : B(H) \to \mathbb{R}$ by

$$f(F) = r_j \sum_{i=1}^{j} \operatorname{Re}\langle Fx_i, y_i\rangle \text{ for all } F \in B(H)\,.$$

The functional f exposes A in S_c. Indeed, $f(A) = j$ and for any $F \in S_c$,

$$\begin{array}{rcl} c_1|\langle Fx_1, y_1\rangle| + \cdots + & c_j|\langle Fx_j, y_j\rangle| & \leq 1\,, \\ c_1|\langle Fx_2, y_2\rangle| + \cdots + & c_j|\langle Fx_1, y_1\rangle| & \leq 1\,, \\ \vdots & \vdots & \vdots \\ c_1|\langle Fx_j, y_j\rangle| + \cdots + & c_j|\langle Fx_{j-1}, y_{j-1}\rangle| & \leq 1\,. \end{array} \tag{1}$$

Summing up the j inequalities, we get $f(F) \leq r_j\sum_{i=1}^{j}|\langle Fx_i, y_i\rangle| \leq j$.

If $f(F) = j$ for $F \in S_c$, then $\operatorname{Re}\langle Fx_i, y_i\rangle = |\langle Fx_i, y_i\rangle|$ for all i, and all inequalities in (1) become equalities. Since $c_1 > c_j$, all $\langle Fx_i, y_i\rangle$ are equal and have the value r_j^{-1}. Let P and Q be projections onto the subspaces span$\{x_1, \ldots, x_j\}$ and span$\{y_1, \ldots, y_j\}$ respectively. Apply [4, Theorem 4.3.26] to the operator $|QFP|$, we get

$$\begin{aligned} s_1(F) &\geq r_j^{-1}\,, \\ s_1(F) + s_2(F) &\geq 2r_j^{-1}\,, \\ &\vdots \\ s_1(F) + \cdots + s_j(F) &\geq jr_j^{-1}\,. \end{aligned}$$

On the other hand, we may replace each $|\langle Fx_i, y_i\rangle|$ by $s_i(F)$ in (1) to get $r_j\sum_{i=1}^{j} s_i(F) \leq j$. Hence $r_j\sum_{i=1}^{j} s_i(F) = j$. Again, as $c_1 > c_j$, all $s_i(F)$ are equal to r_j^{-1}. The other s-numbers must be zero. As $\langle Fx_i, y_i\rangle = s_i(F) = r_j^{-1}$ for all i, $F = A$ by the following version of [8, Lemma 4]:

Lemma 3.9. *Let $F \in B(H)$ be of finite rank n. If for orthonormal sets $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$, $\langle Fx_i, y_i\rangle = s_i(F)$ for every i, then*

$$F = \sum_{i=1}^{n} s_i(F)\langle \cdot, x_i\rangle y_i \, .$$

We now show that A is strongly exposed. It is a modification of the preceding argument. Let $\{A_n\}$ be a sequence in S_c such that $f(A_n) \to f(A)$, or

$$r_j \sum_{i=1}^{j} \operatorname{Re} \langle A_n x_i, y_i\rangle \to j.$$

Replacing F by A_n in system (1) above, we get

$$r_j \sum_{i=1}^{j} |\langle A_n x_i, y_i\rangle| \to j \, .$$

Indeed for each i, $\langle A_n x_i, y_i\rangle \to r_j^{-1}$. This is obtained by showing that every convergent subsequence of $\{\langle A_n x_i, y_i\rangle\}$ has limit r_j^{-1}. Again let P and Q be projections onto the subspaces $\operatorname{span}\{x_1, \dots, x_j\}$ and $\operatorname{span}\{y_1, \dots, y_j\}$ respectively. Then $\|(I-Q)A_n(I-P)\| \to 0$; for otherwise $\|A_n\|_c > 1$ for some large n. We claim that $QA_nP \to A$, $(I-Q)A_nP$ and $QA_n(I-P) \to 0$, and hence $A_n \to A$.

Note that all the QA_nP's are (essentially) mappings between fixed finite-dimensional spaces. To show $QA_nP \to A$, we need only show that A is the only accumulation point. For simplicity, let $QA_nP \to B$. Then $\langle Bx_i, y_i\rangle = r_j^{-1}$ for all i. It is also clear that $\|B\|_c \le 1$ and a similar argument as for F above shows that $B = A$.

If $(I-Q)A_nP \not\to 0$, there exist an $\varepsilon > 0$ and a sufficiently large n such that $\|(I-Q)A_nP\| > \varepsilon$ and $\langle A_n x_i, y_i\rangle > r_j^{-1} - \varepsilon'$ (to be determined) for all i. Without loss of generality we may assume that there is a unit vector y, orthogonal to all y_i's such that $|\langle A_n x_1, y\rangle| > \varepsilon$. Let

$$y_1' = \frac{\langle A_n x_1, y_1\rangle y_1 + \langle A_n x_1, y\rangle y}{\sqrt{|\langle A_n x_1, y_1\rangle|^2 + |\langle A_n x_1, y\rangle|^2}} \, .$$

Then $\langle A_n x_1, y_1'\rangle > \sqrt{(r_j^{-1} - \varepsilon')^2 + \varepsilon^2}$. Choosing ε' small enough, we obtain by Fan's inequality (see [8, Lemma 3]) that $\|A_n\|_c > 1$, which is a contradiction. □

For $p > 1$, we have

Theorem 3.10. *An operator A is an extreme point of $S_{c,p}$ if and only if*

$$A = \sum_{i=1}^{j} s_i \langle \cdot, x_i\rangle y_i + s_{j+1} U \, ,$$

where $1 \le j < k$, $\sum_{i=1}^{k} c_i s_i^p = 1$, where $s_i = s_{j+1}$ for $i \ge j+1$, and U is a maximal partial isometry from $\{x_1, \dots, x_j\}^\perp$ into $\{y_1, \dots, y_j\}^\perp$.

Note that the above description includes A is a scalar multiple of some maximal partial isometry.

Proof. ($\Longleftarrow$) Suppose A is of the above form with U an isometry from $\{x_1, \dots, x_j\}^\perp$ into $\{y_1, \dots, y_j\}^\perp$. We shall show that A is an extreme point of $S_{c,p}$. Let $A = \frac{1}{2}(B + C)$ for $B, C \in S_{c,p}$. Take an orthonormal set $S_1 = \{x_1, \dots, x_j, \dots, x_k\}$. The set $S_2 = \{y_1, \dots, y_j, Ux_{j+1}, \dots, Ux_k\}$ is also orthonormal. Let P and Q be projections onto $\operatorname{span} S_1$ and $\operatorname{span} S_2$ respectively. We have

$$QAP = \sum_{i=1}^{j} s_i \langle \cdot, x_i \rangle + s_{j+1} \sum_{i=j+1}^{k} \langle \cdot, x_i \rangle U x_i \,,$$

and $QAP = \frac{1}{2}(QBP + QCP)$. As the (c, p)-norm on M_k is strictly convex, (this is essentially strict convexity of $\mathbb{C}^k$ under p-norm,) $QBP = QCP = QAP$. If R denotes the projection onto $\operatorname{span}\big(\{y_1, \dots, y_j\} \cup \operatorname{Im} U\big)$, we conclude that $RB = RC = A$. Now $s_i = s_i(RB) \leq s_i(B)$ for all i. As

$$1 \geq \sum_{i=1}^{k} c_i s_i(B)^p \geq \sum_{i=1}^{k} c_i s_i^p = 1 \,,$$

$s_i(B) = s_i$ for all i. We have $(I - R)B = 0$, or $B = A$. Similarly, $C = A$. If U is a co-isometry, the same argument shows that A is also an extreme point of $S_{c,p}$.

($\Longrightarrow$) Let $A \in \operatorname{ext} S_{c,p}$. We contend that the s-numbers eventually equal a constant, which must be $s_\infty(A)$. Otherwise every s-number of A is an eigenvalue of $|A|$ and there is a large $n (> k)$ for which $s_{n+1}(A) < s_n(A)$. Take a corresponding eigenvector x_{n+1} of $s_{n+1}(A)$ and let $B = \langle \cdot, x_{n+1} \rangle A x_{n+1}$. For sufficiently small $\varepsilon > 0$, we have $\|A \pm \varepsilon B\|_{c,p} = 1$. This contradicts $A \in \operatorname{ext} S_{c,p}$. Note that the above reasoning indeed requires $s_k(A) = s_\infty(A)$. Now a similar argument as in Lemma 3.6 shows (i) there is no other value in $\sigma(|A|)$, except perhaps 0, and (ii) A is of the required form. □

4. Isometric isomorphisms for $\|\cdot\|_{c,p}$

The main result of this section is the following

Theorem 4.1. *Suppose $T : B(H) \to B(H)$ is a linear isomorphism such that $\|T(A)\|_{c,p} = \|A\|_{c,p}$ for every $A \in B(H)$. Then there are unitary operators U and V such that either*

$$T(A) = UAV \text{ for every } A \in B(H) \,,$$

or

$$T(A) = UA^tV \text{ for every } A \in B(H) \,,$$

where A^t denotes the transpose of A with respect to an orthonormal basis fixed in advance.

Proof. ($\Longleftarrow$) Clear.

($\Longrightarrow$) Suppose $T : B(H) \to B(H)$ is surjective and $\|T(A)\|_{c,p} = \|A\|_{c,p}$ for all A. By Rias [9, Lemma 3], T is of the given form if (and only if) T preserves maximal partial isometries. As $r_k^{-\frac{1}{p}} R_{\max} \subseteq \operatorname{ext} S_{c,p}$, which is fixed by T, we have to single out $r_k^{-\frac{1}{p}} R_{\max}$ from other extreme points. For $p = 1$, the set is precisely the non-strongly exposed points and we are done. For $p > 1$, the following Lemma 4.2 concludes our proof. □

Lemma 4.2. *Let A be an extreme point of $S_{c,p}$. Then A is a scalar multiple of a maximal partial isometry if and only if A can be decomposed into $A = B + C$ with the property that*

$$\|\lambda B + \mu C\|_{c,p} = \max\{|\lambda|, |\mu|\}$$

for any complex numbers λ and μ.

Proof. ($\Longrightarrow$) Clear.

($\Longrightarrow$) Suppose $A = B + C$ with the said condition. By our description of extreme points of $S_{c,p}$ (Theorem 3.8), it suffices to show that $s_1(A) = s_k(A)$. Now it is plain that $\|B\|_{c,p} = \|C\|_{c,p} = 1$ and $\|A+B\|_{c,p} = \|A+C\|_{c,p} = 2$. Hence $\|A+B\|_{c,p} = \|A\|_{c,p} + \|B\|_{c,p}$. Moreover

$$\sum_{i=1}^{j} s_i(A+B) \leq \sum_{i=1}^{j} s_i(A) + \sum_{i=1}^{j} s_i(B) \quad (j = 1, 2, \dots).$$

We have

$$\begin{aligned} 2 &= \big(c_1 s_1(A+B)^p + \cdots + c_k s_k(A+B)^p\big)^{\frac{1}{p}} \\ &\leq \big(c_1(s_1(A) + s_1(B))^p + \cdots + c_k(s_k(A) + s_k(B))^p\big)^{\frac{1}{p}} \\ &\leq \big(c_1 s_1(A)^p + \cdots + c_k s_k(A)^p\big)^{\frac{1}{p}} + \big(c_1 s_1(B)^p + \cdots + c_k s_k(B)^p\big)^{\frac{1}{p}} \\ &= 2\,. \end{aligned}$$

It follows from [4, Lemma II.3.5] and Minkowski's inequality that

$$\big(s_1(A), \dots, s_k(A)\big) = \big(s_1(B), \dots, s_k(B)\big)$$

and

$$\big(s_1(A+B), \dots, s_k(A+B)\big) = 2\big(s_1(A), \dots, s_k(A)\big)\,.$$

If $s_1(A) > s_k(A)$, then $s_1(A)$ is an s-number of A of finite multiplicity, say l. Clearly the largest s-numbers of B and $A + B$ also have multiplicity l. Now

$$L = \big\{x \in H : \|(A+B)x\| = \|A+B\|\|x\|\big\}$$

is a subspace of H of dimension l, and the same is true if we replace $A + B$ by A and B respectively. Take any $x \in L$, we have

$$2\|A\|\|x\| = \|A+B\|\|x\| = \|(A+B)x\| = \|Ax + Bx\| \leq \|Ax\| + \|Bx\| \leq 2\|A\|\|x\|\,.$$

Hence $Ax = Bx$ and $\|Ax\| = \|A\|\|x\|$. As both subspaces are of dimension l,

$$L = \big\{x \in H : \|Ax\| = \|A\|\|x\|\big\}\,.$$

If we substitute C for B in the above argument, we also get $Ax = Cx$ for every $x \in L$. But then $Ax = (B + C)x = 2Ax$, which is absurb. $\square$

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References

1. J. Arazy, *The isometries of C_p*, Israel J. Math. **22** (1975), 247–256.
2. J.B. Conway, *A Course in Functional Analysis*, 2nd ed., Springer-Verlag, New York, 1990.
3. L. Fialkow and R. Loebl, *Elementary mappings into ideals of operators*, Illinois J. Math. **28** (1984), 555–578.
4. I.C. Gohberg and M.G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Transl. Math. Monographs, Vol. 18, Amer. Math. Soc., Providence, R.I., 1969.
5. R. Grone and M. Marcus, *Isometries of matrix algebras*, J. Algebra **47** (1977), 180–189.
6. R. Grząślewicz, *Exposed points of the unit ball of $\mathcal{L}(H)$*, Math. Z. **193** (1986), 595–596.
7. R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.
8. C.K. Li and N.K. Tsing, *Duality between some linear preserver problems II. Isometries with respect to c-spectral norms and matrices with fixed singular values*, Linear Algebra Appl. **110** (1988), 181–212.
9. M. Rais, *The Unitary group preserving maps (the infinite dimensional case)*, Linear and Multilinear Algebra **20** (1987), 337–345.
10. A.R. Sourour, *Isometries of norm ideals of compact operators*, J. Funct. Anal. **43** (1981), 69–77.

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