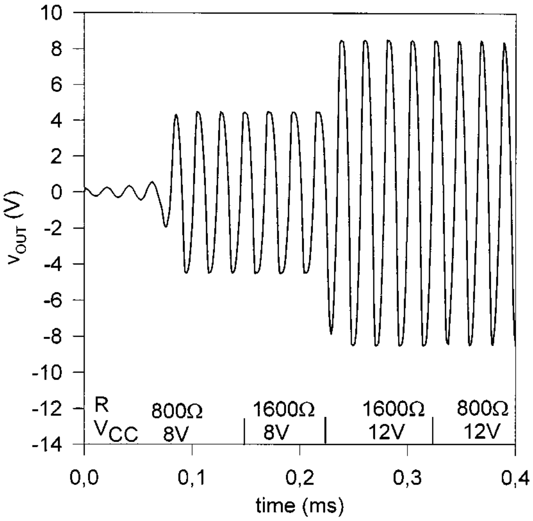

Fig. 11. SPICE results obtained on the AD844-based *LC* sinusoidal oscillator described in Fig. 10.

still increased, the oscillation magnitude also increases but the signal shape is less and less sinusoidal. When the R-value is decreased, the oscillation suddenly stops when R is equal to 807 Ω. Just before the oscillator stops oscillating, the oscillation magnitude was still equal to 5.5 V. So, this experimental study confirms that the oscillator described in Fig. 9 cannot stabilize the magnitude of its oscillation signal. The oscillation magnitude value is only controlled by the power supply. So, an external nonlinear network will be required to stabilize the magnitude of the oscillation signal.

2) Results Obtained With SPICE: Once again SPICE confirms the results obtained on the real device. Fig. 11 displays the response of the oscillator described in Fig. 10 for two values of resistor R ($R = 800$ Ω and $R = 1600$ Ω) and two values of the power supplies ($V_{CC} = 8$ V and $V_{CC} = 12$ V) in the following sequence: a) $0 < t < 150$ μs $R = 800$ Ω and $V_{CC} = 8$ V, b) 150 μs $< t <$ 225 μs $R = 1600$ Ω and $V_{CC} =$ 8 V, c) 225 μs $< t <$ 340 μs $R = 1600$ Ω and $V_{CC} =$ 12 V and finally d) 340 μs $< t <$ 400 μs $R =$ 800 Ω and $V_{CC} =$ 12 V. This figure confirms that the power supply alone determine the amplitude.

V. Conclusion

Using a nonlinear analysis, it was shown that OTA-based and CFOA-based *LC* oscillators have not the same behavior. For the OTA-based oscillator, the OTA nonlinearity can stabilize the magnitude of the oscillation signal whereas, for the CFOA-based oscillator, the CFOA nonlinearity prevents the stabilization phenomenon. Experimental results confirm the theoretical analysis.

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Robust Mixed H_2/H_∞ Filtering With Regional Pole Assignment for Uncertain Discrete-Time Systems

Fuwen Yang and Y. S. Hung

***Abstract*—This paper deals with the robust mixed H_2/H_∞ filtering problem with regional pole assignment for linear uncertain discrete-time systems in the presence of two sets of exogenous disturbance inputs. A general framework for solving this problem is established using a linear matrix inequality (LMI) approach in conjunction with regional pole constraints, and H_2 and H_∞ optimization characterization. Necessary and sufficient conditions for the solvability of the problem are given in terms of a set of feasible LMIs. A numerical example is provided to illustrate the effectiveness of the proposed design algorithm.**

***Index Terms*—Linear matrix inequality, quadratically D-stable, regional pole assignment, robust mixed H_2/H_∞ filtering.**

I. Introduction

State estimation of dynamic systems in the presence of both process and measurement noises is a very important problem in engineering applications. One landmark design approach is the Kalman filtering (also called H_2 filtering), which minimizes the H_2 norm of the estimation error under the assumption that the noise processes have known power spectral densities [1], [14]. In practice, however, the noise processes often have unknown or uncertain spectral densities. This difficulty has been overcome by reformulating the estimation problem in an H_∞ filtering framework during the last few years [10], [13], [23].

H_∞ filtering offers robustness performance that is significantly better than H_2 filtering. But H_∞ filtering is so conservative as to lead to a large intolerable estimation error variance when the system is driven by white noise signals [15]. The mixed H_2/H_∞ filtering problem that simultaneously considers the presence of two sets of exogenous signals (i.e., the deterministic disturbance input with bounded energy and the stochastic disturbance input with known statistics), was first introduced in [3] as an attempt to capture the benefits of both pure H_2 and H_∞ filters. It allows us to make trade-offs between the performance of the H_2 filter and the performance of the H_∞ filter. However, unlike the H_2 and H_∞ filtering problems that

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have computable solutions, there is no compact solution to the mixed problem.

So far, there have been several approaches to solve the mixed H_2/H_∞ filtering problem. In [3], Bernstein and Haddad first transformed the mixed H_2/H_∞ filtering problem into an auxiliary minimization problem, and then by using *Lagrange multiplier technique*, gave the solution which led to an upper bound on the H_2 filtering error variance by solving a set of coupled Riccati and Lyapunov equations. In [5], [24], a *time domain game approach* was proposed to solve the mixed H_2/H_∞ filtering problem through a set of coupled Riccati equations. Khargonekar *et al.* [17] and Rotstein *et al.* [21] have used a *convex optimization approach* to obtain the solutions involving affine symmetric matrix inequalities.

On the other hand, the mixed H_2/H_∞ filter design is primarily concerned with optimal performance (corresponding to the H_2 performance) and robustness (corresponding to the H_∞ performance) of the filter, and does not explicitly consider the transient property of the estimation dynamics. As is well known, the dynamics of a linear system is related to the location of its poles. By constraining the filter's poles to lie inside a prescribed region of the open unit disk, the filter designed would have the expected transient performance. It is worth emphasizing that, in the past few years, the controller design problem with regional pole assignment has been extensively studied. In particular, Chilali and Gahinet [6] studied in detail the design of state- or output-feedback H_∞ controllers that satisfy additional constraints on the regional pole location, and the results were further extended in [7] to uncertain systems described by a polytopic state-space model. In [2], the mixed H_2/H_∞ control problem with regional pole assignment was considered for deterministic continuous-time systems. It should be pointed out that, comparing to the controller design case, the corresponding filter design problem with pole assignment in a desired region has gained much less attention, not to mention the case of the robust H_2 and/or H_∞ filtering problem [9], [10], [12], [25], [26]. A primary result obtained for robust H_∞ filtering with *special* pole constraints has been given in [20]. This situation motivates our present investigation.

In this paper, we study the robust mixed H_2/H_∞ filtering problem with regional pole assignment. The approach developed in this paper is different from that proposed in [2], where the Lagrange multiplier technique was used. Instead, the linear matrix inequality (LMI) approach is adopted. Since LMIs intrinsically reflect constraints rather than optimality, they tend to offer more flexibility for combining several constraints. Specifically, we transform all the performance specifications into unified LMI formulations. Therefore, the overall problem remains convex, and the desired filter parameters can be directly obtained by solving the LMIs using the existing LMI Toolbox.

The notation used here is fairly standard. $\otimes$ denotes the Kronecker product. $\|\cdot\|_p$ stands for $\mathbf{H}_p$-norm in Hardy space. $\mathrm{Tr}(M)$ represents the trace of matrix M. In symmetric block matrices, $*$ is used as an ellipsis for terms induced by symmetry. $\overline{\lambda}$ means the conjugate of λ. $diag\{M_1, M_2, \ldots\}$ denotes a block diagonal matrix whose diagonal blocks are given by M_1, M_2, etc. The dimension of an identity matrix will be omitted in the analysis when no confusion can arise.

II. Problem Statement

Consider a linear discrete-time system with parameter uncertainty described by

$$\begin{cases} x(k+1) = (A+\Delta A)x(k) + B_1 w(k) + B_2 v(k) \\ y(k) = (C+\Delta C)x(k) + D_1 w(k) + D_2 v(k) \\ z_\infty(k) = L_\infty x(k) \\ z_2(k) = L_2 x(k) \end{cases} \tag{1}$$

where $x(k) \in R^n$ is the state, $y(k) \in R^p$ is the measured output, $z_\infty(k) \in R^{m1}$ represents a combination of the states to be estimated (with respect to H_∞-norm constraints), and $z_2(k) \in R^{m2}$ represents another combination of the states to be estimated (with respect to H_2-norm constraints). $w(k) \in R^{p1}$ is a disturbance input with bounded energy and stationary power, which belongs to $L_2[0, \infty]$, and $v(k) \in R^{p2}$ is a zero-mean Gaussian white noise process with unit covariance. A, C, B_1, B_2, D_1, D_2, L_∞, and L_2 are known real matrices with appropriate dimensions, whereas ΔA and ΔC are perturbation matrices representing parameter uncertainties. We will assume that ΔA and ΔC are time-invariant of the form

$$\begin{bmatrix} \Delta A \\ \Delta C \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \Gamma E \tag{2}$$

where H_1, H_2 and E are known constant matrices of appropriate dimensions, and $\Gamma \in R^{i\times j}$ is a perturbation matrix which satisfies

$$\Gamma^T \Gamma \le I. \tag{3}$$

It will be assumed that the initial state $x(0)$ is known, and without loss of generality, we will take $x(0) = 0$.

Assumption 1: The system (1) is stable for all admissible perturbations ΔA. []

Now consider the following filter for the system (1):

$$\begin{cases} \hat{x}(k+1) = F\hat{x}(k) + Gy(k) \\ \hat{z}_\infty(k) = \hat{L}_\infty \hat{x}(k) \\ \hat{z}_2(k) = \hat{L}_2 \hat{x}(k) \end{cases} \tag{4}$$

where $\hat{x}(k) \in R^n$ is the estimated state, $\hat{z}_\infty(k) \in R^{m1}$ is an estimate for $z_\infty(k)$, $\hat{z}_2(k) \in R^{m2}$ is an estimate for $z_2(k)$, and F, G, $\hat{L}_\infty$ and $\hat{L}_2$ are filter parameters to be determined. Notice that the filter structure (4) is not dependent upon the parameter uncertainties.

Define

$$x_e(k) = \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix}. \tag{5}$$

A state-space model describing the augmented system formed from the system (1) and the filter (4) is expressed as

$$\begin{cases} x_e(k+1) = (A_e + \Delta A_e)x_e(k) + B_{e1} w(k) + B_{e2} v(k) \\ e_\infty(k) = z_\infty(k) - \hat{z}_\infty(k) = C_\infty x_e(k) \\ e_2(k) = z_2(k) - \hat{z}_2(k) = C_2 x_e(k) \end{cases} \tag{6}$$

where

$$\begin{cases} A_e = \begin{bmatrix} A & 0 \\ GC & F \end{bmatrix} \\ \Delta A_e = \begin{bmatrix} H_1 \\ GH_2 \end{bmatrix} \Gamma \begin{bmatrix} E & 0 \end{bmatrix} =: H_e \Gamma E_e \\ B_{e1} = \begin{bmatrix} B_1 \\ GD_1 \end{bmatrix} \\ B_{e2} = \begin{bmatrix} B_2 \\ GD_2 \end{bmatrix} \\ C_\infty = \begin{bmatrix} L_\infty & -\hat{L}_\infty \end{bmatrix} \\ C_2 = \begin{bmatrix} L_2 & -\hat{L}_2 \end{bmatrix}. \end{cases} \tag{7}$$

Let

$$\begin{cases} T_\infty(z) = C_\infty(zI - A_e - \Delta A_e)^{-1} B_{e1} \\ T_2(z) = C_2(zI - A_e - \Delta A_e)^{-1} B_{e2} \end{cases} \tag{8}$$

be, respectively, the transfer function from $w(k)$ to the error state $e_\infty(k)$ (corresponding to the H_∞-norm consideration), and the transfer function from $v(k)$ to the error state $e_2(k)$ (corresponding to the H_2-norm consideration).

For the purpose of regional pole assignment, we now recall the concept of LMI region proposed in [7]. An LMI region is any subset D inside the open unit disk that can be described as follows:

$$D = \left\{ \lambda \in C \colon f_D(\lambda) = L + \lambda M + \bar{\lambda} M^T < 0 \right\} \tag{9}$$

where L and M are real matrices such that $L^T = L$. The matrix-valued function $f_D(\lambda)$ is called the characteristic function of D. As explained in [7], with different choices of the matrices L and M, the LMI region D defined in (9) can be used to represent many kinds of popular pole regions, such as disk, vertical strips, horizontal strips, conic sector, etc.

Now, we are in the position to introduce the notion of *quadratically D-stable* for the uncertain system (6).

Definition 1 [7]: The uncertain system (6) is said to be *quadratically D-stable* if there exists a symmetric positive-definite matrix X such that for all admissible perturbations ΔA and ΔC, the following matrix inequality

$$L \otimes X + M \otimes (X(A_e + \Delta A_e)) + M^T \otimes \left((A_e + \Delta A_e)^T X \right) < 0 \tag{10}$$

is true, where the LMI region D is defined in (9). []

Remark 1: It has been revealed in [7] that, if (10) is satisfied, then all poles of the uncertain time-invariant matrix $A_e + \Delta A_e$ are constrained to lie within the specified LMI region D. []

The following well-known lemmas for characterizing H_2- and H_∞-norm constraints, are needed in the derivation of our main results.

Lemma 1 [11], [18]: Let the constant $\gamma > 0$ be given. The uncertain system (6) is quadratically stable and $\|T_\infty(z)\|_\infty < \gamma$, if and only if there exists a symmetric positive-definite matrix X such that for all admissible perturbations ΔA and ΔC, the following matrix inequality:

$$\begin{bmatrix} -X & 0 & X(A_e + \Delta A_e) & XB_{e1} \\ 0 & -\gamma I & C_\infty & 0 \\ (A_e + \Delta A_e)^T X & C_\infty^T & -X & 0 \\ B_{e1}^T X & 0 & 0 & -\gamma I \end{bmatrix} < 0 \tag{11}$$

is satisfied. []

Lemma 2 [8], [19]: Let the constant $\beta > 0$ be given. The uncertain system (6) is quadratically stable and $\|T_2(z)\|_2 < \beta$, if and only if there exist symmetric positive-definite matrices X and Q such that for all admissible perturbations ΔA and ΔC, the following three inequalities:

$$\begin{bmatrix} -X & X(A_e + \Delta A_e) & XB_{e2} \\ (A_e + \Delta A_e)^T X & -X & 0 \\ B_{e2}^T X & 0 & -I \end{bmatrix} < 0$$

$$\begin{bmatrix} X & C_2^T \\ C_2 & Q \end{bmatrix} > 0 \tag{12}$$

$$\mathrm{Tr}(Q) < \beta^2 \tag{13}$$

hold.

Proof: The results are obtained directly from [8], [19] by Schur complement. []

In the light of Definition 1, Lemmas 1 and 2, our filtering problem can be cast as the following optimization problem:

$$\min_{X>0,\, Q>0,\, F,\, G,\, \hat{L}_2,\, \hat{L}_\infty} \mathrm{Tr}(Q) \quad \text{subject to (10)–(12).} \tag{14}$$

The problem (14) is to find the filter (4) to minimize the upper bound of the H_2 performance subject to the H_∞ performance constraint and the poles constraints for the uncertain system (6), which will be referred to as the robust mixed H_2/H_∞ filtering problem with regional pole assignment. Note that at this stage, such a problem is not a convex one yet, since the parameter uncertainties ΔA and ΔC are involved in the conditions (10)–(12), which make the problem more complicated. Our goal in the next section will be to derive *necessary and sufficient* conditions, in the form of LMIs, for the solutions of the aforementioned filter design problem.

III. The Solution to Robust Mixed H_2/H_∞ Filtering Problem With Regional Pole Assignment

In this section, we will give the solution to the robust mixed H_2/H_∞ filtering problem with regional pole assignment based on an LMI approach. The following lemma will be required for developing the main results.

Lemma 3 [4], [27]: Let $M = M^T$, H and E be real matrices of appropriate dimensions, with Γ satisfying (3), then

$$M + H\Gamma E + E^T \Gamma^T H^T < 0 \tag{15}$$

if and only if there exists a positive scalar $\varepsilon > 0$ such that

$$M + \varepsilon E^T E + \frac{1}{\varepsilon} HH^T < 0 \tag{16}$$

or equivalently

$$\begin{bmatrix} M & H & \varepsilon E^T \\ H^T & -\varepsilon I & 0 \\ \varepsilon E & 0 & -\varepsilon I \end{bmatrix} < 0. \tag{17}$$

Proof: The proof of the first conclusion can be found in [4], [27]. The equivalence between (16) and (17) follows immediately from Schur complement. □

Theorem 1: Let D be an arbitrary LMI region contained inside the open unit disk and let (9) be its characteristic function. The problem (14) is solvable, if and only if there exist symmetric positive-definite matrices R, S, Q, and matrices Q_1, Q_2, Q_3, Q_4 and positive scalars ε_1, ε_2, and ε_3 such that the LMIs in (18)–(20), shown at the bottom of the next page, are feasible, where

$$\begin{aligned} \Xi := L \otimes \begin{bmatrix} S & S \\ S & R \end{bmatrix} &+ M \otimes \begin{bmatrix} SA & SA \\ RA + Q_2 C + Q_1 & RA + Q_2 C \end{bmatrix} \\ &+ M^T \otimes \begin{bmatrix} SA & SA \\ RA + Q_2 C + Q_1 & RA + Q_2 C \end{bmatrix}^T \end{aligned} \tag{21}$$

and the constant matrices M_1, M_2 are obtained from the factorization $M = M_1^T M_2$. Here, M_1 and M_2 have full column rank. Moreover, if the LMIs (18)–(20) are feasible, the desired filter parameters can be determined by

$$\begin{cases} F = X_{12}^{-1} Q_1 (S - R)^{-1} X_{12} \\ G = X_{12}^{-1} Q_2 \\ \hat{L}_\infty = Q_3 (S - R)^{-1} X_{12} \\ \hat{L}_2 = Q_4 (S - R)^{-1} X_{12} \end{cases} \tag{22}$$

where the matrix X_{12} comes from the factorization $I - RS^{-1} = X_{12}Y_{12}^T < 0$.

Proof: Factorizing the matrix M as $M = M_1^T M_2$ and using the property of the Kronecker product that $(AC)\otimes(BD) = (A\otimes B)(C\otimes D)$, (10) can be rewritten as

$$\begin{aligned} L \otimes X + M \otimes (XA_e) + M^T \otimes (A_e X)^T \\ + \left(M_1^T \otimes (XH_e)\right)(I\otimes\Gamma)(M_2 \otimes E_e) \\ + (M_2^T \otimes E_e^T)(I\otimes\Gamma^T)\left(M_1 \otimes (H_e^T X)\right) < 0. \end{aligned} \tag{23}$$

By applying Lemma 3 to (23), (11) and (12) to eliminate the uncertainty Γ, we obtain the following LMIs on the positive-definite matrix $X > 0$ and the positive scalar parameters ε_1, ε_2, and ε_3 as shown in (24)–(27) at the bottom of the next page.

Recall that our goal is to derive the expressions of the filter parameters from (24)–(27). To do this, we partition X and X^{-1} as

$$X = \begin{bmatrix} R & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \quad X^{-1} = \begin{bmatrix} S^{-1} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix} \tag{28}$$

where the partitioning of X and X^{-1} is compatible with that of A_e defined in (7).

Now define

$$T_1 = \begin{bmatrix} S^{-1} & I \\ Y_{12}^T & 0 \end{bmatrix} \quad T_2 = \begin{bmatrix} I & R \\ 0 & X_{12}^T \end{bmatrix} \tag{29}$$

which imply that $XT_1 = T_2$ and $T_1^T X T_1 = T_1^T T_2$.

Again define the change of filter parameters

$$\begin{cases} Q_1 := X_{12} F Y_{12}^T S \\ Q_2 := X_{12} G \\ Q_3 := \hat{L}_\infty Y_{12}^T S \\ Q_4 := \hat{L}_2 Y_{12}^T S. \end{cases} \tag{30}$$

By applying the congruence transformations $diag\{I\otimes T_1, I, I\}$ to (24), $diag\{T_1, I, T_1, I, I, I\}$ to (25), $diag\{T_1, T_1, I, I, I\}$ to (26), $diag\{T_1, I\}$ to (27) first, and then the congruence transformations $diag\{diag\{I \otimes S, I\}, I, I\}$ to (24), $diag\{diag\{S, I\}, I, diag\{S, I\}, I, I, I\}$ to (25), $diag\{diag\{S, I\}, diag\{S, I\}, I, I, I\}$ to (26), $diag\{diag\{S, I\}, I\}$ to (27), (18)–(20) follow directly from (24)–(27). Furthermore, if the LMIs (18)–(20) are feasible, they imply that

$$\begin{bmatrix} -S & -S \\ -S & -R \end{bmatrix} < 0, \quad \text{i.e.,} \begin{bmatrix} S^{-1} & I \\ I & R \end{bmatrix} > 0.$$

It follows directly from $XX^{-1} = I$ that $I - RS^{-1} = X_{12}Y_{12}^T < 0$. Hence, one can always find square and nonsingular X_{12} and Y_{12} [22]. Therefore, (22) is obtained from (30), which concludes the proof. □

It follows from Theorem 1 that, the problem (14) can now be successfully recast as the following convex optimization problem:

$$\min_{R>0,S>0,Q>0,Q_1,Q_2,Q_3,Q_4,\varepsilon_1,\varepsilon_2,\varepsilon_3} \mathrm{Tr}(Q) \quad \text{subject to (18)–(20).} \tag{31}$$

$$\begin{bmatrix} \Xi & M_1^T \otimes \begin{bmatrix} SH_1 \\ RH_1 + Q_2H_2 \end{bmatrix} & M_2^T \otimes \begin{bmatrix} \varepsilon_1 E^T \\ \varepsilon_1 E^T \end{bmatrix} \\ M_1 \otimes \begin{bmatrix} H_1^T S & (RH_1 + Q_2H_2)^T \end{bmatrix} & -\varepsilon_1 I & 0 \\ M_2 \otimes [\varepsilon_1 E \ \ \varepsilon_1 E] & 0 & -\varepsilon_1 I \end{bmatrix} < 0 \tag{18}$$

$$\begin{bmatrix} -S & -S & 0 & SA & SA & SB_1 & SH_1 & 0 \\ * & -R & 0 & RA+Q_2C+Q_1 & RA+Q_2C & RB_1+Q_2D_1 & RH_1+Q_2H_2 & 0 \\ * & * & -\gamma I & L_\infty - Q_3 & L_\infty & 0 & 0 & 0 \\ * & * & * & -S & -S & 0 & 0 & \varepsilon_2 E^T \\ * & * & * & * & -R & 0 & 0 & \varepsilon_2 E^T \\ * & * & * & * & * & -\gamma I & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_2 I & 0 \\ * & * & * & * & * & * & * & -\varepsilon_2 I \end{bmatrix} < 0 \tag{19}$$

$$\begin{bmatrix} -S & -S & SA & SA & SB_2 & SH_1 & 0 \\ * & -R & RA+Q_2C+Q_1 & RA+Q_2C & RB_2+Q_2D_2 & RH_1+Q_2H_2 & 0 \\ * & * & -S & -S & 0 & 0 & \varepsilon_3 E^T \\ * & * & * & -R & 0 & 0 & \varepsilon_3 E^T \\ * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & -\varepsilon_3 I & 0 \\ * & * & * & * & * & * & -\varepsilon_3 I \end{bmatrix} < 0$$

$$\begin{bmatrix} -S & -S & L_2^T - Q_4^T \\ -S & -R & L_2^T \\ L_2 - Q_4 & L_2 & -Q \end{bmatrix} < 0 \tag{20}$$

On the other hand, in view of (22), we make the linear transformation on the state estimate

$$\vec{x}(t) = X_{12}\hat{x}(t) \tag{32}$$

and then obtain a new representation of the filter as follows:

$$\begin{cases} \vec{x}(k+1) = Q_1(S-R)^{-1}\vec{x}(k) + Q_2 y(k) \\ \hat{z}_\infty(k) = Q_3(S-R)^{-1}\vec{x}(k) \\ \hat{z}_2(k) = Q_4(S-R)^{-1}\vec{x}(k). \end{cases} \tag{33}$$

We can now see from (33) that, the filter parameters can be obtained directly by solving the problem (31).

Remark 2: The problem (31) is a standard LMI problem. It can be solved efficiently via the interior point method [4], [11], [16]. Note that LMIs (18)–(20) are affine in the scalar positive parameters $\varepsilon_1, \varepsilon_2$, and ε_3. Hence, they can be defined as LMI variables in order to increase the possibility of the solutions and decrease conservatism with respect to the perturbation Γ. []

Remark 3: LMI regions are often specified as the intersection of elementary regions, such as vertical strips, horizontal strips, disks or conic sectors. Given LMI regions $D_1, D_2, \ldots, D_N$, the intersection $D = D_1 \cap D_2 \cap \cdots \cap D_N$ has characteristic function $f_D(\lambda) = diag\{f_{D_1}(\lambda), f_{D_2}(\lambda), \ldots, f_{D_N}(\lambda)\}$ and is still a LMI region [6], [7]. Therefore, the LMIs of $f_{D_1}(\lambda), f_{D_2}(\lambda), \ldots, f_{D_N}(\lambda)$, which can be derived from (9), must be feasible so that the corresponding LMI for the intersection of the regions $D_1, D_2, \ldots, D_N$ is solvable. This will be illustrated by an example in the next section. []

IV. An Illustrative Example

Consider linear uncertain discrete-time system described by (1) with

$$A = \begin{bmatrix} -0.3 & 0.3 & -0.6 \\ 0 & 0 & 0.1 \\ 0.2 & 0.8 & 0.4 \end{bmatrix}$$

$$\Delta A = H_1\Gamma N = \begin{bmatrix} 0.1 \\ 0 \\ 0.2 \end{bmatrix} \Gamma \begin{bmatrix} 0.1 & 0 & 0.3 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} -1 \\ 0.2 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -0.6 & 2 \end{bmatrix}$$

$$\Delta C = H_2\Gamma N = 0.1\Gamma \begin{bmatrix} 0.1 & 0 & 0.3 \end{bmatrix}$$

$$D_1 = 0.2$$

$$D_2 = 0.3$$

$$L_\infty = \begin{bmatrix} 1 & 0 & 0.5 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$$

where Γ is a perturbation matrix satisfying (3). We wish to design a filter such that the upper bound $\mathrm{Tr}(Q)$ of $\|T_2(z)\|_2^2$ is minimized subject to $\|T_\infty(z)\|_\infty < \gamma = 15.6$, and the poles are restricted in the intersection of the disk centered at $(-\alpha, 0)$ with radius r, the vertical strip $\mathrm{Re}(\lambda) < -\alpha_1$ and the vertical strip $\mathrm{Re}(\lambda) > -\alpha_2$, where $\alpha = 0$, $r = 0.8$, $\alpha_1 = -0.5$, $\alpha_2 = 0.5$. The pole constraints for the disk centered at $(-\alpha, 0)$ with radius r can be expressed in terms of LMI as (18) with $L = \left[\begin{smallmatrix} -r & \alpha \\ \alpha & -r \end{smallmatrix}\right]$, $M = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $M_1 = [1 \;\; 0]$ and $M_2 = [0 \;\; 1]$, which will be denoted as LMI1. For the vertical strip $\mathrm{Re}(\lambda) < -\alpha_1$, it can be expressed as (18) with $L = 2\alpha_1$, $M = 1$, $M_1 = 1$ and $M_2 = 1$, which will be denoted as LMI2. For the vertical strip $\mathrm{Re}(\lambda) > -\alpha_2$, it can also be expressed as (18) with $L = -2\alpha_2$, $M = -1$, $M_1 = -1$ and $M_2 = 1$, which will be denoted as LMI3.

$$\begin{bmatrix} L \otimes X + M \otimes (XA_e) + M^T \otimes (A_eX)^T & M_1^T \otimes (XH_e) & \varepsilon_1 M_2^T \otimes E_e^T \\ M_1 \otimes (H_e^T X) & -\varepsilon_1 I & 0 \\ \varepsilon_1 M_2 \otimes E_e & 0 & -\varepsilon_1 I \end{bmatrix} < 0 \tag{24}$$

$$\begin{bmatrix} -X & 0 & XA_e & XB_{e1} & XH_e & 0 \\ 0 & -\gamma I & C_\infty & 0 & 0 & 0 \\ A_e^T X & C_\infty^T & -X & 0 & 0 & \varepsilon_2 E_e^T \\ B_{e1}^T X & 0 & 0 & -\gamma I & 0 & 0 \\ H_e^T X & 0 & 0 & 0 & -\varepsilon_2 I & 0 \\ 0 & 0 & \varepsilon_2 E_e & 0 & 0 & -\varepsilon_2 I \end{bmatrix} < 0 \tag{25}$$

$$\begin{bmatrix} -X & XA_e & XB_{e2} & XH_e & 0 \\ A_e^T X & -X & 0 & 0 & \varepsilon_3 E_e^T \\ B_{e2}^T X & 0 & -I & 0 & 0 \\ H_e^T X & 0 & 0 & -\varepsilon_3 I & 0 \\ 0 & \varepsilon_3 E_e & 0 & 0 & -\varepsilon_3 I \end{bmatrix} < 0 \tag{26}$$

$$\begin{bmatrix} X & C_2^T \\ C_2 & Q \end{bmatrix} > 0 \tag{27}$$

According to (31) and Remark 3, the desired filter design problem can transformed into the convex problem

$$\min_{R>0,S>0,Q>0,Q_1,Q_2,Q_3,Q_4,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5} \mathrm{Tr}(Q) \quad \text{subject to (19)–(20) and LMI1–LMI3.} \tag{34}$$

By using the Matlab LMI toolbox, the optimal solution to the convex problem (34) is given by $\mathrm{Tr}(Q) = 2.0312$ with the filter parameters

$$\overline{F} := Q_1(S-R)^{-1} = \begin{bmatrix} 0.0743 & 0.2486 & 0.0435 \\ -0.4962 & 0.3117 & 0.0925 \\ 0.0369 & 0.6424 & -0.1164 \end{bmatrix}$$
$$\overline{G} := Q_2 = \begin{bmatrix} 0.1928 \\ -0.2653 \\ -0.0626 \end{bmatrix}$$
$$\overline{L}_\infty := Q_3(S-R)^{-1} = [\,-2.0884 \quad 1.0063 \quad 0.6934\,]$$
$$\overline{L}_2 := Q_4(S-R)^{-1} = [\,-0.5616 \quad 0.2581 \quad -0.7350\,].$$

If $\gamma = 9.8$, the optimal solution is given by $\mathrm{Tr}(Q) = 3.0623$. It is evident from this example that the proposed LMIs allow much flexibility in making compromise between the H_2 performance and the H_∞ performance.

Furthermore, if no pole constraint is applied to design the filter, the filter poles are changed from $0.2486 + j0.3032$, $0.2486 - j0.3032$, -0.2276 to 0.6549, 0.1333, -0.1515. We can see that one of the filter poles goes outside the region. The filter will not guarantee the expected transient performance. This illustrates the necessity of poles constraints.

V. Conclusion

In this paper, we have considered the robust mixed H_2/H_∞ filtering problem with regional pole assignment for uncertain discrete-time systems. Necessary and sufficient conditions for the solvability of the problem have been given. The design LMI approach has been proposed to overcome the computational difficulty for the mixed H_2/H_∞ filtering problem. The approach presented in this paper can be extended to design robust filters for more complex systems such as sampled-data systems and stochastic parameter systems.

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