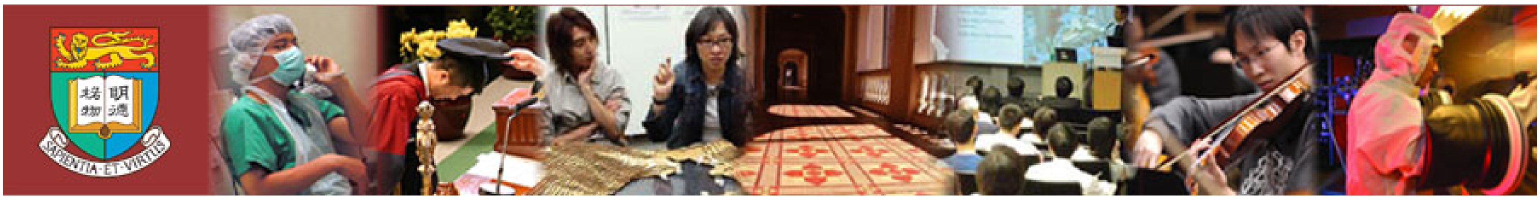

A numerical study of the bearing capacity factor N_γ

D.Y. Zhu, C.F. Lee, and H.D. Jiang

Abstract: Values of the bearing capacity factor N_γ are numerically computed using the method of triangular slices. Three assumptions of the value of ψ, the base angle of the active wedge, are analyzed, corresponding to the following three cases: (1) $\psi = \phi$, the internal friction angle; (2) $\psi = 45° + \phi/2$; and (3) ψ has a value such that N_γ is a minimum. The location of the critical failure surface is presented and the numerical solutions to N_γ for the three cases are approximated by simple equations. The influence of the base angle on the value of N_γ is investigated. Comparisons of the present solutions are made with those commonly used in foundation engineering practice.

Key words: shallow foundation, bearing capacity, bearing capacity factor, limit equilibrium.

Résumé : Les valeurs du coefficient de capacité portante N_γ sont calculées numériquement au moyen de la méthode des tranches triangulaires. Trois hypothèses de la valeur de ψ, l'angle de la base du coin actif, sont analysées, correspondant respectivement aux cas où ψ est égal à l'angle de frottement interne ϕ, 45° + ϕ/2, et le cas de la valeur minimum de N_γ. On présente la localisation de la surface de rupture critique, et les solutions numériques de N_γ sont calculées approximativement par des équations simples pour les trois cas . On étudie l'influence de l'angle de la base sur la valeur de N_γ. On compare les présentes solutions avec celles utilisées communément pour les fondations dans la pratique d'ingénieur.

Mots clés : fondation superficielle, capacité portante, coefficient de capacité portante, équilibre limite.

[Traduit par la Rédaction]

Introduction

For foundation design, the bearing capacity of a shallow strip footing is commonly determined by using the following Terzaghi equation:

$$[1] \qquad q_u = qN_q + cN_c + \frac{1}{2}B\gamma N_\gamma$$

where q_u is the ultimate bearing capacity; c is the cohesion of the soil underneath the footing; q is the surcharge above the base level of the footing; γ is the unit weight of the soil; B is the width of the footing; N_c is the bearing capacity factor related to cohesion c; N_q is the bearing capacity factor related to surcharge q; and N_γ is the bearing capacity factor related to γ (Terzaghi 1943).

Analytical expressions derived from the theory of plasticity give N_c and N_q but not N_γ. Several methods have been employed to compute N_γ, including the limit equilibrium method (Terzaghi 1943; Meyerhof 1951), the method of characteristics (Sokolovski 1960), the limit analysis method (Chen and Davidson 1973), and the finite element method (Griffiths 1982). The values of N_γ given by these methods often vary considerably, and different formulae based on the limit equilibrium method are employed for computing N_γ in the design codes of various countries (Sieffert and Bay-Gress 2000). Discrepancies occur in the results mainly due to the different assumptions made regarding the geometry of the active wedge immediately under the footing and the different procedures used in the computation of passive earth pressure acting on the edge of the active wedge.

The geometry of the active wedge is defined by the base angle ψ (Fig. 1), i.e., the inclination of the wedge with the horizontal, which has been assumed to be equal to ϕ (Terzaghi 1943), the friction angle; 45°+ ϕ/2 (Meyerhof 1963; Vesic 1973); or a selected value such that N_γ is a minimum (Hansen 1970; Vogel and Baracos 1973). The validity and merits of these assumptions have yet to be duly verified. In this paper, N_γ values based on these three assumptions are computed and compared. In previous studies, the passive earth pressure on the edge of the active wedge was obtained using the limit equilibrium method with the assumption of a logarithmic spiral slip line bounding the general failure region. In this paper, the passive pressure is calculated using the method of triangular slices, which is within the framework of the limit equilibrium approach (Zhu and Qian 2000). With this method, the passive failure region is divided into a number of triangular slices and the critical base inclination of each slice is determined based on the principle of optimality. This results in minimal passive earth pressure without any restriction on the shape of the critical failure surface. The passive earth

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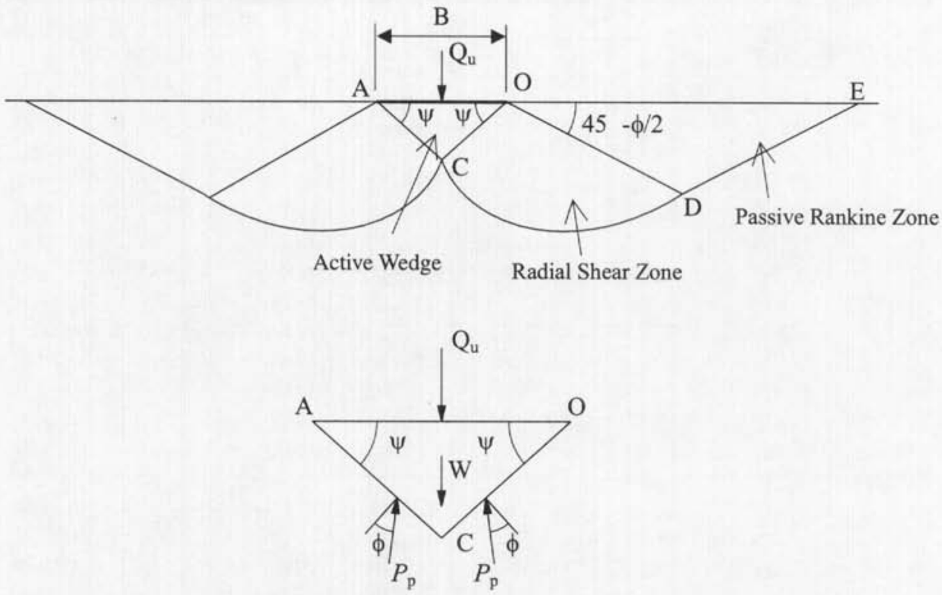
Fig. 1. General shear failure mechanism of soil under the footing.



pressure coefficients thus calculated were found to be in good agreement with those from analytical solutions based on the theory of plasticity.

Procedure

A strip footing with a width B subject to a vertical and central load Q_u on the surface of cohesionless soil is considered as shown in Fig. 1. It is assumed that under the ultimate load, a general symmetrical shear failure surface consisting of three zones is formed: active wedge (OAC in Fig. 1), radial shear zone (OCD), and passive Rankine zone (ODE). The active wedge is an isosceles triangle with a base angle of ψ. The weight W of the active wedge is

$$W = \frac{1}{2}\gamma \overline{OC}^2 \sin 2\psi \quad [2]$$

The passive earth force P_p acting on the edge OC can be expressed as

$$P_p = \frac{1}{2}\gamma k_p \overline{OC}^2 \quad [3]$$

where k_p is the passive earth pressure coefficient, which can be directly determined using the method of triangular slices (Zhu and Qian 2000).

From vertical equilibrium of the active wedge, the ultimate load is given by

$$Q_u = 2P_p \cos(\psi - \phi) - W \quad [4]$$

where ϕ is the internal friction angle.

Substituting eqs. [2] and [3] into eq. [4] and considering the relationship $\overline{OC} = B/(2 \cos \psi)$, we have

$$Q_u = \frac{\gamma B^2}{8\cos^2\psi}[2k_p \cos(\psi - \phi) - \sin 2\psi] \quad [5]$$

From eq. [1], Q_u can also be expressed as

$$Q_u = \frac{1}{2}\gamma N_\gamma B^2 \quad [6]$$

Comparing eqs. [5] and [6], we have

$$N_\gamma = \frac{1}{4\cos^2\psi}[2k_p \cos(\psi - \phi) - \sin 2\psi] \quad [7]$$

Three assumptions of the value of ψ are considered in this analysis corresponding to the following three cases: (1) $\psi = \phi$, (2) $\psi = 45° + \phi/2$, and (3) ψ has a value that makes N_γ a minimum. The values of N_γ in these three cases are denoted $N_\gamma^{(C1)}$, $N_\gamma^{(C2)}$, and $N_\gamma^{(C3)}$, respectively.

Numerical values of N_γ and comparisons

The numerical values of N_γ for $\phi = 1$–$50°$ are presented in Table 1. The variation of N_γ with ϕ is shown in Fig. 2. The numerical values of N_γ can be approximated by the following equations:

$$N_\gamma = (2N_q + 1)(\tan\phi)^{1.35} \quad \text{for case 1} \quad [8]$$

$$N_\gamma = (2N_q + 1)\tan(1.07\phi) \quad \text{for case 2} \quad [9]$$

$$N_\gamma = (2N_q + 1)(\tan\phi)^{1.45} \quad \text{for case 3} \quad [10]$$

where

$$N_q = \tan^2\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \cdot e^{\pi \tan\phi}$$

Table 1 shows that the simple equations [8]–[10] agree well with the numerical results, with a maximum difference not exceeding 10%, which can therefore be used in practice.

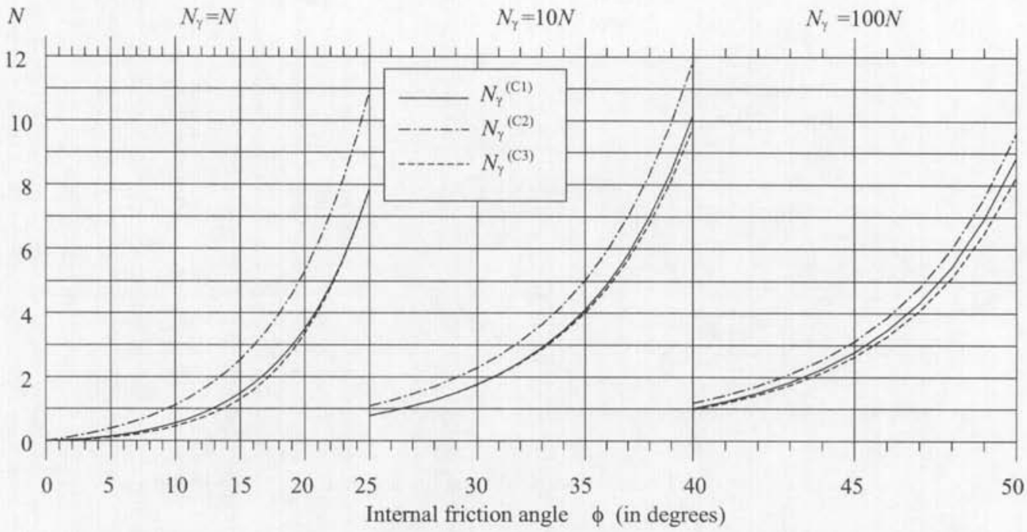
Comparisons of the present solution with those by other investigators are presented in Fig. 3. For case 1, Kumbhojkar (1993) has obtained numerical values of N_γ using the logarithmic spiral method for calculating the passive earth pressure acting on the edge between the active wedge and the radial shear zone. Kumbhojkar's solutions are plotted in Fig. 3,

Table 1. Numerical values of N_γ.

ϕ (°)	Numerical N_γ				Approximate N_γ / numerical N_γ		
	Case 1	Case 2	Case 3	N_q	Case 1	Case 2	Case 3
1	0.012	0.058	0.011	1.094	1.094	1.027	0.812
2	0.031	0.124	0.025	1.197	1.182	1.021	1.028
3	0.058	0.200	0.045	1.309	1.169	1.015	1.121
4	0.094	0.286	0.071	1.432	1.131	1.011	1.146
5	0.139	0.385	0.107	1.568	1.108	1.007	1.131
6	0.195	0.497	0.152	1.726	1.087	1.004	1.111
7	0.262	0.624	0.208	1.879	1.069	1.002	1.095
8	0.343	0.769	0.275	2.058	1.056	1.001	1.081
9	0.440	0.935	0.356	2.255	1.041	1.000	1.068
10	0.552	1.123	0.453	2.471	1.034	1.000	1.059
11	0.684	1.337	0.568	2.710	1.028	1.000	1.051
12	0.842	1.582	0.707	2.973	1.020	1.001	1.041
13	1.023	1.860	0.875	3.264	1.017	1.003	1.027
14	1.238	2.178	1.069	3.586	1.012	1.004	1.020
15	1.488	2.540	1.309	3.941	1.009	1.006	1.006
16	1.775	2.955	1.584	4.335	1.009	1.008	0.998
17	2.114	3.429	1.921	4.772	1.007	1.010	0.984
18	2.509	3.973	2.323	5.258	1.006	1.013	0.971
19	2.964	4.597	2.793	5.798	1.008	1.015	0.961
20	3.499	5.313	3.367	6.399	1.008	1.018	0.947
21	4.123	6.138	4.033	7.071	1.008	1.020	0.937
22	4.852	7.089	4.796	7.821	1.009	1.023	0.932
23	5.701	8.187	5.673	8.661	1.011	1.025	0.932
24	6.698	9.457	6.685	9.603	1.012	1.027	0.935
25	7.867	10.930	7.864	10.662	1.013	1.029	0.939
26	9.240	12.643	9.240	11.854	1.014	1.031	0.944
27	10.856	14.638	10.849	13.199	1.016	1.033	0.950
28	12.761	16.968	12.740	14.720	1.017	1.034	0.956
29	15.013	19.697	14.959	16.443	1.018	1.035	0.963
30	17.682	22.901	17.579	18.401	1.018	1.035	0.970
31	20.851	26.676	20.672	20.631	1.019	1.036	0.977
32	24.625	31.138	24.346	23.177	1.019	1.035	0.984
33	29.136	36.429	28.719	26.092	1.019	1.034	0.990
34	34.542	42.726	33.941	29.440	1.019	1.032	0.997
35	41.048	50.247	40.200	33.296	1.018	1.030	1.003
36	48.902	59.269	47.735	37.752	1.016	1.027	1.008
37	58.426	70.137	56.855	42.920	1.014	1.024	1.013
38	70.025	83.287	67.899	48.933	1.012	1.020	1.018
39	84.210	99.274	81.366	55.957	1.009	1.014	1.022
40	101.653	118.786	97.926	64.195	1.004	1.009	1.025
41	123.203	142.746	118.246	73.896	1.000	1.002	1.027
42	149.981	172.339	143.431	85.373	0.994	0.994	1.028
43	183.451	209.096	174.829	99.013	0.987	0.986	1.029
44	225.420	255.049	214.058	115.307	0.980	0.977	1.029
45	278.540	312.880	263.746	134.872	0.972	0.966	1.027
46	346.205	386.183	326.590	158.500	0.963	0.955	1.024
47	433.047	479.815	407.403	187.204	0.953	0.942	1.020
48	545.391	600.384	511.186	222.297	0.941	0.928	1.015
49	691.958	757.000	646.853	265.494	0.929	0.914	1.008
50	884.930	962.325	824.313	319.053	0.915	0.898	1.000

which shows that his numerical values are, to a certain degree, higher than those from the present solution, especially when the internal friction angle of the soil is large. This discrepancy possibly results from the different approaches used in searching the critical slip surface. Terzaghi (1943) published well-known curves of bearing capacity factors versus ϕ. It is difficult to precisely reproduce his solution. Hence, only three numerical values are plotted in Fig. 3 for comparison. It is shown that Terzaghi's solution is in agreement with that of Kumbhojkar when ϕ is less than 45°.

Fig. 2. Variation of N_γ with ϕ.



For case 2, two widely used formulae were provided by Meyerhof (1963)

[11] $N_\gamma = (N_q - 1)\tan(1.4\phi)$

and Vesic (1973)

[12] $N_\gamma = 2(N_q + 1)\tan\phi$

Figure 3 shows that the present solution is in good agreement with that of Vesic (1973) when ϕ is less than 35°, whereas the solution of Meyerhof (1963) generally gives lower values of N_γ in comparison to the other two solutions.

For case 3, Hansen (1970) gave the following formula for computing N_γ:

[13] $N_\gamma = 1.5(N_q - 1)\tan\phi$

Vogel and Baracos (1973) also gave numerical results for several values of ϕ, which are plotted in Fig. 3 along with those based on eq. [13]. The present solution agrees well with that of Vogel and Baracos. The results provided by eq. [13], based on the solution of Hansen (1970), are lower in value.

Figure 2 shows that the difference in corresponding values of N_γ between case 1 and case 3 is relatively small. Moreover, the influence of the base angle ψ on the value of N_γ computed is not large, as illustrated in Fig. 4. For example, for $\phi = 40°$, when ψ increases from 34° to 61°, N_γ will only decrease by less than 10%.

It must be pointed out that the present solutions for the bearing capacity factor N_γ are in the context of limit equilibrium. Strictly speaking, they are neither lower nor upper bounds to the exact solutions. Since the passive earth pressure coefficient required for calculating N_γ is obtained to a high degree of accuracy by the method of triangular slices, the calculated values of N_γ can be regarded as nearly accurate for the adopted assumption regarding the base angle of the active wedge. Recently, several investigators have developed the least upper-bound solutions to N_γ by means of limit analysis using an optimization technique (Michalowski 1997; Soubra 1999; Zhu 1999). Their results are nearly identical for both the symmetrical and the one-sided failure mechanisms. Comparison between the values of N_γ obtained with the present method and those from the least upper-bound solutions are tabulated in Table 2. It can be seen in Table 2 that the least upper-bound solutions to N_γ are in close agreement with those from the present method for case 2. The values of N_γ in case 3 are the lowest of all three cases and may therefore approximate the lower-bound solutions, since they are minimized with respect to the base angle, and the associated passive earth pressure coefficient is nearly exactly obtained by using the method of triangular slices.

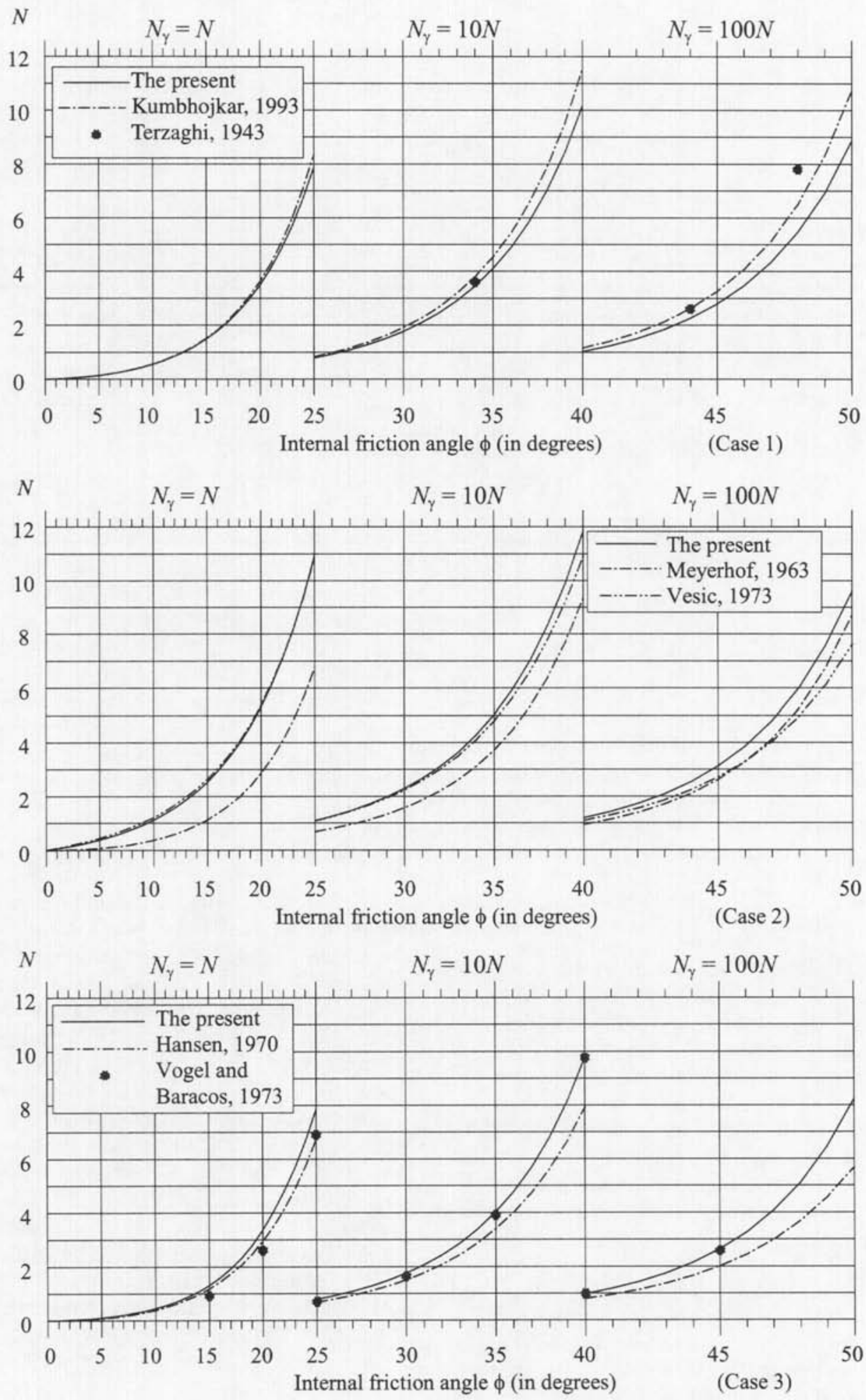
Location of critical failure surfaces

The location of the critical failure surface in soil under ultimate loading is of interest in design. Figure 5 shows that the extent of the failure region in case 2 is larger than those for cases 1 and 3. It is interesting to note that for case 3, when ϕ is equal to or less than 20°, the critical value of ψ is zero. This was first found by Vogel and Baracos (1973). Comparison of the locations of the critical failure surface is shown in Fig. 6. The location of the critical failure surface given by Kumbhojkar (1993) deviates substantially from the present solution, and this results in a large discrepancy in the values of N_γ calculated. Kumbhojkar restricted the center of the logarithmic spiral to the line that makes an angle of 45° – ϕ/2 with the horizontal. Vogel and Baracos (1973) imposed no such restriction on the position of the center of the logarithmic spiral and gave locations of the critical failure surface which are in good agreement with those obtained with the present method of triangular slices.

Conclusions

The method of triangular slices is an effective method for computing the values of the bearing capacity factor N_γ. Three cases of base angle assumption are taken into account in the present analysis. The values of N_γ computed based on the assumption of 45° + ϕ/2 are in close agreement with the least upper-bound solutions by limit analysis, whereas those values which are minimized with respect to the base angle ap-

Fig. 3. Comparison of N_γ values computed in the present study with those from other investigators.



proximately represent the lower bound. The values of N_γ computed based on the Terzaghi assumption of $\psi = \phi$ are slightly larger than the minimum. Simple equations for N_γ are suggested that can be used to approximate the numerical values quite accurately, allowing them to be applied in the design of shallow foundations.

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Fig. 4. Effect of ψ on N_γ.

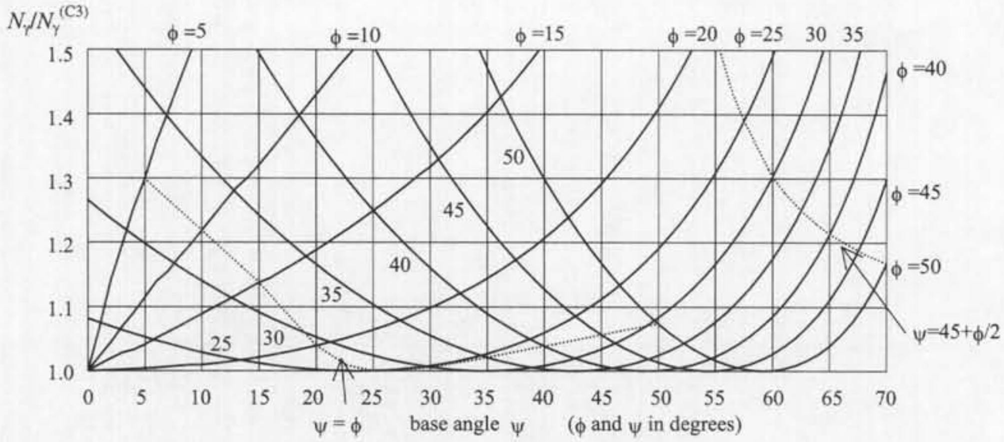


Fig. 5. Critical failure surfaces under ultimate loading.

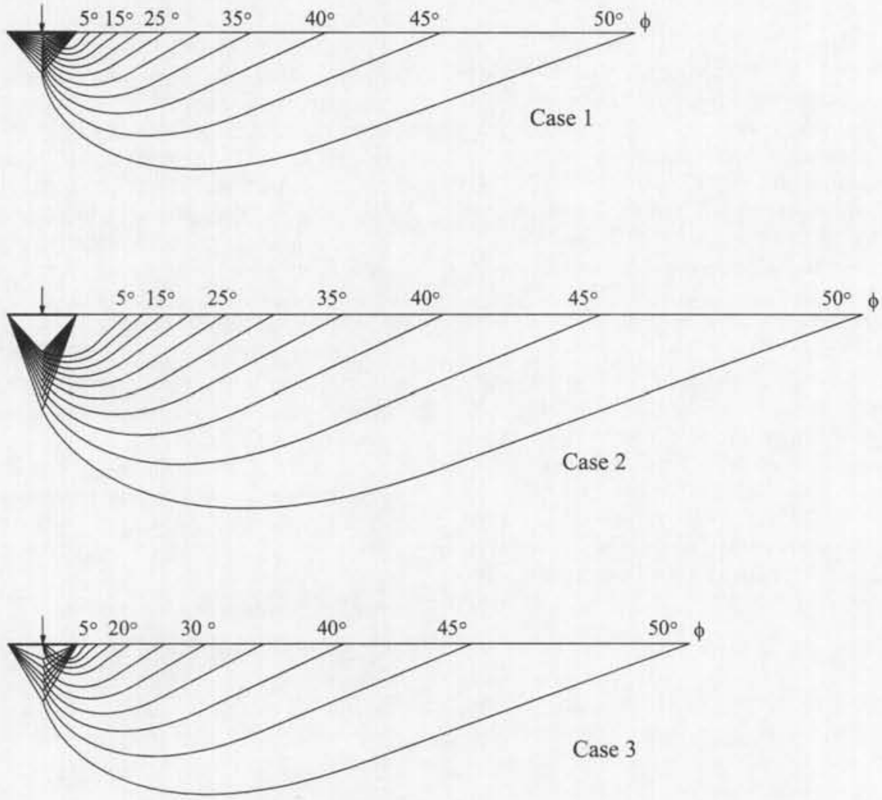
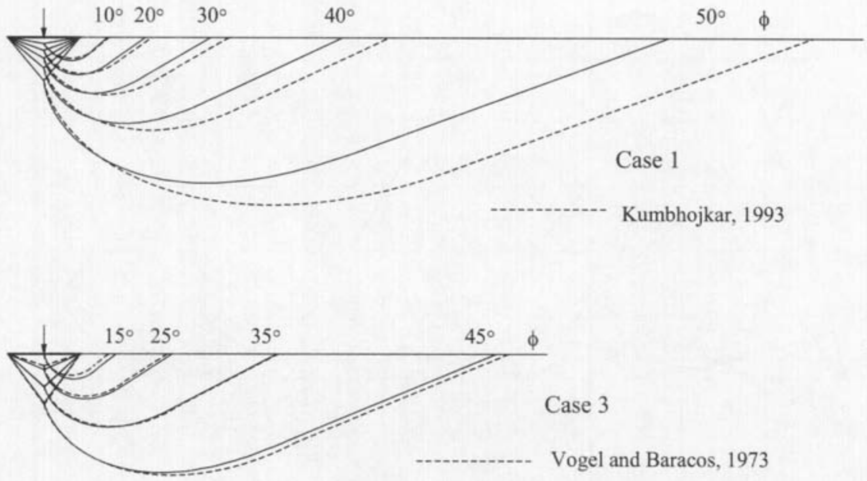


Table 2. Comparison of N_γ values from the present method with those from limit analysis.

φ (°)	Present solutions Case 1	Case 2	Case 3	Least upper-bound solutions Symmetrical mechanism	One-sided mechanism
5	0.139	0.385	0.107	0.181	0.247
10	0.552	1.123	0.453	0.706	0.845
15	1.488	2.540	1.309	1.937	2.095
20	3.499	5.313	3.367	4.466	4.659
25	7.867	10.930	7.864	9.760	10.031
30	17.682	22.901	17.579	21.384	21.805
35	41.048	50.247	40.200	48.654	49.381
40	101.653	118.786	97.926	118.750	120.150
45	278.540	312.880	263.746	322.622	325.766
50	884.930	962.325	824.313	1025.064	1033.480

Fig. 6. Comparison of location of critical failure surfaces.



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