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Dynamical evolution of volume fractions in multipressure multiphase flow models

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Compared to single-pressure models, multipressure multiphase flow models require additional closure relations to determine the individual pressures of the different phases. These relations are often taken to be evolution equations for the volume fractions. We present a rigorous theoretical framework for constructing such equations for compressible multiphase mixtures in terms of submodels for the relative volumetric expansion rates ΔE_i of the phases. These quantities are essentially the rates at which the phases dynamically expand or contract in response to pressure differences, and represent the general tendency of the volume fractions to relax toward values that produce local pressure equilibrium. We present a simple provisional model of this type in which ΔE_i is proportional to pressure differences divided by the time required for sound waves to traverse an appropriate characteristic length. It is shown that the resulting approach to pressure equilibrium is monotonic rather than oscillatory, and occurs instantaneously in the incompressible limit.

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I. INTRODUCTION

This paper is concerned with multifluid models for compressible multiphase flow, i.e., models in which each compressible fluid phase or immiscible material i $(i=1,2,\ldots,N)$ possesses its own continuity, momentum, energy, and state equations. These equations involve the primary dependent variables α_i, ρ_i, $\mathbf{u}_i$, p_i, and e_i, which, respectively, denote the volume fraction, density, velocity, pressure, and specific internal energy of phase i. The volume fractions α_i satisfy the obvious constraint

$$\sum_i \alpha_i = 1, \tag{1}$$

so only $N-1$ of them can be varied independently, and this will be understood in what follows. Such models therefore contain $5N-1$ primary dependent variables.

Multiphase flow models of this type may be broadly classified as single-pressure models, in which all the p_i are equal with a common value p, or multipressure models, in which the p_i of the different phases are in general different. Provided that all the required source and fluctuation terms representing the various interphase interactions and statistical correlations have already been modeled, a single-pressure model constitutes a closed system of $4N$ equations in the $4N$ variables α_i, ρ_i, $\mathbf{u}_i$, e_i, and p, where we reemphasize that the α_i should only be considered and counted as $N-1$ variables. However, a multipressure model constitutes a system of $4N$ equations in $5N-1$ unknowns, and therefore requires an additional $N-1$ closure relations to become a closed system. Until those closure relations have been supplied, the system is indeterminate.

In the mathematical sense of counting equations and unknowns, this indeterminacy is a property of the equation system as a whole. In a physical sense, however, it is natural and helpful to think of the indeterminacy as being associated with the continuity equations, since the N energy equations can be regarded as determining the e_i, the N state equations as determining the p_i, and the N vector momentum equations as determining the N vector velocities $\mathbf{u}_i$. From this point of view, the continuity equations may be regarded as an indeterminate system of N equations in the $2N-1$ unknowns α_i and ρ_i.

For simplicity, we restrict attention to the case in which there is no mass exchange between phases, in which the multiphase continuity equations are given by

$$\frac{\partial(\alpha_i\rho_i)}{\partial t} + \boldsymbol{\nabla}\cdot(\alpha_i\rho_i\mathbf{u}_i) = 0, \tag{2}$$

where ρ_i is the mass density of pure phase i, i.e., the mass of phase i per unit volume occupied by phase i, not per unit total volume. The aforementioned indeterminacy is reflected in the fact that α_i and ρ_i appear in Eqs. (2) only in the form of the product $\tilde{\rho}_i \equiv \alpha_i\rho_i$, but not separately. (Of course, $\tilde{\rho}_i$ is simply the partial mass density of phase i, i.e., the mass of phase i per unit total volume.) This observation suggests two obvious ways of removing the indeterminacy and closing the system: (a) devise a submodel that determines α_i, from which ρ_i can then be obtained as $\tilde{\rho}_i/\alpha_i$, or (b) devise a submodel that determines ρ_i, from which α_i can then be obtained as $\tilde{\rho}_i/\rho_i$. Submodels of type (a) generally take the form of a time evolution or transport equation for α_i. However, submodels of type (b) can also be cast into the same form simply by combining them with Eq. (2), and this is the approach taken here.

Since the α_i obey the constraint of Eq. (1), a submodel that determines them will provide precisely the additional $N-1$ relations required to close the system. Note that a

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single-pressure model can be regarded as a special case of a multipressure model in which the $N-1$ additional closure relations are given by $p_1=p_2=\cdots=p_N$.

Historically, the earliest multipressure multiphase flow models [1–5] were largely motivated by a desire to regularize the behavior of single-pressure models, which often possess complex characteristics and hence constitute an ill-posed initial value problem [6]. At first, it was feared that this behavior would make it difficult or impossible to solve the equations numerically. This fear was gradually assuaged by the realization that various physical and/or numerical regularization mechanisms (e.g., viscosity, surface tension, numerical diffusion) are nearly always present in any case. At least in principle, however, multipressure models also provide a more detailed and accurate description of the physics, and they have remained of more recent interest on that basis as well [7–14].

Many previous multipressure models have been restricted to the important special case of two-phase flow ($N=2$) [1–3,8–11]. In that case, there is only a single independent volume fraction $\alpha_1=1-\alpha_2\equiv\alpha$, the determination of which then requires only a single additional closure relation. The latter is usually taken to be an evolution or transport equation for α, the advective terms in which typically involve a volume fraction propagation velocity $\mathbf{u}_\alpha$. Different models have postulated different transport equations for α and different expressions for $\mathbf{u}_\alpha$. In some models, the transport of α is in fact purely advective [8–10]. When the two phases are incompressible, however, an exact expression for the purely advective $\mathbf{u}_\alpha$ can be derived directly from the continuity equations [10,15]. At least in that special case, $\mathbf{u}_\alpha$ is therefore already known as a function of the primary dependent variables and hence need not, and arguably should not, be modeled. Moreover, the extension of such models to more than two phases presents further difficulties: the proper generalization of $\mathbf{u}_\alpha$ to $N>2$ is not always obvious, and in any case it seems physically unrealistic to assume that the $N-1$ independent α_i are all advected with the same velocity.

As is well known, pressure uniformity or equality is an essential feature of both mechanical and thermodynamical equilibrium. The basic dynamical mechanism by which this equilibration occurs is simply that pressure differences imply unbalanced forces, which of course produce motion. In the present context, these pressure forces cause regions of higher pressure to expand into and compress regions of lower pressure. This process then reduces the higher pressures while increasing the lower ones, thereby driving them both toward equality at some intermediate value. It seems obvious that a properly formulated closure model for the volume fractions in multipressure multiphase mixtures should capture this essential physical tendency for unequal pressures to dynamically equilibrate (cf. Shashkov [16]). Nevertheless, several models have been proposed that do not appear to possess such a tendency [7–10].

Of course, the aforementioned pressure equilibration will not occur instantaneously (except perhaps in the incompressible limit—see Sec. V), and will in general be opposed by various other competing processes that tend to drive the pressures apart, such as Bernoulli effects or selective heating of particular phases. The complete equilibration of pressures should therefore not be expected to occur until the multiphase mixture finally reaches a steady state. In unsteady flows, the p_i will in general still differ, but their dynamical *tendency* to equilibrate always remains operative, and prevents them from differing as much as they otherwise would.

Our purpose here is twofold. First, we present a general theoretical framework for constructing a family of model equations for the time evolution of the volume fractions α_i in compressible multiphase mixtures. These model equations may then be used as closure relations in multipressure multiphase flow models. This framework is rigorously derived from the multiphase continuity equations (2). These equations are cast into a form in which the general tendency of the volume fractions to relax toward values that produce pressure equilibrium is represented by the relative rates ΔE_i at which the phase volumes dynamically expand or contract in response to pressure differences. The different members of this family of models are obtained by specifying different submodels for the quantities ΔE_i.

Second, we present a simple provisional model of this type in which ΔE_i is taken to be directly proportional to pressure differences and inversely proportional to a relaxation time which is identified with the time required for sound waves to traverse an appropriate characteristic length L. It seems natural to identify L with some mean linear dimension of the phase fragments. It is then shown that this model predicts a monotonic approach to pressure equilibration, and that the latter occurs instantaneously in the incompressible limit as the sound speed tends to infinity. This instantaneous equilibration reflects the fact that models of this general type are inherently compressible in nature, and are not intended to describe the much smaller interphase pressure variations that occur in incompressible flows, which are ordinarily negligible compared to the mean pressure of the mixture [17,18].

II. VOLUME FRACTION EVOLUTION

We begin by rewriting Eq. (2) in the equivalent form

$$\frac{\partial\alpha_i}{\partial t}+\boldsymbol{\nabla}\cdot(\alpha_i\mathbf{u}_i)=-\alpha_i\frac{D_i\ln\rho_i}{Dt}, \tag{3}$$

where $D_i/Dt\equiv\partial/\partial t+\mathbf{u}_i\cdot\boldsymbol{\nabla}$ is the convective derivative following phase i. Now if $D_i\ln\rho_i/Dt$ were known, Eq. (3) could be solved for the α_i, and the full equation system would then be closed and determinate as discussed above. Let us therefore consider the significance of the quantity $D_i\ln\rho_i/Dt$, in the hope that this may suggest a suitable way of modeling it. Let $v_i\equiv1/\rho_i$ denote the specific volume of pure phase i. Then

$$\frac{D_i\ln\rho_i}{Dt}=-\frac{D_i\ln v_i}{Dt}=-E_i, \tag{4}$$

where $E_i\equiv(1/v_i)D_iv_i/Dt$ is clearly the intrinsic expansion rate of an infinitesimal Lagrangian fluid element of pure phase i. Now consider a small control volume $V=\Sigma_iV_i$ of the multiphase mixture made up of Lagrangian fluid elements of the individual phases, in which $V_i=\alpha_iV$ is the volume occu-

pied by phase i within V. The overall expansion rate of the mixture is

$$E = \frac{1}{V}\frac{dV}{dt} = \frac{1}{V}\sum_i \frac{dV_i}{dt}. \tag{5}$$

But since the fluid elements are Lagrangian, $(1/V_i)dV_i/dt = E_i$, so that Eq. (5) becomes

$$E = \frac{1}{V}\frac{dV}{dt} = \frac{1}{V}\sum_i V_i E_i = \sum_i \alpha_i E_i. \tag{6}$$

Combining Eqs. (3) and (4), we obtain

$$\frac{\partial \alpha_i}{\partial t} + \nabla \cdot (\alpha_i \mathbf{u}_i) = \alpha_i E_i, \tag{7}$$

and summing this equation over i gives

$$\nabla \cdot \mathbf{u}_v = E, \tag{8}$$

where $\mathbf{u}_v \equiv \Sigma_i \alpha_i \mathbf{u}_i$ is the volume-weighted velocity of the mixture. Thus the overall expansion or contraction rate of the mixture is simply the divergence of its volume-weighted velocity [19]. We may now rewrite Eq. (7) as

$$\frac{\partial \alpha_i}{\partial t} + \nabla \cdot (\alpha_i \mathbf{u}_i) = \alpha_i (E_i - E + E) = \alpha_i \nabla \cdot \mathbf{u}_v + \alpha_i (E_i - E) \tag{9}$$

or

$$\frac{\partial \alpha_i}{\partial t} + \nabla \cdot (\alpha_i \mathbf{u}_i) = \alpha_i \nabla \cdot \mathbf{u}_v + \alpha_i \Delta E_i, \tag{10}$$

where $\Delta E_i = E_i - E$ is the rate of expansion of phase i relative to the overall rate of expansion of the mixture as a whole. We emphasize that Eq. (10) is rigorous, since no approximations have been made in its derivation.

Let us now proceed to consider the significance of the quantities ΔE_i. By virtue of Eq. (6), they satisfy the constraint

$$\sum_i \alpha_i \Delta E_i = 0 \tag{11}$$

so that only $N-1$ of them are independent. Determining the ΔE_i would therefore provide precisely the $N-1$ additional relations required to obtain a closed system of equations. Since $\alpha_i > 0$, Eq. (11) implies that the ΔE_i cannot all have the same sign. Physically, this simply means that phases with $\Delta E_i > 0$ are expanding more rapidly than the mixture as a whole, at the expense of phases with $\Delta E_i < 0$, which are expanding more slowly or compressing. What then determines whether a given phase expands or compresses relative to E? It is physically clear that phases with higher pressures will tend to expand by compressing phases with lower pressures, so that ΔE_i will be positive for the former and negative for the latter. The resulting expansions and compressions will then reduce the pressures of the expanding phases and increase those of the compressing ones, thereby driving the mixture closer to pressure equilibrium.

Of course, these qualitative considerations are insufficient to determine the quantitative positive and negative values of the ΔE_i, but they provide valuable insight upon which quantitative models may subsequently be based. The important and essential point is that Eq. (10) provides a rigorous theoretical framework within which dynamical volume fraction evolution models may be constructed in terms of quantitative submodels for ΔE_i. These submodels should of course exhibit the essential qualitative behavior discussed above, i.e., they should attempt to quantitatively describe the relative rates at which the different phases expand or compress in response to the pressure differences between them.

The task of obtaining a closed model for compressible multipressure multiphase flow has now been reduced to that of devising a suitable submodel for the relative expansion rates ΔE_i. In the next section, we present a provisional submodel of this type which exhibits the qualitative behavior discussed above.

III. A SIMPLE MODEL FOR ΔE_i

We begin by restricting attention to the case $N=2$. It then seems natural to assume that ΔE_1 and ΔE_2 are simply proportional to the pressure difference $p_1 - p_2$ divided by an appropriate characteristic relaxation time τ. Since ΔE_i has the units of $1/t$, the coefficient of proportionality must have the units of $1/p$, and the obvious choice of a characteristic pressure to nondimensionalize $p_1 - p_2$ is the volume-weighted mean pressure p given by

$$p = \sum_i \alpha_i p_i, \tag{12}$$

which reduces to $\alpha_1 p_1 + \alpha_2 p_2$ for $N=2$. We therefore write

$$\Delta E_1 = A_1 \left(\frac{p_1 - p_2}{\tau p} \right), \tag{13}$$

$$\Delta E_2 = A_2 \left(\frac{p_2 - p_1}{\tau p} \right), \tag{14}$$

where A_1 and A_2 are positive dimensionless coefficients that remain to be determined, and the signs of $p_1 - p_2$ in Eqs. (13) and (14) have been chosen to ensure that phase 1 expands ($\Delta E_1 > 0$) and phase 2 contracts ($\Delta E_2 < 0$) when $p_1 > p_2$, and vice versa when $p_2 > p_1$. The values of A_1 and A_2 are constrained by Eq. (11), which requires that

$$\alpha_1 A_1 = \alpha_2 A_2 \equiv A_0. \tag{15}$$

Now if A_0 were simply a constant, A_i and ΔE_i would diverge for small α_i ($i=1,2$). This behavior is not obviously unphysical, but we shall nevertheless prevent it by presuming that A_0 is proportional to both α_1 and α_2, i.e., $A_0 = A\alpha_1\alpha_2$, where A is a positive dimensionless constant of order unity. We then have $A_1 = A\alpha_2$ and $A_2 = A\alpha_1$, so that Eqs. (13) and (14) become

$$\Delta E_1 = A\alpha_2 \left(\frac{p_1 - p_2}{\tau p} \right), \tag{16}$$

$$\Delta E_2 = A\alpha_1\left(\frac{p_2 - p_1}{\tau p}\right). \tag{17}$$

Once τ has been defined, Eqs. (16) and (17) provide an apparently reasonable provisional model for ΔE_i for the case $N=2$.

We now proceed to generalize Eqs. (16) and (17) to the multimaterial case of arbitrary N. The obvious way to accomplish this generalization is to simply assume that ΔE_i is the sum of contributions of the form of Eq. (16) for each other phase j, i.e.,

$$\Delta E_i = \frac{A}{\tau p}\sum_j \alpha_j(p_i - p_j). \tag{18}$$

In principle the term $j=i$ should be excluded from the summation, but this restriction is unnecessary since $p_i - p_j$ vanishes when $i=j$. The rationale for this assumption is apparent: if $p_i > p_j$, phase i will expand at the expense of phase j, thereby making a positive contribution to ΔE_i, whereas if $p_i < p_j$ material j will expand at the expense of material i, thereby compressing material i and making a negative contribution to ΔE_i. These different contributions are clearly additive, because volume changes are additive.

Equation (18) expresses the net expansion rate ΔE_i of phase i as the sum of all the individual expansion and contraction rates due to the pressure differences with respect to each of the other phases. This simple approximation is not unreasonable in the absence of information about the geometrical configuration of the phases, but it cannot be expected to be universally valid. It implicitly assumes that on the average, all phases come into contact with each other, the interfacial contact areas are simply bilinear in the volume fractions, and the pressure differences all relax on the same time scale τ. Any or all of those conditions may not be satisfied in particular geometrical configurations. For example, if the flow is stratified then each phase only comes in contact with its two neighbors, and the sum over j in Eq. (18) should be restricted to the latter. Similarly, if phase i only occurs in the form of bubbles dispersed within a particular continuous phase k, then the sum should be restricted to the single term $j=k$. Equation (18) should therefore be modified accordingly to incorporate any information that may be available about the internal structure of the multiphase mixture (i.e., the "flow regime").

The summation in Eq. (18) can immediately be carried out, with the appealingly simple result

$$\Delta E_i = \frac{A}{\tau p}(p_i - p), \tag{19}$$

where use has been made of Eq. (12). Note that Eq. (19) properly satisfies the constraint of Eq. (11). By construction, it also reduces to Eqs. (16) and (17) when $N=2$, as is easily verified.

All that remains is to specify a physically reasonable relaxation time τ, which we shall identify with the time required for sound waves to travel a characteristic distance L on the order of the mean size of the phase fragments. We therefore set $\tau = L/c$, where c is the sound speed of the mixture, an expression for which is derived in the Appendix. Equation (19) then becomes

$$\Delta E_i = \frac{Ac}{Lp}(p_i - p). \tag{20}$$

Combining Eqs. (10) and (20), we finally obtain

$$\frac{\partial \alpha_i}{\partial t} + \boldsymbol{\nabla}\cdot(\alpha_i \mathbf{u}_i) = \alpha_i \boldsymbol{\nabla}\cdot\mathbf{u}_v + \frac{Ac}{Lp}\alpha_i(p_i - p). \tag{21}$$

Equation (21) is our proposed provisional closure model for the time evolution of the volume fractions in compressible multipressure multiphase flow. It obviously requires knowledge of L, which represents a separate modeling task. In many cases, it will be appropriate to identify L with a characteristic length scale that is already being computed for use in the associated multiphase flow equations (e.g., in the interphase drag terms), or perhaps in an auxiliary turbulence model. In the absence of other information, it may (or may not) be reasonable to identify L with Δx as a temporary expedient, where Δx is the spatial discretization or resolution length of the numerical scheme used to solve the model equations. This would clearly be unsatisfactory as a general procedure, but it should at least produce physically reasonable values of L in situations where Δx is tied to physical characteristic lengths, as is frequently the case.

The model also requires knowledge of the coefficient A, which should be of order unity (and hence more nearly constant than L) but must be determined empirically. The optimal value of A cannot realistically be expected to be a universal constant, but will no doubt vary somewhat from one class of problems to another. Once L has been defined and computed, the natural way to determine A would be to adjust it so that the resulting relaxation time for pressure equilibration (which is analytically estimated in Sec. IV below) agrees with that obtained from experiments, direct numerical simulations, or more detailed theoretical treatments.

We hasten to acknowledge that the idea of representing pressure equilibration by introducing a term proportional to pressure differences into $\partial\alpha_i/\partial t$ is rather obvious, and is certainly not new. In particular, volume fraction evolution equations which differ in form from Eq. (21) but also contain terms proportional to pressure differences have previously been proposed by Ransom and Hicks [1], Saurel and coworkers [11,12], Lallemand *et al.* [13], and Zhang *et al.* [14]. The Hicks model for the case $N=2$ is of particular interest, as it provides a mechanistic interpretation and justification of the form of ΔE_i postulated above. The corresponding term in the Hicks model is $\hat{v}/H$, where $\hat{v}$ and H are a transverse velocity and length scale, respectively [1]. Hicks inferred the form of $\hat{v}/H$ from an approximate solution of the Riemann problem between the states $i=1$ and $i=2$, and thereby obtained an expression of the form $\hat{v}/H = (1/H)(p_1 - p_2)/(\rho_1 c_1 + \rho_2 c_2)$. But $p_i \sim \rho_i c_i^2$, from which it follows that $\hat{v}/H \sim (c/Hp)(p_1 - p_2)$, where c is an appropriately weighted mean sound speed of the mixture. The Rie-

mann approach therefore leads to a ΔE_i of the same order of magnitude as the somewhat more heuristic approach taken in the present development.

Several previous pressure relaxation models of various types are also discussed by Shashkov [16] and references cited therein. Even models not originally constructed or intended for use in multiphase flow may still be suitable, *mutatis mutandis*, as a basis for constructing alternative models of ΔE_i. We harbor no illusions that Eq. (20) is the optimal or best model of this type, and we encourage the continuing search for improved models. We do, however, hope that Eq. (10) will prove useful in that search by reducing the task to that of developing a suitable submodel for the quantities ΔE_i, which are well defined and have a clear physical meaning.

IV. THE APPROACH TO PRESSURE EQUILIBRIUM

Although it is clear that the model of Eq. (21) will tend to drive the pressures p_i toward the common value p, the mathematical character of the equilibration process has not yet been examined. In particular, the question naturally arises as to whether the approach to pressure equilibrium is monotonic or oscillatory, and in the latter case whether the oscillations are damped or persist indefinitely as they would for a frictionless piston separating two gases in a closed cylinder.

A simple qualitative argument based on the piston analogy suggests that Eq. (21) should produce a monotonic approach to pressure equilibrium. The volume fraction of either gas is proportional to the piston displacement, and the force on the piston is proportional to the pressure difference Δp across it, so the equation of motion for a frictionless piston is of the form $d^2\alpha/dt^2 \sim \Delta p$. In contrast, the analogous equation of motion obtained by simplifying Eq. (21) is of the form $d\alpha/dt \sim \Delta p$, which corresponds to the equation of motion of a damped piston in the limit of large friction, in which the inertial or acceleration term can be neglected and the approach to equilibrium becomes monotonic.

This general qualitative behavior is readily confirmed by a mathematical analysis of Eq. (21) for the special case of an isothermal system of two phases with ideal-gas equations of state in which all spatial derivatives vanish. Equation (21) then reduces to

$$\frac{\partial \alpha_1}{\partial t} = \frac{Ac}{Lp}\alpha_1(p_1 - p). \tag{22}$$

In such a system, the equations of state reduce to $p_1 = C_1/\alpha_1$ and $p_2 = C_2/\alpha_2$, where C_1 and C_2 are constants proportional to temperature. It then follows that $p = \alpha_1 p_1 + \alpha_2 p_2$ has the constant value $C_1 + C_2$, so that Eq. (22) becomes

$$\frac{\partial \alpha_1}{\partial t} = (Ac/L)(\alpha_1^e - \alpha_1), \tag{23}$$

where $\alpha_1^e = C_1/(C_1 + C_2)$. The solution of Eq. (23) is simply

$$\alpha_1(t) = \alpha_1^e + [\alpha_1(0) - \alpha_1^e]\exp(-Act/L), \tag{24}$$

which shows that α_1 decays exponentially from its initial value $\alpha_1(0)$ to its equilibrium value α_1^e with a relaxation time $L/(Ac)$.

One might at first be concerned that a monotonic approach to pressure equilibrium seems at variance with one's physical expectations for a process involving pressure waves, which would intuitively be expected to exhibit a damped oscillatory approach to equilibrium. It must be remembered, however, that in most multiphase flows the interfacial geometry is highly complex and irregular. Even two-phase dispersions of spherical particles have an irregular geometry due to the random locations of the particles. In such geometries, the pressure waves will experience multiple reflections and refractions in random directions, and the superposition of a great many such random waves lacking phase coherence will result in the nearly complete cancellation of their oscillatory components, thereby resulting in an essentially monotonic approach to equilibrium. Well-known examples of analogous phenomena in other contexts include spin-spin relaxation in nuclear magnetic resonance and Landau damping in collisionless plasmas. Of course, the time scale for the incoherent acoustic waves to finally decay to thermal energy is ultimately determined by viscous effects, but the above physical picture suggests that the associated viscous time scale is essentially irrelevant. In any case, the viscosity is in general proportional to the mean molecular speed, which is of the same order of magnitude as the sound speed, so that viscous time scales are also proportional to $1/c$, albeit with a possibly large coefficient.

V. THE INCOMPRESSIBLE LIMIT

We now proceed to examine the behavior of the model of Eq. (21) in the incompressible limit as the sound speed tends to infinity. It is immediately apparent by inspection that this limit implies that the pressures instantaneously equilibrate to the common value $p_i = p$. This in turn implies that ΔE_i becomes formally indeterminate in that limit, so that Eq. (21) can no longer be used as a closure relation. However, the equation system remains closed because the resulting equations $p_i = p$ can then be employed as the required closure relations instead. Only $N-1$ of the latter equations are independent because of Eq. (12), so they provide precisely the $N-1$ additional relations required for closure. If desired, the actual numerical values of the ΔE_i can then be determined *a posteriori* by substituting the resulting values of ρ_i back into Eq. (5), but this is unnecessary since the ΔE_i are no longer useful.

An alternative equivalent viewpoint is that in the incompressible limit as $c \to \infty$, the multipressure model effectively reduces to the corresponding single-pressure model, and once this reduction has occurred the model no longer requires the additional closure relations that multipressure models do. It is clear that this behavior will generally obtain for any multipressure model in which the volume fraction evolution equations contain terms proportional to pressure differences divided by an acoustic time scale, i.e., a characteristic time determined by the sound speed. Any model of this type will therefore reduce to the corresponding single-pressure model in the incompressible limit. (Indeed, all such models differing only in the form of the volume fraction equation will reduce to the *same* single-pressure model in the

incompressible limit, since the latter equation becomes indeterminate and effectively drops out in that limit.) Models of this type are therefore essentially compressible in nature, and are inherently incapable of describing any residual interphase pressure differences that may persist in the incompressible limit. If such pressure differences are believed to be significant, their calculation requires the use of alternative models in which the time scale for pressure equilibration is determined by inertial rather than acoustic effects [1–5]. Regardless of the time scale, however, the presence of terms proportional to pressure differences is clearly essential to represent the general tendency for the pressures to dynamically equilibrate. Models that lack such terms therefore omit an essential aspect of the physics.

It should be noted that the incompressible limit, as defined above, does not necessarily imply that the densities ρ_i become constant in that limit. Whether or not this occurs depends entirely on the form of the state equations, which must now be written as expressions for ρ_i as a function of (e_i,p). The pressure dependence becomes negligible for large c, but the energy dependence remains and allows for thermal expansion effects. An entirely analogous situation occurs in single-phase flow [18,20].

In the case when $\rho_i(e_i,p)$ is independent of e_i as well as p, so that the densities ρ_i are indeed given constants, it follows at once from Eq. (4) that $E_i=0$, so that $E=\nabla\cdot\mathbf{u}_v=0$ [19] and $\Delta E_i=0$. Equation (10) then reduces to

$$\frac{\partial\alpha_i}{\partial t}+\nabla\cdot(\alpha_i\mathbf{u}_i)=0, \tag{25}$$

which is no longer indeterminate but remains useless because the continuity equations (2) also reduce to Eq. (25) when the densities are constant, so that Eq. (10) is no longer independent in that case. As discussed above, however, the system remains closed because the required closure relations are provided by the pressure equality conditions $p_i=p$ implied by Eq. (20).

If we further restrict attention to the situation in which all the phase velocities $\mathbf{u}_i$ are equal with the common value $\mathbf{u}$, then the α_i become simply nondiffusive scalars that are passively advected by $\mathbf{u}$. In this case $\mathbf{u}_v=\mathbf{u}$ and Eq. (25) further reduces to

$$\frac{\partial\alpha_i}{\partial t}+\mathbf{u}\cdot\nabla\alpha_i=0, \tag{26}$$

which indeed properly describes the purely kinematic advection of the nondiffusive scalars α_i by the velocity field $\mathbf{u}$.

VI. NUMERICAL CONSIDERATIONS

Numerical solutions to models of the present type are usually obtained by the use of time-marching methods constructed by approximating time derivatives (and usually space derivatives as well) by finite differences. Such methods range from fully explicit (Euler) schemes to fully implicit (backward Euler) schemes, with numerous intermediate variants (semi-implicit, linearly implicit, predictor-corrector, etc.). Such schemes are generally subject to stability restrictions on the time step Δt. If the present model were solved numerically using a simple explicit Euler scheme, its stability condition would be expected to be similar to that obtained by applying the same scheme to Eq. (23), which is simply given by

$$\Delta t<2\ \min\left(\frac{L}{Ac}\right) \tag{27}$$

where the minimum operation is taken over the entire computing region. Equation (27) merely states that Δt must be smaller than the characteristic time for pressure equilibration to occur. Partially and fully implicit schemes usually have less restrictive stability conditions, or are unconditionally stable, but often produce inaccurate solutions if explicit stability limits such as Eq. (27) are greatly exceeded, since this implies that some underlying physical process is not being well resolved in time.

If the remaining fluid dynamical equations are also solved by means of an explicit numerical scheme, then the time step will already be restricted by a Courant stability condition of the form

$$\Delta t<\chi\frac{\Delta x}{c}, \tag{28}$$

where Δx is the spatial discretization length of the scheme, and χ is a dimensionless constant of order unity. Comparison of Eqs. (27) and (28) shows that as long as L is of order Δx or larger, which should normally be the case, Eq. (27) will therefore not require a significant further reduction in Δt.

If a partially or fully implicit scheme is used for the other fluid dynamical equations but not for Eq. (21), then the time step that could otherwise have been used will be further reduced by a stability restriction similar to Eq. (27). This restriction could be circumvented by limiting the maximum allowed changes in α_i from one time level to the next, but that would be tantamount to advancing the α_i using an artificial time step smaller than the true physical time step Δt, or equivalently to an artificial reduction in the coefficient Ac/L, either of which would result in an inaccurate time evolution of the pressure equilibration process. Such limiting procedures should therefore only be used in situations where it is unnecessary to accurately compute the time history of the pressure differences p_i-p.

VII. CONCLUSION

We have presented a rigorous theoretical framework for constructing models for the time evolution of the volume fractions in compressible multipressure multiphase flow. This formulation reduces the problem of constructing such models to that of constructing submodels for the ΔE_i. Within this framework, we presented a simple provisional model of this type in which the ΔE_i are proportional to pressure differences, thereby reflecting the essential tendency for unequal pressures to dynamically equilibrate. This model involves two as yet undetermined coefficients: L and A. The length scale L requires a separate submodel, but in many cases it can probably be identified with a similar length scale

already being computed for use elsewhere in the multiphase flow model. The coefficient A is an empirical parameter of order unity which cannot realistically be expected to be a universal constant. Once L has been obtained, the natural way to determine A is by requiring the resulting pressure equilibration times to agree with the results of experiments, direct numerical simulations, or other theoretical considerations.

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APPENDIX: MIXTURE SOUND SPEED

We shall evaluate the sound speed c of the multiphase mixture under the simplifying assumption that the phases are in pressure and temperature equilibrium. In this case, c can simply be evaluated by means of the well-known relation

$$c^2 = \gamma\left(\frac{\partial p}{\partial \rho}\right)_T, \tag{A1}$$

where $\rho \equiv \Sigma_i \alpha_i \rho_i$, T, and γ are, respectively, the total mass density, temperature, and specific heat ratio of the mixture. As the thermodynamic variables p, ρ, and T vary, the volume fractions α_i will also vary, but the mass fractions $Y_i = \alpha_i \rho_i / \rho$ remain constant. We therefore begin by expressing ρ in terms of ρ_i and Y_i. Since $Y_i/\rho_i = \alpha_i/\rho$, it follows at once that

$$\frac{1}{\rho} = \sum_i \frac{Y_i}{\rho_i}. \tag{A2}$$

Differentiating Eq. (A2) with respect to p at constant T, we obtain

$$\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial p}\right)_T = \sum_i \frac{Y_i}{\rho_i^2}\left(\frac{\partial \rho_i}{\partial p}\right)_T. \tag{A3}$$

But Eq. (A1) also applies to each of the phases separately, so that

$$c_i^2 = \gamma_i\left(\frac{\partial p}{\partial \rho_i}\right)_T, \tag{A4}$$

where c_i and γ_i are, respectively, the sound speed and specific heat ratio of phase i. Combining Eqs. (A1), (A3), and (A4), we obtain

$$c^2 = \frac{\gamma}{\rho^2}\left(\sum_i \frac{Y_i \gamma_i}{\rho_i^2 c_i^2}\right)^{-1}. \tag{A5}$$

The mixture specific heat ratio γ can of course also be evaluated in terms of the thermodynamic properties of the individual phases by means of the elementary relation

$$\gamma = \frac{\sum_i Y_i \gamma_i c_{vi}}{\sum_i Y_i c_{vi}}, \tag{A6}$$

where $c_{vi} \equiv (\partial e_i / \partial T)_{\rho_i}$ is the specific heat at constant volume of phase i.

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