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Variational Theory of Balance Systems

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VARIATIONAL THEORY OF BALANCE SYSTEMS

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ABSTRACT. Abstract. In this work we apply the Poincare-Cartan formalism of the Classical Field Theory to study the systems of balance equations (balance systems). We introduce the partial k-jet bundles $J_p^k(\pi)$ of the configurational bundle $\pi : Y \to X$ and study their basic properties: partial Cartan structure, prolongation of vector fields, etc. A constitutive relation C of a balance system BC is realized as a mapping between a (partial) k-jet bundle $J_p^k(\pi)$ and the extended dual bundle $\Lambda_{2/1}^{n+(n+1)}Y$ similar to the Legendre mapping of the Lagrangian Field Theory.

Invariant (variational) form of the balance system $\mathcal{B}_\mathcal{C}$ corresponding to a constitutive relation $\mathcal{C}$ is studied. Special cases of balance systems -Lagrangian systems of order 1 with arbitrary sources and RET (Rational Extended Thermodynamics) systems are characterized in geometrical terms. Action of automorphisms of the bundle π on the constitutive mappings $\mathcal{C}$ is studied and it is shown that the symmetry group $Sym(\mathcal{C})$ of $\mathcal{C}$ acts on the sheaf of solutions $Sol_\mathcal{C}$ of balance system $\mathcal{B}_\mathcal{C}$. Suitable version of Noether Theorem for an action of a symmetry group is presented together with the special forms for semi-Lagrangian and RET balance systems and examples of energy momentum and gauge symmetries balance laws.

k-jet bundle, balance law, Poincare-Cartan form, Noether Theorem

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1. **Introduction.**

Systems of balance equations (balance system, shortly, BS) for the fields y^μ accompanied by the proper constitutive relations $\mathcal{C}$ are the basic tools of Continuum Thermodynamics. In this work we develop a variational (in the sense of Poincare-Cartan form) theory of balance systems.

In Sec.2 we introduce basic notions: k-jet bundles, contact decomposition of forms, Poincare-Cartan formalism of order 1, balance systems. In Sec.3 the partial jet bundles $J_p^k(\pi)$ of configurational bundles $\pi : Y^{n+m} \to X^n$ are introduced as the appropriate domains of the constitutive relations of the corresponding field theory. These bundles corresponds to an almost product structure on the base manifold M (space-time decomposition is the main example). Partial Cartan structures and prolongation of vector fields from Y to the partial jet bundles are studied. In Sec.4 the constitutive relations $\mathcal{C}$ are defined as the mappings $C : J_p^k(\pi) \to \Lambda_{2/1}^{(n+(n+1))}Y$ to the bundle $\Lambda_{2/1}^{(n+(n+1))}Y$ of $n + (n + 1)$-forms on Y annulated by two π-vertical tangent vectors factorized by the bundles of similar forms annulated by one vertical tangent vector. Lifting $\hat{C} : J_p^k(\pi) \to \Lambda_2^{(n+(n+1))}Y$ of such a mapping induces the $n + (n + 1)$-form $\Theta_{\hat{C}}$ on the partial jet bundle - Poincare-Cartan form of considered field theory. Special forms and examples of such CR are presented: lifting

covering constitutive relations, semi-Lagrangian, RET, relations defined by the La-grangian and a Dissipative potential, etc. In Sec.5 the invariant form of a balance system corresponding to a constitutive relation $\mathcal{C}$ is introduced and studied. Cases of Lagrangian field theories of the first order and of the Rational Extended Thermodynamics as a specific case of the balance systems are described. We study the admissible variations of the fields $y\mu$ splitting the invariant equation into separate balance equations. An appropriate modification of the source term allows us to remove conditions on the admissible variations.

Action of automorphisms of configurational bundle $\pi : Y \to X$ on the constitutive relation C is defined and its effect on the corresponding Poincare-Cartan form and on the sheaf of solutions of the balance system is studies is Sec. 6.

In Sec.7 we prove the appropriate version of the first Noether Theorem associating the new balance laws with an infinitesimal symmetries of a constitutive relation $\mathcal{C}$ and determine when such a balance law is actually, the conservation law. Examples of the energy-momentum balance law and that of the gauge symmetries are presented. In Sec.8 we sketch the application of present scheme to the search and classification of the "secondary balance laws of a given BS, including the entropy balance law. As an example, we present classification of secondary balance laws for the Cattaneo heat propagation balance system and the constitutive restrictions on the CR $\mathcal{C}$ that follows from the II law of thermodynamics.

Short exposition of main results of this paper was presented in the Conference Proceedings [27].

2. Settings and the framework

2.1. **Notations and preliminaries.** Throughout this paper $\pi : Y \to X$ will be a (*configurational*) fibred bundle with a n-dim connected paracompact smooth (C^∞) manifold X as the base and a total space Y, $dim(Y) = n + m$. Fiber of the bundle π is a m-dim connected smooth manifold U.

Base manifold X is endowed with a (pseudo)-Riemannian metric G. Volume form of metric G will be denoted by η. In this paper we will not be dealing with the boundary of a base manifold X, in applications X can be considered as an open subset of R^n or as a compact manifold. As an basic example, we consider the case where $X = T \times B$ is the product of time axis T and an open material manifold (or a domain in the physical space) B.

We will be using fibred charts (W, x^i, y^μ) in the bundle π. Here $(\pi(W), x^i)$ is a chart in X and y^μ are coordinates along the fibers. Tangent frame corresponding to the local chart (W, x^i, y^μ) will be denoted by $(\partial_i = \partial_{x^i}, \partial_\mu = \partial_{y^\mu})$ (shorter notation will be used in more cumbersome calculations), corresponding coframe - (dx^i, dy^μ).

Introduce the contracted forms $\eta_i = i_{\partial_{x^i}}\eta$, $\eta_{ij} = i_{\partial_{x^j}} i_{\partial_{x^i}}\eta$. Below we will be using following relations for the forms η_j, η_{ji} (here and below $\lambda_G = ln(\sqrt{|G|})$):

$$\begin{cases} dx^j \wedge \eta_i = \delta_i^j \eta, \\ dx^j \wedge \eta_{ik} = \delta_\sigma^j \eta_i - \delta_i^j \eta_k, \\ d\eta_i = \lambda_{G,i}\eta, \\ d\eta_{ij} = (\lambda_{G,j}\eta_i - \lambda_{G,i}\eta_j) \end{cases} \qquad . \qquad (2.1)$$

Sections $s : V \to Y$, $V \subset X$ of the bundle π represent the collection of (classical) fields y^μ defined in the domain $V \subset X$. Usually these fields are components of some tensor fields or tensor densities fields (sections of "natural bundles", [4]).

For a manifold M we will denote by $\tau_M : T(M) \to M$ the tangent bundle of a manifold M, by $V(M) \subset T(M)$ the subbundle of of vertical vectors, i.e. vectors $\xi \in T_y(M)$ such that $\tau_{M*y}\xi = 0$.

Denote by $\Lambda^r(M)$ the bundle of exterior r-forms on the manifold M and by $(\Lambda^* = \oplus \Lambda^r(M), d)$ the differential algebra of exterior forms on the manifold M.

2.2. **The k-jet bundles $J^k(\pi)$.** Given a fiber bundle $\pi : Y \to X$ denote by $J^k(\pi)$ the k-jet bundle of sections of the bundle π, [5, 12]. Denote by $\pi_{kr} : J^k(\pi) \to J^r(\pi), k \geqq r \geqq 0$ the natural projections between the jet bundles of different order and by $\pi^k : J^k \to X$ the projection to the base manifold X. Projection mappings $\pi_{k(k-1)} : J^k(\pi) \to J^{k-1}(\pi)$ in the tower of k-jet bundles

$$\ldots \to J^k(\pi) \to J^{k-1}(\pi) \to \ldots \to Y \to X$$

are *affine bundles* modeled by the vector bundle $\bigwedge^k T^*(X) \otimes_{\to J^{k-1}(\pi)} V(\pi) \to J^{k-1}(\pi)$.

Denote by $J^\infty(\pi)$ the infinite jet bundle of bundle π - inverse limit of the projective sequence $\pi_{k(k-1)} : J^k(\pi) \to J^{k-1}(\pi)$. Space $J^\infty(\pi)$ is endowed with the structure of inverse limit of differentiable manifolds with the natural sheaves of vector fields, differential forms etc. making the projections $\pi_{\infty k} : J^\infty(\pi) \to J^k(\pi)$ smooth surjections. See [12, 28] for more about structure and properties of k-jet bundles.

For a mutliindex $I = \{i_1, \ldots, i_n\}, i_k \in \mathbb{N}$ denote by ∂^I the differential operator in $C^\infty(X)$ $\partial^I f = \partial_{x^1}^{i_1} \cdot \ldots \partial_{x^n}^{i_n}$. To every fibred chart (V, x^i, y^μ) in Y there corresponds the fibred chart $(x^i, y^\mu, z_i^\mu, |I| = \sum_s i_s \leqq k)$ in the domain $V^k = \pi_{k0}^{-1}(V) \subset J^k(\pi)$. This chart is defined by the condition $z_I^\mu(j_x^k s) = \partial^I s^\mu(x)$.

For $k = 1, \ldots, \infty$ the space of k-jet bundle $J^k(\pi) \to X$ is endowed with the **Cartan distribution** Ca^k defined by the basic **contact forms**

$$\omega^\mu = dy^\mu - z_i^\mu dx^i, \ldots \omega_I^\mu = dz_I^\mu - z_{I+1_j}^\mu dx^j, \ |I| < k; \ dz_I^\mu, \ |I| = k. \qquad (2.2)$$

These forms generate the contact ideal $C^k \subset \Lambda^*(J^k(\pi))$ in the algebra of all exterior forms. Denote by $I(C^k)$ the differential ideal of contact forms. This ideal is generated by the basic contact forms (2.2) and by the forms dz_I^μ, $|I| = k$, see [11]. In the case where $k = \infty$ the basic contact forms dz_I^μ are absent form the list of generators of ideal $I(C^k)$.

A p-form is called l-contact if it belongs to the l-th degree of this ideal $(C^k)^l \subset \Lambda^*(J^k(\pi))$. 0-contact forms are also called **horizontal (or π_k-horizontal) forms** (or, sometimes, *semi-basic forms*). We denote by $kCon$ the k-contact forms that appear in calculations. For $k = 1$ we will omit index 1.

Let $0 \leqq s < k$. A form $\nu \in \Lambda^*(J^k(\pi))$ is called π_{ks}-**horizontal** if it belongs to the subalgebra $C^\infty(J^k(\pi))\pi_{ks}^*\Lambda^*(J^s(\pi)) \subset \Lambda^*(J^k(\pi))$.

Remind now the following basic result (D. Krupka,[9])

Theorem 1. *Let a form $\nu \in \Lambda^q(J^k(\pi))$ be $\pi_{k(k-1)}$-horizontal (i.e. $\nu = \pi_{k(k-1)}\nu_*, \nu_* \in \Lambda^q(J^{k-1}(\pi))$). Then there is unique* contact *decomposition of the form ν*

$$\nu = \nu_0 + \nu_1 + \ldots + \nu_q, \qquad (2.3)$$

where $\nu_i, 0 \leq i \leq q$ is a i-contact form on $J^k(\pi)$. Form ν_0 is called the horizontal part of the form ν_* (and of the form ν as well).

Related to the contact decomposition is the decomposition of differential operator d as the sum of **horizontal and vertical differentials** d_h, d_v: for a q-form $\nu \in \Lambda^*(J^k)(\pi))$ its differential $d\nu$ lifted into the $J^{k+1}(\pi)$ is presented as the sum of horizontal and contact (vertical) terms:

$$d\nu = d_h\nu + d_v\nu.$$

,see [5, 13]. Operators d_h, d_v are more naturally defined in the space $\Lambda^*(J^\infty)(\pi))$.

We recall that these operators have the following homology properties

$$d_h^2 = d_V^2 = d_V d_h + d_h d_V = 0.$$

In particular, for a function $f \in C^\infty(J^\infty(\pi))$ (depending on the jet variables z_I^μ up to some degree, say $|I| \leq k$),

$$df = (d_i f)dx^i + \sum_{|I| \geq 0} f_{,z_I^\mu}\omega_I^\mu, \tag{2.4}$$

where

$$d_i f = \partial_{x^i} f + \sum_{I||I| \geq 0} z_{I+1_i}^\mu \partial_{z_I^\mu} f \tag{2.5}$$

is the **total derivative** of the function f by x^i. The series in the formulas (2.4-5) contains finite number of terms: $|I| \leq k$.

2.3. Lagrangian Poincare-Cartan formalism, k=1.
Here we remind the basic notions of infinitesimal form of Lagrangian Field Theory of the first order (on $J^1(\pi)$) based on the use of Poincare-Cartan form,[4, 5, 16].

Volume form η permits to construct the **vertical endomorphism**

$$S_\eta = (dy^\mu - z_i^\mu dx^i) \wedge \eta_j \otimes \frac{\partial}{\partial z_j^\mu} \tag{2.6}$$

which is a tensor field of type $(1, n)$ on the 1-jet bundle $J^1(\pi)$.

For a Lagrangian n-form $L\eta$, L being a (smooth) function on the manifold $J^1(\pi)$ the Poincaré-Cartan n-form are defined as follows:

$$\Theta_L = L\eta + S_\eta^*(dL), \tag{2.7}$$

where S_η^* is the adjoint operator of S_η. In coordinates we have

$$\Theta_L = (L - z_i^\mu \frac{\partial L}{\partial z_i^\mu})\eta + \frac{\partial L}{\partial z_i^\mu} dy^\mu \wedge \eta_i, \tag{2.8}$$

An *extremal* of L is a section s of the bundle π such that for any vector field $\hat{\xi}$ on the manifold $J^1(\pi)$,

$$(j^1(s))^*(i_{\hat{\xi}} d\Theta_L) = 0, \tag{2.9}$$

where $j^1(s)$ is the first jet prolongation of s. A section s is an extremal of L if and only if it satisfies to the system of Euler-Lagrange Equations (see, for instance, [2, 5])

$$\frac{\partial(L \circ j^1(s)\sqrt{|G|})}{\partial y^\mu} - \frac{d}{dx^i}\left(\frac{\partial(L \circ j^1(s)\sqrt{|G|})}{\partial z_i^\mu}\right) = 0, \ 1 \leq \mu \leq m. \tag{2.10}$$

2.4. Bundles $\Lambda_p^r Y$ and canonical forms. Introduce the bundle $\Lambda_p^r(Y)$ of the exterior forms on Y which are annulated if p of its arguments are vertical ([10, 16]:

$$\omega^r \in \Lambda_p^r(Y) \Leftrightarrow i_{\xi_1} \ldots i_{\xi_p} \omega^r = 0, \ \xi_i \in V(Y).$$

We will be using these bundles for $r = n, n+1, n+2$ and $p = 1, 2$.

Elements of the space $\Lambda_1^n Y$ are semibasic n-forms locally expressed as $p(x, y)\eta$.

Elements of the space $\Lambda_2^n Y$ have, in a fiber coordinates (x^i, y^μ) the form

$$p(x, y)\eta + p_\mu^i dy^\mu \wedge \eta_i.$$

This introduces coordinates (x^i, y^μ, p) on the manifold $\Lambda_1^n Y$ and (x^i, y^μ, p, p_μ^i) on the manifold $\Lambda_2^n Y$.

For the case where $r = n+1$ the forms $dy^\mu \wedge \eta$ form the basis of fibers of the bundle $\Lambda_2^{n+1}(Y) \to Y$ while the bundle $\Lambda_1^{n+1}(Y)$ is *zero bundle*.

Introduce the notation

$$\Lambda_p^{n+(n+1)}(Y) = \Lambda_p^n(Y) \oplus \Lambda_p^{n+1}(Y), p = 1, 2,$$

for the direct sum of the bundles on the right side.

It is clear that $\Lambda_1^k(Y) \subset \Lambda_2^k(Y)$. Therefore we have the embedding of subbundles $\Lambda_1^{n+(n+1)}(Y) \subset \Lambda_2^{n+(n+1)}(Y)$ and can define the factor-bundle

$$\Lambda_{2/1}^{n+(n+1)}(Y) = \Lambda_2^{n+(n+1)}(Y)/\Lambda_1^{n+(n+1)}(Y),$$

with the projection $q : \Lambda_2^{n+(n+1)}(Y) \to \Lambda_{2/1}^{n+(n+1)}(Y)$.

On the bundles $\Lambda_2^{n+(n+1)}(Y)$ there is defined the canonical form ([16]) with the coordinate expression

$$\Theta_2^{n+(n+1)} = \Theta_2^n + \Theta_2^{n+1} = p\eta + p_\mu^i dy^\mu \wedge \eta_i + p_\mu dy^\mu \wedge \eta. \tag{2.11}$$

On the factor-bundle $\Lambda_{2/1}^{n+(n+1)}(Y)$ the form

$$\widetilde{\Theta}_{2/1}^{n+(n+1)} = p_\mu^i dy^\mu \wedge \eta_i + p_\mu dy^\mu \wedge \eta$$

is defined *mod η*.

2.5. Legendre transformations, $k = 1$. Let $L \in C^\infty(J^1(\pi))$ be a Lagrangian function. We define the fiber mapping (over Y) $leg_L : J^1(\pi) \to \Lambda_2^n Y$, as follows:

$$leg_L(j_x^1 s))(X_1, \ldots, X_n) = (\Theta_L)_{j_x^1 s}(\tilde{X}_1, \ldots, \tilde{X}_n),$$

where $j_x^1 s \in J^1(\pi), X_i \in T_{s(x)}Y$ and $\tilde{X}_i \in T_{j_x^1 s(X)}J^1(\pi)$ are such that $\pi_*(\tilde{X}_i) = X_i$. In local coordinates, we have

$$leg_L(x^i, y^\mu, z_i^\mu) = (x^i, y^\mu, p = L - z_i^\mu \frac{\partial L}{\partial z_i^\mu}, p_\mu^i = \frac{\partial L}{\partial z_i^\mu}).$$

The Legendre transformation $Leg_L : J^1(\pi) \to \Lambda_{2/1}^{n+(n+1)}(Y)$ is defined as the composition $Leg_L = q \circ leg_L$. In coordinates,

$$Leg_L(x^i, y^\mu, z_i^\mu) = (x^i, y^\mu, p_\mu^i = \frac{\partial L}{\partial z_i^\mu}).$$

Recall [16] that the Legendre transformation $Leg_L : J^1(\pi) \to \Lambda_{2/1}^{n+(n+1)}(Y)$ is a local diffeomorphism if and only if L is regular, i.e. when the *vertical Hessian* $\frac{\partial^2 L}{\partial z_i^\mu \partial z_j^\nu}$ is *nondegenerate*.

2.6. **Con-differential.** Below we will be using the con-differential $\tilde{d}$ used in topology, see. for instance, [7].

Let M^n be an n-dimensional smooth manifold. Con-differential is defined on couples of exterior forms $\alpha^k + \beta^{k+1} \in \Lambda^{k+(k+1)}(M)$:

$$\begin{cases} \tilde{d} : \Lambda^{k+(k+1)}(M) = \Lambda^k(M) \oplus \Lambda^{k+1}(M) \to \Lambda^{(k+1)+(k+2)}(M) = \Lambda^{k+1}(M) \oplus \Lambda^{k+2}(M) : \\ \tilde{d}(\alpha^k + \beta^{k+1}) = ((-d\alpha + \beta) + d\beta). \end{cases}$$
$$(2.12)$$

Lemma 1. $\tilde{d} \circ \tilde{d} = 0$.

Proof. We have

$$\tilde{d}\tilde{d}(\alpha^k + \beta^{k+1}) = \tilde{d}((-d\alpha + \beta) + d\beta) = [-d(-d\alpha + \beta) + d\beta] + d(d\beta) = -d\beta + d\beta + 0.$$

$\square$

Some other properties of this differential and of the corresponding complex $(\Lambda^{k+(k+1)}(M) = \Lambda^k(M) \oplus \Lambda^{k+1}(M), \tilde{d})$ are presented in [25].

2.7. **Balance systems.** Let the base manifold X be the material or physical spacetime X with (local) coordinates $x^1 = t$, $x^A, A = 1, 2, 3$. Typical system of balance equations for the fields y^μ is determined by a choice of flux fields F_μ^i, $\mu = 0, 1, 2, 3$ (including the densities for $i = 0$) and source fields Π_μ as functions on a jet space $J^k(\pi)$. Choice of the functions $F_\mu^i, \Pi_\mu \in C^\infty(J^k(\pi))$ is codified in physics as the choice of the **constitutive relation** of a given material, [22, 23]. After such a choice has been done, the *closed* system of balance equations for the fields y^i

$$(F_\mu^i \circ j^k s)_{;\mu} = \Pi_\mu \circ j^k s, \ \mu = 1, \ldots, m \qquad (2.13)$$

can be solved for a section $s : V \to Y$ if one add to the balance system the appropriate boundary (including initial) conditions.

Introducing the horizontal forms in $J^k(\pi)$ - $F_\mu = F_\mu^i \eta_i, \Pi_\mu = \Pi_\mu \eta$ these equations can be written in the form

$$j^1 \ {}^*(s)[dF_\mu - \Pi_\mu] = 0.$$

Using the con-differential $\tilde{d}$ (see above) the balance system (2.13) can be written in the compact form

$$j^1 \ {}^*(s)\tilde{d}[F_\mu + \Pi_\mu] = 0, \ i = 1, \ldots, m. \qquad (2.14)$$

Remark 1. Similarly to the definition of a conservation law on a contact manifold given in [3] one can define a balance law of order k as a $n + (n + 1)$-form $\sigma = F^n + \Pi^{n+1}$ on the space $J^k(\pi)$ (which usually is π_k-horizontal) where F^n is defined mod closed forms from $Z^n(J^k(\pi))$ such that

$$\tilde{d}\sigma = dF^n - \Pi^{n+1} \in I(C^k). \qquad (2.15)$$

In Continuum Thermodynamics balance laws typically are present as the closed system of equations for the dynamical fields. It is shown below that the Poincare-Cartan formalism allows to generate the system of balance laws in the number equal to the number of dynamical fields y^μ.

Example 1. Five fields fluid thermodynamical system. In the *5 fields fluid* thermodynamical system ([22, 23]), the velocity vector field $\mathbf{v}$, density scalar field ρ and scalar field of internal energy ϵ (or total energy $e = \frac{1}{2}\rho\|v\|^2 + \epsilon$) are considered as the basic dynamical fields y^μ. Corresponding balance system has the form of the five balance equations - mass conservation law, linear momentum balance law and the energy balance law:

$$\begin{cases} \partial_t \rho + \partial_{x^B}(\rho v^B) = 0, \\ \partial_t(\rho v^A) + \partial_{x^B}(\frac{1}{2}\rho v^A v^B + t^{AB}) = f^A, \ A = 1,2,3, \\ \partial_t(\frac{1}{2}\rho\|v\|^2 + e) + \partial_{x^B}\left((\frac{1}{2}\rho\|v\|^2 + e)v^B + t^B_C v^C - h^B\right) = f_B v^B + r. \end{cases} \tag{2.16}$$

Here $\mathbf{t}$ is the (1,1)-stress tensor, $\mathbf{f}$ is the 1-form of the bulk force, $\mathbf{h}$ is the heat flux vector filed and the scalar function r - radiation heating source.

Example 2.

3. Partial k-jet bundles

In this section we introduce the "partial jet bundles" of the configurational bundle $\pi : Y \to X$ - factor bundles of the conventional jet bundles $J^k(\pi)$. These bundles are convenient for the description of balance systems whose constitutive relations depends on some but not all derivatives of fields y^μ. We start with the definition of the partial 1-jet bundle defined by a subbundle $K \subset T(X)$.

3.1. Bundle $J^1_K(\pi)$.

Definition 1. *Let $\pi_{XY} : Y \to X$ be a fiber bundle and let $K \subset T(X)$ be a subbundle.*

(1) *Let $x \in U \subset X$, $s_1, s_2 \in \Gamma(U,\pi)|s_1(x) = s_2(x)$. Sections s_1, s_2 are called K-**equivalent of order 1 at a point** $x \in X$: $s_1 \sim_{K_x} s_2$ if $s_{1*x}|_{K_x} = s_{2*x}|_{K_x}$.*

(2) *$J^1_{K\ x}(\pi)$- space of classes of $\sim_{K_x}$ at a point $x \in X$.*

(3) *$J^1_K(\pi) = \cup_{x\in X} J^1_{K\ x}(\pi)$ - the space of partial 1-jets of sections of π.*

In the next proposition we collect basic properties of bundles $J^1_K(\pi)$. Proof of this Proposition is straightforward (see [25]).

Proposition 1.   (1) *Bundle $J^1_K(\pi) \to Y$ is the affine bundle modeled on the vector bundle $\pi^*(K^*) \otimes V(\pi) \to Y$.*

(2) *There is a canonical surjection of affine bundles $w_K : J^1(\pi) \to J^1_K(\pi)$.*

(3) *Let $T(X) = K(X) \oplus K'(X)$ be an almost product structure (AP), then there is the commutative diagram*

$$\begin{array}{ccc} J^1(\pi) & \xrightarrow{\ w_K\ } & J^1_K(\pi) \\ w_{K'}\Big\downarrow & & \pi_{10\ K}\Big\downarrow \\ J^1_{K'}(\pi) & \xrightarrow{\ \pi_{10\ K'}\ } & Y, \end{array}$$

which realizes the conventional 1-jet bundle $J^1(\pi)$ as the fiber product of the bundles $J^1_K(\pi), J^1_{K'}(\pi)$.

(4) *Let $T(X) = K \oplus K'$ be an integrable AP-structure, (x^j, x^k) - local chart integrating the $AP-$ structure (i.e. K_x is the linear span of vector fields ∂_j, K' - is the linear span of vector fields ∂_k). One forms*

$$\omega^\mu = dy^\mu - \sum_j z_j^\mu dx^j$$

generate the "partial" Cartan distribution C_K on $J_K^1(\pi)$ in the sense that a section q of the bundle $J_K^1(\pi)$ is the (partial) 1-jet of a section $s : X \to Y$ if and only if $q^*(\omega^\mu)|_K = 0$, $\forall \mu = 1, \dots, m$.

(5) *All constructions above are covariant with respect to the automorphisms ϕ of the bundle π such that projection $\bar{\phi}$ of automorphism ϕ to the base X leaves the AP structure $T = K \oplus K'$ invariant.*

Remark 2. Partial Cartan structure exist in the case of a general AP structure on the manifold X, see [25] for the proof and construction of the basic contact forms.

Example 3. Basic example is, of course, the space-time decomposition where $X = T \times B$ is the product of the classical time axis and the 3-dim space manifold B (material or physical) and where $T(X) = \langle \partial_t \rangle \oplus T(B)$. In such a case $x^1 = t$ and $x^i, i = 2, 3, 4$ are spacial coordinates. We take $K = T(B) \subset T(X)$ to be the subbundle of derivatives in spacial directions.

Example 4. Mathematically trivial but very important physically is the case of the Rational Extended Thermodynamics, [21], where the configurational space is extended enough so that the constitutive relations do not depend on the derivatives of the basic fields y^μ. To include this case in our scheme we take $K = \{0\}$. Then, the bundle $J_{\{0\}}^1(\pi) \to Y$ has zero dimensional fiber.

3.2. **Space-time splitting bundle** $J_S^1(\pi)$. Employing the construction of the bundle $J_K^1(\pi)$ for the product structure $T(X) = \langle \partial_t \rangle \oplus T(B)$ we define now the partial 1-jet bundle $J_S^1(\pi)$. We assume that the generic fiber U of the bundle $\pi : Y \to X$ is the direct product of subspaces of fields y^i entering the constitutive relation without derivatives, with time derivative only, with spacial derivatives only and with all derivatives respectively:

$$[1, \dots, m] = S_0 \cup S_t \cup S_x \cup S_{xt} \Rightarrow U = U_0 \times U_t \times U_x \times U_{tx}.$$

Proposition 2.   (1) *Bundle $\pi : Y \to X$ is the fiber product of the bundles $\pi_0 : Y_0 \to X$, $\pi_t : Y_t \to X, \dots$ with fibers $U_0, U_t, \dots$ over the base manifold $X : Y = Y_0 \underset{X}{\times} Y_t \underset{X}{\times} Y_x \underset{X}{\times} Y_{xt}$.*

(2) *The bundle $J_S^1(\pi)$ is the fiber product of **affine bundles***

$$J_S^1(\pi) = 0(Y_0) \underset{Y}{\times} J_{\langle \partial_t \rangle}^1(Y_t) \underset{Y}{\times} J_{\langle \partial_{x^A} \rangle}^1(Y_x) \underset{Y}{\times} J^1(Y_{tx}).$$

(3) *Bundle $J_S^1(\pi) \to Y$ is the factor-bundle of the bundle $J^1(\pi)$.*

(4) *Partial Cartan distribution C_S is generated by the 1-forms*

$$\omega^\mu = dy^\mu - z_t^\mu dt, \ \mu \in S_t, \ \omega^\mu = dy^\mu - z_A^\mu dx^A, \ \mu \in S_x; \omega^\mu = dy^\mu - z_i^\mu dx^i, \ \mu \in S_{tx}$$

in the sense that a section $\sigma : X \to J_S^1(\pi)$ is integrable: $\sigma = j_p^1(s)$ for a section $s : X \to Y$ if and only if for all i. $\sigma^* \omega^\mu = 0$ if restricted to the corresponding distribution.

Remark 3. Denote by $Aut_S(\pi)$ the subgroup of the group $Aut(\pi)$ consisting of the transformations $\phi \in Aut_S(\pi)$ such that $\bar{\phi}$ preserves the space-time product structure and, in addition, ϕ is the fiber product of automorphisms of the bundles $\pi_0, \pi_t, \pi_x \pi_{tx}$. Construction of the bundle $J_S^1(\pi)$ is invariant with respect to the transformations $\phi \in Aut_S(\pi)$.

3.3. **Prolongation of vector fields to the partial 1-jet bundles.** Below we will use the prolongations of the automorphisms ϕ of the bundle π whose projection $\bar{\phi}$ to X preserves the AP structure $T(X) = K \oplus K'$ (respectively, an S-structure) to the partial 1-jet bundle $J_K^1(\pi)$ (respectively to $J_S^1(\pi)$). Infinitesimal version of this procedure is the prolongation of a π-projectable vector field ξ whose projection $\bar{\xi}$ generates a (local) phase flow preserving the subbundle K (respectively, S-structure). In the case of a conventional k-jet bundle $J^k(\pi)$ this procedure is well known ([5, 13, 24]). In our situation the results are similar. We formulate corresponding results in the infinitesimal case for the bundle $J_K^1(\pi)$ with an integrable AP structure. For more general case of an arbitrary AP-structure we refer to [25].

Definition 2. (1) *Denote by $\mathcal{X}_K(\pi)$ the Lie algebra of π-projectable vector fields ξ in Y such that the projection $\bar{\xi}$ of ξ to X preserves the distribution $K \subset T(M)$: $\bar{\phi}_{t*}K = K$ for the local flow $\bar{\phi}_t$ of $\bar{\xi}$. This condition is equivalent to the infinitesimal condition $\mathcal{L}_{\bar{\xi}}K \subset K$.*
(2) *Denote by $\mathcal{X}_{K \oplus K'}(\pi)$ the Lie algebra of π-projectable vector fields ξ in Y such that the field $\bar{\xi}$ generated by ξ in X preserves the AP-structure, [15] $T(M) = K \oplus K'$: $\bar{\phi}_{t*}K = K, \bar{\phi}_{t*}K' = K'$ for the local flow $\bar{\phi}_t$ of $\bar{\xi}$.*

Lemma 2. *Let the AP-structure $T(X) = K \oplus K'$ is integrable and let (x^j, x^k) be a (local) integrating chart (i.e. $K = < \partial_{x^j} >$; $K' = < \partial_{x^k} >$). Then*
(1) *A π-projectable vector field $\xi = \xi^i(x)\partial_{x^i} + \xi^\mu(x,y)\partial_{y^\mu}$ belongs to $\mathcal{X}_K(\pi)$ if and only if*

$$\bar{\xi} = \xi^i(x^j, x^k)\partial_{x^i} = \xi^j(x)\partial_{x^j} + \xi^k(x^k)\partial_{x^k},$$

i.e. if the components $\xi^k(x)$ do not depend on the variables x^j.
(2) *A π-projectable vector field $\xi = \xi^i(x)\partial_{x^i} + \xi^\mu(x,y)\partial_{y^\mu}$ belongs to $\mathcal{X}_{K \oplus K'}(\pi)$ (preserves the almost product structure $T(X) = K \oplus K'$) if and only if*

$$\bar{\xi} = \xi^i(x)\partial_{x^i} = \xi^j(x^{j_1})\partial_{x^j} + \xi^k(x^k)\partial_{x^k},$$

Proof. We have $[\bar{\xi}, \partial_{x^j}] = -(\partial_{x^j} \cdot \xi^{j_1})\partial_{x^{j_1}} - (\partial_{x^j} \cdot \xi^k)\partial_{x^k}$. This vector field belongs to K if and only if $\partial_{x^j} \cdot \xi^k = 0$ for all j and k. The second statement is proved in the same way. $\square$

Proposition 3. *Let the AP-structure $T(X) = K \oplus K'$ is integrable and let (x^j, x^k) be a (local) integrating chart .*
(1) *For a π-projectable vector field $\xi = \xi^i(x)\partial_{x^i} + \xi^\mu(x,y)\partial_{y^\mu} \in \mathcal{X}_K(\pi)$ the following properties are equivalent*
   (a) *There exist a vector field $\xi^1 \in \mathcal{X}(J_K^1(\pi))$ such that*
      (i) *Local flow of the vector field ξ^1 preserves the partial Cartan distribution Ca_K (ξ^1 is the* Lie field *in terminology of [13]).*
      (ii) *$\pi_{10 \ *}\xi^1 = \xi$.*
   (b) *Vector field ξ has, in a local integrating chart (x^j, x^k) the form*

$$\xi = \xi^j(x^{j_1})\partial_{x^j} + \xi^k(x^{k_1})\partial_{x^k} + \xi^\mu(x^j, y)\partial_{y^\mu}. \tag{3.1}$$

In particular the projection $\bar{\xi}$ of the vector field ξ in X **preserves the almost product structure** $K \oplus K'$.

(c) Vice versa, any Lie vector field on $J^1_K(\pi)$ is the prolongation of a vector field $\xi \in \mathcal{X}_{K \oplus K'}(Y)$ of the form (3.1) (see [13] Ch.2).

(2) In the case where these conditions are fulfilled the vector field ξ^1 is unique and has the form

$$\xi^1 = \xi^j(x^{j_1})\partial_{x^j} + \xi^k(x^{k_1})\partial_{x^k} + \xi^\mu(x^j, y)\partial_{y^\mu} + \left(d_j\xi^\mu - z^\mu_{j_1}\frac{\partial\xi^{j_1}}{\partial x^j} \right)\partial_{z^\mu_j} \qquad (3.2)$$

(3) Mapping $\xi \to \xi^1$ is the homomorphism of Lie algebras: $[\xi, \eta]^1 = [\xi^1, \eta^1]$ for all $\xi, \eta \in \mathcal{X}_{K,K'}(\pi)$.

Proof. Let $\hat{\xi} = \xi^i(x)\partial_{x^i} + \xi^\mu(x,y)\partial_{y^\mu} + \lambda^\mu_j\partial_{z^\mu_j}$ be a prolongation to the partial jet bundle $J^1_K(\pi)$ of the vector field ξ. Then, condition of the preservation of the partial Cartan structure is equivalent to the condition that for all the generators $\omega^\mu_K = dy^\mu - \sum_j z^\mu_j dx^j$ of the contact ideal of exterior forms, $\mathcal{L}_{\hat{\xi}}\omega^\mu_K = \sum_\nu q^\mu_\nu \omega^\nu_K$, for some functions $q^\mu_\nu \in C^\infty(J^1_K(\pi))$. We calculate

$$\mathcal{L}_{\hat{\xi}}\omega^\mu_K = (di_{\hat{\xi}} + i_{\hat{\xi}}d)(dy^\mu - \sum_j z^\mu_j dx^j) = d[\xi^\mu - z^\mu_j\xi^j] + i_{\hat{\xi}}(-dz^\mu_j \wedge dx^j) =$$

$$= d\xi^\mu - \xi^j dz^\mu_j - z^\mu_j d\xi^j - \lambda^\mu_j dx^j + \xi^j dz^\mu_j = \xi^\mu_{,x^i}dx^i + \xi^\mu_{,y^\nu}dy^\nu - z^\mu_j[\xi^j_{,x^{j_1}}dx^{j_1} + \xi^j_{,x^k}dxk] -$$

$$- \lambda^\mu_j dx^j = \sum_\nu q^\mu_\nu(dy^\nu - \sum_j z^\nu_j dx^j), \quad (3.3)$$

or

$$(\xi^\mu_{,x^k} - z^\mu_j\xi^j_{,x^k})dx^k + \xi^\mu_{,y^j}dy^j + [\xi^i_{,x^j} - \lambda^\mu_j - z^\mu_{j_1}\xi^{j_1}_{,x^j}]dx^j = \sum_\nu q^\mu_\nu(dy^\nu - \sum_j z^\nu_j dx^j).$$

This equality is fulfilled if and only if we have

$$\begin{cases} \xi^\mu_{,x^k} - z^\mu_j\xi^j_{,x^k} = 0, \\ q^\mu_\nu = \xi^\mu_{,y^\nu}, \\ \xi^\mu_{,x^j} - \lambda^\mu_j - z^\mu_{j_1}\xi^{j_1}_{,x^j} = -q^\mu_\nu z^\nu_j. \end{cases}$$

Since neither ξ^μ nor ξ^j depend on z^μ_i first system is equivalent to the requirement that both ξ^μ and ξ^j are independent on x^k. Then the second condition determines q^μ_ν and third - $\lambda^\mu_j = \xi^\mu_{,x^j} + \xi^\mu_{,y^\nu}z^\nu_j - z^i_{j_1}\xi^{j_1}_{,x^j}$ and the prolongation $\hat{\xi}$ takes the form described in the Proposition. $\qquad\square$

Similar results have place for the bundles $J^1_S(\pi)$ defined by a space-time splitting S of the fields y^μ.

Proposition 4. (1) A vector field $\xi \in \mathcal{X}_S(\pi)$ preserves the AP-structure $T(X) = T(B) \oplus <\partial_t>$ if and only if $\bar{\xi} = \xi^i(x,t)\partial_{x^i} = \xi^A(x)\partial_{x^A} + \xi^t(t)\partial_t$,

(2) For any S-admissible π-projectable vector field $\xi \in \mathcal{X}_S(\pi)$ following statements are equivalent

(a) There is a vector field $\xi^1 \in \mathcal{X}(J^1_S(\pi))$ such that

(i) Vector field $\xi^1 \in \mathcal{X}(J^1_S(\pi))$ is π_{10}-projectable and $\pi_{10*}(\xi^1) = \xi$,

(ii) Local flow of the vector field ξ^1 preserves the partial Cartan distribution Co_S at $J^1_S(\pi)$.

(3) *Mapping $\xi \to \xi^1$ is the homomorphism of Lie algebras: $[\xi,\eta]^1 = [\xi^1,\eta^1]$ for all $\xi,\eta \in \mathcal{X}_K(Y)$.*

Detailed structure of vector fields ξ preserving the S-structure and form of their prolongations ξ^1 to the bundle $J_S^1(\pi)$ is described in [25], Ch.2, Prop. 11.

3.3.1. *Prolongation of infintesimal automorphisms $\xi \in \mathcal{X}_p(\pi)$ to the bundles $\Lambda_r^k(Y)$, $\Lambda_{2/1}^{n+(n+1)}(J^1(\pi))$.*

Definition 3. *(Definition-Proposition, [17], Def.3.3. for a case of euclidian metric G) Let α be a pullback to $\Lambda_2^n Y$ of a π-semibasic form $\alpha = \alpha^j(x,y)\eta_\nu$ on Y. Let $\xi \in \mathcal{X}(Y)$.*

(1) *Then there exist and is unique a vector field $\xi^{*\alpha}$ on Λ_2^n satisfying to the following conditions*
   (a) *Vector field $\xi^{*\alpha}$ is $\pi_{\Lambda_2^n Y\ Y}$-projectable and*

$$\pi_{\Lambda_2^n Y\ Y\ *}\xi^{*\alpha} = \xi,$$

   (b)

$$\mathcal{L}_{\xi^{*\alpha}}\Theta_2^n = d\alpha.$$

(2) *In an adopted chart (x^μ, y^i) the vector field $\xi^{*\alpha}$ has the form*

$$\xi^{*\alpha} = \xi + \xi^{*\alpha\ p}\partial_p + \xi^{*\alpha\ p_\mu^i}\partial_{p_\mu^i},\ \text{where}$$

$$\xi^{*\alpha\ p} = -p\left(\frac{\partial\xi^i}{\partial x^i} + \xi^i\lambda_{G,i}\right) - p_\mu^i\frac{\partial\xi^\mu}{\partial x^i} + \left(\frac{\partial\alpha^i}{\partial x^i} - \alpha^j\lambda_{G,j}\right) = -p\cdot div_G(\bar{\xi}) - p_\mu^\mu\frac{\partial\xi^\mu}{\partial x^i} + div_G(\bar{\alpha});$$

$$\xi^{*\alpha\ p_\mu^i} = p_\mu^j\frac{\partial\xi^i}{\partial x^j} - p_\nu^i\frac{\partial\xi^\nu}{\partial y^\mu} - p_\mu^i\left(\frac{\partial\xi^j}{\partial x^j} + \xi^j\lambda_{G,j}\right) + \frac{\partial\alpha^i}{\partial y^\mu} = p_\mu^j\frac{\partial\xi^i}{\partial x^j} - p_\nu^i\frac{\partial\xi^\nu}{\partial y^\mu} - p_\mu^i\cdot div_G(\bar{\xi}) + \frac{\partial\alpha^i}{\partial y^\mu}.$$
$$(3.4)$$

     Here $\bar{\xi} = \xi^i\partial_{x^i},\ \overline{\alpha} = \alpha^j(x,y)\partial_{x^j}$.
(3) *Let a vector field $\xi \in \mathcal{X}(Y)$ be π-projectable. Then the 0-lift ξ^{*0} of ξ coincide with the flow prolongation ξ^{1*} defined above.*

Below we will use the lift of a projectable vector field ξ to the bundle $\Lambda_{2/1}^{(n)+(n+1)}Y$.
Next result allows to lift ξ to the infinitesimal transformation on the bundle $\Lambda_2^{(n+1)}Y$ leaving invariant the canonical form $p_\mu\omega^\mu \wedge \eta$.

Proposition 5. *For any π-projectable vector field $\xi \in X(\pi)$ there exists unique projectable (to Y) vector field $\xi^{*(n+1)}$ on the bundle $\Lambda_2^{(n+1)}Y$ that leaves the canonical form $p_\sigma dy^\sigma \wedge \eta$ invariant. That vector is given by the relation*

$$\xi^{*(n+1)} = \xi + \xi^{p_\sigma}\partial_{p_\sigma},\ \xi^{p_\sigma} = -p_\sigma(\xi^i\lambda_{G,i} + \frac{\partial\xi^i}{\partial x^i}) - p_\nu\frac{\partial\xi^\nu}{\partial y^\sigma} = -p_\sigma div_G(\bar{\xi}) - p_\nu\frac{\partial\xi^\nu}{\partial y^\sigma},$$
$$(3.5)$$

   where $\lambda_G = ln(|G|)$.

Proof. We have, for a vector field of the form $\xi^{*(n+2)} = \xi + \xi^{p_\sigma}\partial_{p_\sigma}$

$$\mathcal{L}_{\xi^{*(n+1)}}(p_\mu dy^\mu \wedge \eta) =$$

$$= (di_{\xi^{*(n+1)}} + i_{\xi^{*(n+1)}}d)(p_\mu dy^\mu \wedge \eta) = d[p_\mu \xi^\mu \eta - p_\mu \xi^i dy^\mu \wedge \eta_i] + i_{\xi^{*(n+1)}}(dp_\mu \wedge dy^\mu \wedge \eta) =$$

$$= [\xi^\mu dp_\mu \wedge \eta + p_\mu d\xi^\mu \wedge \eta - \xi^i dp_\mu \wedge dy^\mu \wedge \eta_i - p_\mu d\xi^i \wedge dy^\mu \wedge \eta_i + p_\mu \xi^i \lambda_{G,i} dy^\mu \wedge \eta] +$$

$$+ [\xi^{p_\mu} dy^\mu \wedge \eta - \xi^\mu dp_\mu \wedge \eta + \xi^i dp_\mu \wedge dy^\mu \wedge \eta_i] = [\xi^\mu dp_\mu \wedge \eta + p_\mu \xi^\mu_{,y^\nu} dy^\nu \wedge \eta - \xi^i dp_\mu \wedge dy^\mu \wedge \eta_i +$$

$$+ p_\mu \xi^i_{,i} \wedge dy^\mu \wedge \eta + p_\mu \xi^i \lambda_{G,i} dy^\mu \wedge \eta] + [\xi^{p_\mu} dy^\mu \wedge \eta - \xi^\mu dp_\mu \wedge \eta + \xi^i dp_\mu \wedge dy^\mu \wedge \eta_i] =$$

$$= [\xi^{p_\mu} + p_\nu \xi^\nu_{,y^\mu} + p_\mu(\xi^i_{,i} + \xi^i \lambda_{G,i})]dy^\mu \wedge \eta. \quad (3.6)$$

Here we have used the relation $d\eta_i = \lambda_{G,i}\eta$.

Equating the obtained expression to zero we get the expression for ξ^{p_μ} as in the Proposition. $\qquad\square$

Combining the last result with the prolongation ξ^{*0} from the Definition-Proposition 10 and with the prolongation $\xi^{*(n+1)}$ from the previous Proposition and using factorization by $\Lambda_1^{(n)+(n+1)}Y$ we get the following

Corollary 1. *For any projectable vector field $\xi \in X(\pi)$ there exists unique projectable vector field ξ^{1*} in the space $\Lambda_2^{n+(n+1)}Y$ - prolongation of ξ, preserving the $n + (n+1)$ form $p\eta + p_\mu^\mu dy^\mu \wedge \eta_i + p_\nu dy^\nu \wedge \eta$.*

Vector field ξ^{1*} projects to $\Lambda_{2/1}^{(n+1)+(n+2)}Y$ to the vector field $\tilde{\xi}^1$ preserving the form $p_\mu^\mu dy^\mu \wedge \eta_i + p_\nu dy^\nu \wedge \eta \mod \eta$.

Below we will also need to define prolongations of projectable vector fields $\xi \in \mathcal{X}_p(\pi)$ to the space of the $n + (n+1)$-forms $\Lambda_{2/1}^{n+(n+1)}(J^1(\pi))$ on the 1-jet bundle $J^1(\pi) \to Y$. Next statement, whose proof is presented in the Appendix 1, describes these prolongations. Local coordinates in this bundle corresponding to a fibred chart (W, x^i, y^μ) is defined at the presentation of elements of the fibers:

$$p\eta + p_i^\mu dy^\mu \wedge \eta_i + q_\mu \omega^\mu \wedge \eta + q_\mu^i \omega_i^\mu \wedge \eta. \quad (3.7)$$

Proposition 6. *For any projectible vector field $\xi = \xi^i \partial_i + \xi^\mu \partial_\mu + \xi_i^\mu \partial_{z_i^\mu} \in \mathcal{X}(\pi_0^1)$ on the space $J_p^1(\pi)$, there exist unique vector field $\xi^* \in \mathcal{X}(\Lambda_2^{n+1}(J^1(\pi)))$ such that*

(1) *Canonical $n+1$-form $Q^{n+1} = (q_\mu \omega^\mu + q_\mu^i \omega_i^\mu) \wedge \eta$ (see ()) is invariant with respect to the flow of vector field ξ^*: $\mathcal{L}_{\xi^*}Q^{n+1} = 0$.*

(2) *This vector field is given by the following expression*

$$\xi^* = \xi + (-q_\mu \xi^\mu_{,y^\nu} - q_\mu^i \xi^\mu_{i,y^\nu} - q_\nu div_G(\bar{\xi}))\partial_{q_\nu} + (-q_\mu \xi^\mu_{,z_j^\nu} - q_\mu^i \xi^\mu_{i,z_j^\nu} - q_\nu^i div_G(\bar{\xi}))\partial_{q_\nu^j}. \quad (3.8)$$

3.4. Partial higher order jet bundles $J_p^k(\pi)$. Here we introduce the higher order partial jet bundles defined by an AP-structure $T(X) = K \oplus K'$ and, more generally, by an S-structure. **In the rest of the paper we will understand by $J_p^k(\pi), k = 1, \ldots, \infty$ one of the bundles** $J^k(\pi), J_K^k(\pi), J_S^k(\pi), J_0(\pi)$. One can introduce more general notion of higher order (and infinite) partial jet bundle of a bundle π, but this will be done elsewhere.

Let Z_+^n be the set of multiindices $\{I = (i_1, \ldots, i_n)|i_l \in N\}$. Let $T(X) = K \oplus K'$ be an integrable AP-structure on X, and let $U, (x^j, j = 1, \ldots n_K; x^k, k = n_K + 1, \ldots n)$ be a (local) integrating chart in X. Distribution K (respectively K') in this chart is generated by vector fields ∂_{x^j} (respectively, by vector fields ∂_{x^k}). Let $N_K =$

$\{(1,0,\ldots,0),(0,1,\ldots,0)\ldots(0,\ldots,0,1_{n_K},0,\ldots)\subset Z_+^n\}$ is the set of multiindices corresponding to the derivatives in K. Finally let $\hat{N}_K = N_K + Z_+^n$ be the set of all multiindices larger or equal to the multiindices from N_K with respect to the natural order in Z_+^n.

Definition 4. *Let $\pi_{XY} : Y \to X$ be a fiber bundle and let $K \subset T(X)$ be a subbundle.*

   (1) *Let $x \in U \subset X$, $s_1, s_2 \in \Gamma(U,\pi)|s_1(x) = s_2(x)$. Sections s_1, s_2 are called K-**equivalent of order k at a point** $x \in X$: $s_1 \sim_{K_x}^k s_2$ if $\partial^I s_1(x) = \partial^I s_2(x)$, $\forall I \in \hat{N}_K$.*
   (2) *$J_{K\ x}^k(\pi)$- space of classes of $\sim_K^k$ at a point $x \in X$.*
   (3) *$J_K^k(\pi) = \cup_{x \in X} J_{K\ x}^k(\pi)$ - the space of partial k-jets of sections of π.*
   (4) *Infinite jet bundle $J_K^\infty(\pi)$ is the inverse limit of the bundles $J_K^k(\pi)$ under the natural projections $\pi_{k(k-1)} : J_K^k(\pi) \to J_K^{k-1}(\pi)$.*

Speaking simply, bundle $J_K^k(\pi)$ carry information about the derivatives $\partial_{x^j} s$ of section $s \in \Gamma(\pi)$ and all derivatives $\partial^I \partial_{x^j} s$ up to the order k.

In the next proposition we collect some basic properties of bundles $J_K^1(\pi)$ that will be used below.

Proposition 7.   (1) *Bundles $\pi_{k(k-1)} : J_K^k(\pi) \to J_K^{k-1}(\pi)$ form the inverse sequence of the affine bundles.*
   (2) *There is a canonical surjection of affine bundles $w_K : J^k(\pi) \to J_K^k(\pi)$ such that the diagram*

$$
\begin{array}{ccc}
J^k(\pi) & \xrightarrow{\ w_K\ } & J_K^k(\pi) \\
\pi_{k(k-1)} \downarrow & & \pi_{k(k-1)} \downarrow \\
J^{k-1}(\pi) & \xrightarrow{\ w_K\ } & J_K^{k-1}(\pi)
\end{array}
$$

is commutative.
   (3) *Let $T(X) = K \oplus K'$ be an integrable AP-structure, (x^j, x^k) - local chart integrating the $AP-$ structure. The 1-forms*

$$
\begin{cases}
\omega^\mu = dy^\mu - \sum_{j \in K} z_j^\mu dx^j;\ i \in \overline{1,m}, \\
\omega_J^\mu = dz_J^\mu - \sum_l z_{Jl}^\mu dx^l,\ J \in \hat{N}_K,\ 0 < |J| < k,\ \mu \in \overline{1,m}
\end{cases}
$$

generate the partial Cartan distribution C_K^k *on the bundle $J_K^k(\pi) \to X$.*
   (4) *The "total derivative" operator $d_i = \partial_{x^i} + \sum_{(i,\mu) \in P} z_i^\mu \partial_{y^\mu} + \sum_{(I,\mu) \in P} z_{I+i}^\mu \partial_{z_I^\mu}$ is defined on the functions $C^\infty(J_K^k(\pi))$ and has the same properties as the usual total derivatives (commutativity, etc). Here and below we use the notation $(\mu, I) \in P$ for the set of variables $z_I^\mu, |I| > 0$ that are present in the partial k-jet bundle $J_p^k(\pi)$. The set P if indices is invariant under the admissible automorphisms of $J_p^k(\pi)$*
   (5) *A section q of the bundle $J_K^k(\pi)$ is the (partial) k-jet of a section $s : X \to Y$ if and only if $q^*(\omega_J^\nu)|_K = 0$ for all 1-forms listed in p.3.*
   (6) *Any π-projectable vector field $\xi \in \mathcal{X}(Y)$ preserving the AP-structure $T(X) = K \oplus K'$ can be uniquely prolonged (by the flow prolongation) to the projectable vector field $\hat{\xi}^k \in X(J_K^k(\pi))$ preserving the partial Cartan distribution C_K^k. For a vector field $\xi = \xi^i \partial_{x^i} + \xi^\mu \partial_{y^\mu}$ the k-th order prolongation*

has the form (see [24]*, Thm. 2.36)*

$$pr^{(k)}\xi = \xi + \sum_{(\mu, I) \in P} \xi_\mu^I \partial_{z_I^\mu}, \ \ \phi_\mu^I = d_I(\xi^\mu - \sum_i \xi^i z_i^\mu) + \sum_i \xi^i z_{I+i}^\mu. \qquad (3.9)$$
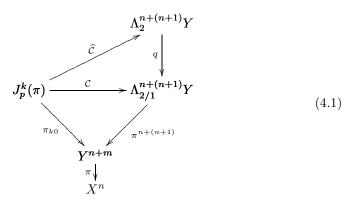
(7) *Contact decomposition (2.3) of the forms is valid for the tower of partial jet-bundles* $J_K^k(\pi)$.

Proof of almost all statements in this Proposition is straightforward, using the standard structural properties of the higher order jet bundles, see [12, 24]. In the last statement the prolongation is the standard flow prolongation of an admissible vector field, see Proposition 3 above. and Lemma before it.

For a space-time splitting S of the space of fields y^i, see Sec.3.2, one can similarly define the bundles $J_S^k(\pi), J_S^\infty(\pi)$, their partial Cartan structure and establish the prolongation properties of π-projectable vector fields $\xi \in \mathcal{X}(Y)$ preserving the S-structure of the bundle π and the existence of the contact decomposition of exterior forms. We will use these results without further references.

4. Constitutive relations (CR) and their Poincare-Cartan forms

Here we introduce the constitutive and covering constitutive relations of order k as generalized Legendre transformation from the (partial) k-jet bundle $J_p^k(\pi)$ to the extended multisymplectic bundles of $n + (n + 1)$-forms on the manifold Y, see next commutative diagram

$$
\begin{array}{c}
\Lambda_2^{n+(n+1)}Y \\
\nearrow{\scriptstyle \widehat{\mathcal{C}}} \qquad \downarrow{\scriptstyle q} \\
J_p^k(\pi) \xrightarrow{\ \mathcal{C}\ } \Lambda_{2/1}^{n+(n+1)}Y \\
\searrow{\scriptstyle \pi_{k0}} \qquad \swarrow{\scriptstyle \pi^{n+(n+1)}} \\
Y^{n+m} \\
\downarrow{\scriptstyle \pi} \\
X^n
\end{array}
\qquad (4.1)
$$

Definition 5.   (1) *A **covering constitutive relation (CCR)** $\hat{C}$ of order k is morphism of bundles $\hat{C} : J_p^k(\pi) \to \Lambda_2^{n+(n+1)}Y$ over Y:*

$$\widehat{\mathcal{C}}(x^i, y^\mu, z_i^\mu, \ldots, z_{i_1\ldots i_k}^\mu) = (x^i, y^\mu; p; F_\mu^i; \Pi_\mu), \ p, F_\mu^i, \Pi_\mu \in C^\infty(J_p^k(\pi)).$$

The Poincare-Cartan form of the CCR $\hat{C}$ *is the form*

$$\Theta_{\hat{C}} = \hat{\mathcal{C}}^*(\Theta_2^n + \Theta_2^{n+1}) = p\eta + F_\mu^i dy^i \wedge \eta_i + \Pi_\mu dy^\mu \wedge \eta. \qquad (4.2)$$

(2) *A **constitutive relation (CR)** C of order k is a morphism of bundles $C : J_p^k(\pi) \to \Lambda_{2/1}^{n+(n+1)}Y$ over Y:*

$$C(x^i, y^\mu, z_i^\mu, \ldots, z_{i_1\ldots i_k}^\mu) = (x^i, y^\mu; F_\mu^i; \Pi_\mu), \ F_\mu^i, \Pi_\mu \in C^\infty(J_p^k(\pi)).$$

The Poincare-Cartan form of the CR $\mathcal{C}$ *is the form defined mod* η

$$\Theta_C = \mathcal{C}^*(\Theta_2^n + \Theta_2^{n+1} \ mod \ \eta) = F_\mu^i dy^\mu \wedge \eta_i + \Pi_\mu dy^\mu \wedge \eta \ mod \ \eta.$$

When the order of a CR or CCR is one ($k = 1$) we will omit the words "of order 1".

Any covering constitutive relation defines the corresponding constitutive relation by the projection - $C = q \circ \hat{C}$. On the other hand, there are many ways to lift a constitutive relation to the CCR. One of them (for $k = 1$) is to use the connection on the bundle π, see [14]. In between all possible CCR corresponding to a given CR there is one privileged (as we will see in the next section)

Definition 6. *The **lifted CCR** $\tilde{C}$ **of a constitutive relation** C (of order k) is defined by*

$$\Theta_{\tilde{C}} = F_\mu^i \omega^\mu \wedge \eta_i + \Pi_\mu \omega^\mu \wedge \eta = -(\sum_{(\mu,j)\in P} z_j^\nu F_\nu^j)\eta + F_\mu^i dy^\mu \wedge \eta_i + \Pi_\mu dy^\mu \wedge \eta. \quad (4.3)$$

As the next Lemma shows, the Poincare-Cartan form of the lifted CCR can be obtained with the help of the vertical endomorphism S_η. Proof of this Lemma is straightforward.

Lemma 3. *Let $k = 1$. For a 1-form $\chi = F_\mu^i dz_i^\mu + \Pi_\mu dy^\mu \in \Lambda^1(J^1(\pi))$, one has*

$$\Theta_{\tilde{C}} = S_\eta^*(F_\mu^i dz_i^\mu + \Pi_\mu dy^\mu).$$

Remind, that the Poincare-Cartan form Θ_L of the the Lagrangian form $L\eta$ is an example of the *Lepage form* corresponding to the Lagrangian L (ref). Having defined the Poincare-Cartan form of a balance system with the covering constitutive relations $\hat{C}$ it is interesting to see when such a form may be a Lepage form and if yes, to which Lagrangian it corresponds. Next result provides the answer to this question.

Lemma 4. *The term $\Theta_{\hat{C}}^n$ of the Poincare-Cartan form (9.4) of a covering constitutive relation $\hat{C}$ is the* Lepage form *in $J^k(\pi)$ if and only if $\hat{C}$ is the CCR of order one, i.e. that $p, F_\mu^i \in \pi_1^{k*}C^\infty(J^1(\pi))$. If this condition is fulfilled, the associated Lagrangian of the Lepage form $\Theta_{\hat{C}}^n$ is equal to $L = p + \sum_{(\mu,i)\in P} z_i^\mu F_\mu^i$.*

Proof. To prove the first statement we write the form $\Theta_{\hat{C}}^n$ as follows

$$\Theta_{\hat{C}}^n = p\eta + F_\mu^i dy^\mu \wedge \eta_i = (p + z_i^\mu F_\mu^i)\eta + F_\mu^i \omega^\mu \wedge \eta_i,$$

and denote $\tilde{p} = p + z_i^\mu F_\mu^i$. Now we calculate

$$\pi_k^{(k+1)*}d\Theta_{\hat{C}}^n = \pi_k^{(k+1)*}(d\tilde{p} \wedge \eta + dF_\mu^i \wedge \omega^\mu \wedge \eta_i + F_\mu^i d\omega^\mu \wedge \eta_i - F_\mu^i \lambda_{G,i}\omega^\mu \wedge \eta) =$$

$$= (d_h\tilde{p} + d_v\tilde{p}) \wedge \eta + (d_hF_\mu^i + d_vF_\mu^i) \wedge \omega^\mu \wedge \eta_i - F_\mu^i dz_j^\mu \wedge dx^j \wedge \eta_i - F_\mu^i \lambda_{G,i}\omega^\mu \wedge \eta) =$$

$$= (\sum_{(\nu,I),|I|\geqq 0} \tilde{p}_{,z_I^\nu}\omega_I^\nu)\wedge\eta + d_jF_\mu^i dx^j \wedge\omega^\mu\wedge\eta_i + (\sum_{(\nu,I),|I|\geqq 0} F_{\mu,z_I^\nu}^i\omega_I^\nu)\wedge\omega^\mu\wedge\eta_i - F_\mu^i dz_i^\mu\wedge\eta - F_\mu^i\lambda_{G,i}\omega^\mu\wedge\eta =$$

$$= -(d_iF_\mu^i + F_\mu^i\lambda_{G,i})\omega^\mu\wedge\eta + \tilde{p}_{,y^\nu}\omega^\nu\wedge\eta + (\tilde{p}_{,z_i^\mu} - F_\mu^i)\omega_i^\mu\wedge\eta + \sum_{(\nu,I),|I|>1} \tilde{p}_{,z_I^\nu}\omega_I^\nu\wedge\eta + 2Con.$$

In the last expression first four terms present 1-contact part, the last one - 2-contact part $(\sum_{(\nu,I),|I|\geqq 0} F_{\mu,z_I^\nu}^i\omega_I^\nu) \wedge \omega^\mu \wedge \eta_i$. 1-contact part is π_0^{k+1}-horizontal if and only if conditions

$$\begin{cases} \tilde{p}_{,z_I^\nu} = 0, \ |I| > 1, \\ \tilde{p}_{,z_i^\mu} - F_\mu^i = 0, \ \forall(\mu,i). \end{cases}$$

First condition requires that $\tilde{p} \in \pi_1^{k*} C^\infty(J^1(\pi))$. If this fulfilled, then the second condition requires that $F_\mu^i \in \pi_1^{k*} C^\infty(J^1(\pi))$ and therefore the CCR $\hat{C}$ is of the order one. What is left is that the second condition tells that the density/flux part F_μ^i components of constitutive relations come from the "Lagrangian" $\tilde{p}$. Below such constitutive relations will be called "semi-Lagrangian". $\qquad\square$

4.1. **Examples.** Present now several examples of different types of constitutive relations or covering constitutive relations and corresponding Poincare-Cartan forms. In all these examples $k = 1$.

Example 5. A **Lagrange constitutive relation of order 1** is defined by a function $L \in C^\infty(J^1(\pi))$:

$$C_L(x^i, y^\mu, z_i^\mu) = (p_\mu^i = F_\mu^i = \frac{\partial L}{\partial z_i^\mu}; \Pi_\mu = \frac{\partial L}{\partial y^\mu}).$$

Balance system for such a constitutive relation (see next section) coincide with the conventional system of Euler-Lagrange Equations (2.10) of the first order Lagrangian Field Theory.

Remark 4. For $k = 1$ Euler-Lagrange Equations (2.10) have the form of a balance system with canonically defined densities and flux components. For $k > 1$ Euler-Lagrange equations can also be written as a balance system but in this case there are different ways to specify flux and even density components due to the presence of higher derivatives. This non-uniqueness is similar to the non-uniqueness of a choice of Lepage form for higher order Lagrangian Variational Theory, see [4]. That why in the Lagrangian case we take $k = 1$ only.

Example 6. A **semi-Lagrangian CCR** $\hat{C}_{L,Q}$ is defined by a functions $L, Q_\mu, \mu = 1, \ldots, m \in C^\infty(J^1(\pi))$:

$$\hat{C}_{L,Q_\mu}(x^i, y^\mu, z_i^\mu) = (p = L - z_i^\mu L_{,z_{,i}^\mu}, p_\mu^i = \frac{\partial L}{\partial z_i^\mu}; \Pi_\mu = Q_\mu(x^i, y^\mu, z_i^\mu)).$$

Example 7. $L + D$**-system**. Let L Lagragian, D ("dissipative potential") $\in C^\infty(J_p^1(\pi))$. Let the time derivatives of the fields y^μ - z_0^μ are present in the partial 1-jet bundle $J_p^1(\pi)$. The **CR** $\mathcal{C}_{L,D}$ is defined by its Poincare-Cartan form

$$\Theta_{L,D} = L_{z_i^\mu} dy^\mu \wedge \eta_i + (D_{z_{,0}^\mu} - L_{,y^\mu}) dy^\mu \wedge \eta.$$

balance system for such constitutive relation have the form

$$\frac{\delta L}{\delta y^\mu} = \frac{\partial D}{\partial \dot{y}^\mu}$$

which is well known in the Continuum Thermodynamics, [18, 19].

Remark 5. If we would like to define directly the analog of Lagrangian or semi-Lagrangian relation on a partial 1-jet bundle we would have a situation where an absence of a variable z_i^μ from the partial 1-jet bundle leads to the nullity of the corresponding flux component. This restricts an application of a conventional semi-Lagrangian CR defined on a partial jet bundles. On the other hand, it is possible that the CR are *partly variational* in the sense that part of the flux components are defined by a "partial Lagrangian" while other components are to be defined independently, see below, Prop.10.

Example 8. Consider a case where only spacial but not time derivatives of fields y^μ enters the constitutive relations: $J_p^1(\pi) = J_{T(B)}^1(\pi)$. Then it is possible that for a function $L \in C^\infty(J_{T(B)}^1(\pi))$, $F_\mu^A = \frac{\partial L}{\partial z_A^\mu}$, $A = 2, \ldots, n$; $\Pi_\mu = \frac{\partial L}{\partial y^\mu}$ but densities F_μ^0 are to be defined independently. Then,

$$\Theta_C = F_\mu^0 dy^\mu \wedge \eta_0 + \frac{\partial L}{\partial z_A^\mu} dy^\mu \wedge \eta_A + \frac{\partial L}{\partial y^\mu} dy^\mu \wedge \eta.$$

The system of balance equations corresponding to this CR is, of course, first order by time derivatives.

Five fields fluid system (Sec.2.7) is an example of such balance system.

Example 9. Vector-potential CR. (RET case, dual variables) In this case $J_p^1(\pi) = \{\cdot\}$ has a zero dimensional (point) fiber over Y. Let $h = h^i(x, y)\eta_i$ be a semi-basic (n-1)-form on Y. Define the **CR**

$$C_h(x^i, y^\mu) = (p_\mu^i = \frac{\partial h^i}{\partial y^\mu}; \Pi_\mu = \Pi_\mu(x, y)).$$

4.2. **Form K_C and the Helmholtz coindition.** Let C be a constitutive relations of first order with the lifted Poincare-Cartan form $\Theta_{\widetilde{C}} = F_\mu^i \omega^\mu \wedge \eta_i + \Pi_\mu \omega^\mu \wedge \eta$. Associate with this constitutive relation the following contact (n+1)-form on $J^2(\pi)$.

$$K_C = (F_\mu^i \omega_i^\mu + \Pi_\mu \omega^\mu) \wedge \eta. \tag{4.4}$$

Calculate differential of this form:

$$dK_C = [(\partial_{y^\nu} \Pi_\mu) dy^\nu \wedge dy^\mu + (\partial_{y^\nu} F_i^\mu - \partial_{z_j^\nu} \Pi_\mu) dy^\nu \wedge dz_i^\mu + (\partial_{z_j^\nu} F_i^\mu) dz_j^\nu \wedge dz_i^\mu] \wedge \eta. \tag{4.5}$$

Now we use the following form of Poincare Lemma

Lemma 5. *Let $f_i, i = 1, \ldots, k$; g_j, $j = 1, \ldots, s$ be functions of all variables $z^i, i = 1, \ldots, k$; y^j, $j = 1, \ldots, s$ such that the form $K = f_i dz^i + g_j dy^j$ is closed: $dK = 0$. Then, locally, $K = dL$, i.e. $f_i = \partial_{z^i} L$, $g_j = \partial_{y^j} L$ for some smooth function $L = L(z^i, y^j)$.*

Applying this Lemma to the second form of equality (4.5) we get the following version of (local) Helmholtz condition (comp. [**?**]) for a balance system of order one to be Euler-Lagrange system for some Lagrangian $L \in C^\infty(J^1(\pi))$.

Proposition 8. *For a constitutive relations C of order 1 the following properties are equivalent:*

  (1) *Form K_C is closed: $dK_C = 0$,*
  (2) *C is locally Lagrangian constitutive relation: $C = C_L$ for a (locally defined) function $L \in C^\infty(J^1(\pi))$.*

Remark 6. Construction of the form K_C is related to the mapping $\omega^\mu \wedge \eta_i \to \omega_i^\mu \wedge \eta$ of $J^1(\pi) \to J^2(\pi)$. It would be interesting to construct such a mapping from $J^k(\pi)$ to $J^{k+1}(\pi)$ for $k > 1$.

4.3. **Variational sequence and the balance systems.** Condition of the Proposition 8 for a balance system to be Lagrangian points to the possibility to interpret and study the balance systems using the variational bicomplex, see [1, 24], or the variational sequence, [29]. In this section we realize this possibility. In presenting the variational bicomplex we will follow [29].

We will be using the augmented variational bicomplex in the form

$$
\begin{array}{ccccccccc}
0 & & 0 & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
R & & R & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow \Omega^0(M) \xrightarrow{\pi_\infty^*} E_0^{0,0} \xrightarrow{d_v} E_0^{1,0} \dashrightarrow E_0^{p,0} \xrightarrow{d_v} E_0^{p+1,0} \xrightarrow{\cdots} \\
d\downarrow \quad d_h\downarrow \quad -d_h\downarrow \quad (-1)^p d_h\downarrow \quad (-1)^{p+1}d_h\downarrow \\
0 \longrightarrow \Omega^1(M) \xrightarrow{\pi_\infty^*} E_0^{0,1} \xrightarrow{d_v} E_0^{1,1} \dashrightarrow E_0^{p,1} \xrightarrow{d_v} E_0^{p+1,1} \xrightarrow{\cdots} \\
\cdots\downarrow \quad \cdots\downarrow \quad \cdots\downarrow \quad \cdots\downarrow \quad \cdots\downarrow \\
0 \longrightarrow \Omega^{n-1}(M) \xrightarrow{\pi_\infty^*} E_0^{0,n-1} \xrightarrow{d_v} E_0^{1,n-1} \dashrightarrow E_0^{p,n-1} \xrightarrow{d_v} E_0^{p+1,n-1} \xrightarrow{\cdots} \\
d\downarrow \quad d_h\downarrow \quad -d_h\downarrow \quad (-1)^p d_h\downarrow \quad (-1)^{p+1}d_h\downarrow \\
0 \longrightarrow \Omega^n(M) \xrightarrow{\pi_\infty^*} E_0^{0,n} \xrightarrow{d_v} E_0^{1,n} \dashrightarrow E_0^{p,n} \xrightarrow{d_v} E_0^{p+1,n} \xrightarrow{\cdots} \\
\downarrow \quad \pi\downarrow \quad I\downarrow \quad I\downarrow \quad I\downarrow \\
0 \longrightarrow E_1^{0,n} \xrightarrow{e_1} E_1^{1,n} \dashrightarrow E_1^{p,n} \xrightarrow{e_1} E_1^{p+1,n} \xrightarrow{\cdots} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \quad 0 \quad 0 \quad 0
\end{array}
$$
$$(4.6)$$

In this diagram, $E_0^{0,q} = \Omega^{0,q}(J^\infty(\pi))$ - the bundle of smooth horizontal q-forms on $J^\infty(\pi)$, $E_0^{p,q} = C^p \bigwedge \Omega^{0,q}(J^\infty(\pi))$ is the bundle of p-vertical and q-horizontal $p+q$-forms. Terms of the lowest line represent the fist derived complex

$$
E_1^{p,n} = E_0^{p,n}/d_h(E_0^{p,n-1}) = C^p \bigwedge \Omega^{0,n}/d_h(C^p \bigwedge \Omega^{0,n-1}).
$$

The mappings $e_1 : E_1^{p,n} \to E_1^{p+1,n}$, $e_1([\alpha]) = [d_v\alpha]$ participate in the last horizontal line of the bicomplex as well as in the second half of the induced variational sequence (where Euler-Lagrange mapping $\mathcal{E} = e_1 \circ \pi$, π being the quotient projection defining $E_1^{0,n}$)

$$
0 \to \mathbf{R} \to E_0^{0,0} \xrightarrow{d_h} E_0^{1,0} \xrightarrow{d_h} \dots E_0^{0,n-1} \xrightarrow{d_h} E_0^{0,n} \xrightarrow{\mathcal{E}} E_1^{1,n} \xrightarrow{e_1} E_1^{2,n} \dots\dots\dots E_1^{p,n} \xrightarrow{e_1} E_1^{p+1,n} \cdots
$$
$$(4.7)$$

Operator I is the *interior Euler* operator $I : \Omega^{s,n}(J^\infty(\pi)) \to \Omega^{s,n}(J^\infty(\pi))$, defined by ([1])

$$
I(\omega) = \frac{1}{s}\omega^i \wedge \left[i_{\partial_i}\omega - d_\mu(i_{\partial_{z_\mu^i}}\omega + d_{\mu_1\mu_2}(i_{\partial_{z_{\mu_1\mu_2}^i}}\omega) - \dots \right].
$$
$$(4.8)$$

Interior euler operator is closely related to the Euler-Lagrange operator $\mathcal{E}$, namely (see [1], Ch. 2), for any Lagrangian n-form $\lambda = L\eta$

$$
E(\lambda) = I \circ d_v(\lambda).
$$
$$(4.9)$$

That is why the next proposition is hardly surprising

Proposition 9. *Let* $\mathcal{C}$ *be a CR and* $K_{\mathcal{C}} = \Pi_i \omega^i \wedge \eta + F_i^\mu \omega_\mu^i \wedge \eta$ *- corresponding K-form introduced in (4.4). Then, the balance system* ★ *is equivalent to the equation*

$$j^* s i_{\xi^1} I(K_{\mathcal{C}}) = 0, \tag{4.10}$$

for all $\xi \in \mathcal{V}(\pi)$.

Proof.

$$I(K_{\mathcal{C}}) = \omega^i \wedge [\Pi_i \eta - d_\mu(F_i^\mu \eta)] = \omega^i \wedge [\Pi_i - d_\mu F_i^\mu - F_i^\mu \lambda_{G,\mu}]\eta. \tag{4.11}$$

Here we have used the form $\eta = \sqrt{|G|} dx^1 \wedge \ldots \wedge dx^n$.

Applying i_{ξ^1} for any variational vector field $\xi \in \Gamma(\mathcal{V}(\pi))$ and taking pullback by the mapping $J^* s$ we get equation () in the form

$$\omega^i(\xi) \circ j^1 s [d_\mu F_i^\mu + F_i^\mu \lambda_{G,\mu} - \Pi_i] \circ j^\infty s \eta = 0.$$

Fulfillment of this equality for all (or many enough) variational vector fields ξ. is equivalent to the statement that s is the solution of the balance system ★.   □

5. Variational form of balance systems

In this section we present the invariant variational form of balance system and the separation of this invariant form into the m separate balance laws by independent variations. In difference to the Lagrangian case, in general, one has to put a condition on the variations that can be used for separating equations. In cases of semi-Lagrangian, order 1 or RET balance systems that condition is fulfilled for all variations as in the conventional Lagrangian theory. In general this situation can be remedied by modifying the source term $\Pi_\mu dy^\mu \wedge \eta$ - addition to it some contact form (see below, Sec.5.2). We prove the main result for the case of a constitutive relation of arbitrary finite order, so, for simplicity we will consider that a CR $\mathcal{C}$ is defined on the infinite partial jet bundle $J_p^\infty(\pi)$ but is π_k^∞-projectable for some $k < \infty$.

5.1. **Invariant form of balance systems.** We start with an arbitrary covering constitutive relation $\widehat{C}$ and change the sign of source term, i.e. we consider

$$\Theta_{\widehat{C}_-} = p\eta + F_\mu^i dy^\mu \wedge \eta_i - \Pi_\mu dy^\mu \wedge \eta.$$

For a section $s \in \Gamma_V(\pi)$, $V \subset X$ we request the fulfilment of the equation (Invariant Balance System, shortly IBS)

$$j^k(s)^*(i_\xi \tilde{d}\Theta_{\hat{C}_-}) = 0, \ \xi \in \mathcal{X}(J_p^k(\pi)) \qquad (IBS) \tag{5.1}$$

for a large enough family of variations ξ - sufficient for separation of individual balance laws (see Def.7 below).

Thus, we take the $n + (n+1)$-form $\Theta_{\hat{C}_-}$ of the form (4.2) and apply first the Con-differential $\tilde{d}$ and then i_ξ for a vector field $\xi = \xi^j \partial_j + \xi^\nu \partial_{y^\nu} + \sum_{|I|>1} \xi_I^\mu \partial_{z_I^\mu}$.

Adopting the summation by repeated indices we recall that only z_I^μ or derivatives by these variables with $(\mu, I) \in P$ are present in the formulas. We introduce the notation $d\eta_i = \lambda_{G,i} \eta$. It will be convenient to include variables y^μ into the family of variables z_I^μ for $|I| = 0$ taking $z^i = y^i$.

We will also use the contact splitting (2.3) of the lift of differential of a function p to $J_p^\infty(\pi)$ and similar contact decomposition for the flux components F_ν^j.

We have,

$$\tilde{d}\Theta_{\hat{C}_-} = d(p\eta + F^i_\mu dy^\mu \wedge \eta_i) + \Pi_\mu dy^\mu \wedge \eta = dp \wedge \eta + dF^i_\mu \wedge dy^\mu \wedge \eta_i - F^\mu_\mu dy^\mu \wedge \lambda_{G,i}\eta + \Pi_\mu dy^\mu \wedge \eta = \tag{5.2}$$

Now we continue the calculation replacing in the last two terms in (5.2) the dy^μ by ω^μ and using $dy^\nu \wedge \eta_j = \omega^\nu \wedge \eta_j + z^\nu_k dx^k \wedge \eta_j = \omega^\nu \wedge \eta_j + z^\nu_j \eta$

$$= [(d_i p)dx^i - \sum_{(\mu,I)\in\hat{P}} p_{,z^\mu_I}\omega^\mu_I] \wedge \eta +$$

$$+ [(d_i F^j_\nu)dx^i - \sum_{(\mu,I)\in\hat{P}} (F^j_\nu)_{,z^\mu_I}\omega^\mu_I] \wedge dy^\nu \wedge \eta_j - \lambda_{G,i}F^i_\mu \omega^\mu \wedge \eta + \Pi_\mu \omega^\mu \wedge \eta =$$

$$= (d_i F^j_\nu)dx^i \wedge dy^\mu \wedge \eta_j - \lambda_{G,i}F^i_\mu \omega^\mu \wedge \eta + \Pi_\mu \omega^\mu \wedge \eta - [\sum_{(\mu,I)\in\hat{P}} p_{,z^\mu_I}\omega^\mu_I] \wedge \eta +$$

$$- [\sum_{(\mu,I)\in\hat{P}} (F^j_\nu)_{,z^\mu_I}\omega^\mu_I] \wedge (\omega^j \wedge \eta_j + z^\nu_j \eta) =$$

$$= (-d_j F^j_\nu)\omega^\nu \wedge \eta - \lambda_{G,i}F^i_\mu \omega^\mu \wedge \eta + \Pi_\mu \omega^\mu \wedge \eta - \sum_{(\mu,I)\in\hat{P}} [p_{,z^\mu_I} + z^\nu_j(F^j_\nu)_{,z^\mu_I}]\omega^\mu_I \wedge \eta + 2Con =$$

$$= [-d_\mu F^\mu_i) - \lambda_{G,\mu}F^\mu_i + \Pi_i]\omega^i \wedge \eta + F^\nu_j \omega^j_\nu \wedge \eta + [\sum_{(\mu,I)\in\hat{P}} (p + z^\nu_j F^j_\nu)_{,z^\mu_I}\omega^\mu_I \wedge \eta + 2Con. \tag{5.3}$$

Here we have used the equality $(d_i F^j_\nu)dx^i \wedge dy^\mu \wedge \eta_j = -(d_j F^j_\nu)dy^\mu \wedge \eta = -(d_j F^j_\nu)\omega^\mu \wedge \eta$. Term $2Con$ in the last formula represents 2-contact form $[\sum_{(\mu,I)\in P}(F^j_\nu)_{,z^\mu_I}\omega^\mu_I] \wedge \omega^j \wedge \eta_j$.

Last formula proves the first statement of the next

Theorem 2. *Let $\hat{C}$ be a CCR defined in a domain of the partial k-jet bundle $J^k_p(\pi)$, $k \geqq 1$. Then,*

(1) *We have the following contact decomposition*

$$\tilde{d}\Theta_{\hat{C}_-} = [-d_i F^i_\mu) - \lambda_{G,i}F^i_\mu + \Pi_\mu]\omega^\mu \wedge \eta + F^j_\nu \omega^\nu_j \wedge \eta -$$

$$- \left[\sum_{(\mu,I)\in\hat{P}} (p + \sum_{(\nu,j)\in P} z^\nu_j F^j_\nu)_{,z^\mu_I}\omega^\mu_I \right] \wedge \eta + 2Con. \tag{5.4}$$

*Term in the brackets is the **vertical differential** $d_v(p + z^\nu_j F^j_\nu)$ of the function $p + z^\nu_j F^j_\nu$, see [5, 13]. Internal sum is taken over all indices (ν,j) such that $z^\nu_j \in J^1_p(\pi)$ is in the 1-jet projection of $J^k_p(\pi)$, while the outside sum is taken over all (μ,I) with z^μ_I in $J^k_p(\pi)$.*

(2) *Let $\xi \in \mathcal{X}(J^k_p(\pi))$ be any vector field. Then,*

$$i_\xi \tilde{d}\Theta_{\hat{C}_-} = -\omega^1_{\hat{C}}(\xi)\eta - \omega^{k+1}_{\hat{C}}(\xi^{k+1})\eta + Con, \tag{5.5}$$

for an arbitrary prolongation ξ^{k+1} to the vector field ξ to $J_p^{k+1}(\pi)$, where

$$\begin{cases} \omega_{\hat{C}}^1(\xi) = \omega^\mu(\xi)[d_i F_\mu^i + \lambda_{G,i} F_\mu^i - \Pi_\mu], \\ \omega_{\hat{C}}^{k+1}(\xi^{k+1}) = \sum_{(\nu,j)\in P} F_\nu^j \omega_j^\nu(\xi^{k+1}) - \left[\sum_{(\mu,I)\in \hat{P}}(p + \sum_{(\mu,j)\in P} z_j^\nu F_\nu^j)_{,z_I^\mu} \omega_I^\mu(\xi^{k+1})\right]. \end{cases}$$
$$(5.6)$$

Here ξ^2 is the projection of ξ to $J_p^2(\pi)$ for $k \geqq$ and the prolongation to this subbundle for $k = 1$.

(3) *For the lifted CCR $\tilde{C}$ of a constitutive relation C and an arbitrary vector field $\xi \in \mathcal{X}(J_p^k(\pi))$,*

$$i_\xi \tilde{d}\Theta_{\tilde{C}_-} = -\omega_{\hat{C}}^1(\xi)\eta - \sum_{(\nu,j)\in P} F_\nu^j \omega_j^\nu(\xi^{k+1})\eta + Con \qquad (5.7)$$

(4) *Decompositions (5.4,5.7) are covariant with respect to a change of admissible coordinate system $(x^i, y^\mu) \rightarrow (x'^i = h^i(x^j), y'^\mu = h^\mu(x^j, y^\nu)$ preserving the "partial" structure (both in $K \oplus K'$ and S cases).*

Proof. Previous discussion proves the validity of the first decomposition. Second and third statements follow from the first one.

Last statement follows from the tensorial behavior of both components of decompositions with regard of such coordinate changes. $\qquad \square$

Remark 7. Quantities $Q_\mu = \omega^\mu(\xi) = \xi^\mu - z_i^\mu \xi^i$ form the *characteristic of the vector field* $\xi = \xi^i \partial_i + \xi^\mu \partial_{y^\mu}$ in the sense of [24], Ch.2, or the *generating section* in terms of [13].

Remark 8. Equality (5.7) contains the jet variables of the second order z_{iI}^μ. Yet, **only ω_i^μ with $(i, \mu) \in P$ are present in the formula** (5.7)! For instance in the RET case *all these terms are absent* from (5.7).

It is seen from the formula (5.7) that the lifted covering constitutive relation is privileged in the sense that the formula for $\tilde{d}\Theta_{\hat{C}_-}$ simplifies essentially for lifted CCR $\tilde{C}_-$.

In order that the equation resulting from taking the pullback by $j^k(s)$ in (5.4) **would not depend on the variables not in $J_p^k(\pi)$** we require that all the coefficients of these variables would be zero. This leads to the condition one has to put to the allowed variations ξ (see Def.7 below).

Consider now two cases where no restrictions to the variations of Poincare-Cartan form appears.

Proposition 10. *Let $\hat{C}$ be a CCR **of order k**. Then,*

$$\omega_{\hat{C}}^{k+1} = \sum_{(\nu,j)\in P} F_\nu^j \omega_j^\nu - \left[\sum_{(\mu,I)\in \hat{P}}(p + \sum_{(\nu,j)\in P} z_j^\nu F_\nu^j)_{,z_I^\mu} \omega_I^\mu\right] \equiv 0 \qquad (5.8)$$

*if and only if for some $L \in C^\infty(J_p^1(\pi))$, $F_\mu^i = \partial_{z_i^\mu}L$, $(\mu, i) \in P$. i.e. CR C is **(locally) semi-P-Lagrangian (lagrangian by variables $z_i^\mu|(i,\mu) \in P$)** and does not depend on the variables $z_I^\mu, |I| > 1$. In this case, $p = L - \sum_{(\mu,i)\in P} z_i^\mu \partial_{z_i^\mu}L + l(x, y)$ with an arbitrary $l(x, y) \in C^\infty(Y)$.*

Proof. Combining terms with $|I| = 1$ in the sum with the first term in (5.8) and using linear independence of basic contact forms we split condition (5.8) into the following two groups of conditions

$$\begin{cases} 1.\ F_\nu^j - \partial_{z_j^\nu}(p + \sum_{\mu,k) \in P} z_k^\mu F_\mu^k) = 0,\ \forall\ (j, \nu) \in P, \\ 2.\ (p + \sum_{(j,\nu) \in P} z_j^\nu F_\nu^j)_{,z_I^\mu} = 0,\ \forall\ (\mu, I) \in \tilde{P},\ |I| > 1. \end{cases}.$$

Rewrite the equalities of the first group in the form

$$\partial_{z_j^\mu} p = -z_k^\mu \partial_{z_j^\mu} F_\mu^k. \tag{5.9}$$

To see that $\Rightarrow$ holds we notice that provided left statement is true, the right sides of these equalities satisfy to the mixed derivative test

$$\partial_{z_k^\nu}(-z_i^\lambda \partial_{z_j^\mu} F_\lambda^i) = \partial_{z_j^\mu}(-z_i^\lambda \partial_{z_k^\nu} F_\lambda^i),$$

or $\partial_{z_k^\nu} F_\mu^l = \partial_{z_l^\mu} F_\nu^k$. It follows from this equality valid for all couples of indices $(\nu, k), (\mu, l) \in P$ that there exists a function $L \in C^\infty(J_p^k(\pi))$ such that $F_\nu^j = \partial_{z_j^\nu} L, \forall (\nu, j) \in P$. Substituting this to the first family of equalities we see that the function $p + \sum_{\mu,k) \in P} z_k^\mu L_{,z_k^\mu} - L$ does not depend on $z_i^\mu, (\mu, i) \in P$. Therefore,

$$p = L - \sum_{(\mu,k) \in P} z_k^\mu L_{,z_k^\mu} + l(x, y, z_l^\nu,\ (\mu, l) \notin P).$$

Moving sum in the right side to the left and using second group of equations (5.9) we see that the function $L + l$ does not depend on the variables $z_I^\nu,\ (\nu, I) \in \hat{P} \smallsetminus P$. Using this function instead of L we get the conclusion of Proposition.

To prove the opposite - reverse the arguments. $\square$

Notice that the conclusion of this proposition put no restrictions on the components $F_i^\mu, (\mu, i) \notin P$.

In the case of a semi-Lagrangian CR there are no restriction to the variations in the IBS equation (5.1) provided one uses the proper lift of the constitutive relation to the CCR (see previous section, Example 2). More specifically,

Theorem 3. *Let $\mathcal{C}_{L,\Pi}$ be a semi-P-Lagrangian (by flux components in P) constitutive relation of order 1: $F_\mu^i = L_{,z_i^\mu},\ (\mu, i) \in P$ where $L \in C^\infty(J_p^1(\pi))$. Let $\hat{C}_{L,\Pi}$ be a CCR covering $\mathcal{C}$ with $p = L - \sum_{(\mu,i) \in P} z_i^\mu \partial_{z_i^\mu} L$. Then the following statements are equivalent*

(1) *For a section $s \in \Gamma(\pi)$ and for all $\xi \in \mathcal{X}(J_p^1(\pi))$*

$$j_p^1(s)^* i_\xi \tilde{d}\Theta_{\hat{C}_{L,\Pi}\ -} = 0.$$

(2) *Section s is the solution of the system of balance equations*

$$j^1(s)^* E_L = \sum_{i|(\mu,i) \in P} \left[(L_{,z_i^\mu} \circ j_p^1(s))_{;i} - L_{,y^\mu} \circ j_p^1(s) \right] + \sum_{(\mu,i) \notin P} (F_\mu^i)_{;i} = \Pi_\mu(j_p^1(s)). \tag{5.10}$$

Remark 9. In a case of a Lagrangian Field Theory of higher order ($k > 1$) the Euler-Lagrange system of equations does not have unique representation as a balance system. Perhaps this is the reason why the last results are limited to the Lagrangian system of the first order. Probably introduction of more general form of balance systems allows to include higher order Lagrangian systems in this scheme

in such a way that the arbitrary vector fields ξ in the k-jet bundles are admissible as variations.

Similarly, there are no restriction to the variation in the RET case where CR is the section of the bundle $\pi^{n+(n+1)} : \Lambda_{2/1}^{n+(n+1)} Y$ (see [25]).

Theorem 4. *Let $\mathcal{C}$ be a constitutive relation of the RET type and $\tilde{C}$ - corresponding lifted CCR. Then, for a given section $s \in \Gamma(\pi)$*

$$s^*(i_\xi \tilde{d}\Theta_{\hat{C}_-}) = 0 \ \forall \ \xi \in \mathcal{X}(Y) \Leftrightarrow \text{Section } s \text{ is the solution of } \bigstar.$$

In general case we have to introduce the class of *admissible variations*.

Definition 7. *A vector field $\xi \in \mathcal{X}(U)$, $U \subset J_p^k(\pi)$ is called C-admissible ($\xi \in \mathcal{X}_{\mathcal{C}}(U)$) if*

(1) *In the case $k = 1$, for some (=any) prolongation of ξ to the vector field $\xi^2 \in \mathcal{X}(J_p^2(\pi_{21}^{-1}(U))$*

$$\omega_{\tilde{C}}^2(\xi^2) = \sum_{(\mu,i)\in P} (\xi_i^\mu - \xi^k z_{ik}^\mu) F_\mu^i = 0.$$

(2) *For $k > 1$, for some (=any) prolongation of ξ to the vector field $\xi^{k+1} \in \mathcal{X}(J_p^{k+1}(\pi_{(k+1)1}^{-1}(U))$*

$$\omega_{\tilde{C}}^{k+1}(\xi) = \sum_{(\mu,i)\in P} (\xi_i^\mu - \xi^k z_{ik}^\mu) F_\mu^i = 0.$$

Remark 10. Condition of C-admissibility is much more restrictive in the case $k = 1$ then in the case $k > 1$ since in the last case F_k^i may depend on z_{ik}^μ.

As the next result shows, locally any CR is separable.

Lemma 6. *Let $W \subset Y$ be the domain of a fiber chart (x^i, y^μ). Any CR C is **separable** in $W^k = \pi_{k0}^{-1}(W)$, more specifically, $\forall y \in W$, $z \in W^k, \pi_{k0}(z) = y$, the mapping $\mathcal{X}_C(W^k) \ni \xi \to \{\omega_z^\mu(\xi)\} \in R^m$ is the epimorphism.*

Proof. Consider vertical vector fields that are constant along the fiber $U \cap W$. $\square$

As a result, in general case we have the following result

Theorem 5. *For any CR $\mathcal{C}$ and a domain $U \subset J_p^k(\pi)$ the following statements for a section $s \in \Gamma(\pi)(U)$, $U \subset X$ are equivalent:*

(1) *$\mathcal{C}$ is separable in U and for all $\xi \in \Gamma(U, \mathcal{V}_{\mathcal{C}}(J_p^k(\pi)))$,*

$$j^k(s)^*(i_\xi \tilde{d}\Theta_{\tilde{C}_-}) = 0. \hspace{2cm} (IBS)$$

(2) *Section $s \in \Gamma(U, \pi)$ is the solution of the system*

$$(F_\mu^i \circ j_p^k(s))_{,x^i} + F_\mu^i \circ j_p^k(s)\lambda_{G,i} = \Pi_\mu(j_p^k(s)), \ \mu = 1, \ldots, m. \hspace{1cm} (\bigstar)$$

5.2. **Contact source correction of the lifted PC-form $\widetilde{\Theta}_{\hat{C}}$ and the balance equations.** In this subsection we show that one may modify the Poincare-Cartan form $\Theta_{\tilde{C}}$ by adding a contact term to the source in such a way that applying the procedure of variation described above to the PC-form modified in this way one can remove the restriction s ont the variational vector fields.

Let $\Lambda_2^{n+(n+1)}(J_p^1(\pi)) \to J_p^1(\pi)$ be the bundle of $n+(n+1)$-forms on the partial 1-jet bundle $J_p^1(\pi)$ annulated by substitution of any two π^1- vertical vector fields. An element of a fiber of this bundle have the form

$$p\eta + p_\mu^i dy^\mu \wedge \eta_i + p_\mu dy^\mu \wedge \eta + k_\mu^{ij} dz_i^\mu \wedge \eta_j + q_\mu^i dz_i^\mu \wedge \eta. \qquad (5.11)$$

Introduce on this manifold the following $n+(n+1)$-form:

$$Q = p\eta + p_\mu^i dy^\mu \wedge \eta_i + p_\mu dy^\mu \wedge \eta + q_\mu^i dz_i^\mu \wedge \eta = p\eta + p_\mu^i dy^\mu \wedge \eta_i + p_\mu \omega^\mu \wedge \eta + q_\mu^i \omega_i^\mu \wedge \eta \qquad (5.12)$$

which differs from the canonical form $\Theta_2^n + \Theta_2^{n+1}$ by the last term only. It is easy to see that this form behaves covariantly under the change of fibred coordinates.

Let $\hat{C} : J_p^k(\pi) \to \Lambda_2^{n+(n+1)}(Y)$ be a covering constitutive relation of the form $\hat{C}(x, y, z) = p\eta + F_\mu^i dy^\mu \wedge \eta_i - \Pi_\mu dy^\mu \wedge \eta$. Mapping $\hat{C}$ can be lifted to the mapping

$$\hat{C}^1 : J_p^k(\pi) \to \Lambda_2^{n+(n+1)}(J_p^1(\pi)) : (x, y, z) \to p\eta + F_\mu^i dy^\mu \wedge \eta_i - K_\mathcal{C},$$

where the "source" form $K_\mathcal{C} = \Pi_\mu \omega^\mu \wedge \eta + F_\mu^i \omega_i^\mu \wedge \eta$ was introduced in the previous section.

Now we will modify the Poincare-Cartan form $\Theta_{\hat{C}}$ on the (partial) jet bundle $J_p^k(\pi)$ for $k > 1$ and on the partial 2-jet bundle $J_p^2(\pi)$ for $k = 1$ by adding an extra source term to get the modified covering Poincare-Cartan form $\Theta_{\hat{C}}$ by the formula

$$\widehat{\Theta}_{\hat{C}} = \hat{C}^{1\ *}Q = p\eta + F_\mu^i dy^\mu \wedge \eta_i - \Pi_\mu \omega^\mu \wedge \eta - F_\mu^i \omega_i^\mu \wedge \eta. \qquad (5.13)$$

Applying to the form $\Theta_{\hat{C}}$ arguments leading to (5.4-5.7) we get, for a lift ξ^2 to the bundle $J_p^2(\pi)$ of a vector field $\xi \in \mathcal{X}(J_p^1(\pi))$ for $k = 1$ and for a lift $\xi^{k+1} \in \mathcal{X}(J_p^{k+1}(\pi))$ of the vector field ξ^k on $J_p^k(\pi)$ for $k > 1$

$$i_{\xi^2}\tilde{d}\widehat{\Theta}_{\hat{C}} = \{-\omega^i(\xi)[d_i F_\mu^i + \lambda_{G,i} F_\mu^i - \Pi_\mu - \partial_{y^\mu}(p + z_j^\nu F_\nu^j)] - \omega_i^\mu(\xi^2)[F_\mu^i - \partial_{z_i^\mu}(p + z_j^\nu F_\nu^j)]\}\eta +$$
$$+ F_\mu^i \omega_i^\mu(\xi^2)\eta - F_\mu^i \omega_i^\mu \wedge i_\xi\eta + Con =$$
$$= \{-\omega^\mu(\xi)[d_i F_\mu^i + \lambda_{G,i} F_\mu^i - \Pi_\mu - \partial_{y^\mu}(p + z_j^\nu F_\nu^j)] + \omega_i^\mu(\xi^2)[\partial_{z_i^\mu}(p + z_j^\nu F_\nu^j)]\}\eta + Con, \qquad (5.14)$$

since the second to the last last form in the previous expression is contact. For the *lifted covering constitutive relation* $\tilde{C}$ (see Def.6) this expression simplifies:

$$i_{\xi^2}\tilde{d}\widehat{\Theta}_{\tilde{C}} = -\omega^\mu(\xi)[d_i F_\mu^i + \lambda_{G,i} F_\mu^i - \Pi_\mu]\eta + Con. \qquad (5.15)$$

Thus, we have proves the following

Theorem 6. *For a given constitutive relation $\mathcal{C}$ of order k and a section $s \in \Gamma(\pi)$ of the bundle π the following statements are equivalent:*

(1) *For any vector field $\xi \in \mathcal{X}(J_p^k(\pi))$ and its arbitrary prolongation to the π_k^{k+1}-projectable vector field ξ^{k+1} on $J_p^{k+1}(\pi)$,*

$$j^{k+1\ *}(s)i_{\xi^{k+1}}\tilde{d}\widehat{\Theta}_{\tilde{C}_-} = 0, \qquad (5.16)$$

(2) *Section s is the solution of the system* $\bigstar$: $\left(F_\mu^i \circ j^k(s)\right)_{;i} = \Pi_\mu \circ j^{k+1}(s)$, $\mu = 1, \ldots, m$.

5.3. **Dual form of a balance system.** Reversing in the IBS equation (5.1) the order of contraction with a vector field ξ^k and acting by the Iglesias differential (but keeping the sign of the source term) we get the equation

$$j^{k+1}(s)^* \tilde{d} i_{\xi^k} \Theta_{\hat{C}} = 0. \tag{5.17}$$

Calculations similar to one performed above for the equation (5.1) allows to prove the following

Proposition 11. *Let $\hat{C}$ be a CCR and let $\Theta_{\hat{C}} = p\eta + F_i^\mu dy^i \wedge \eta_\mu + \Pi_i dy^i \wedge \eta$ be the corresponding Poincare-Cartan form. Then, for any projectable vector field $\xi \in X_p(\pi)$,*

$$\tilde{d} i_{\xi^k} \Theta_{\hat{C}} = [\omega^\mu(\xi)[d_i F_\mu^i + F_\mu^i \lambda_{G,i} - \Pi_\mu] + \xi^i(d_i + \lambda_{G,i}) \cdot (p + \sum_{(\mu,j) \in P} z_j^\mu F_\mu^j) +$$

$$+ (p + \sum_{(\mu,j) \in P} z_j^\mu F_\mu^j) d_i \xi^i + F_\mu^i d_i \omega^\mu(\xi)\eta. \tag{5.18}$$

For the lifted CCR $\tilde{C}$, where $p = -\sum_{(\mu,i) \in P} z_i^\mu F_\mu^i$, the previous formula takes the form

$$\tilde{d} i_{\xi^k} \Theta_{\hat{C}} = \left[\omega^\mu(\xi)[d_i F_\mu^i + F_\mu^i \lambda_{G,i} - \Pi_\mu] + F_\mu^i d_i \omega^\mu(\xi)\right] \eta \tag{5.19}$$

Remark 11. Notice that the condition $F_\mu^i d_i \omega^\mu(\xi) = 0$ is equivalent, for the vertical vector fields $\xi \in V(\pi)$ to the admissibility condition of Def.7.

Corollary 2. *Let C be a constitutive relation of order $k \geq 1$ and let $\tilde{C}$ be the corresponding lifted CCR. Then for a section $s \in \Gamma(U, \pi)$, $U \subset X$ defined in an open subset $U \subset X$ such that C is separable in $\pi^{-1}(U)$, the following conditions are equivalent*

(1) *For all C-admissible vector fields $\xi \in \pi^{-1}(U)$*

$$j^{k+1}(s)^* \tilde{d} i_{\xi^k} \Theta_{\hat{C}} = 0.$$

(2) *s is the solution of the balance system $\bigstar$:*

$$\left(F_\mu^i \circ j^1(s)\right)_{;i} = \Pi_\mu \circ j^1(s), \ \mu = 1, \ldots, m.$$

Remark 12. Expression in the left side of equation (5.17) applied to the $\Theta_{\hat{C}}^n$ represents the second, boundary term in the Cartan formula for the Lie derivative

$$\mathcal{L}_\xi \Theta_{\tilde{C}}^n = i_\xi d\Theta_{\tilde{C}}^n + d i_\xi \Theta_{\tilde{C}}^n.$$

It is interesting that one can use both these terms to get the balance system $\bigstar$.

6. Geometrical transformations and the symmetry groups of constitutive relations.

Let $\hat{C}$ be a CCR with the domain $J_p^k(\pi)$, let $\phi \in Aut_p(\pi)$ be an automorphism of the bundle π preserving the partial structure. Denote by ϕ^k the flow lift of ϕ to to the manifold $J_p^k(\pi)$ and by ϕ^{1*} - its canonical flow lift to the space $\Lambda_2^{n+(n+1)} Y$

Define the action of ϕ on the CCR $\hat{C}$ by the "conjugation"

$$\hat{C}^\phi = \phi^{1*} \circ \hat{C} \circ \phi^{k\ -1}. \tag{6.1}$$

Similarly one defines the action of ϕ on a CR C, using the projection $\widetilde{\phi}^{1*}$ of ϕ^{1*} to $\Lambda_{2/1}^{n+(n+1)}Y$ instead of ϕ^{1*}, see [12, 17]. Immediate check shows that $\hat{C}^{\phi\circ\psi} = (\hat{C}^\psi)^\phi$.

Since the transformation ϕ^{1*} preserves the canonical multisymplectic form on $\Lambda_2^{n+(n+1)}Y$ it is easy to see that for the CCR $\hat{C}^\phi$ one has

$$\Theta_{\hat{C}^\phi} = \phi^{k\ -1*}\Theta_{\hat{C}}. \tag{6.2}$$

6.1. Geometrical symmetries of CR and CCR.

Definition 8. (1) *An automorphism $\phi \in Aut_p(\pi)$ is called a* **geometrical symmetry transformation** *of a CR C (resp. of a CCR $\hat{C}$) if*

$$\mathcal{C}^\phi = \mathcal{C} \quad (\text{respectively}, \ \widehat{\mathcal{C}}^\phi = \widehat{\mathcal{C}}).$$

(2) *Let $\xi \in \mathcal{X}_p(\pi)$ be a π-projectable vector field. ξ is called a* **geometrical infinitesimal symmetry** *of the CR $\mathcal{C}$ (resp. of a CCR $\widehat{\mathcal{C}}$) if (see [12], or below for the definition of a Lie derivative along the mapping)*

$$L_{(\xi^k, \widetilde{\xi}^{1*})}\mathcal{C} = 0,$$

(respectively, $L_{(\xi^k, \xi^{1*})}\widehat{\mathcal{C}} = 0$).

In the next proposition we collected some simple properties of geometrical symmetries and infinitesimal symmetries of constitutive relations and covering constitutive relations. These properties follows directly from definitions and the fact that the transformation ϕ^{1*} preserves the canonical multisymplectic form on $\Lambda_2^{n+(n+1)}Y$. In this proposition we put

$$W_p = J_p^k(\pi) \times \Lambda_{2/1}^{n+(n+1)}(Y), \hat{W}_p = J_p^k(\pi) \times \Lambda_2^{n+(n+1)}(Y)$$

Proposition 12. *For a CR $\mathcal{C}$ (respectively for a CCR $\hat{C}$)*

(1) *Geometrical symmetries ϕ of C (resp. $\hat{C}$) form the subgroup $Sym(\mathcal{C}) \subset Aut_p(\pi)$.*

(2) *Infinitesimal symmetries of $\mathcal{C}$ (resp. $\hat{C}$) form the Lie algebra $\mathfrak{g}(\mathcal{C}) \subset \mathcal{X}_p(\pi)$ with the bracket of vector fields in Y as the Lie algebra operation.*

(3) *A vector field $X \in \mathcal{X}(W_p)$ (resp. $X \in \mathcal{X}(\hat{W}_p)$) is the generator of the 1-parametrical group of generalized symmetries of $\mathcal{C}$ (resp. $\hat{C}$) if and only if it is tangent to the graph $\Gamma_\mathcal{C}$ (resp. $\Gamma_{\widehat{c}}$).*

(4) *A vector field $\xi \in \mathcal{X}_p(\pi)$ is an infinitesimal symmetry of $\mathcal{C}$ if (and only if) the (local) phase flow diffeomorphisms $\Phi_t^\xi = \psi_t \times \phi_{-t}^k$ of W_p (resp. of $\hat{W}_p$) defined by the prolongation of ξ maps $\Gamma_\mathcal{C}$ into itself satisfies to the relation*

$$\Phi_t^\xi(\Gamma_\mathcal{C}) = \Gamma_\mathcal{C} \ \text{resp.}\ , \Phi_t^\xi(\Gamma_{\widehat{c}}) = \Gamma_{\widehat{c}}.$$

i.e. if the (local) phase flow ϕ_t of vector field ξ is formed by the geometrical symmetry transformations of $\mathcal{C}$ (resp. of $\widehat{\mathcal{C}}$).

(5) *Let ξ be an infinitesimal symmetry of a CR C (respectively of a covering constitutive relation $\hat{C}$). Then, for the local phase flow ϕ_t of ξ one has*

$$\Theta_{C^{\phi_t}} = \phi_t^{k\ -1\ *}\Theta_C = \Theta_C \Leftrightarrow \mathcal{L}_{\xi^l}\Theta_C = 0,$$

(respectively, $\mathcal{L}_{\xi^k}\Theta_{\hat{C}} = 0$).

In the next statement we collect the basic properties of action of automorphisms $\phi \in Aut_p(\pi)$, including geometrical symmetries of constitutive relations, on the balance systems B_C. For a domain $U \subset X$ denote by $Sol_C(U)$ the space of solutions $s : U \to Y$ of the balance system $\bigstar$. correspondence $U \to Sol(U)$ determine the sub-sheaf on the manifold X. Denote by Sol_C the sheaf corresponding to the introduced subsheaf.

Theorem 7. (1) *Mapping $\xi \to \phi_* \xi$ maps the space of C-admissible vector fields $\mathcal{X}_C(W)$ at a domain W onto the space of C^ϕ-admissible vector fields $X_{C^\phi}(\phi(W))$ in the domain $\phi(W)$ for any open subset $W \subset J_p^k(\pi)$. This mapping defines the isomorphism of (pre)-sheaves of admissible vector fields $\mathcal{X}_C \rightleftarrows \mathcal{X}_{C^\phi}$.*

(2) *Mapping $s \to s^\phi = \phi \circ s \circ \bar{\phi}^{-1}$, on the sections of the configurational bundle $\pi : Y \to X$ maps the space of solutions $Sol_C(W)$ in the domain $W \subset X$ onto the space of solutions $Sol_{C^\phi}(\phi(W))$ of the balance system $\mathcal{B}_{C^\phi}$ in the domain $\phi(W) \subset X$:*

$$Sol_C(W) \rightleftarrows Sol_{C^\phi}(\phi(W)).$$

(3) *Let $\phi \in Aut_p(\pi)$ be a symmetry of the CR C. Then the mapping $s \to s^\phi = $ maps the sheaf Sol_C of solutions of the balance system $\bigstar$ into itself.*

Proof of this Theorem is based on the following statement

Lemma 7. *Let $\hat{C}$ be a CCR with the domain $J_p^k(\pi)$ and the Poincare-Cartan form $\Theta_{\hat{C}} = p\eta + F_i^\mu dy^i \wedge \eta_\mu + \Pi_i dy^i \wedge \eta$ and let $\widehat{\phi} \in Aut_p(\pi^k)$ be an automorphism of the double bundle $J_p^k(\pi) \to Y \to X$. Then*

$$[\widehat{\phi}^* \Theta_{\hat{C}}] = {}^{\widehat{\phi}} p\eta + {}^{\widehat{\phi}} F_i^\mu dy^i \wedge \eta_\mu + {}^{\widehat{\phi}} \Pi_i dy^i \wedge \eta, \tag{6.3}$$

where

$$\begin{cases} {}^{\widehat{\phi}} p = [p \circ \widehat{\phi} + (F_i^\mu \circ \widehat{\phi})(\phi_{,x^\nu}^i J(\bar{\phi}^{-1})_\mu^\nu] \cdot det J(\bar{\phi}), \\ {}^{\widehat{\phi}} F_i^\mu = det J(\bar{\phi}) J(\bar{\phi}^{-1})_\nu^\mu (F_j^\nu \circ \widehat{\phi})\phi_{,y^i}^j, \\ {}^{\widehat{\phi}} \Pi_i = det J(\bar{\phi})(\Pi_j \circ \widehat{\phi})\phi_{,y^i}^j. \end{cases} \tag{6.4}$$

Proof. (of Lemma). We notice that $\bar{\phi}^* \eta = (det J(\bar{\phi}))\eta$ where $det J(\bar{\phi})$ is the Jacobian of the (local) diffeomorphism $\bar{\phi}$ defined by the volume form η. On the other hand

$$\bar{\phi}^* \eta_\mu = \bar{\phi}^* i_{\partial_{x^\mu}} \eta = i_{\bar{\phi}_*^{-1} \partial_{x^\mu}} \bar{\phi}^* \eta = det J(\bar{\phi}) J(\bar{\phi}^{-1})_\mu^\nu \eta_\nu$$

since $\bar{\phi}_*^{-1} \partial_{x^\mu} = J(\bar{\phi}^{-1})_\mu^\nu \partial_{x^\nu}$.

Altogether we have

$$\widehat{\phi}^* \Theta_{\hat{C}} = \widehat{\phi}^* [p\eta + F_i^\mu dy^i \wedge \eta_\mu + \Pi_i dy^i \wedge \eta] = (p \circ \widehat{\phi}) \cdot \bar{\phi}^* \eta +$$
$$+ (F_i^\mu \circ \widehat{\phi}) d(\phi^i)(x, y) \wedge \bar{\phi}^* \eta_\mu + \Pi_i \circ \widehat{\phi} d(\phi^i) \wedge \bar{\phi}^* \eta =$$
$$= p \circ \widehat{\phi} \cdot det J(\bar{\phi})\eta + (F_i^\mu \circ \widehat{\phi})(\phi_{,x^\sigma}^i dx^\sigma + \phi_{,y^j}^i dy^j) \wedge [det J(\bar{\phi}) J(\bar{\phi}^{-1})_\mu^\nu \eta_\nu] +$$
$$+ \Pi_i \circ \widehat{\phi}(\phi_{,x^\sigma}^i dx^\sigma + \phi_{,y^j}^i dy^j) \wedge det J(\bar{\phi})\eta = [det J(\bar{\phi}) J(\bar{\phi}^{-1})_\mu^\nu (F_i^\mu \circ \widehat{\phi})\phi_{,y^j}^i] dy^j \wedge \eta_\nu +$$
$$+ [det J(\bar{\phi})(\Pi_i \circ \widehat{\phi})\phi_{,y^j}^i] dy^j \wedge \eta + [p \circ \widehat{\phi} \cdot det J(\bar{\phi}) + (F_i^\mu \circ \widehat{\phi})[\phi_{,x^\nu}^i det J(\bar{\phi}) J(\bar{\phi}^{-1})_\mu^\nu] \eta.$$
$$\tag{6.5}$$

Splitting the terms we get the result stated in Lemma. $\square$

Proof. **Proof of Theorem 7.** Transformation ϕ, generating the transformation of a constitutive relations $C \to \mathcal{C}^\phi$, transforms the (1,n)-component F_i^μ of C into the corresponding component $F_i^{\phi\ \mu} = det J(\bar{\phi}^{-1}) J(\bar{\phi})_\nu^\mu (F_j^\nu \circ (\phi^1)^{-1}) \phi_{,y^i}^{-1\ j}$ of the CR C^ϕ, see last Lemma. On the other hand the mapping ϕ_* maps a vector field $\hat{\xi} = \xi^\mu \partial_{x^\mu} + \xi^i \partial_{y^i} + \xi_\mu^i \partial_{z_\mu^i} + \ldots \in \mathcal{X}(J_p^k(\pi))$ into the vector field with the local z_μ^i-component having the form

$$\phi_* \xi = \ldots + \phi_{,y^i}^j J(\bar{\phi}^{-1})_\nu^\mu \xi_\mu^i \partial_{z_\nu^j} + \ldots,$$

see Appendix II in [25]. As a result for the value of the pairing we have

$$F_i^{\phi\ \mu}(\phi_* \xi)_\mu^i = \left(F_i^\mu \xi_\mu^i \circ \phi^1\right) \cdot det J(\bar{\phi}^{-1}).$$

Since $det J(\bar{\phi}^{-1}) > 0$, expressions for contraction of flux components and components of lifted vector field in left and right sides vanish simultaneously. This proved the first statement in Theorem 6.

To prove the second statement notice that

$$(j_p^1(s^\phi))^* i_{\phi_* \xi} \tilde{d}\Theta_{\widehat{\mathcal{C}}^\phi} = [\phi^1\ {}^*(j_p^1(s))]^* i_{\phi_* \xi} \tilde{d}\Theta_{\widehat{\mathcal{C}}^\phi} = (j_p^1(s))^* \circ \phi^1\ {}^* i_{\phi_* \xi} \tilde{d}\Theta_{\widehat{\mathcal{C}}^\phi} =$$
$$= (j_p^1(s))^* \circ i_{\phi_*^1\ {}^{-1} \phi_* \xi} \tilde{d}\phi^1\ {}^* \Theta_{\widehat{\mathcal{C}}^\phi} = (j_p^1(s))^* \circ i_\xi \tilde{d}\Theta_{(\widehat{\mathcal{C}}^\phi)_-^{\phi^{-1}}} = (j_p^1(s))^* \circ i_\xi \tilde{d}\Theta_{\widehat{\mathcal{C}}_-}.$$
$$(6.6)$$

Since the mapping $\xi \to \phi_* \xi$ is, by the first statement the isomorphism of the pre-sheaves of admissible vector fields, expression in the right side is zero for all $\xi \in \mathcal{X}_\mathcal{C}(W)$ if and only if the expression in the left side is zero for all vector fields $\xi \in \mathcal{X}_{\mathcal{C}^\phi}(\phi(W))$ and, therefore $s \in Sol_\mathcal{C}(W)$ if and only if $s^\phi \in Sol_{\mathcal{C}^\phi}(\phi(W))$.

For the proof of the third statement we use the symmetry condition in the form presented at the end of Definition 8 and notice that in this case $\mathcal{C}^\phi = \mathcal{C}$ so the mapping $s \to s^\phi$ maps the space $Sol_\mathcal{C}(W)$ isomorphically onto the space $Sol_\mathcal{C}(\phi(W))$. $\square$

6.2. **Infinitesimal symmetries of the constitutive relations.** Here we consider the infinitesimal transformations of a CCR $\hat{C}$ and these of a CR $\mathcal{C}$ of the first order. We start with an arbitrary covering constitutive relation $\hat{C} : z = (x^\mu, y^i, z_\mu^i) \to (p(z), F_i^\mu(z), \Pi_k(z))$.

Let

$$\hat{\xi} = \xi^\mu(x)\partial_{x^\mu} + \xi^i(x,y)\partial_{y^i} + \xi_\mu^i(x,y,z)\partial_{z_\mu^i} = \xi + \xi_\mu^i(x,y,z)\partial_{z_\mu^i}$$

be an arbitrary infinitesimal automorphism of the double bundle $J_p^1(\pi) \to Y \to X$. Here we introduced notation ξ for the projection of $\hat{\xi}$ to Y.

In particular, let $\xi = \xi^\mu(x)\partial_{x^\mu} + \xi^i(x,y)\partial_{y^i} \in \mathcal{X}_p(\pi)$ be an *infinitesimal automorphism* (vector field) of the bundle π, i.e. a projectable vector field in Y satisfying to the conditions of Sec.3.3 for lifting to the partial 1-jet bundle $J_p^1(\pi)$. Let, as before, ξ^1 be its prolongation to the projectable contact vector field in $J_p^1(\pi)$. Then, as an example of vector fields $\hat{\xi}$ one may consider the flow lift (see (3.2))

$$\xi^1 = \xi^\mu(x)\partial_{x^\mu} + \xi^i(x,y)\partial_{y^i} + \left(d_\mu \xi^i - z_\nu^i \frac{\partial \xi^\nu}{\partial x^\mu}\right)\partial_{z_\mu^i},$$

of ξ to $J_p^1(\pi)$. Here summation in the last term is taken over the z_μ^i that are present in the partial 1-jet bundle. In the RET case we do not need to introduce any prolongation.

Projection ξ of a vector field $\hat{\xi}$ defines the prolongation ξ^{1*} to the projectable vector field in $Z* = \Lambda_2^{(n+1)+(n+2)}Y$ preserving canonical multisymplectic form(s) $\Theta_2^n+\Theta_2^{n+1}$ (See Sec.7) and, the prolongation $\hat{\xi}^{1*}$ to the bundle $\Lambda_2^{(n+1)+(n+2)}(J^1(\pi))$:

$$\xi^{1*} = \xi^\mu(x)\partial_{x^\mu} + \xi^i(x,y)\partial_{y^i} + \left(-p \cdot div_G(\bar{\xi}) - p_i^\mu \frac{\partial \xi^i}{\partial x^\mu}\right)\partial_p +$$

$$+ \left(p_i^\nu \frac{\partial \xi^\mu}{\partial x^\nu} - p_j^\mu \frac{\partial \xi^j}{\partial y^i} - p_i^\mu div_G(\bar{\xi})\right)\partial_{p_i^\mu} + \left(-q_k div_G(\bar{\xi}) - q_j \frac{\partial \xi^j}{\partial y^k} - q_i^\mu \xi_{\mu,y^k}^i\right)\partial_{q_k} +$$

$$+ \left(-q_i^\mu \xi_{\mu,z^j}^i - q_j^\nu div_G(\bar{\xi})\right)\partial_{q_j^\nu}. \quad (6.7)$$

We have used here the equality $\xi_{,z_\nu^j}^i = 0$ valid for automorphisms of the double bundle $J_p^1(\pi) \to Y \to X$. It is easy to see that vector field ξ^{1*} is projectable to $\Lambda_{2/1}^{(n+1)+(n+2)}J_p^1(\pi)$. This projection - $\widetilde{\xi^{1*}}$ has, in fibred chart, expression (6.7) with the third term omitted.

Let now $\hat{\phi}_t$ be a local flow in $J_p^1(\pi)$ of the vector field $\hat{\xi}$ and ψ_t be a local flow in $\Lambda_2^{(n+1)+(n+2)}(J^1(\pi))$ of the vector field ξ^{1*} (respectively $\widetilde{\psi}_t$ be a local flow in $\Lambda_{2/1}^{(n+1)+(n+2)}Y$ of the vector field $\widetilde{\xi}^{1*}$).

Derivating by t at $t = 0$ the expression for the transformed mapping $\psi_t \circ \hat{C} \circ \hat{\phi}_{-t}$ (respectively for $\psi_t \circ \hat{C} \circ \hat{\phi}_{-t}$) we get the *generalized Lie derivative* of mapping $\hat{C}$ with respect to the vector fields $(\hat{\xi}, \xi^{1*})$ (see [12], Chapter 11) - the vector field over the mapping $\hat{C} : J_p^1(\pi) \to \Lambda_2^{(n+1)+(n+2)}Y$ (respectively $C : J_p^1(\pi) \to \tilde{Z}$ for CR C):

$$\mathcal{L}_{(\hat{\xi},\widetilde{\xi}*)}\hat{C} = \widetilde{\xi}^* \circ \hat{C} - C_*(\hat{\xi}). \quad (6.8)$$

In local fibred coordinates we have for $\mathcal{L}_{(\hat{\xi},\widetilde{\xi}*)}\hat{C}$ the following expression

$$\mathcal{L}_{(\hat{\xi},\xi^{1*})}\hat{C} = -\left[\hat{\xi} \cdot p + div_G(\bar{\xi}) \cdot p + F_i^\mu \frac{\partial \xi^i}{\partial x^\mu}\right]\partial_p - \left[\hat{\xi} \cdot F_k^\mu + div_G(\bar{\xi})F_k^\mu + F_j^\mu \frac{\partial \xi^j}{\partial y^k} - F_k^\nu \frac{\partial \xi^\mu}{\partial x^\nu}\right]\partial_{p_k^\mu} -$$

$$-\left[\hat{\xi} \cdot \Pi_k + div_G(\bar{\xi})\Pi_k + \Pi_j \frac{\partial \xi^j}{\partial y^k} + F_i^\mu \xi_{\mu,y^k}^i\right]\partial_{q_k} - \left[\hat{\xi} \cdot F_j^\nu + F_j^\nu div_G(\bar{\xi}) + F_i^\mu \xi_{\mu,z_\nu^j}^i\right]\partial_{q_j^\nu}, \quad (6.9)$$

for a CR $\mathcal{C}$ the term with ∂_p is absent from the expression of $L_{(\hat{\xi},\widetilde{\xi}1*)}\mathcal{C}$.

Condition that the generalized Lie bracket (6.9) is zero for a CCR $\hat{C}$ and a vector field $\xi \in X(Y)$, has the form of a system of differential equation of the first order for the components of the covering constitutive relation $\hat{C}$:

$$\begin{cases} \hat{\xi} \cdot p + div_G(\bar{\xi})p + F_i^\mu \frac{\partial \xi^i}{\partial x^\mu} = 0, \\ \hat{\xi} \cdot F_k^\mu + div_G(\bar{\xi})F_k^\mu + F_j^\mu \frac{\partial \xi^j}{\partial y^k} - F_k^\nu \frac{\partial \xi^\mu}{\partial x^\nu} = 0, \forall \mu, k, \\ \hat{\xi} \cdot \Pi_k + div_G(\bar{\xi})\Pi_k + \Pi_j \frac{\partial \xi^j}{\partial y^k} + F_i^\mu \xi_{\mu,y^k}^i = 0, \forall k = 1, \ldots, m, \\ \hat{\xi} \cdot F_k^\mu + F_k^\mu div_G(\bar{\xi}) + F_i^\nu \xi_{\nu,z_\mu^k}^i = 0, \; \forall k, \mu. \end{cases} \quad (6.10)$$

Vector field $\hat{\xi} = \xi^\mu \partial_\mu + \xi^i \partial_i + \xi^i_\mu \partial_{z^i_\mu}$ in these equations acts on the components of the vector functions in the space $J^1_p(\pi)$.

If we subtract forth equation of this system from the second one we get

$$F^\mu_j \frac{\partial \xi^j}{\partial y^k} - F^\nu_k \frac{\partial \xi^\mu}{\partial x^\nu} - F^\nu_i \xi^i_{\nu, z^k_\mu} = 0. \tag{6.11}$$

This equation may replace the forth equation in the previous system. Yet, for $\hat{\xi} = \xi^1$, $\xi \in \mathcal{X}_\pi(Y)$ this equation is identically fulfilled. To see this we remind (see ()) that in this case $\xi^j_\nu = d_\nu \xi^i - z^j_\mu \xi^\mu_{,\nu} = \xi^j_{,\nu} + z^i_\nu \xi^j_{,y^i} - z^j_\sigma \xi^\sigma_{,\nu}$ and, as a result, $\xi^i_{\nu, z^k_\mu} = \delta^i_k \delta^\mu_\nu \xi^j_{,y^i} - \delta^j_k \delta^\mu_\sigma \xi^\sigma_{,\nu} = \delta^\mu_\nu \xi^j_{,y^k} - \delta^j_k \xi^\mu_{,\nu}$. Substituting this in the equation () we will see that it is fulfilled identically.

Thus, for the case when $\hat{\xi} = \xi^1$ for a vector field $\xi \in \mathcal{X}_p(\pi)$ the use of modified CCR instead of the usual one does not add forth condition to the first three in the system (30.6) although it adds an extra term to the third equation.

We can rewrite this system of equations as the system of conditions *for a vector field $\xi \in \mathcal{X}_p(\pi)$ to be an infinitesimal symmetry of the CCR $\hat{C}$:*

$$\begin{cases} I: \ [p_{,z^i_\mu} \xi^i_\mu + F^\mu_i \xi^i_{,\mu} + p_{,i} \xi^i] + [p div_G(\bar{\xi}) + p_{,\mu} \xi^\mu] = 0, \\ II^\mu_k: \ [F^\mu_{k,z^i_\nu} \xi^i_\nu + F^\mu_i \xi^i_{,y^k} + F^\mu_{k,i} \xi^i] + [F^\mu_k div_G(\bar{\xi}) - F^\nu_k \xi^\mu_{,\nu} + F^\mu_{k,\nu} \xi^\nu] = 0, \\ III_k: \ [\Pi_{k,z^i_\mu} \xi^i_\mu + \Pi_i \xi^i_{y^k} + \Pi_{k,y^i} \xi^i] + [div_G(\bar{\xi}) \Pi_k + \Pi_{k,\mu} \xi^\mu] + F^\mu_i \xi^i_{\mu,y^k} = 0. \end{cases} \tag{6.12}$$

In the case of a *modified* CCR (see Sec.5.2) last condition contains both flux and source terms while for the original CCR last equation contains only source. Similar conditions an infinitesimal symmetry of a CR $\mathcal{C}$ are obtained from these by removing the first subsystem of equations.

Consider now the form these conditions takes in two special cases of vector fields:

Example 10. π-vertical vector fields.

For a **vertical vector field** $\xi = \xi^i \partial_{y^i} \in \mathcal{V}(\pi)$ (infinitesimal gauge transformation) the system (12.18) takes the following form

$$\begin{cases} I: \ p_{,z^i_\nu} d_\nu \xi^i + F^\mu_i \xi^i_{,\mu} + p_{,y^i} \xi^i = 0, \\ II^\mu_k: \ \left(F^\mu_{k,z^j_\sigma} d_\sigma + F^\mu_j \frac{\partial}{\partial y^k}\right) \xi^j + F^\mu_{k,y^j} \xi^j = 0, \ \mu = 0, \ldots, n; k = 1, \ldots, m, \\ III_k: \ \left(\Pi_{k,z^j_\sigma} d_\sigma + \Pi_j \frac{\partial}{\partial y^k}\right) \xi^j + \Pi_{k,y^j} \xi^j + F^\mu_i \xi^i_{\mu,y^k} = 0, \ k = 1, \ldots, m. \end{cases} \tag{6.13}$$

Example 11. ν-lifted vector fields, ν-trivial.

Consider now mathematically simple but practically important case of a trivial bundle $\pi: Y = X \times U \to X$, fix a fibred chart $(W; x^\mu, y^i)$ and consider the zero connection ν_0 in this bundle defined in the domain $W \subset Y$ of the chart above by $\Gamma^i_\mu = 0$. The ν_0-horizontal lift of a basic vector field ∂_μ is $\hat{\partial}_\mu = \partial_\mu$ with zero vertical components. Its flow prolongation to $J^i_p(\pi)$ has the same form: $\hat{\partial}^1_\mu = \partial_\mu$.

Conditions (6.12) for the vector field $\hat{\partial}_\mu$ to define geometrical symmetry of a CCR $\hat{C}$ takes the form

$$\begin{cases} I: \ p_{,\mu} + \lambda_{G,\mu} p = 0, \\ II^\nu_k: \ F^\nu_{k,\mu} + \lambda_{G,\mu} F^\nu_k = 0, \\ III_k: \ \Pi_{k,\mu} + \lambda_{G,\mu} \Pi_k = 0. \end{cases} \tag{6.14}$$

since $div_G(\partial_\mu) = \lambda_{G,\mu}$. These conditions require that the *densities* of fluxes, source components and p are invariant under the translation along x^μ-axis. In a case where $|G|$-const these conditions reduces to the simple independence of all components of CCR $\hat{C}$ on the variable x^μ.

Next result establish relation between the infinitesimal symmetries of a CCR and those of the corresponding CR.

Proposition 13.   (1) *Any infinitesimal symmetry ξ (respectively geometrical symmetry) of a CCR $\hat{C}$ generate (by the projection $\Lambda_2^{(n+1)+(n+2)} \to \Lambda_{2/1}^{(n+1)+(n+2)}$ the infinitesimal symmetry (respectively geometrical symmetry) of $\mathcal{C}$.*

(2) *Let $\tilde{C}$ be the **lifted** CCR of a CR $\mathcal{C}$. Then the previous mapping defines the **bijection** between infinitesimal symmetries (generalized symmetries $\hat{\xi}$) of C and $\tilde{C}$.*

Proof. First statement follows from the fact that second and third subsystems of the conditions (6.12) do not depend on p.

Second statement follows from the fact that for the lifted CCR, where $p = -z_\mu^i F_i^\mu$, multiplying equation II_k^μ by z_μ^k and sum by k, μ we get exactly equation I. $\square$

6.3. **Homogeneous constitutive relations.** If the state space of a theory contains enough background or dynamical fields to make the constitutive relation $\mathcal{C}$ free from the explicit dependence of $\mathcal{C}$ on $(t,x) \in X$ (general relativity, theory of uniform materials and RET (see [21]) are three examples), then the corresponding balance system simplifies and while studying it one does not need to introduce assumptions on the character of the dependence of the balance system on a space-time point. Definition given below is an invariant way to distinguish a class of such CR.

Any local chart x^μ in X defined the local (translational) action of R^n in X associating with the basic vectors e_μ the vector fields ∂_{x^μ}. Vice versa, any n-dimensional commutative subalgebra $\mathfrak{h}$ of the Lie algebra of vector fields $\mathcal{X}(U)$, U being an open connected subset of X, defines the locally transitive action of R^n in U and, therefore, a local chart in a neighborhood of any point in U.

Proposition 14. - **Definition.** *Let ν be a connection in the bundle π. We will call a constitutive relation $\mathcal{C}$ ν-**homogeneous** if the following **equivalent** properties of the constitutive relation C are valid:*

(1) *For any point $z \in J_p^1(\pi)$ there exists a local chart in a neighborhood U_x, $x = \pi^1(z)$ such that the (n+1)-Poincare-Cartan form $\Omega_{\mathcal{C}}$ of the CR $\mathcal{C}$ is invariant under the local flows $\phi_t^{\hat{\xi}^1}$ of the lifts $\hat{\xi}^1$ of ν-horizontal vector fields $\hat{\xi}, \xi \in \mathfrak{h}$ in the neighborhood of $y = \pi_{10}(z)$:*

$$\mathcal{L}_{\hat{\xi}^1}\Omega_{\mathcal{C}} = 0 \; mod \; \textbf{Con}.$$

(2) *For all $\xi \in \mathfrak{h}$, the ν-horizontal lift $\hat{\xi}$ is the infinitesimal symmetry of the constitutive mapping $\mathcal{C}$ in sense of Definition 8.*

(3) *The graph $\Gamma_C \subset J_p^1 \times \tilde{Z}$ of mapping $\mathcal{C}$ is invariant under the flow generated by (flow) lifts of ν-horizontal vector fields $\hat{\xi}, \xi \in \mathfrak{h}$.*

Proof. Equivalence of statements of this proposition follows directly from the Proposition 11. $\square$

Similar definition and results formulated in the last Proposition can be formulated and proved for a CCR $\hat{C}$.

Remark 13. In a case where connection ν is flat, the association $\xi \to \hat{\xi}^1$ defines the Lie algebra homomorphism $\mathfrak{h} \to Aut(\pi^1) \subset \mathcal{X}(J_p^1)$.

Using first the ν-horizontal lift of vector fields from X to Y and then the flow lift to $J_p^1(\pi)$ (Sec.3) one can get the ν-homogeneity conditions of a CCR $\hat{C}$, those of lifted CCR $\tilde{C}$ or those of the CR C.

7. Noether Theorem.

In this section we present the (first) Noether theorem for balance systems corresponding to the action of a Lie group $G \subset Sym(C) \subset Aut_p(\pi)$ of the geometrical symmetries of a CR $\mathcal{C}$. We denote by $\mathfrak{g}$ - the Lie algebra of the group G, $\mathfrak{g}^*$ - its dual space, $\xi \to \xi_Y \in \mathcal{X}(Y)$ - action mapping on the manifold Y and ξ^k be its lift to the partial k-jet bundle $J_p^k(\pi)$, see Sec. 3.4.

7.1. **Momentum Mapping and general Noether Theorem.** Condition that the group G is the symmetry group of the CR $\mathcal{C}$ (and its lifted CCR $\tilde{C}$) has the infinitesimal form (see previous section)

$$\mathcal{L}_{\xi_Y^k}(\Theta_{\tilde{C}}^n + \Theta_{\tilde{C}}^{n+1}) = 0. \tag{7.1}$$

This condition splits into two corresponding to the order of the forms:

$$\begin{cases} \mathcal{L}_{\xi_Y^k}\Theta_{\tilde{C}}^n = (i_{\xi^k}d + di_{\xi^k})\Theta_{\tilde{C}}^n = 0, \\ \mathcal{L}_{\xi_Y^k}\Theta_{\tilde{C}}^{n+1} = (i_{\xi^k}d + di_{\xi^k})\Theta_{\tilde{C}}^{n+1} = 0. \end{cases} \tag{7.2}$$

For the first condition we have, using formula (5.7) of Theorem 2;

$$di_{\xi^k}\Theta_{\tilde{C}}^n = -i_{\xi^k}d\Theta_{\tilde{C}}^n = -i_{\xi^k}[\tilde{d}\Theta_{\tilde{C}_-} - \Theta_C^{n+1}] =$$

$$= \omega_C^1(\xi) + \sum_{(\nu,j)\in P} F_\nu^j\omega_j^\nu(\xi^{k+1})\eta - i_{\xi^k}(\Pi_\mu\omega^\mu \wedge \eta) + Con =$$

$$= \omega^\mu(\xi)[d_iF_\mu^i + \lambda_{G,i}F_\mu^i - \Pi_\mu] + \sum_{(\nu,j)\in P} F_\nu^j\omega_j^\nu(\xi^{k+1})\eta - \Pi_\mu\xi^\mu\eta + Con. \tag{7.3}$$

Let now $s : X \to Y$ be a solution of the balance system $\mathcal{B}_\mathcal{C}$. Taking pullback of the last equality with respect to the $j_p^{k+1}s$ we see that the first and last terms in the right side vanishes and we get

$$j^k(s)^*di_{\xi^k}\Theta_{\tilde{C}}^n = dj^k(s)^*i_{\xi^k}\Theta_{\tilde{C}}^n = -(\Pi_\mu\xi^\mu)\circ j^k(s)\eta + \left[\sum_{(\nu,j)\in P} F_\mu^j\omega_j^\mu(\xi^{k+1})\right]\circ j^{k+1}(s)\eta. \tag{7.4}$$

Now we recall definition of the **multimomentum mapping**, see [6, 14]

Definition 9. A multimomentum mapping $J^{\hat{C}} : J_p^k(\pi) \to \Lambda^n(X) \otimes \mathfrak{g}^*$ of a CCR $\hat{C}$ is defined as

$$J^{\hat{C}}(z)(\xi_Y^k) = -i_{\xi_Y^k}\Theta_{\hat{C}}^n(z). \tag{7.5}$$

Lemma 8.

$$J^{\hat{C}}(z)(\xi_Y^k) = -[(p + z_j^\mu F_\mu^j)\xi^i + F_\mu^i \omega^\mu(\xi)]\eta_i + DCon \qquad (7.6)$$

where DCon is the sheaf of contact forms whose differential is also contact.

Proof. We have

$$i_{\xi_Y^k}\Theta_{\hat{C}}^n(z) = i_\xi\Theta_{\hat{C}}^n(z) = p\xi^i\eta_i + F_\mu^i\xi^\mu\eta_i - F_\mu^i\xi^j dy^\mu \wedge \eta_{ij} =$$
$$(p\xi^i + \xi^\mu F_\mu^i)\eta_i - F_\mu^i\xi^j(\omega^\mu + z_k^\mu dx^k)\wedge\eta_{ij} = (p\xi^i + \xi^\mu F_\mu^i)\eta_i - F_\mu^i\xi^j\omega^\mu\wedge\eta_{ij} - F_\mu^i\xi^j z_k^\mu(\delta_j^k\eta_i - \delta_i^k\eta_j) =$$
$$= (p\xi^i + \xi^\mu F_\mu^i)\eta_i - F_\mu^i\xi^j\omega^\mu \wedge \eta_{ij} - (F_\mu^i\xi^j z_j^\mu\eta_i - F_\mu^i\xi^j z_i^\mu\eta_j) =$$
$$= [p\xi^i + \xi^\mu F_\mu^i - F_\mu^i\xi^j z_j^\mu + F_\mu^j\xi^i z_j^\mu]\eta_i - F_\mu^i\xi^j\omega^\mu \wedge \eta_{ij} =$$
$$= [(p + z_j^\mu F_\mu^j)\xi^i + F_\mu^i\omega^\mu(\xi)]\eta_i - F_\mu^i\xi^j\omega^\mu \wedge \eta_{ij}.$$

Now we notice that not just the last term in this equality is contact but its differential is contact as well, namely

$$d[F_\mu^i\xi^j\omega^\mu \wedge \eta_{ij}] = Con - F_\mu^i\xi^j dz_k^\mu \wedge dx^k \wedge \eta_{ij} = Con - F_\mu^i\xi^j dz_k^\mu \wedge (\delta_j^k\eta_i - \delta_i^k\eta_j) =$$
$$= Con - F_\mu^i\xi^j(\omega_k^\mu + z_{kl}^\mu dx^l) \wedge (\delta_j^k\eta_i - \delta_i^k\eta_j) = Con - F_\mu^i\xi^j z_{kl}^\mu(\delta_j^k\delta_i^l - \delta_i^k\delta_j^l)\eta =$$
$$= Con - F_\mu^i\xi^j(z_{ij}^\mu - z_{ij}^\mu) = Con.$$

□

Using the formula (7.4) we obtain the proof of the first statement of the next Theorem. Proof of other statements of Theorem is straightforward.

Theorem 8. *Noether Theorem (general).* *Let $\mathcal{C}$ be a constitutive relation defined on a partial k-jet bundle $J_p^k(\pi)$, let $\hat{C}$ be an arbitrary covering CCR (with an arbitrary p) of $\mathcal{C}$ and $\tilde{C}$ - lifted CCR of $\mathcal{C}$. Let a Lie group $G \subset Sym(\tilde{C}) \subset Aut(\pi)$ be a geometrical symmetry group of the flux part Θ_C^n of CR $\mathcal{C}$. Then,*

(1) *For all $\xi \in \mathfrak{g}$ and for all solutions $s \in Sol(X, \mathcal{B}_C)$ of the balance system ★,*

$$d[J^{\hat{C}}(j^k(s)(x))(\xi_Y^k)] = -dj^k \,^*(s)(x)[((p + z_j^\mu F_\mu^j)\xi_Y^i + F_\mu^i\omega_Y^\mu(\xi))\eta_i] =$$

$$= (\Pi_\mu\xi_Y^\mu) \circ j^k(s) - \left[\sum_{(\nu,j)\in P} F_\nu^\nu\omega_j^\nu(\xi_Y^{k+1})\right] \circ j^{k+1}(s). \quad (7.7)$$

(2) *If $\hat{C} = \tilde{C}$, then the term $(p + z_j^\mu F_\mu^j)\xi_Y^i$ in the left side vanishes.*
(3) *If $\mathfrak{g}_Y \subset \mathcal{X}_C(Y)$ on Y, last term in the right side drops out.*
(4) *The same formula holds for a Lie group of projectable Cartan symmetries $G \subset Aut_p(\pi_{k0})$ of the space $J_p^k(\pi)$ and its Lie algebra $\mathfrak{g}_Y \subset \mathcal{X}(J_p^k(\pi))$.*
(5) *If $\mathfrak{g}_Y \subset \mathcal{X}_C(Y)$ and if the system B_C is the conservative laws system (i.e. if $\Pi_\mu = 0$, $\mu = 1, \ldots, m$), then $\forall\xi \in \mathfrak{g}$ and for all $s \in Sol(C)$ the **Noether conservation law holds**:*

$$d[j^k \,^*(s)(x)F_\mu^i\omega_Y^\mu(\xi)\eta_i] = ((F_\mu^i\omega_Y^\mu(\xi)) \circ j^k \,^*(s)(x))_{;i} = 0.$$

7.2. **Semi-Lagrangian and RET cases.** In the special cases of semi-P-Lagrangian and RET constitutive relations the formulation of Noether Theorem is conveniently simplifies.

Theorem 9. *Noether Theorem for* $C_{L,\Pi}, k = 1$. *Let $\hat{C}_{L,\Pi}$ be a semi-P-Lagrangian CCR with*

$$\Theta_{\hat{C}} = (L - z_i^\mu L_{z_i^\mu})\eta + \sum_{(\mu,i)\in P} L_{z_i^\mu} dy^\mu \wedge \eta_i + \sum_{(\mu,i)\notin P} F_\mu^i dy^\mu \wedge \eta_i + \Pi_\mu dy^\mu \wedge \eta.$$

Let $\xi \in \mathcal{X}_p(Y)$ be an infinitesimal variational symmetry of the CR $\hat{C}_{L,\Pi}$. Then, for all $s \in Sol(\mathcal{C})$ of the system $\bigstar$

(1)

$$d[(j^1(s))^* J^{\hat{C}_{L,\Pi}}(z)(\xi^1)] = -[(L(j^1 s(x))\xi^i + \sum_{(\mu,i)\in P} (L_{z_i^\mu}\omega^\mu(\xi^1)) \circ j^1 s(x) +$$

$$+ \sum_{(\mu,i)\notin P} (F_\mu^i \omega^\mu(\xi^1)) \circ j^1 s(x)]_{;i}\eta = (\omega^\mu(\xi^1)\Pi_\mu) \circ j^1(s))^*\eta. \quad (7.8)$$

(2) *If $J_p^1(\pi) = J^1(\pi)$ is the full 1-jet bundle, last equality takes the form*

$$-[(L(j^1 s(x))\xi^i + (L_{z_i^\mu}\omega^\mu(\xi^1)) \circ j^1 s(x)]_{;i} = (\omega^\mu(\xi^1)\Pi_\mu) \circ j^1(s))^*.$$

(3) *If, in addition, $\Pi_\mu = 0$, $\mu = 1, \ldots, m$, then $\forall \xi \in \mathfrak{g}$ and for all solutions $s \in \Gamma(\pi)$ the Noether conservation law holds:*

$$d[(j^1(s))^* J^{\hat{C}_{L,\Pi}}(z)(\xi^1)] = [(L(j^1 s(x))\xi^\mu + (L_{z_i^\mu}\omega^\mu(\xi^1)) \circ j^1 s(x)]_{;i}\eta = 0.$$

Last statement of this theorem is the standard form of Noether Theorem in Lagrangian Field Theory.

Theorem 10. *Noether Theorem, RET,* $k = 0$. *Let $\tilde{C}$ be a lifted covering constitutive relation of the RET type with $\Theta_{\tilde{C}} = F_\mu^i dy^\mu \wedge \eta_i + \Pi_\mu dy^\mu \wedge \eta$, and let $\xi \in \mathcal{X}(Y)$ be a variational symmetry of $\tilde{C}$. Then for all solutions $s \in \Gamma(\pi)$ of the balance system $\bigstar$,*

$$d[(j^1(s))^* F_\mu^i \omega^\mu(\xi^1)\eta_i)] = (\omega^\mu(\xi^1)\Pi_\mu) \circ j^1(s))^*\eta.$$

7.3. **Full invariance and the source charge.** Let now condition (7.1) is fulfilled i.e. G is symmetry of *both flux **and** modified source term* $\Theta_C^{n+1} = \Pi_\mu\omega^\mu\wedge\eta + F_\mu^i\omega_i^\mu\wedge\eta$ of a covering constitutive relation $\hat{C}$. Notice that this term depends only on the $\mathcal{C}$ but not on the lifting of the CR $\mathcal{C}$ to the covering constitutive relation.

Then, for any $\xi \in \mathfrak{g} \subset \mathcal{X}(J_p^k(\pi))$, lifting ξ and the Poincare-Cartan form to $J_p^{k+1}(\pi)$ and using the fact that any (n+1)-form on $J_p^{k+1}(\pi)$ is contact we have

$$\pi_k^{(k+1)*} di_{\xi^k}\Theta_{\hat{C}}^{n+1} = -i_{\xi^{k+1}} d\Theta_{\hat{C}}^{n+1} = -i_{\xi^{k+1}}(d\Pi_\mu\wedge\omega^\mu\wedge\eta + \Pi_\mu d\omega^\mu\wedge\eta + d(F_\mu^i)\wedge\omega_i^\mu\wedge\eta + F_\mu^i d\omega_i^\mu\wedge\eta)) =$$

$$= -i_{\xi^k}[(d_h\Pi_\mu + d_v\Pi_\mu) \wedge \omega^\mu \wedge \eta + d_h F_\mu^i \wedge \omega_i^\mu \wedge \eta + d_v F_\mu^i \wedge \omega_i^\mu \wedge \eta] =$$

$$= -i_{\xi^k}[(d_i\Pi_\mu dx^i + d_v\Pi_\mu) \wedge \omega^\mu \wedge \eta + d_i F_\mu^i dx^i \wedge \omega_i^\mu \wedge \eta + d_v F_\mu^i \wedge \omega_i^\mu \wedge \eta] =$$

$$= -i_{\xi^k}[d_v\Pi_\mu) \wedge \omega^\mu \wedge \eta + d_v F_\mu^i \wedge \omega_i^\mu \wedge \eta] = -i_{\xi^k}2Con = Con. \quad (7.9)$$

Here, as before Con is a contact form while 2Con - two-contact form (see Sec.2). We have used here the equalities $d\omega^\mu \wedge \eta = -dz_{ij}^\mu \wedge dx^j \wedge \eta = 0$ and the same for

$d\omega_i^\mu \wedge \eta$. During this calculation we repeatedly used the equality $dx^j \wedge \eta = 0$ *but we have not used the balance system.*

Reminding the decomposition $d = d_h + d_v$ as the sum of vertical and horizontal differentials we see that the (n+1)-form

$$Q(\xi) = i_{\xi^k}\Theta_{\hat{C}}^{n+1} = i_{\xi^k}[\Pi_\mu\omega^\mu \wedge \eta + F_\mu^i\omega_i^\mu \wedge \eta] = i_{\xi^k}K_{\mathcal{C}} \qquad (7.10)$$

has contact differential and therefore, **zero horizontal differential**: $d_h Q(\xi) = 0$. Thus, it defines the **class of cohomology** $[Q(\xi)]$ of the *horizontal complex* $(\bigwedge^*(J_p^\infty(\pi)), d_h)$:

$$[Q(\xi) = i_{\xi^k}K_{\mathcal{C}}] \in H_{hor}^{n+1}(J_p^\infty(\pi)))$$

If the class $[Q]$ is zero, then $Q(\xi) = d_h\Phi(\xi) + Con$ is the sum of horizontal differential of a form $\Phi(\xi)$ and a contact form and, therefore is also the sum of a differential $d\Phi(\xi)$ and the contact form

$$Q(\xi) = d\Phi(\xi) + Con.$$

Applying now the pullback by $j^k(s)$ (and $j^{k+1}(s)$ where appropriate) and recalling that we have not used the balance equation in the calculation above, we prove the following

Theorem 11. *Let, in addition to the conditions of Theorem 8, $\mathfrak{g}$ is the Lie algebra of the variational (infinitesimal) symmetries of the source part of the constitutive relation, i.e. (7.2) is true. Then*

(1)
$$di_\xi\Theta_{\hat{C}}^{n+1} = Cont \Rightarrow dj^k(s)^* i_\xi\Theta_{\hat{C}}^{n+1} = 0$$

  for all sections s of the configurational bundle π.

(2) *Form $Q(\xi) = \pi_k^{(k+1)*} i_{\xi^k}\Theta_{\hat{C}}^{n+1} = i_{\xi^k}K_{\mathcal{C}}$ defines the **class of cohomology** $[Q(\xi)]$ of the horizontal complex $(\bigwedge^*(J_p^\infty(\pi)), d_h)$, [13] - $\mathfrak{g}$-**charge of the source** $\tilde{\Pi} = (\Pi_\mu\omega^\mu + F_\mu^i\omega_i^\mu) \wedge \eta$.*

(3) *If the class $[Q(\xi)]$ is zero for all $\hat{\xi} \in \mathfrak{g}$, there exists a form $\Phi_{\mathcal{C}}(\hat{\xi})$ linearly depending on $\hat{\xi}$ (and its derivatives) such that $Q(\hat{\xi}) = d\Phi_C(\hat{\xi}) + Con$ is the sum of a differential $d(\Phi_{\mathcal{C}}(\hat{\xi}))$ and the contact form. In this case locally (and in a topologically trivial domain, globally)*

$$j^k(s)^* i_{\hat{\xi}}\Theta_{\hat{C}}^{n+1} = dj^{k+1}(s)^*\Phi_{\mathcal{C}}(\hat{\xi})$$

  *for this $\mathfrak{g}^*$-valued n-form $\Phi_{\mathcal{C}}$ (that is natural to call the $\mathfrak{g}$-**potential of the source** Θ_C^{n+1}).*

Corollary 3. *If G is the Lie group of symmetries of a constitutive relation C and if the class $[Q]$ in the last Proposition is trivial, then there exists the (locally defined) $\mathfrak{g}^*$-valued n-form $\Phi_{\mathcal{C}} : \xi \to \Lambda^n(J_p^k(\pi))$ such that in the conditions of Theorem 11 the following **conservation law** will holds*

$$d[J^{\hat{\mathcal{C}}}(j^k(s)(x)(\xi) - j^{k+1}(s)^*\Phi_{\mathcal{C}}(\hat{\xi})] = 0$$

for all solutions $s \in \Gamma(\pi)$ of the balance system $\mathcal{B}_{\mathcal{C}}$ and all $\hat{\xi} \in \mathfrak{g}$.

7.4. Examples: Energy-Momentum Balance law, case of gauge symmetries.

Example 12. Energy-Momentum Balance Law, lifted CCR. Let ν be a connection in the bundle $\pi : Y \to X$ with the connection form $K_\nu = \partial_{y^\mu} \otimes (dy^\mu - \Gamma_i^\mu dx^i)$.

Then, the horizontal lift of the basic vector field ∂_{x^i} in Y is $\xi_i = \partial_{x^i} + \Gamma_i^\mu \partial_{y^\mu}$. Its lift to the Lie vector field in $J_p^k(\pi)$ has the form

$$\xi_i^k = \partial_{x^i} + \Gamma_i^\mu \partial_{y^\mu} + \sum_{(\mu,j) \in P} d_j \Gamma_i^\mu \partial_{z_j^\mu} + \dots.$$

Remark 14. Connection ν is assumed to be compatible with the partial structure of $J_p^1(\pi)$. In a case of $K \oplus K'$-structure this means that the ν-horizontal lift ξ to Y of any vector field $\bar{\xi} \in \mathcal{X}(X)$ preserving the $K \oplus K'$-structure should be such that its flow prolongation to $J_p^1(\pi)$ is possible and preserves the (partial) Cartan structure. In the case of an integrable AP-structure this requires that the components Γ_i^μ of the connection ν depends on the variables $x^j, \partial_j \in K$ and on y^μ (see [25], Sec.8.2) but not on the complemental variables in X.

Consider a ν-homogeneous constitutive law $\mathcal{C}$ and the corresponding balance system $\mathcal{B}_{\mathcal{C}}$. Let $\tilde{C}$ be the lifted CCR corresponding to $\mathcal{C}$ and $\Theta_{\tilde{C}_-} = F_\mu^i \omega^\mu \wedge \eta_i - \Pi_\mu dy^\mu \wedge \eta - F_\mu^i \omega_i^\mu \wedge \eta$ - corresponding (corrected) Poincare-Cartan form. Calculate now

$$i_{\hat{\partial}_i^1} \Theta_{\tilde{C}}^n = i_{\hat{\partial}_i^1} F_\mu^k \omega^\mu \wedge \eta_k = F_\mu^k (\Gamma_i^\mu - z_i^\mu) \eta_\sigma - F_i^\sigma \omega^\mu \wedge \eta_{ki}.$$

Next one has,

$$di_{\hat{\partial}_i^1} \Theta_{\tilde{C}}^n = d_k (F_\mu^k (\Gamma_i^\mu - z_i^\mu)) \eta + F_\mu^k (\Gamma_i^\mu - z_i^\mu) \lambda_{G,k} \eta - F_\mu^k d\omega^\mu \wedge \eta_{ki} + Con = (d_k + \lambda_{G,k})[F_\mu^k (\Gamma_k^\mu - z_k^\mu)] \eta + con,$$

since $j(s)^* d\omega^\mu = dj^* s\omega^\mu = 0$.

For the right side of balance law (7.7) we have

$$\Pi_\mu \omega^\mu (\hat{\partial}_i^1) \eta + F_\mu^k \omega_i^\mu (\xi^2) \eta = [\Pi_\mu (\Gamma_i^\mu - z_i^\mu) + F_\mu^k (d_k \Gamma_i^\mu - z_{ki}^\mu)] \eta,$$

since $\omega^\mu (\hat{\xi}_i) = \Gamma_i^\mu - z_i^\mu$.

Applying the pullback by the $j^2 s$ we get for the right side expression

$$[\Pi_\mu (j^1 s)(\Gamma_i^\mu - z_i^\mu) \circ j^1 (s) + F_\mu^k \circ j^1 s (d_k \Gamma_i^\mu \circ j^2 s - s_{,ki}^\mu)] \eta.$$

As a result, the balance law (7.7) takes, for the vector field $\hat{\xi}_i$, the form

$$(d_k + \lambda_{G,k})[F_\mu^k (\Gamma_k^\mu - z_k^\mu)] \circ j^2 s = \Pi_\mu (j^1 s)(\Gamma_k^\mu - z_k^\mu) \circ j^1 (s) + F_\mu^k \circ j^1 s (d_k \Gamma_i^\mu \circ j^2 s - s_{,ki}^\mu). \tag{7.11}$$

Thus, the Energy-Momentum Tensor for the constitutive relation $\mathcal{C}$ has, thus, the form

$$T_i^j = F_\mu^j (\Gamma_i^\mu - z_i^\mu). \tag{7.12}$$

In the left side of (7.11) stays the covariant divergence of the energy-momentum tensor. Right side of (7.11) represents the **source/dissipation term**.

Calculating derivative in the left side of (31.17) and canceling similar terms we reduce it to the equality

$$[(d_k + \lambda_{G,k})F_\mu^k](\Gamma_i^\mu - z_i^\mu)] \circ j^2 s = \Pi_\mu(j^1 s)(\Gamma_i^\mu - z_i^\mu) \circ j^1(s) \Leftrightarrow$$
$$\Leftrightarrow [(d_k + \lambda_{G,k})F_\mu^k - \Pi_\mu](\Gamma_i^\mu - z_i^\mu)] \circ j^2 s = 0. \quad (7.13)$$

This gives the explicit presentation of the components of the energy-momentum balance law as the linear combination of the original balance laws with variable coefficients $(\Gamma_i^\mu - z_i^\mu)$.

Example 13. Energy-Momentum balance law, semi-P-Lagrangian case.
Let $C_{L,\Pi}$ be a semi-P-Lagrangian covering constitutive relation (see Theorem 3). Let us specialize the Noether balance law given in Theorem 8 above for the case of the vector fields $\xi_i = \partial_i + \Gamma_i^\mu \partial^\mu$ introduced in the previous Example. This balance law takes, for a solution $s : X \to Y$ of the balance system $\bigstar$, the form

$$[(L\delta_{ii}^j + \sum_{(\mu,j)\in P} L_{,z_j^\mu} \cdot (\Gamma_i^\mu - z_i^\mu) + \sum_{(\mu,j)\notin P} F_\mu^i \cdot (\Gamma_i^\mu - z_i^\mu)) \circ j^1(s)(x)]_{;i} = -(\Pi_\mu \cdot (\Gamma_i^\mu - z_i^\mu)) \circ j^1(s)(x).$$
$$(7.14)$$

In particular, for the case of the full 1-jet bundle $J^1(\pi)$ and of zero connection $\Gamma_i^\mu \equiv 0$ this balance law takes the conventional form

$$[(L(j^1(s)(x))\delta_i^j - L_{,z_j^\mu}(j^1(s)(x))s_{,i}^\mu)]_{;i} = (\Pi_i z_i^\mu)(j^1(s)(x)). \quad (7.15)$$

Example 14. Pure gauge symmetry transformation. Let $\xi = \xi^\mu \partial_{y^\mu}$ be a vertical (pure gauge) symmetry transformation of a constitutive relation C. Then, the Noether balance law corresponding to the vector field ξ has the form

$$[(\xi^\mu F_\mu^i)(j_p^1(s))]_{;x^i} = (\xi^\mu \Pi_\mu)(j_p^1(s)) \quad (7.16)$$

- *the secondary balance law* defined by the vector field $\xi \in g(C)$, see Lemma 9 below.

8. Secondary balance laws and the entropy principle.

Let the system of PDE ($\bigstar$) be a balance system corresponding to a constitutive relation C. A natural question that generalizing the "entropy principle" of Continuum Thermodynamics ([21]) is - are there, except of the linear combinations of balance equations of the system $\star$, *nontrivial* (see below) balance laws defined on the same bundle $J_p^1(\pi)$ where the CR C is defined *that are satisfied by all the solutions of the balance system $\star$*:

Definition 10. *Let C be a constitutive relation defined on a partial k-jet bundle $J_p^k(\pi)$ and let $\mathcal{B}_C$ be corresponding balance system ($\bigstar$). We call a balance law*

$$(K^i \circ j_p^q(s))_{,i} = Q \circ j_p^{q+1}(s). \quad (8.1)$$

*of the order q i.e. given by a (n+1)+(n+2)-form $\sigma = K^i \eta_i + Q\eta$ on the space $J_p^q(\pi) + J_p^{q+1}(\pi)$ - the **secondary balance law** for the system $\bigstar$ if **any** solution $s : X \to Y$ of the balance system ($\bigstar$) is at the same time solution of the balance law (8.1).*

Following is the list of four classes of secondary balance laws of the balance systems.

(1) An interesting class of the secondary balance laws, including the linear combinations of the balance laws of the system ($\bigstar$), is determined by the following

Lemma 9. *Let a vertical vector field $\xi = \xi^\mu \partial_{y^\mu} \in V(\pi)$ is such that the condition $FDiv(\xi) = F_i^\mu d_i \xi^\mu = 0$ is fulfilled. Then the balance law*

$$j_p^1 {}^*(s)d(\xi^\mu F_i^\mu \eta_i) = \xi^\mu \Pi_\mu \eta \Leftrightarrow ((\xi^\mu F_i^\mu) \circ j^1(s))_{,x^i} = (\xi^\mu \Pi_\mu) \circ j^1(s)$$

belongs to the space $\mathcal{BL}_\mathcal{C}$.

Proof. Follows from $d(j_p^1 {}^*(s)\xi^\mu F_i^\mu \eta_i) = \xi^\mu d(j_p^1 {}^*(s)F_i^\mu \eta_i) + j_p^1 {}^*(s)FDiv(\xi)\eta$.
$\square$

(2) If a Lie group G is the (geometrical) symmetry group of the balance system $\bigstar$, it determines the family of the balance laws corresponding to the elements of Lie algebra $\mathfrak{g}$, see previous section. If the second order source part of these balance laws vanishes, we get the subspace $\mathcal{BL}_{\mathcal{C},\mathfrak{g}}$ of the space $\mathcal{BL}_\mathcal{C}$.

(3) The **entropy principle** of Thermodynamics (see ([20, 21, 22]) requires that the entropy balance

$$h_{,i}^i = \Sigma, \tag{8.2}$$

with the entropy density h^0, entropy flux $h^A, A = 1, 2, 3$, and entropy production plus source $\Sigma = \Sigma_s + \Sigma_p$ belongs to the space $\mathcal{BL}_\mathcal{C}$ of the balance system of a given theory. This requirement place a serious restrictions on the form of constitutive relation $\mathcal{C}$ and leads to the construction of a dual system in terms of Lagrange-Liu fields λ^μ considered below (see [21]). Even more serious constitutional restriction is the proper II law of thermodynamics requiring that the entropy production Σ_p is nonnegative, [21, 22].

(4) If an integrable dynamical system can be formulated as a system of balance equations (KdV equation is an example) then all the higher order conservation laws are the secondary balance laws of this balance system.

8.1. **Secondary balance laws for the RET balance systems.** As an illustration of the geometrical approach to the balance systems and the example of application we present here some results from the paper [26] (in preparation). In the Rational Extended Thermodynamics developed by I.Muller, T.Ruggeri, and I. Shish-Liu, [21] the space of the dynamical fields y^μ is chosen to be large enough so that the constitutive relation C is defined on the bundle Y and, in our terms, is represented by the section of the bundle $\Lambda_{2/1}^{n+(n+1)}(Y) \to Y$.

A constitutive relation $\mathcal{C}$ is called regular if the matrix $\frac{\partial F_\mu^0}{\partial y^\mu}$ is *nondegenerate.* For a regular CR $\mathcal{C}$ one can introduce new field variables $w^\mu = F_\mu^0$ and to write the balance system in the form $\partial_t w^\mu + \partial_{x^i} F_i^\mu = \Pi_\mu, \ \mu = 1, \ldots, m.$

Theorem 12.
Let $\mathcal{C}$ be a a constitutive relation of the RET type with $F_\mu^0 = y^\mu$.

(1) *There is a bijection between*
    (a) *Functions $h^0(y^\mu)$ satisfying to the (overdetermined) system of PDE*

$$(\eta_\sigma^A \partial_{y^\nu} - \eta_\nu^A \partial_{y^\sigma})h^0 = 0, \ \sigma \neq \nu, A = 1, 2, \ldots, n, \tag{8.3}$$

where $\eta_\sigma^A = F_{\mu,y^\sigma}^A(x,y)\partial_{y^\mu}$ and such that the (vertical) Hessian

$$Hess(h^0)(x_0, y_0) = \left(\frac{\partial^2 h^0}{\partial y^\mu \partial y^\nu}\right)|_{x=x_0, y=y_0}$$

is nondegenerate in a neighborhood of a point (x_0, y_0), and

(b) Secondary balance laws () with the functionally independent Lagrange-Liu multipliers $\lambda^\mu = \frac{\partial h^0}{\partial y^\mu}$ in a neighborhood of a point (x_0, y_0).

(2) In terms of local vertical variables λ^μ the density, flux and the source of the corresponding secondary balance law are

$$\begin{cases} K^0(x, y) = h^0(\lambda(y)), \\ K^\nu(x, y) = \frac{h^0(\lambda)}{\partial \lambda^\nu}(\lambda(y)), \ \nu = 1, 2, 3, \\ Q(x, y) = \lambda^\mu(y)\Pi_\mu(x, y). \end{cases} \qquad (8.4)$$

8.2. Example: Cattaneo heat propagation law.

Balance equations of this model have the form

$$\begin{cases} \partial_t(\rho\epsilon) + div(q) = 0, \\ \partial_t(\tau q) + \nabla\Lambda(\vartheta) = -q. \end{cases} \qquad (8.5)$$

Since ρ is not considered here as a dynamical variable, we merge it with the field ϵ and from now on and till the end it will be omitted. On the other hand, in this model the the energy ϵ depends on temperature ϑ and on the heat flux q (see [8], Sec.2.1.2 or, by change of variables, temperature $\vartheta = \vartheta(\epsilon, q)$ will be considered as the function of dynamical variables.

The secondary balance laws for Cattaneo model (including the original balance laws) have the form

$$\begin{pmatrix} K^0 \\ K^1 \\ K^2 \\ K^3 \\ Q \end{pmatrix} = a^0 \begin{pmatrix} \epsilon \\ q^1 \\ q^2 \\ q^3 \\ 0 \end{pmatrix} + \sum_A k^A \begin{pmatrix} \tau(\vartheta)q^A \\ \delta_A^1\Lambda(\vartheta) \\ \delta_A^2\Lambda(\vartheta) \\ \delta_A^3\Lambda(\vartheta) \\ -q^A \end{pmatrix} + \begin{pmatrix} f_0 \\ m^1 \\ m^2 \\ m^3 \\ 0 \end{pmatrix} +$$

$$+ \alpha \begin{pmatrix} \hat\lambda^0\epsilon - \int_?^\vartheta \lambda_{,\vartheta}^0\epsilon^{eq}ds + \tau(\vartheta)\Lambda_{,\vartheta}^{-1}[\frac{1}{2}\lambda_\vartheta^0\|q\|^2 + \hat{K}_{,\vartheta}^A(\vartheta)q^A] \\ \hat\lambda^0(\vartheta)q^1 + \hat{K}^1(\vartheta) \\ \hat\lambda^0(\vartheta)q^2 + \hat{K}^2(\vartheta) \\ \hat\lambda^0(\vartheta)q^3 + \hat{K}^3(\vartheta) \\ -\Lambda_{,\vartheta}^{-1}(\hat\lambda_\vartheta^0\|q\|^2 + \hat{K}_{,\vartheta}^A(\vartheta)q^A) \end{pmatrix}. \qquad (8.6)$$

First and second balance laws in the system (8.6) are the balance laws of the original Cattaneo system. Third one one is the trivial balance law (see Sec.2).

Restrictions placed by the system (8.3) leads to the expression of internal energy

$$\epsilon = \epsilon^{eq}(\vartheta) + \frac{\tau_{,\vartheta}}{2\Lambda_{,\vartheta}}\|q\|^2 - \frac{\tau(\vartheta)}{\lambda_{,\vartheta}^0}\left[\frac{1}{2}\left(\frac{\lambda_{,\vartheta}^0}{\Lambda_{,\vartheta}}\right)_{,\vartheta}\|q\|^2 + \left(\frac{\tilde{K}_{,\vartheta}^A}{\Lambda_{,\vartheta}}\right)_{,\vartheta}q^A\right]. \qquad (8.7)$$

Forth column gives the balance law with the production term

$$-\Lambda_{,\vartheta}^{-1}(\hat{\lambda}_\vartheta^0\|q\|^2 + \hat{K}_{,\vartheta}^A(\vartheta)q^A) = -\Lambda_{,\vartheta}^{-1}\hat{\lambda}_\vartheta^0\left[\sum_A(q^A + \frac{\hat{K}_{,\vartheta}^A(\vartheta)}{2\hat{\lambda}_\vartheta^0})^2 - \sum_A\left(\frac{\hat{K}_{,\vartheta}^A(\vartheta)}{2\hat{\lambda}_\vartheta^0}\right)^2\right]$$

(8.8)

For a fixed ϑ this expression may have constant sign as the function of q^A if and only if all $\hat{K}_{,\vartheta}^A(\vartheta) = 0$. Therefore this is possible only if the internal energy (8.7) has the form

$$\epsilon = \epsilon^{eq}(\vartheta) + \left[\frac{\tau_{,\vartheta}}{2\Lambda_{,\vartheta}} - \frac{\tau(\vartheta)}{2\hat{\lambda}_{,\vartheta}^0}\left(\frac{\hat{\lambda}_{,\vartheta}^0}{\Lambda_{,\vartheta}}\right)_{,\vartheta}\right]\|q\|^2$$

(8.9)

with some function $\hat{\lambda}^0(\vartheta)$. As a result, Cattaneo model has the secondary balance law

$$\partial_t\left[\hat{\lambda}^0\epsilon - \int^\vartheta\hat{\lambda}_{,\vartheta}^0\epsilon^{eq}ds + \frac{1}{2}\tau(\vartheta)\Lambda_{\vartheta}^{-1}\hat{\lambda}_\vartheta^0\|q\|^2\right] + \partial_{x^A}\left[\hat{\lambda}^0(\vartheta)q^A\right] = -\Lambda_{,\vartheta}^{-1}\hat{\lambda}_\vartheta^0\|q\|^2.$$

(8.10)

with the production term that may have constant sign - nonnegative, provided

$$\Lambda_{,\vartheta}^{-1}\hat{\lambda}_\vartheta^0 \leqq 0.$$

(8.11)

Last inequality is the II law of thermodynamics for Cattaneo heat propagation model.

9. Conclusion

In this work we've presented a variational theory of system of balance equations, realization of the constitutive relations of such system as an abstract Legendre transformation, invariant form of the balance system and the Noether Theorem associating with the geometrical symmetries of the constitutive relation the corresponding balance law (and in the appropriate cases - the conservation law). Further development of this scheme including algebraic and geometrical structures related to a balance laws, some functorial properties of developed formalism, more detailed relations to the variational bicomplex will be presented elsewhere. Applications to the thermodynamical systems: 5-field thermodynamical system, 13-fields systems, description of secondary balance laws using the exterior differential systems (see [3]) will be presented in the forthcoming paper [26].

10. Appendix 1. Proof of Proposition 6.

Proof. Let $\xi = \xi^i\partial_i + \xi^\mu\partial_\mu + \xi_\mu^i\partial_{z_i^\mu}$ be any vector field in $J^1(\pi)$. Present any vector field in the space $\Lambda_2^{n+1}(J^1(\pi))$ in the form

$$\xi^* = \xi^i\partial_i + \xi^\mu\partial_\mu + \xi_\mu^i\partial_{z_i^\mu} + \xi^{q_\mu}\partial_{q_\mu} + \xi^{q_\mu^i}\partial_{q_\mu^i}.$$

We have:

$$\mathcal{L}_{\xi^*} Q^{n+1} = (d i_{\xi^*} + i_{\xi^*} d)[(q_\mu dy^\mu + q_i^i dz_i^\mu) \wedge \eta] =$$
$$= d[(q_\mu \xi^\mu + q_\mu^i \xi_i^\mu) \wedge \eta - (q_\mu dy^\mu + q_\mu^i dz_i^\mu) \wedge \xi^i \eta_i] + i_{\xi^*}[(dq_\mu \wedge dy^\mu + dq_\mu^i \wedge dz_i^\mu) \wedge \eta] =$$
$$= \xi^\mu dq_\mu \wedge \eta + q_\mu d\xi^\mu \wedge \eta + \xi_i^\mu dq_i^i \wedge \eta + q_\mu^i d\xi_i^\mu \wedge \eta -$$
$$- (dq_\mu \wedge dy^\mu + dq_\mu^i \wedge dz_i^\mu) \wedge \xi^i \eta_i + (q_\mu dy^\mu + q_\mu^i dz_i^\mu) \wedge (\xi_{,i}^i \eta + \xi^i \lambda_{G,i} \eta)] +$$
$$+ i_{\xi^*}[(dq_\mu \wedge dy^\mu + dq_\mu^i \wedge dz_i^\mu) \wedge \eta] =$$
$$= \xi^\mu dq_\mu \wedge \eta + q_\mu d\xi^\mu \wedge \eta + \xi_i^\mu dq_i^i \wedge \eta + q_\mu^i d\xi_i^\mu \wedge \eta -$$
$$- (dq_\mu \wedge dy^\mu + dq_\mu^i \wedge dz_i^\mu) \wedge \xi^i \eta_i + (q_i dy^\mu + q_\mu^i dz_i^\mu) \wedge div_G(\bar\xi) \eta)] +$$
$$+ [(\xi^{q_\mu} dy^\mu + \xi^{q_\mu^i} dz_i^\mu) \wedge \eta] + [(-\xi^\mu dq_\mu - \xi_i^\mu dq_\mu^i) \wedge \eta] + [(dq_\mu \wedge dy^\mu + dq_\mu^i \wedge dz_i^\mu) \wedge \xi^i \eta_i].$$
$$(10.1)$$

It is easy to see that all terms containing dq_μ and dq_μ^i will cancel and we get

$$\mathcal{L}_{\xi^*} Q^{n+1} = q_\mu d\xi^\mu \wedge \eta + q_\mu^i d\xi_i^\mu \wedge \eta + div_G(\bar\xi)(q_\mu dy^\mu + q_\mu^i dz_i^\mu) \wedge \eta)] +$$
$$+ [(\xi^{q_\mu} dy^\mu + \xi^{q_\mu^i} dz_i^\mu) \wedge \eta]. \quad (10.2)$$

Calculating differentials, leaving only vertical differential due to the presence of η as common factor and equating result to zero we finally get

$$\mathcal{L}_{\xi^*} Q^{n+1} = [q_\mu d_v \xi^\mu + q_\mu^i d_v \xi_i^\mu + div_G(\bar\xi)(q_\mu dy^\mu + q_\mu^i dz_i^\mu) + (\xi^{q_\mu} dy^\mu + \xi^{q_\mu^i} dz_i^\mu)] \wedge \eta = 0$$
$$(10.3)$$

Using the fact that the ξ is projectable to X, so that ξ^μ depend only on X and using relations

$$d_v f = f_{,y^\nu} \omega^j + f_{,z_j^\nu} \omega_j^\nu$$

we reduce last relation to the following

$$q_\mu (\xi_{,y^\nu}^\mu \omega^j + \xi_{,z_j^\nu}^\mu \omega_j^\nu) + q_\mu^i (\xi_{i,y^\nu}^\mu \omega^\nu + \xi_{\mu,z_j^\nu}^\mu \omega_j^\nu) + div_G(\bar\xi)(q_\mu \omega^\mu + q_\mu^i \omega_i^\mu) + (\xi^{q_\mu} \omega^\mu + \xi^{q_\mu^i} \omega_i^\mu)] \wedge \eta =$$
$$= [q_\mu \xi_{,y^\nu}^\mu + q_\mu^i \xi_{i,y^\nu}^\mu + div_G(\bar\xi) q_\mu + \xi^{q_\nu}] \omega^j \wedge \eta + [q_\mu \xi_{,z_j^\nu}^\mu + q_\mu^i \xi_{i,z_j^\nu}^\mu + div_G(\bar\xi) q_\mu^i + \xi^{q_\mu^i}] \omega_j^\nu \wedge \eta = 0$$
$$(10.4)$$

From this it follows that the prolongation of ξ to the $\Lambda_2^{n+1}(J_p^1(\pi))$ with the properties listed in the Proposition exists, is unique and given by

$$\begin{cases} \xi^{q_\nu} = -q_\mu \xi_{,y^\nu}^\mu - q_\mu^i \xi_{i,y^\nu}^\mu - q_\nu div_G(\bar\xi), \\ \xi^{q_\nu^i} = -q_\mu \xi_{,z_j^\nu}^\mu - q_\mu^i \xi_{i,z_j^\nu}^\mu - q_\nu^i div_G(\bar\xi). \end{cases} \quad (10.5)$$

$\square$

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