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Eva Thanheiser
Portland State University, evat@pdx.edu

Ian Whitacre
Florida State University

George J. Roy
University of South Carolina

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**Mathematical Content Knowledge for Teaching Elementary Mathematics:
A Focus on Whole-Number Concepts and Operations**

Eva Thanheiser
Portland State University

Ian Whitacre
Florida State University

George J. Roy
University of South Carolina

ABSTRACT: This report represents part of a recent effort to summarize the state of knowledge of prospective elementary teachers' (PTs') mathematics content knowledge and the development thereof. Extensive reviews of the research literature were conducted by a recent PME-NA Working Group across various content areas. This report focuses on whole number and operations. Research in this area is scarce. What we do know from the literature is that PTs' knowledge of whole number and operations is insufficient and in need of improvement. PTs reason about whole numbers and operations in ways that are tied to the standard algorithms. At the same time, they are hard-pressed to explain why these algorithms work. PTs tend to overgeneralize about operations and to overlook important distinctions. Some of the research reviewed helps us to understand the nuances of PTs' conceptions and can help to inform instruction. Further research is needed to (a) better understand PTs' conceptions when they enter our programs, and (b) better understand how PTs' conceptions develop.

Keywords: whole number, operation, number concepts, mathematical knowledge for teaching, mathematical content knowledge, preservice teachers, prospective teachers, elementary, teacher education

Introduction

Consider a prospective elementary teacher (PT) solving $527 - 135$, using the standard algorithm and explaining regrouping as follows:

You put a 1 over next to the number and that gives you 10. . . . I don't get how the 1 can become a 10. One and 10 are two different numbers. How can you subtract 1 from here and then add 10 over here? Where did the other 9 come from?

This PT clearly followed the correct procedure and arrived at the correct answer, but she was not able to provide an explanation for why this solution method results in a correct answer. Figure 1 shows her written work.

$$\begin{array}{r} 4 \cancel{5} 27 \\ - 135 \\ \hline 392 \end{array}$$

Figure 1. A PT's explanation of regrouping in $527 - 135$ (Thanheiser, 2009, p. 251).

Now consider another PT's reflection describing her inability to explain regrouping:

I learned [at the beginning of my elementary mathematics methods class] that there was a lot more to the concept [of number and place value] than I was aware of. I am able to use math effectively in my everyday life, such as balancing my checkbook, but when I was presented with questions as to why I carry out such procedures as *carrying*¹ and *borrowing* in addition and subtraction, I was stuck. I could not explain why I followed any of these procedures or rules. I just knew how to do them. This came as a huge shock to me considering I did well in most of my math classes. I felt terrible that I could not explain simple addition and subtraction.

Both of these PTs have determined that they want to teach children, yet at this point neither of them would be able to conceptually help an elementary-aged child make sense of why regrouping works when using the standard algorithms taught in the United States.

¹ Note that students in the United States often term regrouping in the context of addition *carrying* and regrouping in the context of subtraction *borrowing*.

Moreover, solving a problem using the algorithms is not sufficient knowledge for teaching mathematics to children. In the United States, the National Council of Teachers of Mathematics (NCTM, 2000a) and the *Common Core State Standards for Mathematics* (CCSSM) (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) call for children to develop a conceptual understanding (Hiebert & Lefevre, 1986) of the mathematics they encounter. Procedural fluency is one of several aspects of being mathematically proficient (National Research Council, 2001); the other four aspects are conceptual understanding, strategic competence, adaptive reasoning, and productive disposition. In order to be equipped to support students' development of mathematical proficiency, inservice teachers and PTs also need such an understanding of mathematics. Researchers have highlighted the need for teachers to have a deep and multifaceted understanding of the mathematics they teach (Hill, Ball, & Schilling, 2008; Ma, 1999). Less clear, however, is how improvement in teachers' knowledge can be accomplished.

At the core of elementary school mathematics is the teaching of number concepts and operations. NCTM (2000a) stressed that all pre K–12 students should “[a] understand numbers, ways of representing numbers, relationships among numbers, and number systems; [b] understand the meanings of operations and how they relate to one another; [and c] compute fluently and make reasonable estimates” (p. 32). A conceptual understanding of number and operations underlies learning of all future mathematics and other STEM subjects. “Number pervades all areas of mathematics. The other four Content Standards [other than Number and Operations] as well as all five Process Standards are grounded in number” (NCTM, 2000b, ¶1). In the CCSSM, “Number and Operation in Base

Ten” is one of the focal domains in each grade from K through 5, followed by “The Number System” in Grades 6–8 and “Number and Quantity” in high school.

Even with this strong focus on number throughout the K–12 curriculum, children in the United States and other countries “experience considerable difficulty constructing appropriate number concepts of multidigit numeration and appropriate procedures for multidigit arithmetic” (Verschaffel, Greer, & De Corte, 2007, p. 565). Rather than developing desirable number concepts and strategies, children often learn standard algorithms, which they view as involving concatenated single digits, rather than numbers of ones, tens, and hundreds (Fuson et al., 1997).

Research has also shown that elementary teachers and PTs in the United States and Australia continue to lack a conceptual understanding in this important area (Ball, 1988; Ma, 1999; Southwell & Penglase, 2005; Thanheiser, 2009, 2010). To be in a position to help PTs develop more sophisticated conceptions, mathematics educators need to (a) understand the conceptions with which PTs enter our classrooms, so that we can build on those conceptions (Bransford, Brown, & Cocking, 1999); and (b) understand how those conceptions can develop. As the authors of *The Mathematical Education of Teachers* stated, “The key to turning even poorly prepared prospective elementary teachers into mathematical thinkers is to work from what they *do* know” (Conference Board of the Mathematical Sciences [CBMS], 2001, p. 17). In order to work from what PTs know, we must first find out what they know.

In our summary work, we examined the current knowledge in the field of mathematics education concerning PTs’ conceptions of whole numbers and operations and the development thereof. We present a summary of the research in three parts:

1. *A Historical Look*, which represents a summary of the research literature prior to 1998.
2. *A Current Perspective*, based on research articles published between 1998 and 2011.
3. *A View of the Horizon*, based on 2012 journal articles, as well as 2011 and 2012 proceedings of the International Group for Psychology of Mathematics Education (PME) and North American Chapter of the International Group for Psychology of Mathematics Education (PME-NA).

Methods

The authors met as part of a larger Working Group (see introductory article to this Special Issue) focusing on summarizing the current knowledge of the field on PTs' content knowledge and the development thereof. The larger Working Group set the parameters for the search in general. In this section, we describe the methods that pertain to this particular article. We began by searching the ERIC database for combinations of the following search terms: *prospective, preservice, or pre-service* with any of *whole number, operation, place value, multidigit, algorithm, or number sense*.² Each combination of search terms was entered into the ERIC database. We searched separately for articles published prior to 1998 and for articles published from 1998 to 2011 in order to get an overview of the research that occurred during those periods.

All results were checked for a focus on PTs' content knowledge of whole numbers and operations. We read the title and abstract to determine whether each paper fit our

² Other search terms, e.g., *elementary education* and *whole number*, yielded no additional relevant results.

criteria. If the title and abstract did not suffice to make a determination of fit, then we read the whole paper. We included all papers that met the following criteria:

- Focused on our target group of PTs.
 - We also included prospective middle school teachers because some certification programs focus on K–8, and not all countries follow the same school system. We excluded papers focusing on prospective high school teachers.
 - We included papers focusing on both pre- and inservice teachers (i.e., mixed groups) but excluded papers focusing only on inservice teachers.
- Focused on content knowledge of whole numbers and operations. We included papers that did not exclusively focus on whole numbers and operations, but we focused our summaries of these on the findings that speak to PTs' conceptions of whole numbers and operations. We excluded papers that focused on beliefs or general content knowledge.
- Published research studies in peer-reviewed research journals. Our larger Working Group (Thanheiser et al., 2010) identified 23 journals to include in our reviews for the section focusing on the years 1998–2011 (i.e., *A Current Perspective*). (See introductory article to this Special Issue for more details).

The section focusing on the years prior to 1998 (i.e., *A Historical Look*) followed the same methods. For the section looking forward (i.e., *A View of the Horizon*), we followed the same methods starting in 2012 for journal articles. We also searched the 2011 and 2012 proceedings of the annual conferences of PME and PME-NA for relevant papers. It was our assumption that new research is more likely to be presented at conferences since there is a

time delay between conducting research and publishing papers. For this search, we read all paper titles in the relevant category. For example, Chapter 6 of the 2012 PME-NA proceedings focuses on “Teacher Education and Knowledge—Preservice.” We read all paper titles in this chapter to identify candidates to include in our review, based on the same criteria for article content as described above.

Once the research articles were identified, we read each to make a final determination of whether it should be included in the review. Questions and disagreements were discussed and resolved. In the end, we identified a total of 28 articles that were relevant to our search—26 peer-reviewed journal articles and 2 conference proceedings. The pre-1998 historical search identified 7 relevant peer-reviewed journal articles. The 1998–2011 article search identified 18 relevant peer-reviewed journal articles. The search for *A View of the Horizon* yielded 1 relevant peer-reviewed journal article and 2 peer-reviewed conference proceedings. We then read and summarized each of the three groups of articles.

Within the groups of articles belonging to *A Historical Look* and *A Current Perspective*, we identified categories to help organize our summaries of the literature. These categories were not decided a priori; rather, they emerged in the course of our review through a process of constant comparative analysis (Strauss & Corbin, 1998). These analyses were focused within the group of articles and were influenced by the number and nature of the articles in the group. *A Historical Look* consists of seven articles, almost all of which focus on multiplication and division. This being the case, we made fine-grained distinctions regarding what content knowledge was investigated (e.g., *understanding of the long-division algorithm*). As a result, in some cases, there is only one article per category.

The articles belonging to *A Current Perspective* are more abundant and cover a broader range of topics than those belonging to *A Historical Look*. The grain size and focus of our categories reflect this. For example, *PTs' reasoning about alternative algorithms or nonstandard strategies* is broader than the categories identified in *A Historical Look*, and it includes four articles. The categories in *A Current Perspective* reflect the broadening range of recent research related to PTs' content knowledge. For example, PTs' reasoning about alternative algorithms or nonstandard strategies was not a focus of any of the articles in *A Historical Look*.

With only three articles in the section *A View of the Horizon*, it did not make sense to categorize them. We simply summarized each article.

Results and Discussion

We first present *A Historical Look*, which represents a summary of the research literature prior to 1998. Next, we present *A Current Perspective*, based on research articles published between 1998 and 2011. Finally, we present *A View of the Horizon*, based on 2011 and 2012 PME and PME-NA proceedings and one article.

A Historical Look

What was known about PTs' understanding of whole numbers and operations prior to 1998? It is important to look at articles published prior to 1998 in order to understand the history of research in this area. It enables us to characterize the state of the field prior to our current perspective. This review is based on research articles published in mathematics education journals before 1998. Only seven such research articles were found. A summary of articles is included in Table 1.

What was known relates primarily to multiplication and division. In particular, the following five categories were identified:

1. PTs' reasoning about division story problems (Simon, 1993; Tirosh & Graeber, 1991; Vest, 1978).
2. PTs' reasoning about the properties of multiplication and division (Graeber, Tirosh, & Glover, 1989; Tirosh & Graeber, 1989).
3. PTs' understanding of the long-division algorithm (Simon, 1993).
4. PTs' understanding of divisibility and multiplicative structure (Zazkis & Campbell, 1996).
5. PTs' conceptions of zero (Wheeler, 1983).

Below, we report the results of our literature review. The results are organized according to the categories listed above.

Table 1

Articles Written Prior to 1998 Dealing With PTs' Knowledge of Whole Numbers and Operation

Authors	Year	Number of PTs Studied	PTs' Level	Country	Methodology
Graeber, Tirosh, & Glover	1989	129 PTs	Content or Methods course	USA	Survey for 129 Interview for 33 of the 129 PTs were asked to solve story problems for multiplication and division
Tirosh & Graeber	1991	80 PTs	Content course	USA	Survey; PTs were asked to (a) write expressions to match given story problems, (b) write story problems corresponding to given division expressions
Tirosh & Graeber	1989	136 PTs	Content or Methods course	USA	Survey; PTs were explicitly asked for misconceptions about multiplication and division and then asked to solve problems
Simon	1993	33 PTs	Methods course	USA	Open response written instrument for 33 PTs and interviews for 8 of the PTs – PTs were asked to write story problems for division and make sense of long division
Wheeler & Feghali	1983	52 PTs	Methods course	USA	Survey and interviews
Vest	1978	87 PTs	Content course	USA	Survey; PTs were asked to write story problems for division
Zazkis & Campbell	1996	21 PTs	Content course	USA	Interviews; The authors used a variety of tasks related to elementary number theory

PTs' reasoning about division story problems. Three studies investigated PTs' reasoning about the relationship between division and story problems. Studies by Vest (1978), Simon (1993), and Tirosh and Graeber (1991) all relate to PTs' reasoning about partitive and quotitive division story problems. Partitive problems involve the forming of equal-sized groups. In these story problems, a total amount or number of things is given, along with a desired number of equal groups. The question is how much or how many things should go in each group. Quotitive problems involve a predetermined group size. A total amount or number of things is given, along with a group size. The question that results from these situations is how many such groups can be formed. For example, in a context of children sharing candies, a partitive problem would give a total number of candies and a number of children and ask how many candies each child would receive, given that the candies are to be shared fairly. In the same context, a quotitive problem would give a number of candies that each child should receive and ask how many children can receive candy. It is important for children to be able to explore partitive and quotitive problems and to see both as related to the division operation (Carpenter, Fennema, Franke, Levi, & Empson, 1999).³ For this reason, it is just as important for PTs as it is for practicing teachers (Carpenter, Fennema, Peterson, & Carey, 1988) to make sense of these problem types and to be able to clearly distinguish between them.

Vest (1978) surveyed 87 PTs enrolled in a content course in the southern part of the United States to investigate their preferences for the type of division story problem, partitive or quotitive.⁴ When asked to write a division story problem, 59 of 87 PTs wrote a

³ Carpenter et al. used the language "partitive" and "measurement" problems.

⁴ Vest used the language "partitioning" and "measurement" problems.

partitive problem, while only 6 of 87 wrote a quotitive problem. The remaining 22 responses were categorized as “Other.” In another task, participants were given a page from an elementary textbook, in which whole-number division was introduced through measurement situations, which are quotitive in nature. Participants were asked to write a story problem that they would use to introduce that page. Again, participants favored partitive problems. Of the 89 participants who responded to this task, 62 wrote a partitive division story problem, while only 12 wrote a quotitive problem. This is a striking finding. It does not merely show that PTs preferred partitive problems in general; it shows that they would inappropriately choose partitive problems to introduce a lesson on quotitive division.

The above results might indicate that PTs simply do not see a difference between partitive and quotitive problems. However, Vest (1978) found that the same PTs were able to distinguish between problems of the two types. Given the simple instruction to label problems according to whether they asked “How many sets?” or “How many in each set?” the study participants categorized an average of 95.4% of story problems correctly. The PTs also did not express an explicit preference for one type of problem over the other. Nonetheless, when PTs were asked to produce their own story problems, partitive problems were overwhelmingly more common. PTs’ apparent preference for partitive problems is a concern because they will need to support their students in coming to relate to division to *both* partitive and quotitive problems.

A study of Simon (1993) corroborated Vest’s (1978) findings. Simon’s study involved 33 PTs enrolled in a methods course in the United States. When asked to write division story problems involving given numbers, the majority of the participants also

wrote problems that reflected a partitive, rather than quotitive, meaning of division. Specifically, 74% of the problems created were partitive, and only 17% were quotitive. Simon found that most participants were able to relate partitive story problems to division of whole numbers. On the other hand, the quotitive meaning was more elusive. Many assumed the partitive meaning and had difficulty when it did not fit well.

Tirosh and Graeber (1991) investigated the effect of division problem type (e.g., partitive or quotitive) on PTs' performance. They surveyed 80 PTs who were enrolled in either a content or methods course for elementary education majors in the southeastern United States. When asked to write expressions to match given story problems, the participants were less successful on quotitive than on partitive problems. The PTs also performed worse on problems in which the divisor was greater than the dividend. Both effects were statistically significant. When asked to write story problems corresponding to given division expressions, when the divisor was a whole number, the majority of participants wrote a partitive division problem that correctly matched the given expression. There were three such items. The percentage of correct partitive story problems ranged from 63 to 78%. Only 1 to 3% of correct responses were quotitive problems. Given an expression in which the divisor was not a whole number (e.g., $4 \div 0.5$), participants did not attempt to write a partitive problem, and only 44% correctly wrote a quotitive story problem. Tirosh and Graeber concluded, "Many preservice teachers are familiar with the partitive interpretation of division but have limited access to the measurement [quotitive] interpretation" (p. 162).

PTs' reasoning about properties of multiplication and division. One study focused on understanding properties of multiplication and division. Graeber, Tirosh, and

Glover (1989) documented PTs' misconceptions related to these properties. The researchers surveyed 129 PTs who were enrolled in either a content or methods course for elementary education majors in the southeastern United States. They then interviewed 33 of these PTs. The authors found that the PTs had difficulty with story problems in which multiplication did not "make bigger" or division did not "make smaller." For example, they performed worse when solving multiplication tasks if the multiplier was a decimal, rather than a whole number. When solving story problems, which required them to determine the appropriate operation to use, participants' choices were often influenced by the relative sizes of the given numbers, as opposed to the relationships between quantities. For example, on the four given multiplication story problems that had a decimal operator less than 1, more than 25% of the PTs incorrectly wrote a division expression, rather than a multiplication expression (Graeber et al., 1989). Explicit, incorrect beliefs about division were more common. In interviews with 33 of the PTs, Graeber et al. found that 22 of them reversed the roles of dividend and divisor when given story problems in which the divisor was greater than the dividend. The authors reported, "All 22 claimed that in division the larger number should be divided by the smaller number" (p. 99).

Tirosh and Graeber (1989) surveyed 136 PTs enrolled in either a content or methods course for elementary education majors in the United States. When asked directly, 87% of the PTs in the study responded correctly to questions concerning whether a product would always be greater than the factors (Tirosh & Graeber, 1989). However, in practice, many of the PTs reasoned in ways that evinced the influence of the belief that multiplication makes bigger. In response to a set of four survey questions regarding the properties of division, 72% of participants answered at least one of the True/False

questions incorrectly. For example, 52 (i.e., 38%) of the participants who responded to the statement “In division problems, the quotient must be less than the dividend” incorrectly answered that the statement was true.

The misconceptions of multiplication and division that were identified concerned problems involving rational numbers. PTs’ generalizations that multiplication makes bigger and division makes smaller hold true for whole numbers, except in the special cases involving 0 and 1. So, PTs’ reasoning about multiplication and division seem to be strongly connected to their experiences with whole numbers. In the whole-number domain, their reasoning is essentially correct. Thus, if we restrict our view to reasoning about whole-number operations, PTs may appear to be equipped to support students’ learning. However, children’s learning of mathematics in the early grades should prepare them for continued learning as they mature. If PTs’ overgeneralize about multiplication and division, their future students may make the same overgeneralizations and face the same difficulties as PTs when it comes to operations involving rational numbers.

PTs’ understanding of the long-division algorithm. Simon’s (1993) study involving 33 PTs investigated their understand of the long-division algorithm. Given a dividend and divisor and a calculator to use, 76% of participants were unable to find the remainder. The PTs also had difficulty explaining the meaning of the remainder in a division calculation. Many of them related the remainder to a fraction or decimal in inappropriate ways. They knew the long-division algorithm but were unable to explain its steps conceptually, and their justifications appealed to the procedure itself. Simon reported that participants were unable to connect a meaning of division with symbolic representations of division calculations, regardless of whether the calculations were

performed by long division or with a calculator. Simon characterized “prospective teachers’ mathematical knowledge as procedural and sparsely connected” (p. 252).

PTs’ understanding of divisibility and multiplicative structure. One study, by Zazkis and Campbell (1996), investigated PTs’ understanding of divisibility and the multiplicative structure of natural numbers. This study involved 21 PTs enrolled in a mathematics content course in the United States. The authors used a variety of tasks related to elementary number theory in interviews with the PTs. They found that the PTs tended to reason about divisibility procedurally, in terms of performing the division operation itself, rather than on the basis of multiplicative composition of number. The authors reported, “A minority (6 out of 21) of the participants in this study group were able to consistently discuss and demonstrate an understanding of divisibility as a property of, or relation between, natural numbers” (p. 546).

Given $M = 3^3 \times 5^2 \times 7$ and asked whether M was divisible by 7, participants thought that they needed to compute M and then divide by 7 to find out. The researchers observed that the PTs tended to be unsure of claims regarding divisibility in the absence of a specific quotient. For instance, even if a PT thought that M was divisible by 7, he or she was uncomfortable making such a claim without knowing what M divided by 7 actually equaled. The PTs also made reference to and use of divisibility rules, which were sometimes misremembered or misapplied, and they had difficulty reasoning about divisibility without the use of such rules. Participants also had difficulty in generating numbers with desired properties; they tended to guess and check, rather than to construct numbers in ways that would guarantee those properties.

PTs' conceptions of zero. One study reported on PTs' understanding of zero and of division by zero. The study, by Wheeler and Feghali (1983), involved 52 PTs enrolled in a methods course in the United States. The authors investigated the PTs' conceptions of zero, using a written instrument and individual interviews. The authors report that the PTs did not have an adequate understanding of zero. Most of the participants incorrectly answered items of the form $a \div b$, where $b = 0$. Most said that $0 \div 0 = 0$. In a classification task involving some cards with various images on them and some cards that were blank, most PTs rejected using blank cards as a category for classification. The participants were interested in the attributes of the images on the cards, and they viewed blank cards as being without attributes, rather than as having the attribute of being blank. The PTs described zero in a variety of ways, including as (a) a symbol, (b) a number, and (c) nothing. When asked directly whether zero was a number, most said that it was, but 15% of the participants disagreed. For example, one PT said, "Zero is not a number because it has no value" (p. 152).

Summary of the historical look. Our database search revealed seven research articles published in mathematics education journals prior to 1998 that addressed PTs' conceptions of whole numbers and operations. According to these reports, PTs favor the partitive over the quotitive meaning of division. They are more likely to write partitive story problems, except when the divisor is not a whole number (Simon, 1993; Tirosh & Graeber, 1991; Vest, 1978). PTs can recognize the difference between partitive and quotitive story problems, and they can find the solutions to problems of both types (Tirosh & Graeber, 1991; Vest, 1978); however, they perform worse on quotitive problems, and they perform worse on problems in which the divisor is greater than the dividend (Tirosh

& Graeber, 1991). When asked to make their beliefs about the properties of multiplication explicit, PTs tend to respond correctly (e.g., to indicate the multiplication does not always “make bigger”). However, their responses to various tasks reflect the influence of overgeneralizations about multiplication (Tirosh & Graeber, 1989).

When it comes to division, PTs often explicitly make incorrect claims, such as that the divisor must be less than the dividend (Graeber et al., 1989). In addition, PTs bring a range of procedural and conceptual knowledge to bear on division-related tasks; however, their knowledge of division is disconnected (Simon, 1993). Their understanding of the long-division algorithm tends to be procedural, and they have difficulty relating that procedure to real-world situations. PTs also reason procedurally about divisibility and often feel the need to perform calculations in order to answer questions regarding divisibility (Zazkis & Campbell, 1996). PTs have limited conceptions of zero. Some do not regard it as a legitimate number, and many PTs answer questions involving division by zero incorrectly.

Reflections on the historical look. The pre-1998 research literature characterized PTs’ knowledge as inadequate and partially incorrect. Descriptions emphasized PTs’ limited understandings and reliance on procedures. PTs were described as holding misconceptions, which led, at least some of the time, to incorrect answers. We learn from these reports that PTs’ knowledge of whole numbers and operations—especially multiplication and division—was in need of improvement. At the same time, this research literature is limited in its guidance regarding how to support PTs to develop more sophisticated mathematical understandings. The reports provide snapshots of PTs’ content knowledge, and these descriptions do not emphasize ways in which PTs may be able to

build on what they know to improve their understanding of whole numbers and operations.

The historical look also leaves us with many unanswered questions regarding specific content knowledge that was not addressed. The literature focused on multiplication and division and did not address addition or subtraction. It did not address PTs' conceptions of whole numbers themselves. In particular, their understanding of place value was not explored. Also, researchers did not report on PTs' understanding of number theory beyond divisibility. For instance, PTs' reasoning about oddness and evenness were not directly addressed. Perhaps the most noteworthy finding is simply how little the field knew about PTs' knowledge of whole numbers and operations prior to 1998.

A Current Perspective

With number being such a pervasive topic in elementary school mathematics, surprisingly few papers have focused on PTs' conceptions of whole numbers and operations. Our search for research literature on PTs' understanding of whole numbers and operations, spanning the time from 1998 to 2011, resulted in 18 articles (see Table 2).

Table 2

Articles Written About PTs' Understanding of Whole Numbers and Operations, Spanning the Time From 1998 to 2011

Authors	Year	Number of PTs Studied	PTs' Level	Country	Methodology
Chapman	2007	20 PTs	Content course	Canada	Group tasks that allowed the PTs to reflect on the operations in order to develop a deeper understanding
Crespo & Nicol	2006	32 PTs	Methods course	Canada/ USA	Task involving division
Glidden	2008	381 PTs	Content course	USA	Tasks involving the order of operations
Green, Piel, & Flowers	2008	53/39 PTs	Child Development course	Canada/ USA	Survey
Kaasila, Pehkonen, & Hellinen	2010	269 PTs	Math Education course	Finland	Task involving nontraditionally posed division problem
Liljedahl, Chernoff, & Zazkis	2007	90 PTs	Content course		Tasks using a computer-based microworld
Harkness & Thomas	2008	71 PTs	Content course	USA	Case study of a student sharing an invented algorithm
Lo, Grant, & Flowers	2008	38 PTs	Content course	USA	Video of class sessions
McClain	2003	24 PTs	Methods course	USA	Survey
Menon	2003	77 PTs	Methods course	USA	Set of tasks involving two-digit multiplication

(continued)

Table 2—continued

Authors	Year	Number of PTs Studied	PTs' Level	Country	Methodology
Menon	2004	142 PTs	Methods course	USA	10-item number sense test
Menon	2009	64 PTs	Methods course	USA	Survey
Thanheiser	2009	15 PTs	Content course	USA	Interviews
Thanheiser	2010	33 PTs	Methods course	USA	Survey and interviews
Tsao	2005	12 PTs	Content course	USA	Interviews
Yackel, Underwood, & Elias	2007	45 PTs	Content course	USA	Video of class sessions
Yang	2007	15 PTs		Taiwan	Interviews
Zazkis	2005	116 PTs	Content course	Canada	Task involving prime numbers

Of the research papers reviewed, the following five categories emerged:

1. PTs' conceptions of number and the development thereof (McClain, 2003; Thanheiser, 2009, 2010; Yackel, Underwood, & Elias, 2007).
2. PTs' reasoning about alternative algorithms or nonstandard strategies (Harkness & Thomas, 2008; Kaasila, Pehkonen, & Hellinen, 2010; Lo, Grant, & Flowers, 2008; Menon, 2003, 2009).
3. PTs' number sense (Menon, 2004; Tsao, 2005; Yang, 2007; Zazkis, 2005).

4. PTs' conceptions of arithmetic operations and order of operations (Chapman, 2007; Crespo & Nicol, 2006; Glidden, 2008).
5. Addressing PTs' misconceptions through the use of manipulatives or computer microworlds (Green, Piel, & Flowers, 2008; Liljedahl, Chernoff, & Zazkis, 2007).

Below, we report the results of our literature review. The results are organized according to the categories listed above.

PTs' conceptions of number and the development thereof. Two research studies focused on PTs' conceptions of number (Thanheiser, 2009, 2010) and two research studies focused on the development thereof (McClain, 2003; Yackel et al., 2007). Thanheiser (2009) interviewed 15 PTs in the United States before their first content course for teachers. The interview data allowed for the identification and categorization of PTs' conceptions of multidigit whole numbers into four major groups: thinking in terms of (a) *reference units*, (b) *groups of ones*, (c) *concatenated-digits plus*, and (d) *concatenated-digits only*. See Table 3 for the definition and distribution of the conceptions among the PTs in that study.

Thanheiser (2009) found that two thirds of the PTs in that study saw the digits in a number incorrectly in terms of ones, at least some of the time. This conception prohibits the PTs from making sense of regrouping. And while the groups-of-ones conception is a correct conception, it also limits what a PT will be able to explain. While PTs may be able to correctly explain the regrouped 1 in Figure 1 as 100 *ones*, they may struggle to explain that the 1 represents 10 *tens* and thus combined with the 2 *tens* represents 12 *tens*. Thus, while five of the PTs held a correct conception, only three of those five held a conception that

enabled them to explain all aspects of regrouping, including why we “make the 1 a 10” when we move it over.

Table 3

Definition and Distribution of Conceptions in the Context of the Standard Algorithm for the 15 U.S. PTs in Thanheiser’s (2009) Study (p. 263)

Conception	# of PTs
<i>Reference units.</i> PTs with this conception reliably ⁵ conceive of the reference units for each digit and relate reference units to one another, seeing the 3 in 389 as 3 <i>hundreds</i> or 30 <i>tens</i> or 300 <i>ones</i> , the 8 as 8 <i>tens</i> or 80 <i>ones</i> , and the 9 as 9 <i>ones</i> . They can reconceive of 1 <i>hundred</i> as 10 <i>tens</i> , and so on.	3
<i>Groups of ones.</i> PTs with this conception reliably conceive of all digits correctly in terms of groups of <i>ones</i> (389 as 300 <i>ones</i> , 80 <i>ones</i> , and 9 <i>ones</i>) but not in terms of reference units; they do not relate reference units (e.g., 10 <i>tens</i> to 1 <i>hundred</i>).	2
<i>Concatenated-digits plus.</i> PTs with this conception conceive of <i>at least one</i> digit as an incorrect unit type, at least on occasion. They struggle when relating values of the digits to one another (e.g., in 389, 3 is 300 <i>ones</i> but the 8 is only 8 <i>ones</i>).	7
<i>Concatenated-digits only.</i> PTs holding this conception conceive of <i>all</i> digits in terms of <i>ones</i> (e.g., 548 as 5 <i>ones</i> , 4 <i>ones</i> , and 8 <i>ones</i>).	3

Thanheiser (2009) also examined PTs’ conceptions in various contexts. One of these contexts was a time task. PTs were given an artifact of children’s mathematical thinking in which the child had incorrectly applied the standard subtraction algorithm in a time context (see Figure 2). Of the 15 PTs in the study, 9 initially thought that the child’s application of the standard algorithm was correct. Eight of those 9 PTs eventually changed their mind after calculating the time difference another way. However, only 8 of the 15 PTs

⁵ *Reliably* in these definitions means that after the PTs were first able to draw on a conception in their explanations in a context, they continued to do so in that context.

were able to explain why the application of this algorithm was incorrect (i.e., regrouping 100 rather than 60) and alter the algorithm to make it work for a time situation.

Last week, in a third-grade classroom, the children were given the following problem:

You were on a train that left Los Angeles at 2:53 p.m. and arrived in Phoenix at 7:08 p.m. How long were you on the train?

One student solved the problem as follows:

$$\begin{array}{r} 7:08 \\ - 2:53 \\ \hline 4:55 \end{array}$$

He explained, "I took the 3 from the 8; that is 5. Then I couldn't take the 5 from the 0, so I borrowed 1 from the 7 and put it by the zero. Then I took 5 from 10; that's 5. Then I took 2 from 6, and that's 4."

Figure 2. Time task (Thanheiser, 2009, p. 259).

In a different task, PTs were asked to relate hundreds and millions (i.e., how many hundreds are in a million?) and in that context were asked to relate tens and hundreds (i.e., 10 tens are a hundred) and hundreds and thousands (i.e., 10 hundreds are a thousand). Six of the 15 PTs, at least in some instances, claimed that $100 \times 100 = 1,000$. Thanheiser (2009) explained this mistake as possibly being based on an overgeneralization of the pattern $10 \times 10 = 100$ (e.g., multiply a reference unit by itself to get the next larger one) resulting in $100 \times 100 = 1,000$. Thanheiser also noted that this notion would make it hard to see the regularity in our base-ten number system. In summary, Thanheiser found that PTs who held one of the concatenated-digits conceptions struggled when asked to explain why things worked, whereas PTs who held one of the correct conceptions were able to explain these things. This was true in the context of the standard algorithms, as well as in alternate contexts.

In a follow-up study, Thanheiser (2010) surveyed 33 PTs enrolled in a math methods course in the United States. In this investigation of PTs' interpretations of regrouped digits, Thanheiser (a) replicated the earlier results that most PTs held one of the concatenated-digits conceptions, even at the end of their teacher education programs; and (b) refined the *concatenated-digits plus* conception into three further categories:

1. Regrouped digits are consistently explained as 10, regardless of whether it is in the context of addition or subtraction.
2. Regrouped digits are explained consistently depending on context (i.e., 10 in subtraction, 1 in addition, or vice versa).
3. Changed interpretations of the regrouped digit depending on the question posed (i.e., regrouped 1 in the tens' place in the context of addition as 10 or 1 in different tasks).

In this study, only 3 of 33 PTs were able to correctly explain the values of the regrouped digits in both addition and subtraction contexts. Of the remaining 30 PTs, 5 saw the values of all regrouped digits as 1, consistent with the concatenated-digits conception. The distribution of the remaining 25 PTs who fell into the concatenated-digits-plus category can be seen in Table 4.

Table 4

Conceptions of the 33 PTs in the Context of Standard Algorithms (Detailed) in Thanheiser (2010)

Conception Across Addition and Subtraction Tasks	Number of PTs
One of the two correct conceptions (reference units or groups of ones)	3
Concatenated digits plus Refined conception: <ul style="list-style-type: none"> – Regrouped digits consistently explained as 10 (regardless of whether it is in the context of addition or subtraction) (7 PTs) – Regrouped digits explained consistently depending on context (i.e., 10 in subtraction, 1 in addition or vice versa) (10 PTs) – Changed interpretations of the regrouped digit depending on the question posed (i.e., regrouped 1 in the ten’s place in the context of addition as 10 or 1 in different tasks) (8 PTs) 	25
Concatenated digits only	5

A surprising result in Thanheiser’s (2010) study was that eight PTs changed their explanation of the regrouped digits from one problem to the next. While they would interpret the regrouped 1 as 10 or 1 in one addition problem, they would interpret it differently in another (see Figure 3). For example, PTs may interpret the circled 1 in the first problem in Figure 3 as 1, but the circled 1 in the second problem in Figure 3 as 10, thus changing how they interpret the regrouped digit in the tens’ place in the context of addition.

$$\begin{array}{r}
 \textcircled{1} \\
 389 \\
 + 475 \\
 \hline
 864
 \end{array}
 \qquad
 \begin{array}{r}
 \textcircled{1} \\
 259 \\
 + \quad 38 \\
 \hline
 297
 \end{array}$$

Figure 3. Comparing regrouped digits in the tens' place in the context of addition (Thanheiser, 2010, p. 249).

Two studies focused on the development of PTs' conceptions of place value (McClain, 2003; Yackel et al., 2007). Both studies examined the PTs' development of conceptions by working with them in a context involving an alternate base (base eight). McClain (2003) asked 24 PTs enrolled in the second of two methods courses in the United States to work in the Candy Factory context (Cobb, Yackel, & Wood, 1992), in which eight candies were packed into a roll of candies and eight rolls were packed into a box of candies. While McClain asked PTs to work in the context of boxes, rolls, and pieces of candies, she did not ask PTs to use base-eight notation. In earlier work, she had found that the PTs were distracted by being asked to use base-eight notation and focused more on that than on the mathematics of quantifying, adding, and subtracting numbers. With the Candy Factory context, McClain found that PTs initially focused on pictures to represent numbers but then invented a notational form using B for boxes, R for rolls and P for pieces. McClain focused on grouping and regrouping to help the PTs understand place value and the multiplicative structure of the system. At the end of the sequence, PTs were asked to buy or sell candies to help them understand addition and subtraction. McClain found that PTs invented "nontraditional yet personally meaningful algorithms for addition and subtraction to symbolize their activity" (p. 298). The goal of this sequence was to help PTs develop a reference-units conception (cf. Thanheiser, 2009) and thus see a box not just as a box, but

simultaneously as eight rolls, as well as 64 candies, and then draw on that number concept to develop algorithms and a deeper understanding of the numbers and of the algorithms.

McClain (2003) examined the development of PTs' conceptions and compared it to the development of children's conceptions of base ten. She found that the PTs' development mirrored that of children. She stated, "This finding also has broader implications—that the broad base of research conducted in elementary classrooms can feed forward to inform efforts at supporting the development of PTs' content knowledge" (p. 301). As a result, PTs also came to the realization that in order to teach for conceptual understanding, they themselves would need to possess this type of understanding.

Yackel, Underwood, and Elias (2007) also used the Candy Factory context in base eight with 45 PTs in a content course in the United States. They examined how PTs learned to count in base eight and used that as an underpinning for operating on numbers in base eight. In contrast to McClain (2003), Yackel et al. did use base-eight language (e.g., they named a unit of eight as "one-e"). They spent a considerable amount of time in counting to lay the foundations for operating on numbers. One of their foci was to help PTs coordinate units of different rank (i.e., develop reference-units conceptions). They point out that the focus on counting not only helped the PTs make sense of the counting sequence and how it is learned by children, but it also helped the PTs make sense of early arithmetic. They note that it is often surprising to PTs as well as teacher educators how much sense making can happen in early arithmetic.

PTs' reasoning about alternative algorithms or nonstandard strategies. Five studies focused on exploring PTs' reasoning about alternative algorithms (Harkness & Thomas, 2008; Kaasila et al., 2010; Lo et al., 2008; Menon, 2003, 2009). Harkness and

Thomas (2008) worked with 71 PTs in three sections of a freshmen content course in the United States. They reported that PTs' understanding of invented algorithms is more procedural than conceptual. The PTs were presented with a case study of a student sharing an invented algorithm in front of her class and being told by her teacher that it is incorrect (Corwin, 1989). Then the PTs were asked to explore the validity of the invented algorithm (see Figure 4). Only 7 of 71 PTs were able to explain why the invented algorithm works. An additional 15 PTs showed some understanding but were not able to give a complete explanation. The remaining 49 PTs drew on procedural understanding to give explanations. For example, they used arguments such as that the 10 from the upper line was moved to the lower line.

34	34
<u>x23</u>	<u>x23</u>
102	92
<u>680</u>	<u>690</u>
782	782
"Downwards"	"Upwards"

Figure 4. Standard downwards and invented upwards algorithms (Harkness & Thomas, 2008, p. 129).

While the PTs struggled to make sense of the upwards method, they still empathized with the student in the case, either by relating to similar experiences in their past, highlighting that their current class allows alternative methods, or hoping that they will be able to allow for alternative methods in their own future classrooms. The PTs also disagreed with the teachers' choices in the case. Finally, the PTs highlighted how impressed they were by the child in the case. In addition, Harkness and Thomas found that it was

difficult to get PTs to attend to the details of the mathematics; if they expected PTs to write about the mathematics, the authors needed to explicitly ask them to do so.

Menon (2003) reported on PTs' responses to a set of tasks involving two-digit multiplication. A total of 77 PTs in two sections of a methods course in the United States were shown three different ways of performing two-digit multiplication. The PTs responded to each task individually and then discussed their ideas in small groups of four or five students. The first task concerned the standard algorithm as it relates to partial products. The partial products in 65×34 were mislabeled, not taking place value into account, to draw the conclusion that the product consisted of 3 groups of 65 plus 4 groups of 65. PTs were asked whether they agreed with the description of the partial products. The instructor then pointed out that 34×65 represented 34 groups of 65, so that 27 groups of 65 had not been accounted for. PTs were asked to respond. The author reported that, when responding individually, 39% of PTs said that nothing was missing from the product.

In Menon's (2003) second task, 65×34 was computed from left to right. That is, the work showed 1,950 (i.e., the product of 30 and 65) on the first row and 260 (i.e., the product of 4 and 65) on the second row. PTs were asked whether this alternative method would work, and why or why not. The author reported that only 52% of PTs gave a correct explanation.

The third task involved yet another way of computing the same product. Three partial products were shown: 1,820, 240, and 150 (i.e., $(4 \times 5) + (30 \times 60)$, 4×60 , and 30×5). PTs were shown only the computed partial products. They were asked to determine the origins of these, to decide whether or not this algorithm was generalizable, and to justify their answers. The author reported that only 39% of the participants

produced a correct explanation. The author noted that the frequencies of correct responses from groups of PTs were considerably greater than for individuals. Thus, discussing the ideas in groups often led to a correct group response.

In another study, Menon (2009) surveyed PTs to investigate their understanding of multidigit multiplication. A written instrument was administered to 64 PTs enrolled in a middle school mathematics methods course in the United States. The author found that 95% of the PTs correctly computed 456×78 . However, only 75% were able to write a correct word problem corresponding to this computation. The author gives two examples of incorrect responses. One was a division, rather than multiplication, story problem. The other showed lack of awareness of the distinct roles of multiplier and multiplicand: "There are 456 pencils, and 78 erasers in the classroom. If we multiply the 456 pencils and the 78 erasers, how many pencils and erasers will we have in total?" (p. 3). This PT seemed to rather directly translate the computation to a story involving pencils and erasers without taking into account what it would mean to multiply pencils by erasers. Evidently, the PT had in mind a meaning for multiplication as finding a total number of things, but the PT did not provide a rate in the story, and as a result the suggested multiplication was nonsensical. The vast majority (86%) of the PTs' explanations for the algorithm were largely procedural, and their ideas for helping a child learn to compute the product were likewise mostly procedural (72%). Menon described the PTs in this study as generally displaying the kind of understanding of multiplication that was required of them as students, noting that this understanding is inadequate for teaching multidigit multiplication.

Lo, Grant, and Flowers (2008) worked with 38 PTs enrolled in a content course in the United States. The authors found that the PTs' ability to develop and justify reasoning

strategies for multiplication develops slowly and presents several challenges. In their study, they describe a four-day lesson designed to help PTs develop a deeper understanding of multiplication. The researchers focused on both the development of and the justification of reasoning strategies. They found PTs struggled with both. Lo et al. hypothesize that the PTs struggled with the development of reasoning strategies because (a) the PTs lacked the multiplicative structure, and (b) the PTs lacked the understanding that there is more to a multiplication problem than finding the answer. One of the tasks they used was to ask students to multiply 24×38 by starting with $20 \times 40 = 800$ and adjusting the result. Lo et al. argue that PTs struggle with justifications for four reasons:

1. The PTs think justification is a description of the steps.
2. The PTs think justification is drawing a picture.
3. The PTs struggled in relating the picture to their reasoning, especially with the area model.
4. The PTs struggled coordinating the equal groups interpretation with their strategy.

Lo et al. (2008) also found that PTs struggled in recognizing the difference between procedural and conceptual descriptions of solutions to multiplication problems. It was not clear whether PTs needed more time or different kinds of experiences to continue to develop their understandings. As a result, the researchers suggest more research be conducted to investigate this. They also emphasized that we need to highlight the “ineffectiveness of memorizing and applying rules/procedures without understanding why they work” (p. 20).

Kaasila, Pehkonen, and Hellinen (2010) examined PTs'⁶ understanding of a nontraditionally posed division problem. The participants were 269 Finnish PTs enrolled in a mathematics education course. The problem that the researchers posed was, "We know that $498 \div 6 = 83$. How could you conclude from this relationship without using long-division algorithm what $491 \div 6 =$ is?" (p. 247). According to the authors, "This problem especially measures conceptual understanding, adaptive reasoning, and procedural fluency" (p. 247). Kaasila et al. found that 45% of the PTs were able to produce complete or almost correct solutions, and 30% produced complete and correct solutions. Of those PTs who answered correctly, almost all drew on both subtraction and division in their reasoning. Kaasila et al. (2010) identified four difficulties that the remaining 70% of the PTs had:

- (1) staying on the integer level (difficulties especially in conceptual understanding),
 - (2) inability to handle the remainder of the division (difficulties especially in procedural fluency),
 - (3) difficulties in understanding the relationships between different operations (problems especially in conceptual understanding), and
 - (4) inadequate reasoning strategies (difficulties especially in adaptive reasoning).
- (p. 257)

PTs' number sense. Four studies focused on number sense (Menon, 2004; Tsao, 2005; Yang, 2007; Zazkis, 2005). Tsao (2005) and Yang (2007) both found that PTs, especially the ones who struggled, relied on procedures rather than using number sense to solve problems. Tsao's study involved PTs enrolled in six sections of a mathematics content course in the United States. He found that the PTs were not ready to be immersed into a curriculum that reflects the vision of less emphasis on paper-and-pencil computation and more emphasis on number sense and mental arithmetic, as described in the NCTM

⁶ They also examined secondary students; however, in this report we leave out that part of the study.

Standards. Tsao compared six randomly selected high-ability PTs (scoring in the top 10% on a 25-item number sense test) and six randomly selected low-ability PTs (scoring in the bottom 10%). The data indicate that the high-ability students were more successful on each type of number sense item than the low-ability students. The items were intended to assess five components of number sense—number magnitude, use of benchmarks, decomposition/recomposition, relative effect of operations on numbers, and flexibility with numbers and operations. Compared to high-ability students, the low-ability students in this study (a) tended to use rule-based methods more frequently when answering interview items; and (b) preferred the use of standard, written computation algorithms rather than the use of “number sense based” strategies. The high-ability students tended to use benchmarks and to apply “number sense based” knowledge. Results also indicate that items including fractions were more difficult than whole number and decimal items for both groups of students.

Yang (2007) interviewed 15 PTs from a university in southern Taiwan. He examined strategies used by PTs when responding to number sense-related items. Yang defines number sense as consisting of the following four categories: (a) understanding the meanings of numbers, operations, and their relationships; (b) recognizing relative number size; (c) judging the reasonableness of a computational result by using strategies of estimation; and (d) developing and using benchmarks appropriately. Yang found that for each category, about two thirds of participants relied on rule-based methods to answer the questions. Thus, PTs, especially those in the low-ability group, tended to reason procedurally.

Menon (2004) worked with 142 PTs in four sections of a methods course in the United States. The PTs took a 10-item multiple-choice test designed to measure their number sense. The test consisted of items intended to measure their ability to (a) make mathematical judgments, (b) develop useful and effective strategies for numerical situations, and (c) understand number and operations related to fractions and decimals.⁷ A student would be considered having number sense only if he or she provided both a correct response and a correct explanation to an item. Menon stated that a majority of the PTs were able to make mathematical judgments by being aware of the mathematical context while not blindly perform computations. However, Menon also noted that many of the PTs were unable to provide explanations describing the relationship between the numbers used to arrive at a solution.

Zazkis (2005) worked with 116 PTs enrolled in a content course for elementary teacher certification in Canada. After a unit on elementary number theory, the PTs were posed the question whether the product of 151×157 was a prime number. Incorrect responses included: (a) two prime numbers multiplied together would result in another prime; (b) the last digit of 23,707 is 7, so the product is prime; and (c) the sum of the digits equals 19 and 19 is prime. Furthermore, Zazkis indicated that although 74 of the PTs correctly identified that the product was a composite number, only 52 of them were able to justify their reasoning using the definition of a prime or composite number. Zazkis summarized that the underlying feature of these shortcomings was PTs not understanding that the product of two whole numbers will have more than two factors.

⁷ We include only whole number items.

PTs' conceptions of arithmetic operations and order of operations. Three studies focused on understanding of operations (Chapman, 2007; Crespo & Nicol, 2006; Glidden, 2008). Chapman (2007) examined 20 PTs' understanding of arithmetic operations. The PTs were enrolled in an elementary mathematics content course in Canada. The PTs' initial knowledge of arithmetic operations was "inadequate to teach conceptually and in depth" (p. 347). The PTs' initial knowledge was based upon "procedural understanding of both the mathematical and semantic structure of a problem" (p. 347). Often PTs thought there was only one way to represent an operation. Chapman devised three group tasks that allowed the PTs to reflect on the operations in order to develop a deeper understanding. The first group task asked PTs to create word problems similar to given word problems and to compare different kinds of word problems. See Figure 5 for Part 1 of the task. After they worked on these problems individually, the PTs worked in groups to discuss their answers and then were asked to collaboratively create word problems that reflected the meaning for each of the four operations and then reflect on those.

PART 1:

1. Create an original word problem for an elementary grade. Reflect on and describe what you thought about to create the problem.
 2. Create a word problem that is similar to the following problem.
Golf balls come in packs of 4. A carton holds 25 packs. Marie, the owner of a sporting goods store, ordered 1600 golf balls. How many cartons did Marie order?
Reflect on and describe what you thought about to create the problem.
 3. Compare and contrast this golf balls problem to the following problem:
Jerry, a new elementary school teacher, wants his students to sit in groups on the carpet section of the classroom floor for certain activities. After doing some investigating, he found that the carpet area was 144 square feet and decided to allocate 6 square feet for one student. How many groups will Jerry be able to form if he wants 3 students in a group?
-

Figure 5. Part 1 of Chapman's (2007) task (p. 343).

The second group task asked PTs to examine a given list of word problems representing various situations for each operation and asked them to analyze the word problems by modeling solutions and to reflect on similarities and differences. The third group task asked PTs to compare and contrast the problems given in the first group task, to create their own problems, to review an elementary textbook, and to choose one of the operations to create a lesson plan. The tasks were deemed effective as they allowed PTs to have:

- “relevant, practical, and meaningful examples and possibilities for thinking about the concepts”
- “allowed for simulation of real-world situations”
- “promoted reflection and discourse”
- “facilitated new understandings of familiar concepts.” (p. 384)

Crespo and Nicol (2006) focused on understanding division by zero. They examined 32 PTs enrolled in two methods courses (18 in course A in Canada, 14 in course B in the United States). In course A, PTs watched videos of children who stated that $5 \div 0 = 0$. In course B, PTs reacted to written artifacts stating the same thing. The authors’ stated reason for changing from video to written artifacts was to eliminate distractors from the mathematics, such as PTs focusing on the child’s emotional state or the interviewer’s actions. Crespo and Nicol found that initially almost all PTs in both courses thought that $5 \div 0 = 0$. They stated “the preservice teachers’ initial understandings of 0 and division by 0 were founded more on rule-based and flawed reasoning than on well-reasoned mathematical explanations and that they lacked the experience and inclination to understand or appreciate different ideas and approaches to this topic” (p. 94). Examining

the artifacts and discussing them helped the PTs make sense of division by zero, and at the end of the study, only two PTs remained who thought $5 \div 0 = 0$. The authors also noted that division by zero is often overlooked in prospective teacher education, and with such a high number of PTs entering with incorrect conceptions, we should include this topic into our courses.

Glidden (2008) focused on order of operations and found that PTs in a mathematics content course in the United States held superficial knowledge of the order of operations. He found that many PTs who performed multiplication before addition—correctly followed the order of operations—also performed addition before subtraction and multiplication before division. He hypothesized that they take the mnemonic PEMDAS (i.e., “Please Excuse My Dear Aunt Sally”) too literally. He also showed that almost 80% of the PTs used the incorrect order of operations to execute -3^2 .

Addressing PTs’ misconceptions through the use of manipulatives or computer microworlds. One paper focused on addressing PTs’ misconceptions with the use of manipulatives (Green et al., 2008), and one paper focused on addressing understanding of factors, multiples, and primes using a computer microworld (Liljedahl et al., 2007). Green et al. worked with two sets of PTs in the context of a child development course. There were 53 PTs in the first study, which was conducted in Canada, and 39 PTs in the second study, which was conducted in the United States. Green et al. explored the use of manipulatives and found that manipulative-based instruction resulted in statistically significant decreases in arithmetic misconceptions and statistically significant increases in knowledge of the basic arithmetic operations. The authors reported that the use of manipulatives can effectively reverse most arithmetic misconceptions of PTs and that the

same activities used to reverse misconceptions can also improve the accuracy and depth of arithmetic knowledge. Thus, they conclude, manipulatives can and should be used effectively in PT classrooms.

Liljedahl et al. (2007) worked with 90 PTs enrolled in a content course. The authors engaged the PTs in tasks using a computer-based microworld called Number Worlds to encourage them to reason in new ways about basic concepts in elementary number theory. The microworld represents sets of numbers in grids. The user can determine the set to be represented and can also change the dimensions of the grid. The researchers stated that “about one-half” of the 90 PTs spent time in the computer lab using Number Worlds, and 17 of those who used Number Worlds participated in follow-up interviews. The authors reported that the PTs who used Number Worlds “thickened” their understandings of factors, multiples, and primes. They described new connections that the PTs made based on the visual representation of the microworld. The PTs noticed patterns that related to their previous understandings of factors, multiples, and primes, such as occurrences of multiples at regular intervals. They also developed new understandings that were grounded in visual features of the microworld, such as patterns in the distribution of primes.

Summary of the current perspective. Our review revealed the following five categories of current research examining PTs’ content knowledge of whole numbers and operations: (a) PTs’ understanding of whole-number concepts, and the development thereof; (b) PTs’ reasoning about alternative algorithms or nonstandard strategies; (c) PTs’ number sense; (d) PTs’ conceptions of arithmetic operations and the order of operations; and (e) addressing PTs’ misconceptions through the use of manipulatives or computer microworlds.

Many PTs realize that they need to understand mathematics conceptually in order to teach their future students for conceptual understanding (McClain, 2003). However, PTs tend to approach tasks procedurally because they lack the conceptual understanding required to do otherwise. For example, many PTs exhibit unsophisticated conceptions of digits in whole numbers, which then limits their understanding of regrouping when adding or subtracting (Thanheiser, 2009, 2010). Similarly with multiplication, PTs have difficulty explaining why algorithms work, and their reasoning is not easily improved (Lo et al., 2008). PTs may not recognize the difference between a procedural and conceptual description of a solution (Lo et al., 2008). Related to these difficulties is the finding that PTs tend to rely on procedures, rather than make use of number sense (Menon, 2004; Tsao, 2005; Yang, 2007). Many PTs are unable to describe relationships between numbers to arrive at a solution efficiently (Menon, 2004). Furthermore, many PTs experience difficulty understanding zero (Crespo & Nicol, 2006), and they have superficial understanding of the order of operations (Glidden, 2008). Overall, PTs, especially the ones who struggled, relied heavily on procedural knowledge.

Reflection on the current perspective. More research articles concerning PTs' knowledge of whole numbers and operations appeared between 1998 and 2011 than appeared prior to 1998. However, much remains to be learned about PTs' mathematical thinking in this area. As exemplified by the thinking of two PTs described in the introduction of this paper, researchers have found that PTs rely on memorized procedures involving whole numbers and operations. In addition, many PTs struggle to conceptually explain why the procedures work. Some research has examined how mathematics educators can help PTs develop more sophisticated conceptions, but there is still much

work to do for the mathematics education community to better understand how PTs' conceptions develop and how this development can be facilitated.

We note that several of the current research papers dealt with PTs' conceptions and/or the development thereof. This may suggest that mathematics educators are moving away from a focus on snapshot studies explicating what PTs do and do not know and toward attempting to understand PTs' conceptions and how their knowledge develops. The papers on alternative algorithms and nonstandard strategies address the need to help PTs develop the ability to make sense of children's mathematical thinking so that they will be prepared to do more than present standard procedures to their students. The papers on number sense show that PTs who exhibit better number sense are more able to make conceptual sense of problems. Thus, there is a need to promote PTs' number sense development. The papers on using manipulatives and computer microworlds identify tools that can help PTs make sense of mathematics. In the spirit of working from what PTs know (CBMS, 2001), these articles contribute to the literature on PTs' knowledge of whole numbers and operations. Thus, the current literature helps mathematics educators to be better equipped to support PTs' learning. However, many open questions remain.

A View of the Horizon

Our review of journal articles published in 2012 and papers from PME and PME-NA proceedings for conference years 2011 and 2012 yielded only three relevant results (see Table 5).

Table 5

*Articles Published in 2012 and PME/PME-NA Proceedings From 2011 and 2012
Dealing With PTs' Knowledge of Whole Numbers and Operation*

Authors	Year	Number of PTs Studied	PTs' Level	Country	Methodology
Feldman	2012	59 PTs	Content course	USA	Pre/post interviews with 6 PTs regarding their understanding of number theory, as well as pre/post surveys of 59 PTs
Thanheiser	2012	1 PT	Content course	USA	Two interviews in which one PT was asked to reason about and justify addition and subtraction algorithms in different bases
Whitacre & Nickerson	2012	7 PTs	Content course	USA	Interviews in which PTs were asked to perform mental computation and to justify their strategies

Thanheiser (2012) offers a case study of a PT's understanding of regrouping. The PT seemed to hold all the essential knowledge pieces needed to give a conceptual explanation for regrouping but was unable to do so. Thanheiser hypothesized that this may be due to the PT's lack of strategic knowledge (i.e., knowing when to draw on a piece of information). This point highlights the need to attend not only to conceptual understanding but also to strategic knowledge in PT content courses.

Whitacre and Nickerson (2012) report on PTs' reasoning in the area of whole-number mental computation. Building on the work of Yang (2007) that focused on the strategies that PTs tend to use, Whitacre and Nickerson investigated the mathematical justifications that U.S. PTs offer when using nonstandard mental computation strategies.

The authors describe PTs' justifications for both valid and invalid strategies. They draw distinctions between the mathematical ideas involved in the various justifications in order to clarify how PTs' strategies make sense to the PTs themselves. This analysis sheds light on PTs' reasoning when using nonstandard mental computation strategies.

In particular, Whitacre and Nickerson (2012) report the mathematical ideas used in PTs' justifications for four nonstandard addition strategies and four nonstandard subtraction strategies. This includes justifications for valid and invalid versions of subtrahend compensation. For example, two PTs computed $125 - 49$ mentally to find the amount that a vendor would profit if he bought an item for \$49 and then sold it for \$125. Both PTs rounded 49 to 50, and both knew that $125 - 50$ equaled 75. However, their thinking differed when it came to how to compensate for the initial rounding move. Trina reasoned that she should add 1 to 75 because by adding 1 to 49 she had "pretended [the vendor] used more money than he did," thus decreasing his profit. By contrast, Natalie reasoned that she had added 1 to "the problem" and now had to subtract 1 from the problem in order to compensate (p. 779). Thus, Trina distinguished the roles of minuend and subtrahend, and this enabled her to determine how to compensate correctly. By not making this distinction, Natalie drew the incorrect conclusion regarding how to compensate. These fine-grained distinctions in PTs' justifications reveal the reasoning underlying their strategies.

The only other report concerning PTs' knowledge of whole numbers and operations was a paper of Feldman (2012). Feldman gave a poster presentation, so the information in the proceedings paper is quite limited. He studied PTs' developing understanding of number theory during instruction on number theory in a mathematics content course in

the United States. Feldman used action-process-object-schema theory (Dubinsky, 1991) to analyze participants' interview responses and describe transitions between levels of understanding. He also mentions quantitative data that points to changes in PTs' understanding of number theory.

Summary of the view of the horizon. Although only three papers appeared in 2012 journals and recent PME and PME-NA proceedings, these reports do point to promising directions for research related to PTs' knowledge of whole numbers and operations. Each report involves analyses that move beyond pointing out deficits in PTs' content knowledge. Instead, these papers concern understanding PTs' reasoning in depth and studying the development of that reasoning. The report of Whitacre and Nickerson (2012) derives from Whitacre's (2012) dissertation, which focuses on PTs' number sense development. Note that in this work we did not search for or review dissertations. In the coming years, we hope that relevant dissertations, such as the works of Roy (2008) and others, will lead to valuable contributions to the research literature concerning PTs' knowledge of whole numbers and operations and the development thereof.

Conclusion

We have summarized research literature concerning PTs' knowledge of whole numbers and operations in *A Historical Look*, *A Current Perspective*, and *A View of the Horizon*. Taking a step back to view the history of this research literature, we see a progression. Not only has more research been done and more learned in this area, but there is also evidence of a shift in emphasis. We know that there are inadequacies in PTs' knowledge, and these are cause for concern. Recently, researchers have become more interested in investigating the nuances of PTs' conceptions and the further development of

their conceptions. We see the emphasis on deficits and misconceptions giving way to insightful characterizations of how PTs reason when doing mathematics and how they can make use of what they know as they develop more sophisticated conceptions. We are optimistic about the future of research on PTs' knowledge of whole numbers and operations because the kind of research being done has the potential to illuminate our understanding of PTs' mathematical thinking and to better equip mathematics teacher educators to help PTs make sense of mathematics in new ways.

We conclude with a few suggestions regarding directions for future research:

- There is a need for more research like that of Thanheiser (2009, 2010) that provides insightful characterizations of PTs' conceptions, rather than evaluations of PTs' knowledge that emphasize what they do not know. Such findings can help mathematics teacher educators to better understand PTs' thinking and to envision how PTs' conceptions can develop over time.
- There is a need for more work like that of McClain (2003) and Yackel et al. (2007) that moves beyond snapshot studies of content knowledge to document PTs' learning process in an illuminating manner. Such studies have the potential to advance the field both theoretically and practically by helping mathematics teacher educators to better understand how to support productive learning in courses for PTs.

These and other suggestions are discussed in greater detail in "Mathematical Content Knowledge for Teaching Elementary Mathematics: What Do We Know, What Do We Not Know, and Where Do We Go?" in this Special Issue.

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