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The Multiplicities of a Dual-thin Q -polynomial Association Scheme

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Abstract

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ denote a symmetric association scheme, and assume that Y is Q -polynomial with respect to an ordering E_0, \dots, E_D of the primitive idempotents. Bannai and Ito conjectured that the associated sequence of multiplicities m_i ($0 \leq i \leq D$) of Y is unimodal. Talking to Terwilliger, Stanton made the related conjecture that $m_i \leq m_{i+1}$ and $m_i \leq m_{D-i}$ for $i < D/2$. We prove that if Y is dual-thin in the sense of Terwilliger, then the Stanton conjecture is true.

1 Introduction

For a general introduction to association schemes, we refer to [1], [2], [5], or [9]. Our notation follows that found in [3].

Throughout this article, $Y = (X, \{R_i\}_{0 \leq i \leq D})$ will denote a symmetric, D -class association scheme. Our point of departure is the following well-known result of Taylor and Levingston.

1.1 Theorem. [7] *If Y is P -polynomial with respect to an ordering R_0, \dots, R_D of the associate classes, then the corresponding sequence of valencies*

$$k_0, k_1, \dots, k_D$$

is unimodal. Furthermore,

$$k_i \leq k_{i+1} \quad \text{and} \quad k_i \leq k_{D-i} \quad \text{for } i < D/2. \quad \blacksquare$$

Indeed, the sequence is log-concave, as is easily derived from the inequalities $b_{i-1} \geq b_i$ and $c_i \leq c_{i+1}$ ($0 < i < D$), which are satisfied by the intersection numbers of any P -polynomial scheme (cf. [5, p. 199]).

In their book on association schemes, Bannai and Ito made the dual conjecture.

1.2 Conjecture. [1, p. 205] *If Y is Q -polynomial with respect to an ordering E_0, \dots, E_D of the primitive idempotents, then the corresponding sequence of multiplicities*

$$m_0, m_1, \dots, m_D$$

is unimodal.

Bannai and Ito further remark that although unimodality of the multiplicities follows easily whenever the dual intersection numbers satisfy the inequalities $b_{i-1}^* \geq b_i^*$ and $c_i^* \leq c_{i+1}^*$ ($0 < i < D$), unfortunately these inequalities do not always hold. For example, in the Johnson scheme $J(k^2, k)$ we find that $c_{k-1}^* > c_k^*$ whenever $k > 3$.

Talking to Terwilliger, Stanton made the following related conjecture.

1.3 Conjecture. [8] *If Y is Q -polynomial with respect to an ordering E_0, \dots, E_D of the primitive idempotents, then the corresponding multiplicities satisfy*

$$m_i \leq m_{i+1} \quad \text{and} \quad m_i \leq m_{D-i} \quad \text{for } i < D/2.$$

Our main result shows that under a suitable restriction on Y , these last inequalities are satisfied.

To state our result more precisely, we first review a few definitions. Let $\text{Mat}_X(\mathbb{C})$ denote the \mathbb{C} -algebra of matrices with entries in \mathbb{C} , where the rows and columns are indexed by X , and let A_0, \dots, A_D denote the associate matrices for Y . Now fix any $x \in X$, and for each integer i ($0 \leq i \leq D$), let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ with yy entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } xy \in R_i, \\ 0 & \text{if } xy \notin R_i. \end{cases} \quad (y \in X). \quad (1)$$

The *Terwilliger algebra* for Y with respect to x is the subalgebra $T = T(x)$ of $\text{Mat}_X(\mathbb{C})$ generated by A_0, \dots, A_D and E_0^*, \dots, E_D^* . The Terwilliger algebra was first introduced in [9] as an aid to the study of association schemes. For any $x \in X$, $T = T(x)$ is a finite dimensional, semisimple \mathbb{C} -algebra, and is noncommutative in general. We refer to [3] or [9] for more details. T acts faithfully on the vector space $V := \mathbb{C}^X$ by matrix multiplication. V is endowed with the inner product $\langle \cdot, \cdot \rangle$ defined by $\langle u, v \rangle := u^t \bar{v}$ for all $u, v \in V$. Since T is semisimple, V decomposes into a direct sum of irreducible T -modules.

Let W denote an irreducible T -module. Observe that $W = \sum E_i^* W$ (orthogonal direct sum), where the sum is taken over all the indices i ($0 \leq i \leq D$) such that $E_i^* W \neq 0$. We set

$$d := |\{i : E_i^* W \neq 0\}| - 1,$$

and note that the dimension of W is at least $d + 1$. We refer to d as the *diameter* of W . The module W is said to be *thin* whenever $\dim(E_i^*W) \leq 1$ ($0 \leq i \leq D$). Note that W is thin if and only if the diameter of W equals $\dim(W) - 1$. We say Y is *thin* if every irreducible $T(x)$ -module is thin for every $x \in X$.

Similarly, note that $W = \sum E_i W$ (orthogonal direct sum), where the sum is over all i ($0 \leq i \leq D$) such that $E_i W \neq 0$. We define the *dual diameter* of W to be

$$d^* := |\{i : E_i W \neq 0\}| - 1,$$

and note that $\dim W \geq d^* + 1$. A *dual thin* module W satisfies $\dim(E_i W) \leq 1$ ($0 \leq i \leq D$). So W is dual thin if and only if $\dim(W) = d^* + 1$. Finally, Y is *dual thin* if every irreducible $T(x)$ -module is dual thin for every vertex $x \in X$.

Many of the known examples of Q -polynomial schemes are dual thin. (See [10] for a list.) Our main theorem is as follows.

1.4 Theorem. *Let Y denote a symmetric association scheme which is Q -polynomial with respect to an ordering E_0, \dots, E_D of the primitive idempotents. If Y is dual-thin, then the multiplicities satisfy*

$$m_i \leq m_{i+1} \quad \text{and} \quad m_i \leq m_{D-i} \quad \text{for } i < D/2.$$

The proof of Theorem 1.4 is contained in the next section.

We remark that if Y is bipartite P - and Q -polynomial, then it must be dual-thin and $m_i = m_{D-i}$ for $i < D/2$. So Theorem 1.4 implies the following corollary. (cf. [4, Theorem 9.6]).

1.5 Corollary. *Let Y denote a symmetric association scheme which is bipartite P - and Q -polynomial with respect to an ordering E_0, \dots, E_D of the primitive idempotents. Then the corresponding sequence of multiplicities*

$$m_0, m_1, \dots, m_D$$

is unimodal. ■

1.6 Remark. By recent work of Ito, Tanabe, and Terwilliger [6], the Stanton inequalities (Conjecture 1.3) have been shown to hold for any Q -polynomial scheme which is also P -polynomial. In other words, our Theorem 1.4 remains true if the words “dual-thin” are replaced by “ P -polynomial”.

2 Proof of the Theorem

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ denote a symmetric association scheme which is Q -polynomial with respect to the ordering E_0, \dots, E_D of the primitive idempotents. Fix any $x \in X$ and let $T = T(x)$ denote the Terwilliger algebra for Y with respect to x . Let W denote any irreducible T -module. We define the *dual endpoint* of W to be the integer t given by

$$t := \min\{i : 0 \leq i \leq D, E_i W \neq 0\}. \tag{2}$$

We observe that $0 \leq t \leq D - d^*$, where d^* denotes the dual diameter of W .

2.1 Lemma. [9, p.385] *Let Y be a symmetric association scheme which is Q -polynomial with respect to the ordering E_0, \dots, E_D of the primitive idempotents. Fix any $x \in X$, and write $E_i^* = E_i^*(x)$ ($0 \leq i \leq D$), $T = T(x)$. Let W denote an irreducible T -module with dual endpoint t . Then*

$$(i) \ E_i W \neq 0 \quad \text{iff} \quad t \leq i \leq t + d^* \quad (0 \leq i \leq D).$$

(ii) *Suppose W is dual-thin. Then W is thin, and $d = d^*$. ■*

2.2 Lemma. [3, Lemma 4.1] *Under the assumptions of the previous lemma, the dual endpoint t and diameter d of any irreducible T -module satisfy*

$$2t + d \geq D. \quad \blacksquare$$

Proof of Theorem 1.4. Fix any $x \in X$, and let $T = T(x)$ denote the Terwilliger algebra for Y with respect to x . Since T is semisimple, there exists a positive integer s and irreducible T -modules W_1, W_2, \dots, W_s such that

$$V = W_1 + W_2 + \dots + W_s \quad (\text{orthogonal direct sum}). \quad (3)$$

For each integer j , $1 \leq j \leq s$, let t_j (respectively, d_j^*) denote the dual endpoint (respectively, dual diameter) of W_j . Now fix any nonnegative integer $i < D/2$. Then for any j , $1 \leq j \leq s$,

$$\begin{aligned} E_i W_j \neq 0 &\Rightarrow t_j \leq i && \text{(by Lemma 2.1(i))} \\ &\Rightarrow t_j < i + 1 \leq D - i \leq D - t_j && \text{(since } i < D/2) \\ &\Rightarrow t_j < i + 1 \leq D - i \leq t_j + d_j^* && \text{(by Lemmas 2.1(ii), 2.2)} \\ &\Rightarrow E_{i+1} W_j \neq 0 \text{ and } E_{D-i} W_j \neq 0 && \text{(by Lemma 2.1(i)).} \end{aligned}$$

So we can now argue that, since Y is dual thin,

$$\begin{aligned} \dim(E_i V) &= |\{j : 0 \leq j \leq s, E_i W_j \neq 0\}| \\ &\leq |\{j : 0 \leq j \leq s, E_{i+1} W_j \neq 0\}| \\ &= \dim(E_{i+1} V). \end{aligned}$$

In other words, $m_i \leq m_{i+1}$. Similarly,

$$\begin{aligned} \dim(E_i V) &= |\{j : 0 \leq j \leq s, E_i W_j \neq 0\}| \\ &\leq |\{j : 0 \leq j \leq s, E_{D-i} W_j \neq 0\}| \\ &= \dim(E_{D-i} V) \end{aligned}$$

This yields $m_i \leq m_{D-i}$. ■

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