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On Stochastic Comparisons of Largest Order Statistics in the Scale Model

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On stochastic comparisons of largest order statistics in the scale model

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Abstract

Let $X_{\lambda_1}, X_{\lambda_2}, \ldots, X_{\lambda_n}$ be independent nonnegative random variables with $X_{\lambda_i} \sim F(\lambda_i t)$, $i = 1, \ldots, n$, where $\lambda_i > 0$, $i = 1, \ldots, n$ and F is an absolutely continuous distribution. It is shown that, under some conditions, one largest order statistic $X_{n:n}^{\lambda}$ is smaller than another one $X_{n:n}^{\theta}$ according to likelihood ratio ordering. Furthermore, we apply these results when F is a generalized gamma distribution which includes Weibull, gamma and exponential random variables as special cases.

Keywords: likelihood ratio order; reverse hazard rate order; majorization; order statistics

Mathematics Subject Classification (2010) 62G30 ; 60E15 ; 60K10

1   Introduction

Let F be the distribution function of some nonnegative random variable X. Then the independent random variables $X_{\lambda_1}, X_{\lambda_2}, \ldots, X_{\lambda_n}$ follow the scale model if there exists $\lambda_1 > 0, \ldots, \lambda_n > 0$ such that, $F_i(t) = F(\lambda_i t)$ for $i = 1, \ldots, n$. F is called the baseline distribution and the λ_i's are the scale parameters. Recently, Khaledi et al. (2011) studied conditions under which series and parallel systems consisting of components with lifetimes from the scale family of distributions are ordered in the hazard rate and the reverse hazard rate orderings, respectively. In this paper we revisit this problem and broaden the scope of their results to likelihood ratio ordering, which is stronger than the other orderings.

There is an extensive literature on stochastic orderings among order statistics and spacings when the observations follow the exponential distribution with different scale parameters, see for instance, Kochar and Kirmani (1996), Dykstra et al. (1997), Bon and Păltănea (1999), Khaledi and Kochar (2000), Kochar and Xu (2009), Joo and Mi (2010), Torrado et al. (2010), Torrado and Lillo (2013) and the references therein. Also see a review paper by Kochar (2012) on this topic. A natural way to extend these works is to consider the scale model since it includes the exponential distribution, among others. The scale model, also known in the literature as the proportional random variables (PRV) model, is of theoretical as well as practical importance in various fields of probability and statistics and has been investigated in Pledger and Proschan (1971), Hu (1995) and Torrado and Veerman (2012), among others.

In this article, we focus on stochastic orders to compare the magnitudes of two largest order statistics from the scale model when one set of scale parameters majorizes the other. The new results obtained here are applied when the baseline distributions are generalized gamma distributions. Recall that a random variable X has a generalized gamma distribution, denoted by $X \sim GG(\beta, \alpha)$, when its density function has the following form

$$f(t) = \frac{\beta}{\Gamma(\frac{\alpha}{\beta})} x^{\alpha-1} e^{-x^{\beta}}, t > 0,$$

where $\beta, \alpha > 0$ are the shapes parameters. The importance of this distribution lies in its

flexibility in describing lifetime distributions ensuring their applications in survival analysis and reliability theory. It is of great interest in several areas of application, see for example, Manning et al. (2005), Ali et al. (2008) and Chen et al. (2012). It is well known that generalized gamma distribution includes many important distributions like exponential, Weibull and gamma as special cases. We also present some new results which strengthen some of those established earlier in the literature by Zhao (2011), Misra and Misra (2013) and Zhao and Balakrishnan (2013) for gamma distributions and Torrado and Kochar (2015) for Weibull distributions. Further results on these subjects are contained in, e.g., Lihong and Xinsheng (2005), Khaledi and Kochar (2007), Zhao and Balakrishnan (2011), Balakrishnan and Zhao (2013), Fang and Zhang (2013). It may be mentioned that Gupta et al. (2006) considered monotonicity of the hazard rate and the reverse hazard rates of series and parallel systems when the components are dependent.

The rest of the paper is organized as follows. In Section 2, we introduce the required definitions. Sections 3 and 4 are devoted to investigate the reverse hazard rate and likelihood ratio orderings of largest order statistics considering the general scale model, respectively.

2 Basic definitions

In this section, we review some definitions and well-known notions of majorization concepts and stochastic orders. Throughout this article *increasing* and *non-decreasing* will be used synonymously as *decreasing* and *non-increasing*.

We focus attention in this article on nonnegative random variables. We shall also be using the concept of majorization in our discussion. Let $\{x_{(1)}, x_{(2)}, \ldots, x_{(n)}\}$ denote the increasing arrangement of the components of the vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$.

Definition 2.1 *The vector $\boldsymbol{x}$ is said to be majorized by the vector $\boldsymbol{y}$, denoted by $\boldsymbol{x} \overset{m}{\leq} \boldsymbol{y}$, if*

$$\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}, \quad for \ j = 1, \ldots, n-1 \quad and \quad \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}.$$

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Functions that preserve the ordering of majorization are said to be Schur-convex, as one can see in the following definition.

Definition 2.2 *A real valued function φ defined on a set $\mathcal{A} \in \Re^n$ is said to be* Schur-convex *(Schur-concave) on $\mathcal{A}$ if*

$$\boldsymbol{x} \overset{m}{\leq} \boldsymbol{y} \; on \; \mathcal{A} \Rightarrow \varphi(\boldsymbol{x}) \leq (\geq)\varphi(\boldsymbol{y}).$$

Replacing the equality in Definition 2.1 by a corresponding inequality leads to the concept of weak majorization. One can majorize from above or below. The following definition addresses majorization from above. It is also called *supermajorization.*

Definition 2.3 *The vector $\boldsymbol{x}$ is said to be weakly majorized by the vector $\boldsymbol{y}$, denoted by $\boldsymbol{x} \overset{w}{\leq} \boldsymbol{y}$, if*

$$\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)}, \quad for \; j = 1, \ldots, n.$$

It is known that $\boldsymbol{x} \overset{m}{\leq} \boldsymbol{y} \Rightarrow \boldsymbol{x} \overset{w}{\leq} \boldsymbol{y}$. The converse is, however, not true. For extensive and comprehensive details on the theory of majorization orders and their applications, please refer to the book of Marshall et al. (2011).

Let X and Y be univariate random variables with cumulative distribution functions (c.d.f.'s) F and G, survival functions $\bar{F} (= 1 - F)$ and $\bar{G} (= 1 - G)$, p.d.f.'s f and g, hazard rate functions $h_F (= f/\bar{F})$ and $h_G (= g/\bar{G})$, and reverse hazard rate functions $r_F (= f/F)$ and $r_G (= g/G)$, respectively. The following definitions introduce stochastic orders, which are considered in this article, to compare the magnitudes of two random variables. For more details on stochastic comparisons, see Shaked and Shanthikumar (2007).

Definition 2.4 *We say that X is smaller than Y in the:*

a) *usual stochastic order if $\bar{F}(t) \leq \bar{G}(t)$ for all t and in this case, we write $X \leq_{st} Y$,*

b) *reverse hazard rate order if $G(t)/F(t)$ is increasing in t for which the ratio is well defined, or if $r_F(t) \leq r_G(t)$, for all t, denoted by $X \leq_{rh} Y$,*

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c) likelihood ratio order if $g(t)/f(t)$ is increasing in t for which the ratio is well defined, for all t, denoted by $X \leq_{lr} Y$.

3 Reverse hazard rate ordering results

Let $X_{\lambda_1}, X_{\lambda_2}, \ldots, X_{\lambda_n}$ be independent nonnegative random variables with $X_{\lambda_i} \sim F(\lambda_i t)$, $i = 1, \ldots, n$, where $\lambda_i > 0, i = 1, \ldots, n$ and F is an absolutely continuous distribution. Let f, h and r be the density, hazard rate and the reverse hazard rate functions of F, respectively. The distribution function of $X_{n:n}^{\lambda}$, the largest order statistic formed from $X_{\lambda_1}, X_{\lambda_2}, \ldots, X_{\lambda_n}$ is

$$F_{n:n}^{\lambda}(t) = \prod_{i=1}^{n} F(\lambda_i t),$$

and its reverse hazard rate function is

$$r_{n:n}^{\lambda}(t) = \sum_{i=1}^{n} \lambda_i r(\lambda_i t). \tag{3.1}$$

Khaledi et al. (2011) proved the following result on comparing two parallel systems when the underlying random variables follow the scale model and their scale parameters majorize each other.

Theorem 3.1 *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent nonnegative random variables with $X_{\lambda_i} \sim F(\lambda_i t)$, $i = 1, \ldots, n$, where $\lambda_i > 0$, $i = 1, \ldots, n$ and F is an absolutely continuous distribution. Let r be the reverse hazard rate function of F, respectively. If $t^2 r'(t)$ is increasing in t, then*

$$(\lambda_1, \ldots, \lambda_n) \overset{m}{\leq} (\theta_1, \ldots, \theta_n) \Rightarrow X_{n:n}^{\lambda} \leq_{rh} X_{n:n}^{\theta}.$$

In the next theorem we extend the above result to the case when the two sets of scale parameters weakly majorize each other instead of usual majorization.

Theorem 3.2 *Let $X_{\lambda_1}, X_{\lambda_2}, \ldots, X_{\lambda_n}$ be independent random variables with $X_{\lambda_i} \sim F(\lambda_i t)$ where $\lambda_i > 0$, $i = 1, \ldots, n$. If $t r(t)$ is decreasing in t and $t^2 r'(t)$ is increasing in t, then*

$$(\lambda_1, \ldots, \lambda_n) \overset{w}{\leq} (\theta_1, \ldots, \theta_n) \Rightarrow X_{n:n}^{\lambda} \leq_{rh} X_{n:n}^{\theta}.$$

Proof. Fix $t > 0$. Then the reverse hazard rate of $X_{n:n}^{\lambda}$ as given by (3.1) can be rewritten as

$$r_{n:n}^{\lambda}(t) = \sum_{i=1}^{n} \lambda_i r(\lambda_i t) = \frac{1}{t} \sum_{i=1}^{n} \psi(\lambda_i t),$$

where $\psi(t) = tr(t)$, $t \geq 0$. From Theorem A.8 of Marshall et al. (2011) (p. 59) it suffices to show that, for each $t > 0$, $r_{n:n}^{\lambda}(t)$ is decreasing in each λ_i, $i = 1, \ldots, n$, and is a Schur-convex function of $(\lambda_1, \ldots, \lambda_n)$. By the assumptions, $tr(t)$ is decreasing in t, then the reverse hazard rate function of $X_{n:n}$ is decreasing in each λ_i.

Now, from Proposition C.1 of Marshall et al. (2011) (p. 64), the convexity of $\psi(t)$ is needed to prove Schur-convexity of $r_{n:n}^{\lambda}(t)$. Note that the assumption $t^2 r'(t)$ is increasing in t is equivalent to $r(t) + tr'(t)$ is increasing in t since

$$\left[t^2 r'(t)\right]' = t\left(2r'(t) + tr''(t)\right) = t\left[r(t) + tr'(t)\right]',$$

and $r(t) + tr'(t)$ is increasing in t is equivalent to $tr(t)$ is convex since

$$[tr(t)]' = r(t) + tr'(t).$$

Hence, $\psi(t)$ is convex. ∎

Note that the conditions of Theorem 3.2 are satisfied by the generalized gamma distribution with parameters $\beta \leq 1$ and $\alpha > 0$ as Khaledi et al. (2011) proved that $t^2 r'(t)$ is an increasing function for $X \sim GG(\beta, \alpha)$, when $\beta \leq 1$ and $\alpha > 0$. It is easy to verify that $tr(t)$ is a decreasing function of t when $\beta, \alpha > 0$.

As one natural application, Theorem 3.2 guarantees that, for parallel systems of components having independent generalized gamma distributed lifetimes with parameters $\beta \leq 1$ and $\alpha > 0$, the weakly majorized scale parameter vector leads to a larger system's lifetime in the sense of the reverse hazard rate order.

The generalized gamma distribution includes many important distributions like exponential ($\beta = \alpha = 1$), Weibull ($\beta = \alpha$) and gamma ($\beta = 1$) as special cases. Misra and Misra (2013) proved that in the case of gamma distribution with density function

$$f(t) = \frac{\lambda_i^{\alpha}}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda_i t}, \ \ t > 0,$$

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when $\alpha > 0$ and $n \geq 2$,

$$(\lambda_1, \ldots, \lambda_n) \overset{w}{\leq} (\theta_1, \ldots, \theta_n) \Rightarrow X_{n:n}^\lambda \leq_{rh} X_{n:n}^\theta. \tag{3.2}$$

Note that, when $0 < \alpha \leq 1$, (3.2) can be seen as a particular case of Theorem 3.2 since gamma distribution is a particular case of generalized gamma distribution when $\beta = 1$.

Recently, Torrado and Kochar (2015) established, in Theorem 4.1, the reverse hazard rate ordering between parallel systems based on two sets of heterogeneous Weibull random variables with a common shape parameter and with scale parameters which are ordered according to a majorization order when the common shape parameter α satisfies $0 < \alpha \leq 1$. So Theorem 4.1 in Torrado and Kochar (2015) can be seen as a particular case of Theorem 3.2 because Weibull distribution is a particular case of generalized gamma distribution when $\beta = \alpha$.

4   Likelihood ratio ordering results

In this section, we investigate whether the result of Theorem 3.2 can be strengthened from reverse hazard rate ordering to likelihood ratio ordering. First, we consider the case when $n = 2$ and the scale parameters of the scale model are ordered according to a weekly majorization order.

Theorem 4.1 *Let X_{λ_1}, X_λ be independent nonnegative random variables with $X_{\lambda_1} \sim F(\lambda_1 t)$ and $X_\lambda \sim F(\lambda t)$, where $\lambda_1, \lambda > 0$ and F is an absolutely continuous distribution. Let $Y_{\lambda_1^*}, Y_\lambda$ be independent nonnegative random variables with $Y_{\lambda_1^*} \sim F(\lambda_1^* t)$ and $Y_\lambda \sim F(\lambda t)$, where $\lambda_1^*, \lambda > 0$. Let r be the reverse hazard rate function of F. Assume $tr(t)$ and $tr'(t)/r(t)$ are both decreasing in t. Suppose $\lambda_1^* = \min(\lambda, \lambda_1, \lambda_1^*)$, then*

$$(\lambda_1, \lambda) \overset{w}{\leq} (\lambda_1^*, \lambda) \Rightarrow \frac{r_{2:2}^*(t)}{r_{2:2}(t)} \text{ is increasing in } t.$$

Proof. Let

$$\phi(t) = \frac{r_{2:2}^*(t)}{r_{2:2}(t)} = \frac{\lambda_1^* r(\lambda_1^* t) + \lambda r(\lambda t)}{\lambda_1 r(\lambda_1 t) + \lambda r(\lambda t)},$$

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and its derivative for $t > 0$ is,

$$\phi'(t) \overset{sign}{=} t^3 \left((\lambda_1^*)^2 \, r'(\lambda_1^* t) + \lambda^2 r'(\lambda t) \right) \left(\lambda_1 r(\lambda_1 t) + \lambda r(\lambda t) \right)$$

$$-t^3 \left(\lambda_1^* r(\lambda_1^* t) + \lambda r(\lambda t) \right) \left(\lambda_1^2 r'(\lambda_1 t) + \lambda^2 r'(\lambda t) \right)$$

$$= \lambda_1 \lambda_1^* t^3 \left(\lambda_1^* r'(\lambda_1^* t) r(\lambda_1 t) - \lambda_1 r(\lambda_1^* t) r'(\lambda_1 t) \right)$$

$$+ \lambda_1 \lambda t^3 \left(\lambda r'(\lambda t) r(\lambda_1 t) - \lambda_1 r(\lambda t) r'(\lambda_1 t) \right)$$

$$+ \lambda \lambda_1^* t^3 \left(\lambda_1^* r'(\lambda_1^* t) r(\lambda t) - \lambda r(\lambda_1^* t) r'(\lambda t) \right)$$

$$= \lambda_1 \lambda_1^* t^2 r(\lambda_1^* t) r(\lambda_1 t) \left(\lambda_1^* t \frac{r'(\lambda_1^* t)}{r(\lambda_1^* t)} - \lambda_1 t \frac{r'(\lambda_1 t)}{r(\lambda_1 t)} \right)$$

$$+ \lambda_1 \lambda t^2 r(\lambda t) r(\lambda_1 t) \left(\lambda t \frac{r'(\lambda t)}{r(\lambda t)} - \lambda_1 t \frac{r'(\lambda_1 t)}{r(\lambda_1 t)} \right)$$

$$+ \lambda \lambda_1^* t^2 r(\lambda_1^* t) r(\lambda t) \left(\lambda_1^* t \frac{r'(\lambda_1^* t)}{r(\lambda_1^* t)} - \lambda t \frac{r'(\lambda t)}{r(\lambda t)} \right)$$

$$= \psi(\lambda_1^* t) \psi(\lambda_1 t) \left(-\eta(\lambda_1^* t) + \eta(\lambda_1 t) \right) + \psi(\lambda t) \psi(\lambda_1 t) \left(-\eta(\lambda t) + \eta(\lambda_1 t) \right)$$

$$+ \psi(\lambda_1^* t) \psi(\lambda t) \left(-\eta(\lambda_1^* t) + \eta(\lambda t) \right),$$

where

$$\psi(t) = t r(t) \text{ and } \eta(t) = -t \frac{r'(t)}{r(t)}.$$

Note that $\psi(t) \geq 0$ for all $t \geq 0$ and $\eta(t) \geq 0$ since $r'(t) \leq 0$ because $tr(t)$ is a decreasing function. By the assumptions, we know that $\psi(t)$ is decreasing and $\eta(t)$ is increasing in t. If $\lambda_1^* = \min(\lambda, \lambda_1, \lambda_1^*)$ and $(\lambda_1, \lambda) \overset{w}{\leq} (\lambda_1^*, \lambda)$, then $\lambda_1^* \leq \lambda \leq \lambda_1$ or $\lambda_1^* \leq \lambda_1 \leq \lambda$. When $\lambda_1^* \leq \lambda \leq \lambda_1$, we have

$$\phi'(t) \overset{sign}{=} \psi(\lambda_1^* t) \psi(\lambda_1 t) \left(-\eta(\lambda_1^* t) + \eta(\lambda_1 t) \right) + \psi(\lambda t) \psi(\lambda_1 t) \left(-\eta(\lambda t) + \eta(\lambda_1 t) \right)$$

$$+ \psi(\lambda_1^* t) \psi(\lambda t) \left(-\eta(\lambda_1^* t) + \eta(\lambda t) \right)$$

$$\geq 0,$$

since $\eta(\lambda_1^* t) \leq \eta(\lambda t) \leq \eta(\lambda_1 t)$. When $\lambda_1^* \leq \lambda_1 \leq \lambda$, we get

$$\phi'(t) \geq \psi(\lambda t) \psi(\lambda_1 t) \left(-\eta(\lambda_1^* t) + \eta(\lambda_1 t) \right) + \psi(\lambda t) \psi(\lambda_1 t) \left(-\eta(\lambda t) + \eta(\lambda_1 t) \right)$$

$$+ \psi(\lambda_1 t) \psi(\lambda t) \left(-\eta(\lambda_1^* t) + \eta(\lambda t) \right)$$

$$= 2\psi(\lambda t) \psi(\lambda_1 t) \left(-\eta(\lambda_1^* t) + \eta(\lambda_1 t) \right) \geq 0.$$

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Therefore $r_{2:2}^*(t)/r_{2:2}(t)$ is increasing in t. ∎

In the next result, we extend Theorem 3.2 from reverse hazard rate ordering to likelihood ratio ordering for $n = 2$.

Theorem 4.2 *Let X_{λ_1}, X_λ be independent nonnegative random variables with $X_{\lambda_1} \sim F(\lambda_1 t)$ and $X_\lambda \sim F(\lambda t)$, where $\lambda_1, \lambda > 0$ and F is an absolutely continuous distribution. Let r be the reverse hazard rate function of F. Let $Y_{\lambda_1^*}, Y_\lambda$ be independent nonnegative random variables with $Y_{\lambda_1^*} \sim F(\lambda_1^* t)$ and $Y_\lambda \sim F(\lambda t)$, where $\lambda_1^*, \lambda > 0$. Assume $tr(t)$ and $tr'(t)/r(t)$ are both decreasing in t and $t^2 r'(t)$ is increasing in t. Suppose $\lambda_1^* = \min(\lambda, \lambda_1, \lambda_1^*)$, then*

$$(\lambda_1, \lambda) \stackrel{w}{\leq} (\lambda_1^*, \lambda) \Rightarrow X_{2:2} \leq_{lr} Y_{2:2}.$$

Proof. From Theorem 4.1, we know that $r_{2:2}^*(t)/r_{2:2}(t)$ is increasing in t under the given assumptions. By Theorem 3.2, $(\lambda_1, \lambda) \stackrel{w}{\leq} (\lambda_1^*, \lambda)$ implies $X_{2:2} \leq_{rh} Y_{2:2}$. Thus the required result follows from Theorem 1.C.4 of Shaked and Shanthikumar (2007). ∎

The conditions of Theorem 4.2 hold when the baseline distribution in the scale model is $GG(\beta, \alpha)$ with parameters $\alpha \leq \beta \leq 1$. We know from Khaledi et al. (2011) that for $\alpha, \beta > 0$, the function $tr(t)$ is decreasing in t and for $\beta \leq 1$ and $\alpha > 0$, the function $t^2 r'(t)$ is increasing in t. In Lemma 4.3, we show that the function $tr'(t)/r(t)$ is decreasing in t when $\alpha \leq \beta$.

Lemma 4.3 *Let $X \sim GG(\beta, \alpha)$, $\alpha \leq \beta$, with reverse hazard rate $r(t)$, then $tr'(t)/r(t)$ is a decreasing function.*

Proof. The reverse hazard rate of $GG(\beta, \alpha)$ is

$$r(t) = \frac{t^{\alpha-1} e^{-t^\beta}}{\int_0^t x^{\alpha-1} e^{-x^\beta} dx}.$$

From (A.21) in Khaledi et al. (2011), we know

$$t\frac{r'(t)}{r(t)} = \alpha - 1 - \beta t^\beta - tr(t). \tag{4.1}$$

Differentiating with respect to t, we get

$$\left[t\frac{r'(t)}{r(t)}\right]' = -\beta^2 t^{\beta-1} - r(t) - tr'(t).$$

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Note that, in general, the derivative of any reverse hazard rate with respect to t is

$$r'(t) = \frac{f'(t)}{F(t)} - r^2(t). \tag{4.2}$$

Combining these observations, we have

$$\begin{aligned}
\left[t\frac{r'(t)}{r(t)}\right]' &= -\beta^2 t^{\beta-1} - r(t) - t\frac{f'(t)}{F(t)} + tr^2(t) \\
&= -\beta^2 t^{\beta-1} + r(t)\left(tr(t) - 1 - t\frac{f'(t)}{f(t)}\right).
\end{aligned}$$

From Khaledi et al. (2011), we know that $tr(t)$ is a decreasing function for $\beta, \alpha > 0$ and also that $\lim_{t\to 0} tr(t) = \alpha$ and $\lim_{t\to\infty} tr(t) = 0$, then $tr(t) \leq \alpha$ for all $t > 0$. Then

$$\left[t\frac{r'(t)}{r(t)}\right]' \leq -\beta^2 t^{\beta-1} + r(t)\left(\alpha - 1 - t\frac{f'(t)}{f(t)}\right). \tag{4.3}$$

From (4.1) and (4.2), we get

$$t\frac{r'(t)}{r(t)} = \frac{t}{r(t)}\left(\frac{f'(t)}{F(t)} - r^2(t)\right) = t\left(\frac{f'(t)}{f(t)} - r(t)\right),$$

then

$$\begin{aligned}
t\frac{f'(t)}{f(t)} &= \alpha - 1 - \beta t^\beta - tr(t) + tr(t) \\
&= \alpha - 1 - \beta t^\beta.
\end{aligned}$$

By replacing the above expression in (4.3), we have

$$\begin{aligned}
\left[t\frac{r'(t)}{r(t)}\right]' &\leq -\beta^2 t^{\beta-1} + r(t)\left(\alpha - 1 - (\alpha - 1 - \beta t^\beta)\right) \\
&= -\beta^2 t^{\beta-1} + \beta t^\beta r(t) \\
&= \beta t^\beta \left(-\frac{\beta}{t} + r(t)\right) \leq 0
\end{aligned}$$

since $tr(t) \leq \alpha \leq \beta$. $\blacksquare$

Theorem 4.2 says that the lifetime of a parallel system consisting of two types of generalized gamma components with parameters $\alpha \leq \beta \leq 1$ is stochastically larger according to likelihood ratio ordering when the scale parameters are more dispersed according to weakly majorization.

Note that, when $0 < \alpha \leq 1$, Theorem 3.4 in Zhao (2011) can be seen as a particular case of Theorem 4.2 since gamma distribution is a particular case of generalized gamma distribution when $\beta = 1$.

As an immediate consequence of Theorem 4.2, we have the following result which provides an upper bound of two random variables from a scale model.

Corollary 4.4 *Let $X_{\lambda_1}, X_{\lambda_2}$ be independent nonnegative random variables with $X_{\lambda_i} \sim F(\lambda_i t)$ for $i = 1, 2$. Let Y_1, Y_2 be independent nonnegative random variables with a common distribution $Y_i \sim F(\lambda t)$ for $i = 1, 2$. Assume $tr(t)$ and $tr'(t)/r(t)$ are both decreasing in t and $t^2 r'(t)$ is increasing in t. Suppose $\lambda \leq \min(\lambda_1, \lambda_2)$, then*

$$\lambda \leq \frac{\lambda_1 + \lambda_2}{2} \Rightarrow X_{2:2} \leq_{lr} Y_{2:2}.$$

Next, we extend the study of likelihood ratio ordering between largest order statistics from the two-variable case to multiple-outlier scale models.

Theorem 4.5 *Let $X_1, \ldots, X_n$ be independent nonnegative random variables such that $X_i \sim F(\lambda_1 t)$ for $i = 1, \ldots, p$ and $X_j \sim F(\lambda t)$ for $j = p + 1, \ldots, n$, with $\lambda_1, \lambda > 0$ and F is an absolutely continuous distribution. Let $Y_1, \ldots, Y_n$ n be independent nonnegative random variables with $Y_i \sim F(\lambda_1^* t)$ for $i = 1, \ldots, p$ and $Y_j \sim F(\lambda t)$ for $j = p + 1, \ldots, n$, with $\lambda_1^*, \lambda > 0$. Let r be the reverse hazard rate function of F. Assume $tr(t)$ and $tr'(t)/r(t)$ are both decreasing in t. Suppose $\lambda_1^* = \min(\lambda, \lambda_1, \lambda_1^*)$, then*

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda, \ldots, \lambda}_{q}) \overset{w}{\preceq} (\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda, \ldots, \lambda}_{q}) \Rightarrow \frac{r_{n:n}^*(t)}{r_{n:n}(t)} \text{ is increasing in } t,$$

where $q = n - p$.

Proof. From (3.1) we get the reverse hazard rate function of $X_{n:n}$:

$$r_{n:n}(t) = p\lambda_1 r(\lambda_1 t) + q\lambda r(\lambda t),$$

where $p + q = n$. Let

$$\phi(t) = \frac{r_{n:n}^*(t)}{r_{n:n}(t)} = \frac{p\lambda_1^* r(\lambda_1^* t) + q\lambda r(\lambda t)}{p\lambda_1 r(\lambda_1 t) + q\lambda r(\lambda t)}.$$

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On differentiating $\phi(t)$ with respect to t, we get

$$\phi'(t) \stackrel{\mathrm{sign}}{=} t^3 \left(p\left(\lambda_1^*\right)^2 r'(\lambda_1^* t) + q\lambda^2 r'(\lambda t) \right) \left(p\lambda_1 r(\lambda_1 t) + q\lambda r(\lambda t) \right)$$

$$-t^3 \left(p\lambda_1^* r(\lambda_1^* t) + q\lambda r(\lambda t) \right) \left(p\lambda_1^2 r'(\lambda_1 t) + q\lambda^2 r'(\lambda t) \right)$$

$$= p^2 \lambda_1 \lambda_1^* t^3 \left(\lambda_1^* r'(\lambda_1^* t) r(\lambda_1 t) - \lambda_1 r(\lambda_1^* t) r'(\lambda_1 t) \right)$$

$$+ pq\lambda_1 \lambda t^3 \left(\lambda r'(\lambda t) r(\lambda_1 t) - \lambda_1 r(\lambda t) r'(\lambda_1 t) \right)$$

$$+ pq\lambda \lambda_1^* t^3 \left(\lambda_1^* r'(\lambda_1^* t) r(\lambda t) - \lambda r(\lambda_1^* t) r'(\lambda t) \right)$$

$$= p^2 \lambda_1 \lambda_1^* t^2 r(\lambda_1^* t) r(\lambda_1 t) \left(\lambda_1^* t \frac{r'(\lambda_1^* t)}{r(\lambda_1^* t)} - \lambda_1 t \frac{r'(\lambda_1 t)}{r(\lambda_1 t)} \right)$$

$$+ pq\lambda_1 \lambda t^2 r(\lambda t) r(\lambda_1 t) \left(\lambda t \frac{r'(\lambda t)}{r(\lambda t)} - \lambda_1 t \frac{r'(\lambda_1 t)}{r(\lambda_1 t)} \right)$$

$$+ pq\lambda \lambda_1^* t^2 r(\lambda_1^* t) r(\lambda t) \left(\lambda_1^* t \frac{r'(\lambda_1^* t)}{r(\lambda_1^* t)} - \lambda t \frac{r'(\lambda t)}{r(\lambda t)} \right)$$

$$= p^2 \psi(\lambda_1^* t) \psi(\lambda_1 t) \left(-\eta(\lambda_1^* t) + \eta(\lambda_1 t) \right) + pq\psi(\lambda t)\psi(\lambda_1 t)\left(-\eta(\lambda t) + \eta(\lambda_1 t) \right)$$

$$+ pq\psi(\lambda_1^* t)\psi(\lambda t)\left(-\eta(\lambda_1^* t) + \eta(\lambda t) \right),$$

where

$$\psi\left(t \right) = tr(t) \text{ and } \eta\left(t \right) = -t\frac{r'(t)}{r(t)}.$$

Note that $\psi\left(t \right), \eta\left(t \right) \geq 0$ for all $t \geq 0$. By the assumptions, we know that $\psi\left(t \right)$ is decreasing and $\eta\left(t \right)$ is increasing in t. If $\lambda_1^* = \min(\lambda, \lambda_1, \lambda_1^*)$ and $(\lambda_1, \ldots, \lambda_1, \lambda, \ldots, \lambda) \stackrel{w}{\leq} (\lambda_1^*, \ldots, \lambda_1^*, \lambda, \ldots, \lambda)$, then $\lambda_1^* \leq \lambda \leq \lambda_1$ or $\lambda_1^* \leq \lambda_1 \leq \lambda$. When $\lambda_1^* \leq \lambda \leq \lambda_1$, it is easy to check that $\phi'(t) \geq 0$ since $\eta\left(\lambda_1^* t \right) \leq \eta\left(\lambda t \right) \leq \eta\left(\lambda_1 t \right)$. When $\lambda_1^* \leq \lambda_1 \leq \lambda$, we get

$$\phi'(t) \geq p^2 \psi(\lambda t)\psi(\lambda_1 t)\left(-\eta(\lambda_1^* t) + \eta(\lambda_1 t) \right) + pq\psi(\lambda t)\psi(\lambda_1 t)\left(-\eta(\lambda t) + \eta(\lambda_1 t) \right)$$

$$+ pq\psi(\lambda_1 t)\psi(\lambda t)\left(-\eta(\lambda_1^* t) + \eta(\lambda t) \right)$$

$$= np\psi(\lambda t)\psi(\lambda_1 t)\left(-\eta(\lambda_1^* t) + \eta(\lambda_1 t) \right) \geq 0.$$

Therefore $r_{n:n}^*(t)/r_{n:n}(t)$ is increasing in t. ∎

In the next result, we extend Theorem 4.2 from the two-variable case to multiple-outlier scale models.

Theorem 4.6 *Let $X_1, \ldots, X_n$ be independent nonnegative random variables such that $X_i \sim F(\lambda_1 t)$ for $i = 1, \ldots, p$ and $X_j \sim F(\lambda t)$ for $j = p+1, \ldots, n$, with $\lambda_1, \lambda > 0$ and F is an absolutely continuous distribution. Let $Y_1, \ldots, Y_n$ n be independent nonnegative random variables with $Y_i \sim F(\lambda_1^* t)$ for $i = 1, \ldots, p$ and $Y_j \sim F(\lambda t)$ for $j = p+1, \ldots, n$, with $\lambda_1^*, \lambda > 0$. Let r be the reverse hazard rate function of F. Assume $tr(t)$ and $tr'(t)/r(t)$ are both decreasing in t and $t^2 r'(t)$ is increasing in t. Suppose $\lambda_1^* = \min(\lambda, \lambda_1, \lambda_1^*)$, then*

$$(\underbrace{\lambda_1, \ldots, \lambda_1}_{p}, \underbrace{\lambda, \ldots, \lambda}_{q}) \overset{w}{\leq} (\underbrace{\lambda_1^*, \ldots, \lambda_1^*}_{p}, \underbrace{\lambda, \ldots, \lambda}_{q}) \Rightarrow X_{n:n} \leq_{lr} Y_{n:n}.$$

Proof. From Theorem 4.5, we know that $r_{n:n}^*(t)/r_{n:n}(t)$ is increasing in t when $tr(t)$ and $tr'(t)/r(t)$ are both decreasing in t. Since $(\lambda_1, \ldots, \lambda_1, \lambda, \ldots, \lambda) \overset{w}{\leq} (\lambda_1^*, \ldots, \lambda_1^*, \lambda, \ldots, \lambda)$ and $t^2 r'(t)$ is increasing in t, then $X_{n:n} \leq_{rh} Y_{n:n}$ from Theorem 3.2. Thus the required result follows from Theorem 1.C.4 in Shaked and Shanthikumar (2007). ∎

Note that, when $0 < \alpha \leq 1$, Theorem 3.1 in Zhao and Balakrishnan (2013) can be seen as a particular case of Theorem 4.6 when $\lambda_1^* \leq \lambda_1 \leq \lambda$ since gamma distribution is a particular case of generalized gamma distribution when $\beta = 1$.

Next, we establish the analog of Theorem 4.6 when both the baseline distributions and the scale parameters are different in the multiple-outlier scale models.

Theorem 4.7 *Let $X_1, \ldots, X_n$ be independent nonnegative random variables such that $X_i \sim F(\lambda_1 t)$ for $i = 1, \ldots, p$ and $X_j \sim G(\lambda t)$ for $j = p+1, \ldots, n$, with $\lambda_1, \lambda > 0$ and F is an absolutely continuous distribution. Let $X_1^*, \ldots, X_n^*$ be n independent nonnegative random variables with $X_i^* \sim F(\lambda_1^* t)$ for $i = 1, \ldots, p$ and $X_j^* \sim G(\lambda t)$ for $j = p+1, \ldots, n$, with $\lambda_1^*, \lambda > 0$. Let r_F and r_G be the reverse hazard rate functions of F and G, respectively. Assume $tr_F(t)$ and $tr_F'(t)/r_F(t)$ are both decreasing in t. Suppose $r_F(t)/r_G(t)$ is increasing in t, then*

$$\lambda_1^* = \min(\lambda, \lambda_1, \lambda_1^*) \Rightarrow X_{n:n} \leq_{lr} X_{n:n}^*.$$

Proof. From (3.1) we get the reverse hazard rate function of $X_{n:n}$:

$$r_{n:n}(t) = p\lambda_1 r_F(\lambda_1 t) + q\lambda r_G(\lambda t),$$

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where $p + q = n$. Similarly the reverse hazard rate function of $X^*_{n:n}$ is

$$r^*_{n:n}(t) = p\lambda^*_1 r_F(\lambda^*_1 t) + q\lambda r_G(\lambda t).$$

Observe that $X_j \stackrel{st}{=} X^*_j$ for $j = p+1, \ldots, n$. By the assumptions, we know that $tr_F(t)$ is decreasing in t and $\lambda^*_1 \leq \lambda_1$, then we have $X_{n:n} \leq_{rh} X^*_{n:n}$ since $r_{n:n}(t) \leq r^*_{n:n}(t)$ for all t. From Theorem 1.C.4 in Shaked and Shanthikumar (2007), it is enough to prove that the ratio of their reverse hazard rate functions is increasing, i.e., we need to show that the function

$$\phi(t) = \frac{r^*_{n:n}(t)}{r_{n:n}(t)} = \frac{p\lambda^*_1 r_F(\lambda^*_1 t) + q\lambda r_G(\lambda t)}{p\lambda_1 r_F(\lambda_1 t) + q\lambda r_G(\lambda t)}$$

is increasing in t. On differentiating $\phi(t)$ with respect to t, we get

$$\phi'(t) \stackrel{sign}{=} t^3 \left(p\left(\lambda^*_1\right)^2 r'_F(\lambda^*_1 t) + q\lambda^2 r'_G(\lambda t) \right) \left(p\lambda_1 r_F(\lambda_1 t) + q\lambda r_G(\lambda t) \right)$$
$$- t^3 \left(p\lambda^*_1 r_F(\lambda^*_1 t) + q\lambda r_G(\lambda t) \right) \left(p\lambda^2_1 r'_F(\lambda_1 t) + q\lambda^2 r'_G(\lambda t) \right).$$

Let us denote:

$$\psi_F(t) = tr_F(t), \ \eta_F(t) = -t\frac{r'_F(t)}{r_F(t)}, \ \psi_G(t) = tr_G(t) \text{ and } \eta_G(t) = -t\frac{r'_G(t)}{r_G(t)},$$

then the derivative of $\phi(t)$ can be rewritten as

$$\phi'(t) \stackrel{sign}{=} p^2\psi_F(\lambda^*_1 t)\psi_F(\lambda_1 t)\left(-\eta_F(\lambda^*_1 t) + \eta_F(\lambda_1 t) \right) + pq\psi_G(\lambda t)\psi_F(\lambda_1 t)\left(-\eta_G(\lambda t) + \eta_F(\lambda_1 t) \right)$$
$$+ pq\psi_F(\lambda^*_1 t)\psi_G(\lambda t)\left(-\eta_F(\lambda^*_1 t) + \eta_G(\lambda t) \right).$$

The assumption $r_F(t)/r_G(t)$ is increasing in t is equivalent to $\eta_F(t) \leq \eta_G(t)$ for all t. In addition, we know that $\eta_F(t)$ is increasing in t and $\lambda^*_1 \leq \lambda$ then $\eta_F(\lambda^*_1 t) \leq \eta_F(\lambda t) \leq \eta_G(\lambda t)$ for all t. By the assumptions, we know that $\psi_F(t)$ is decreasing in t and $\lambda^*_1 \leq \lambda_1$, then

$$\phi'(t) \geq p^2\psi_F(\lambda^*_1 t)\psi_F(\lambda_1 t)\left(-\eta_F(\lambda^*_1 t) + \eta_F(\lambda_1 t) \right) + pq\psi_G(\lambda t)\psi_F(\lambda_1 t)\left(-\eta_F(\lambda^*_1 t) + \eta_F(\lambda_1 t) \right)$$
$$= p\psi_F(\lambda_1 t)\left(-\eta_F(\lambda^*_1 t) + \eta_F(\lambda_1 t) \right)\left(p\psi_F(\lambda^*_1 t) + q\psi_G(\lambda t) \right) \geq 0,$$

since $\eta_F(t)$ is increasing in t. Therefore $r^*_{n:n}(t)/r_{n:n}(t)$ is increasing in t. ■

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