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Incompressible Finite Elements via Hybridization. Part I: The Stokes System in Two Space Dimensions

Bernardo Cockburn
University of Minnesota - Twin Cities

Jay Gopalakrishnan
Portland State University, gjay@pdx.edu

INCOMPRESSIBLE FINITE ELEMENTS VIA HYBRIDIZATION. PART I: THE STOKES SYSTEM IN TWO SPACE DIMENSIONS

BERNARDO COCKBURN AND JAYADEEP GOPALAKRISHNAN

ABSTRACT. In this paper, we introduce a new and efficient way to compute exactly divergence free velocity approximations for the Stokes equations in two space dimensions. We begin by considering a mixed method that provides an exactly divergence-free approximation of the velocity and a continuous approximation of the vorticity. We then rewrite this method solely in terms of the tangential fluid velocity and the pressure on mesh edges by means of a new hybridization technique. This novel formulation bypasses the difficult task of constructing an exactly divergence free basis for velocity approximations. Moreover, the discrete system resulting from our method has fewer degrees of freedom than the original mixed method since the pressure and the tangential velocity variables are defined just on the mesh edges. Once these variables are computed, the velocity approximation satisfying the incompressibility condition exactly, as well as the continuous numerical approximation of the vorticity, can at once be obtained locally. Moreover, a discontinuous numerical approximation of the pressure within elements can also be obtained locally. We show how to compute the matrix system for our tangential velocity–pressure formulation on general meshes and present in full detail such computations for the lowest-order case of our method.

1. INTRODUCTION

In this paper, we introduce a new and efficient way to compute exactly divergence free velocity approximations for the Stokes equations in two space dimensions. We proceed as follows. First, we consider the mixed method for the Stokes equations studied in [11, 19]. This method provides a continuous approximation for the vorticity and an exactly divergence-free approximation of the velocity. Then we introduce a new hybridization technique that allows us to reduce the original method to a *mixed method for the Lagrange multipliers* arising from the hybridization, namely, the tangential fluid velocity and the pressure along mesh edges. This novel implementation of the method requires neither the introduction of stream function variables (as in [11, 19]), nor the construction of a globally divergence-free finite element basis. We thus avoid the difficulties in construction of a globally divergence free basis as well as the increase in degrees of freedom that accompanies the introduction of the stream function. Our new *tangential velocity-pressure* formulation has fewer degrees of freedom as both the unknowns are only defined on mesh edges. Moreover, after solving for these unknowns, the original exactly divergence-free numerical approximation of the fluid velocity and the original continuous numerical approximation of the vorticity can be easily computed in an element by element fashion. An approximation to the pressure inside the

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elements can also be computed in this way, a feature made possible by the hybridization procedure.

Let us describe the hybridization technique we propose. Recall that the Stokes equations couple the fluid velocity $\boldsymbol{u}$ and the pressure p by the equations

$$-\boldsymbol{\Delta u} + \mathbf{grad}\, p = \boldsymbol{f}, \qquad \text{on } \Omega, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0, \qquad \text{on } \Omega, \tag{1.2}$$

$$\boldsymbol{u} = \boldsymbol{g}, \qquad \text{on } \partial\Omega. \tag{1.3}$$

Here, $\boldsymbol{f} \in L^2(\Omega)^2$ and $\boldsymbol{g} \in H^{1/2}(\partial\Omega)^2$ are given data. For simplicity, we assume for that $\Omega \subseteq \mathbb{R}^2$ is a (bounded connected) polygon. To define the mixed method, we introduce the vorticity

$$\omega = \operatorname{curl} \boldsymbol{u} := \frac{\partial}{\partial x} u_y - \frac{\partial}{\partial y} u_x, \qquad \text{where} \qquad \boldsymbol{u} = (u_x, u_y),$$

and rewrite the Stokes system as

$$\omega - \operatorname{curl} \boldsymbol{u} = 0 \qquad \text{on } \Omega, \tag{1.4}$$

$$\mathbf{curl}\, \omega + \mathbf{grad}\, p = \boldsymbol{f} \qquad \text{on } \Omega, \tag{1.5}$$

$$\operatorname{div} \boldsymbol{u} = 0, \qquad \text{on } \Omega, \tag{1.6}$$

$$\boldsymbol{u} \cdot \boldsymbol{t} = g_t, \qquad \text{on } \partial\Omega, \tag{1.7}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = g_n, \qquad \text{on } \partial\Omega. \tag{1.8}$$

Here, $g_t = \boldsymbol{g} \cdot \boldsymbol{t}$ and $g_n = \boldsymbol{g} \cdot \boldsymbol{n}$, where $\boldsymbol{n}$ denotes the outward unit normal on $\partial\Omega$ and $\boldsymbol{t}$ the unit tangent vector on $\partial\Omega$ oriented such that Ω is on the left as we move in the direction of $\boldsymbol{t}$ along $\partial\Omega$. Note that to obtain equation (1.5), we made use of the identity

$$-\boldsymbol{\Delta u} = \mathbf{curl}\, \operatorname{curl} \boldsymbol{u} - \mathbf{grad}\, \operatorname{div} \boldsymbol{u},$$

where

$$\mathbf{curl}\, \omega = \left(\frac{\partial \omega}{\partial y}, -\frac{\partial \omega}{\partial x} \right).$$

To give a weak formulation of the above problem, define the spaces

$$\begin{aligned} \mathcal{W} \quad &= H^1(\Omega), \\ \mathcal{V} \quad &= \{ \boldsymbol{v} \in H(\operatorname{div}, \Omega) : \operatorname{div} \boldsymbol{v} = 0 \}, \\ \mathcal{V}(b) &= \{ \boldsymbol{v} \in \mathcal{V} : \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega} = b \}, \end{aligned}$$

for any $b \in H^{-1/2}(\partial\Omega)$. The weak formulation seeks the pair of functions satisfying

$$(\omega, \tau)_\Omega - (\boldsymbol{u}, \mathbf{curl}\, \tau)_\Omega = (g_t, \tau)_{\partial\Omega} \qquad \text{for all } \tau \in \mathcal{W}, \tag{1.9}$$

$$(\boldsymbol{v}, \mathbf{curl}\, \omega)_\Omega = (\boldsymbol{f}, \boldsymbol{v})_\Omega \qquad \text{for all } \boldsymbol{v} \in \mathcal{V}(0). \tag{1.10}$$

Here, $(\cdot, \cdot)_\Omega$ denotes the $L^2(\Omega)$ (or $L^2(\Omega)^2$) inner product. Note that since the velocity test functions are taken in the space $\mathcal{V}(0)$, the pressure is no longer present in this variational formulation. By classical existence results for the Stokes system, it is easy to show that there is a unique solution for the above system of equations provided the compatibility condition

$$(g_n, 1)_{\partial\Omega} = 0 \tag{1.11}$$

is satisfied. We assume throughout that (1.11) holds.

Now the approximate solution is sought in the finite element subspaces of the above defined spaces:

$$\mathcal{W}_h = \{w \in \mathcal{W} : w|_K \in P_{k+1}(K) \text{ for all } K \in \mathcal{T}\},$$
$$\mathcal{V}_h = \{\boldsymbol{v} \in \mathcal{V} : \boldsymbol{v}|_K \in P_k(K)^2 \text{ for all } K \in \mathcal{T}\}.$$

Here $\mathcal{T}$ denotes a finite element triangulation of Ω. Let $\mathcal{V}_h(b) = \mathcal{V}(b) \cap \mathcal{V}_h$ and $g_{n,h}$ be the $L^2(\partial\Omega)$-orthogonal projection of the boundary data g_n onto the space

$$\{\boldsymbol{v}_h \cdot \boldsymbol{n}|_{\partial\Omega} : \boldsymbol{v}_h \in \mathcal{V}_h\}.$$

Then the discrete mixed formulation seeks $(\omega_h, \boldsymbol{u}_h)$ in $\mathcal{W}_h \times \mathcal{V}_h(g_{n,h})$ satisfying

$$(\omega_h, \tau)_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\, \tau)_\Omega = (g_t, \tau)_{\partial\Omega} \qquad \text{for all } \tau \in \mathcal{W}_h, \tag{1.12}$$

$$(\boldsymbol{v}, \mathbf{curl}\, \omega_h)_\Omega = (\boldsymbol{f}, \boldsymbol{v})_\Omega \qquad \text{for all } \boldsymbol{v} \in \mathcal{V}_h(0). \tag{1.13}$$

We will tacitly assume throughout that the space $\mathcal{V}_h(0)$ is not empty. A three dimensional version of the above mixed discretization was studied in [11, 19] where the existence of a unique solution is established. Note that this is a conforming method since

$$\mathcal{W}_h \times \mathcal{V}_h(0) \subset \mathcal{W} \times \mathcal{V}(0) \subset H^1(\Omega) \times \mathcal{V}.$$

This implies, in particular, that in order to implement the method in the above form, we must face the difficult task of constructing bases for the finite dimensional space of globally divergence-free velocities $\mathcal{V}_h(0)$.

The construction of exactly divergence free finite element basis has been a long standing research question [13]. Basis functions for finite dimensional spaces of *weakly* divergence-free functions were constructed in [14], [15], and [23]. However, this construction proved to be extremely difficult to extend to spaces of polynomials of higher degree. Exactly divergence free finite element spaces have been studied, but known results require the use of polynomials of degree four or higher for the two dimensional case [18, 22] and no similar result exists for the three-dimensional case. The difficulty of constructing exactly incompressible finite element spaces was overcome in [11] by setting the divergence free spaces as the curl of an appropriate space of stream functions. Unfortunately, the introduction of the stream function increases degrees of freedom. In contrast, our approach to overcome this difficulty via hybridization actually results in a reduction in degrees of freedom.

Recently, globally divergence-free approximations were devised by using discontinuous Galerkin methods with polynomials of degree one or higher in the framework of the Navier-Stokes equations [9]. To achieve this, the fact that the divergence-free condition is enforced element by element is exploited to construct an element by element post-processing of the discontinuous approximation which automatically results in an exactly divergence-free velocity. A similar technique in the framework of discontinuous Galerkin methods for Darcy flow was developed in [3]. Unfortunately, such approaches cannot be used for conforming mixed methods since it relies on the fact that the discontinuous Galerkin methods enforce the equations element by element.

The main idea of our procedure is to look for approximations in discrete spaces which have *no* continuity constraints across mesh interfaces and introduce new sets of equations that guarantee that the new approximation *coincides* with the original approximation $(\omega_h, \boldsymbol{u}_h)$ given by (1.12)–(1.13). This approach is inspired by hybridization techniques used in the context of of mixed methods for second-order elliptic problems [1, 5, 8, 10]. We proceed in two steps. The objective of the first step is to circumvent construction of divergence free finite element bases. Hence, in this step, we relax the continuity of the normal components of the approximate velocity across inter-element boundaries and use a velocity space of functions with no inter-element continuity. As a direct consequence, the pressure reappears in the equations, but only on the edges if the approximate velocities are divergence-free inside each element. Then, new equations are introduced to enforce the continuity of the normal component of the velocity across inter-element boundaries. A similar hybridization technique, but in the framework of discontinuous Galerkin methods for the Stokes problem, is explored in [7].

The objective of the second step is the eventual elimination of both the original unknowns (velocity and vorticity) from the equations. In order to do this, we must develop a new hybridization technique for the vorticity. Such a hybridization is by far more involved than the previous one since the vorticity is continuous across inter-element boundaries. Indeed, all the previously known hybridization procedures relaxed continuity of spaces with edge (or face in 3D) degrees of freedom. Examples include hybridization techniques for the Raviart-Thomas and BDM methods for scalar second-order elliptic problems which involve finite element subspaces of $H(\mathrm{div}, \Omega)$. Hybridization of the Morley element method for the biharmonic problem [1] also involved such spaces with edge degrees of freedom. However, hybridization techniques to relax continuity constraints of finite element subspaces of $H^1(\Omega)$ with vertex degrees of freedom have remained unknown until now. While this may have led to a widespread belief that methods using this type of spaces are not amenable to hybridization, in this paper, we show otherwise. We show how one can approximate vorticity in a space of functions which have no continuity conditions across element interfaces while imposing the natural continuity properties of the vorticity as an equation of the method.

After the above mentioned hybridizations, we proceed to adapt the methodology introduced in [8] to eliminate the vorticity and velocity from the hybridized method. This elimination is by far not obvious, but is greatly facilitated by the fact that both the vorticity and the velocity are in spaces of functions with no inter-element continuity and by the fact that both the pressure and the tangential velocity are only defined on the mesh edges. This allows us to express the vorticity and velocity in terms of the pressure and tangential velocity. Then, we show how to characterize these Lagrange multipliers as the only solution of a new mixed method. We view this method as a "tangential velocity-pressure discretization" for the Stokes equation wherein the unknowns are all on the mesh edges.

Notice that since the unknowns are defined only on the edges, this system is smaller than the original one. Moreover, once the Lagrange multipliers are obtained, vorticity and velocity approximations can be obtained by *local* element by element computations. An interesting feature of our mixed method for the Lagrange multipliers is that it is possible to further eliminate the pressure Lagrange multiplier and form one Schur complement equation for the tangential velocity Lagrange multiplier. This equation can be easily solved using well established iterative techniques for symmetric positive definite systems.

We should note that ours is not the first paper to give hybridized methods for the Stokes problem. A hybrid formulation involving deviatoric stresses, hydrostatic pressure, and velocity was given in [6, 24]. Note also that some domain decomposition methods result from hybridization performed at the subdomain level. For example, in [4], the method gives rise to an indefinite system for the velocity nodes on the subdomain boundaries and the mean values of the pressure on the subdomains. However, none of the above mentioned methods provide incompressible velocities.

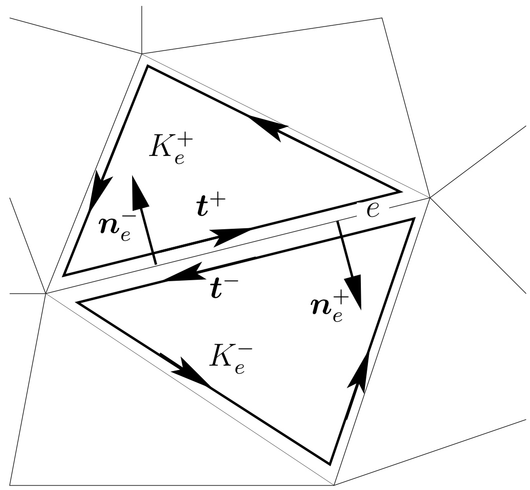

FIGURE 1. Notation for elements, normals, and tangents near an edge e.

The paper is organized as follows. In Section 2, we give a detailed description of the hybridization of the original conforming mixed method. The resulting method is written as a method for the two original variables *and* two additional Lagrange multipliers. Then, in Section 3, we show how to eliminate the former two variables from the equations and characterize the Lagrange multipliers alone as the unique solution of a mixed method. This characterization (Theorem 3.1) is an extension to the Stokes system of what was done for hybridized mixed methods for second-order elliptic problems in [8] and is one of our main results. In Section 4, we construct the bases for the Lagrange multipliers and in Section 5, we discuss some key implementation aspects of the method. These include the construction of the Schur complement matrix for the tangential velocity and the detailed computation of the matrices of the method for the lowest order case. Section 6 concludes the paper.

2. The hybridized mixed method

In this section, we present the hybridization of the mixed method in full detail as described in the introduction. Let us emphasize once again that this is carried out in two steps. The objective of the first is to avoid having to construct finite dimensional spaces of divergence-free velocities. The objective of the second is the eventual elimination of the original variables from the equations. Note that the actual elimination is not carried out until Section 3.

2.1. **First hybridization: Introduction of pressure on the mesh edges.** We begin by relaxing the continuity of the normal component of the approximate velocity $\boldsymbol{u}_h$ across inter-element boundaries. Thus, instead of seeking velocity approximations in the space $\mathcal{V}_h$, we seek approximations in the space

$$V_h = \{\boldsymbol{v} : \boldsymbol{v}|_K \in P_k(K)^2 \text{ and } \operatorname{div}(\boldsymbol{v}|_K) = 0 \text{ for all } K \in \mathcal{T}\}.$$

This forces us to weakly impose (1.5) in a different way. Indeed, if we multiply (1.5) by a test function $\boldsymbol{v}_h \in V_h$ and integrate over the element K, we obtain

$$(\mathbf{curl}\,\omega, \boldsymbol{v}_h)_K + (\mathbf{grad}\,p, \boldsymbol{v}_h)_K = (\boldsymbol{f}, \boldsymbol{v}_h)_K,$$

and hence,

$$(\mathbf{curl}\,\omega, \boldsymbol{v}_h)_K + (p, \boldsymbol{v}_h \cdot \boldsymbol{n})_{\partial K} = (\boldsymbol{f}, \boldsymbol{v}_h)_K.$$

Replacing ω and p by their respective approximations, ω_h and p_h, and adding over the elements of the triangulation, we obtain one equation of the method:

$$(\boldsymbol{v}_h, \mathbf{curl}\, \omega_h)_\Omega + \sum_{e\in\mathcal{E}} (p_h,\, [\![\boldsymbol{v}_h \cdot \boldsymbol{n}]\!])_e = (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega \quad \text{for all } \boldsymbol{v}_h \in V_h.$$

Here, we are using the following notation: For $\boldsymbol{v} \in V_h$ the jump of the normal component of $\boldsymbol{v}$ across inter-element boundaries, denoted by $[\![\boldsymbol{v}\cdot\boldsymbol{n}]\!]$, is defined on the set $\mathcal{E}$ of all edges of the triangulation $\mathcal{T}$ as follows. On every interior edge e in $\mathcal{E}$ shared by two mesh triangles K_e^+ and K_e^- we define

$$[\![\boldsymbol{v}\cdot\boldsymbol{n}]\!]_e = \boldsymbol{v}_e^+ \cdot \boldsymbol{n}_e^+ + \boldsymbol{v}_e^- \cdot \boldsymbol{n}_e^-,$$

where $\boldsymbol{n}_e^+$ and $\boldsymbol{n}_e^-$ denote the outward unit normals on the boundaries of K_e^+ and K_e^-, respectively (see Figure 1) and $\boldsymbol{v}_e^\pm(\boldsymbol{x}) = \lim_{\epsilon\downarrow 0} \boldsymbol{v}(\boldsymbol{x} - \epsilon\, \boldsymbol{n}_e^\pm)$. On edges $e \subset \partial\Omega$, we set $[\![\boldsymbol{v}\cdot\boldsymbol{n}]\!]_e = \boldsymbol{v}|_{\partial\Omega}\cdot\boldsymbol{n}$. By $[\![\boldsymbol{v}\cdot\boldsymbol{n}]\!]$ (without any subscript) we mean the function that is defined on the union of all edges in $\mathcal{E}$ and equals $[\![\boldsymbol{v}\cdot\boldsymbol{n}]\!]_e$ on each edge $e \in \mathcal{E}$.

Now, in accordance with the hybridization paradigm, we impose the continuity of the normal component of the velocity $\boldsymbol{u}_h$ across inter-element boundaries through the equation

$$\sum_{e\in\mathcal{E}} (q_h,\, [\![\boldsymbol{u}_h\cdot\boldsymbol{n}]\!])_e = (g_n, q_h)_{\partial\Omega} \quad \text{for all } q_h \in P_h,$$

where P_h is defined naturally by

$$P_h = \{p : p = [\![\boldsymbol{v}\cdot\boldsymbol{n}]\!] \text{ for some } \boldsymbol{v} \in V_h\}. \tag{2.1}$$

Notice that in the above equation, we are also incorporating the boundary condition on the normal component of the velocity.

Thus, after the first hybridization of the mixed method, we are seeking an approximation $(\omega_h, \boldsymbol{u}_h, p_h) \in \mathcal{W}_h \times V_h \times P_h$ satisfying

$$(\omega_h, \tau_h)_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\, \tau_h)_\Omega = (g_t, \tau_h)_{\partial\Omega} \quad \text{for all } \tau_h \in \mathcal{W}_h, \tag{2.2}$$

$$(\boldsymbol{v}_h, \mathbf{curl}\, \omega_h)_\Omega + \sum_{e\in\mathcal{E}} (p_h,\, [\![\boldsymbol{v}_h\cdot\boldsymbol{n}]\!])_e = (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega \quad \text{for all } \boldsymbol{v}_h \in V_h, \tag{2.3}$$

$$\sum_{e\in\mathcal{E}} (q_h,\, [\![\boldsymbol{u}_h\cdot\boldsymbol{n}]\!])_e = (g_n, q_h)_{\partial\Omega} \quad \text{for all } q_h \in P_h. \tag{2.4}$$

Note that although the original mixed method (1.12)–(1.13) did not involve the pressure variable, the pressure reappears upon hybridization, but only along the mesh edges. We shall call p_h the "pressure Lagrange multiplier". The above discrete formulation has a unique solution, as we show next.

Proposition 2.1. *There is a unique solution* $(\omega_h, \boldsymbol{u}_h, p_h) \in \mathcal{W}_h \times V_h \times P_h$ *for the hybridized mixed method* (2.2)–(2.4) *and the solution components* ω_h *and* $\boldsymbol{u}_h$ *are the same as the solution of* (1.12)–(1.13).

Proof. We show that if g_n, g_t, and $\boldsymbol{f}$ are equal to zero, then $\omega_h, \boldsymbol{u}_h$, and p_h are also zero. First, (2.4) implies $\boldsymbol{u}_h \in \mathcal{V}_h(0)$. Moreover, (2.2)–(2.3) implies that ω_h and $\boldsymbol{u}_h$ satisfy (1.12)–(1.13) with zero data. By uniqueness of solutions of (1.12)–(1.13), we find that $\omega_h = 0$ and $\boldsymbol{u}_h = 0$. This together with (2.3) implies that p_h is zero. Hence there is a unique solution for (2.2)–(2.4). It is easy to see that if ω_h and $\boldsymbol{u}_h$ satisfy (2.2)–(2.4), then they also satisfy (1.12)–(1.13), hence the equivalence of both the problems. ∎

Before proceeding to describe the second hybridization, let us point out that this first hybridization allows us to recover an approximation for the pressure inside the elements in an element by element fashion. To define such an approximation, we follow a technique of [7]. We define the pressure π_h on the triangle K as the element of $P_k(K)$ such that

$$-(\pi_h, \operatorname{div} \boldsymbol{v})_K = (\boldsymbol{f}, \boldsymbol{v})_K - (\mathbf{curl}\, \omega_h, \boldsymbol{v})_K - (\boldsymbol{v} \cdot \boldsymbol{n}, p_h)_{\partial K}, \tag{2.5}$$

for all $\boldsymbol{v}$ in $P_k(K)^2 + \boldsymbol{x}\, P_k(K)$, where $\boldsymbol{n}$ denotes the outward unit normal to K. That Equation (2.5) uniquely defines π_h follows from two facts:

(i) $\operatorname{div} : P_k(K)^2 + \boldsymbol{x}\, P_k(K) \mapsto P_k(K)$ is a surjection;
(ii) If $\operatorname{div} \boldsymbol{v} = 0$ for a $\boldsymbol{v}$ in $P_k(K)^2 + \boldsymbol{x}\, P_k(K)$, then $\boldsymbol{v} \in P_k(K)^2$ and the right hand side of the above equation is zero by the definition of the hybridized method.

The idea of recovering pressure approximations *a posteriori* as in (2.5) from approximations of other variables is old (see e.g. [14]), but because hybridization provides p_h, we are able to compute π_h locally in our case. Thus our method can simultaneously provide approximations to the *velocity, vorticity* and *pressure.*

2.2. **Second hybridization: Introduction of the tangential velocity variable.** Now we relax the continuity of the approximate vorticity ω_h across mesh edges in the interior of the domain. Thus, instead of considering continuous approximations in the space $\mathcal{W}_h$, we formulate a method using the space

$$W_h = \{w : w|_K \in P_{k+1}(K) \text{ for all } K \in \mathcal{T}\}.$$

This forces us to weakly impose the equation (1.4) in a different way. Indeed, if we multiply that equation by a test function $\tau_h \in W_h$ and integrate over the element K, we obtain

$$(\omega, \tau_h)_K - (\boldsymbol{u}, \mathbf{curl}\, \tau_h)_K - (\boldsymbol{u} \cdot \boldsymbol{t}, \tau_h)_{\partial K} = 0,$$

where $\boldsymbol{t}$ denotes the unit tangent vector along ∂K oriented as in Figure 1. Here and elsewhere, we do not explicitly indicate the dependence of $\boldsymbol{t}$ on the underlying boundary (such as ∂K above) to simplify notation. Denoting the tangential component of the velocity $\boldsymbol{u}$ on the inter-element boundaries by

$$\boldsymbol{\lambda} = (\boldsymbol{u} \cdot \boldsymbol{t})\, \boldsymbol{t},$$

we can rewrite the above equation as

$$(\omega, \tau_h)_K - (\boldsymbol{u}, \mathbf{curl}\, \tau_h)_K - (\boldsymbol{\lambda}, \tau_h \boldsymbol{t})_{\partial K} = 0.$$

Next, replacing ω, $\boldsymbol{u}$ and $\boldsymbol{\lambda}$ by their respective approximations ω_h, $\boldsymbol{u}_h$ and $\boldsymbol{\lambda}_h$, we obtain, after adding over the elements K of the triangulation,

$$(\omega_h, \tau_h)_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\, \tau_h)_\Omega - \sum_{e \in \mathcal{E} \setminus \partial\Omega} (\boldsymbol{\lambda}_h, [\![\tau_h \boldsymbol{t}]\!]_e)_e = (g_t, \tau_h)_{\partial\Omega}. \tag{2.6}$$

Here, the "tangential jump" of τ across inter-element boundaries, $[\![\tau \boldsymbol{t}]\!]$, is defined as follows. For every interior edge $e \in \mathcal{E}$ shared by triangles K_e^+ and K_e^-, let

$$[\![\tau \boldsymbol{t}]\!]_e = \tau_e^+ \boldsymbol{t}^+ + \tau_e^- \boldsymbol{t}^-,$$

where, as before, $\tau_e^\pm(\boldsymbol{x}) = \lim_{\epsilon \downarrow 0} \tau(\boldsymbol{x} - \epsilon\, \boldsymbol{n}_e^\pm)$, and $\boldsymbol{t}^+$ and $\boldsymbol{t}^-$ are unit tangent vectors along the boundaries of K_e^+ and K_e^-, respectively, oriented in accordance with our previous notation: unit tangent vectors along the boundary of a domain are given the orientation that leaves the domain on its left (see Figure 1). Hence $\boldsymbol{t}^+ = -\boldsymbol{t}^-$ on e. It is convenient to adopt the convention that the jump $[\![\tau \boldsymbol{t}]\!]$ on the boundary of Ω vanishes:

$$[\![\tau \boldsymbol{t}]\!]_e = 0 \qquad \text{for edges } e \subset \partial\Omega.$$

By $[\![\tau \boldsymbol{t}]\!]$ (without any subscript), we mean the function defined on the union of all edges in $\mathcal{E}$ that equals $[\![\tau \boldsymbol{t}]\!]_e$ on each edge $e \in \mathcal{E}$. With these conventions, we can now write (2.6) as

$$(\omega_h, \tau_h)_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\, \tau_h)_\Omega - \sum_{e\in\mathcal{E}} (\boldsymbol{\lambda}_h, [\![\tau_h \boldsymbol{t}]\!])_e = (g_t, \tau_h)_{\partial\Omega}.$$

Now, proceeding as in the first hybridization, we impose the continuity of the vorticity by using the equation

$$\sum_{e\in\mathcal{E}} (\boldsymbol{\mu}_h, [\![\omega_h \boldsymbol{t}]\!])_e = 0 \qquad \text{for all } \boldsymbol{\mu}_h \in M_h, \tag{2.7}$$

where the space M_h is given by

$$M_h = \{\boldsymbol{\mu} : \boldsymbol{\mu} = [\![\tau \boldsymbol{t}]\!] \text{ for some } \tau \in W_h\}.$$

The above choice is dictated by the fact that a function $w \in W_h$ is continuous if and only if $[\![w \boldsymbol{t}]\!] = 0$. Clearly, if ω_h satisfies (2.7), then it belongs to the space $\mathcal{W}_h \subset H^1(\Omega)$.

Summarizing our considerations so far, the hybridized mixed method gives an approximation $(\omega_h, \boldsymbol{u}_h, \boldsymbol{\lambda}_h, p_h) \in W_h \times V_h \times M_h \times P_h$ defined by

$$(\omega_h, \tau_h)_\Omega - (\boldsymbol{u}_h, \mathbf{curl}\, \tau_h)_\Omega - \sum_{e\in\mathcal{E}} (\boldsymbol{\lambda}_h, [\![\tau_h \boldsymbol{t}]\!])_e = (g_t, \tau_h)_{\partial\Omega}, \tag{2.8}$$

$$(\boldsymbol{v}_h, \mathbf{curl}\, \omega_h)_\Omega + \sum_{e\in\mathcal{E}} (p_h, [\![\boldsymbol{v}_h \cdot \boldsymbol{n}]\!])_e = (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega, \tag{2.9}$$

$$\sum_{e\in\mathcal{E}} (q_h, [\![\boldsymbol{u}_h \cdot \boldsymbol{n}]\!])_e = (g_n, q_h)_{\partial\Omega}, \tag{2.10}$$

$$\sum_{e\in\mathcal{E}} (\boldsymbol{\mu}_h, [\![\omega_h \boldsymbol{t}]\!])_e = 0, \tag{2.11}$$

for all $\tau_h \in W_h, \boldsymbol{v}_h \in V_h, q_h \in P_h, \boldsymbol{\mu}_h \in M_h$. By arguments similar to those used in the proofs of Proposition 2.1, it is easy to prove the following result:

Proposition 2.2. *There is a unique solution* $(\omega_h, \boldsymbol{u}_h, \boldsymbol{\lambda}_h, p_h) \in W_h \times V_h \times M_h \times P_h$ *for the hybridized mixed method* (2.8)–(2.11) *and the solution components* ω_h *and* $\boldsymbol{u}_h$ *satisfy* (1.12)–(1.13).

At this point, the number of unknowns of our method seem to have proliferated and it is far from evident that the hybridizations we just described has any advantage at all. However, in the next section we show that the structure of this hybridized method allows us to easily eliminate the velocity $\boldsymbol{u}_h$ and the vorticity ω_h from the above equations.

3. A characterization of the Lagrange multipliers

In this section, we eliminate the velocity and vorticity unknowns from the equations of the previously given hybridized mixed method using the methodology developed in [8]. As a result, we obtain a characterization of the tangential velocity and pressure Lagrange multipliers.

3.1. **The main result.** We begin by defining *local* maps that lift functions defined on the boundary of the elements of the triangulation into functions on the domain Ω: Define $(w(\boldsymbol{\lambda}), \boldsymbol{u}(\boldsymbol{\lambda})) \in W_h \times V_h$ and $(\mathrm{w}(p), \mathbf{u}(p)) \in W_h \times V_h$ by

$$(w(\boldsymbol{\lambda}), \tau)_K - (\boldsymbol{u}(\boldsymbol{\lambda}), \mathbf{curl}\, \tau)_K = (\boldsymbol{\lambda}, \tau \boldsymbol{t})_{\partial K}, \qquad \text{for all } \tau \in W_h, \tag{3.1}$$

$$(\boldsymbol{v}, \mathbf{curl}\ w(\boldsymbol{\lambda}))_K = 0, \qquad \text{for all } \boldsymbol{v} \in V_h, \tag{3.2}$$

$$(\mathrm{w}(p), \tau)_K - (\mathbf{u}(p), \mathbf{curl}\, \tau)_K = 0, \qquad \text{for all } \tau \in W_h, \tag{3.3}$$

$$(\boldsymbol{v}, \mathbf{curl}\ \mathrm{w}(p))_K = -(p, \boldsymbol{v} \cdot \boldsymbol{n})_{\partial K}, \qquad \text{for all } \boldsymbol{v} \in V_h. \tag{3.4}$$

In addition it is convenient to define the local mappings $(w(g_t), \boldsymbol{u}(g_t))$ and $(\mathsf{w}(\boldsymbol{f}), \mathbf{u}(\boldsymbol{f}))$ in $W_h \times V_h$ as follows:

$$(w(g_t), \tau)_K - (\boldsymbol{u}(g_t), \mathbf{curl}\, \tau)_K = (g_t, \tau)_{\partial K \cap \partial \Omega}, \qquad \text{for all } \tau \in W_h, \tag{3.5}$$

$$(\boldsymbol{v}, \mathbf{curl}\ w(g_t))_K = 0, \qquad \text{for all } \boldsymbol{v} \in V_h \tag{3.6}$$

$$(\mathsf{w}(\boldsymbol{f}), \tau)_K - (\mathbf{u}(\boldsymbol{f}), \mathbf{curl}\, \tau)_K = 0, \qquad \text{for all } \tau \in W_h, \tag{3.7}$$

$$(\boldsymbol{v}, \mathbf{curl}\ \mathsf{w}(\boldsymbol{f}))_K = (\boldsymbol{f}, \boldsymbol{v})_K, \qquad \text{for all } \boldsymbol{v} \in V_h. \tag{3.8}$$

Note that all the four pairs of local maps above are given as solutions of a single mixed problem, but with different right hand sides. That all the four maps are well defined follows from the unique solvability of the mixed problem (which is the original mixed problem restricted to one element). Although all four maps use the same mixed problem, we have chosen to explicitly distinguish each of them so as to delineate the dependence of the final solution on the data components and the Lagrange multipliers.

The main result of this section characterizes the Lagrange multipliers as the unique solution of a variational equation involving the bilinear forms,

$$a(\boldsymbol{\lambda}, \boldsymbol{\mu}) = (w(\boldsymbol{\lambda}), w(\boldsymbol{\mu}))_\Omega, \tag{3.9}$$

$$b(\boldsymbol{\mu}, p) = -\sum_{K \in \mathcal{T}} (\boldsymbol{u}(\boldsymbol{\mu}), \mathbf{curl}\ \mathrm{w}(p))_K, \tag{3.10}$$

$$c(p, q) = (\mathrm{w}(p), \mathrm{w}(q))_\Omega, \tag{3.11}$$

and the linear functionals

$$\ell_1(\boldsymbol{\mu}) = (\boldsymbol{f}, \boldsymbol{u}(\boldsymbol{\mu}))_\Omega - (g_t, w(\boldsymbol{\mu}))_{\partial\Omega}, \tag{3.12}$$

$$\ell_2(q) = (\boldsymbol{f}, \mathbf{u}(q))_\Omega + (g_n, q)_{\partial\Omega} - (g_t, \mathrm{w}(q))_{\partial\Omega}. \tag{3.13}$$

Theorem 3.1. *The Lagrange multiplier* $(\boldsymbol{\lambda}_h, p_h) \in M_h \times P_h$ *of the hybridized mixed method* (2.8)–(2.11) *is the unique solution of*

$$a(\boldsymbol{\lambda}_h, \boldsymbol{\mu}) + b(\boldsymbol{\mu}, p_h) = \ell_1(\boldsymbol{\mu}), \qquad \textit{for all } \boldsymbol{\mu} \in M_h \textit{ and} \tag{3.14}$$

$$b(\boldsymbol{\lambda}_h, q) - c(p_h, q) = \ell_2(q), \qquad \textit{for all } q \in P_h. \tag{3.15}$$

Moreover, the solution components ω_h *and* $\boldsymbol{u}_h$ *of the hybridized mixed method* (2.8)–(2.11) *can be determined locally as follows:*

$$\omega_h = w(\boldsymbol{\lambda}_h) + \mathrm{w}(p_h) + w(g_t) + \mathsf{w}(\boldsymbol{f}), \tag{3.16}$$

$$\boldsymbol{u}_h = \boldsymbol{u}(\boldsymbol{\lambda}_h) + \mathbf{u}(p_h) + \boldsymbol{u}(g_t) + \mathbf{u}(\boldsymbol{f}). \tag{3.17}$$

3.2. **Proof.** To prove the above result, we follow the approach introduced in [8]. Accordingly, the first step will be to use the local maps to rewrite the first two equations of the hybridized method, namely (2.8) and (2.9). This will yield (3.16) and (3.17). Next, the two remaining equations of the hybridized method, namely (2.10) and (2.11), will be used to characterize the pressure and tangential velocity Lagrange multipliers of the method. In order to carry out these steps, we need to obtain a few identities involving the local mappings. This is done in the first lemma below. Then, in a second lemma, we show how to rewrite the equations (2.10) and (2.11) solely in terms of the multipliers. In this way, we eliminate the vorticity and velocity and at the same time obtain a variational characterization of the Lagrange multipliers. Let us now state and prove the lemmas.

Lemma 3.1 (Elementary identities)**.** *On any mesh element $K \in \mathcal{T}$, for any $\boldsymbol{\lambda} \in M_h, \boldsymbol{\mu} \in M_h, p \in P_h$, and $q \in P_h$, we have the following orthogonality properties for the local vorticity maps:*

$$(w(\boldsymbol{\lambda}), \mathrm{w}(p))_K = 0, \tag{3.18}$$

$$(w(\boldsymbol{\lambda}), \mathsf{w}(\boldsymbol{f}))_K = 0, \tag{3.19}$$

$$(w(g_t), \mathrm{w}(p))_K = 0, \tag{3.20}$$

$$(w(g_t), \mathsf{w}(\boldsymbol{f}))_K = 0. \tag{3.21}$$

Moreover, we have the following identities for the bilinear forms a, b and c:

$$a_K(\boldsymbol{\lambda}, \boldsymbol{\mu}) := (w(\boldsymbol{\lambda}), w(\boldsymbol{\mu}))_K = (\boldsymbol{\lambda}, w(\boldsymbol{\mu})\boldsymbol{t})_{\partial K}, \tag{3.22}$$

$$b_K(\boldsymbol{\lambda}, p) := -(\boldsymbol{u}(\boldsymbol{\lambda}), \mathbf{curl}\ \mathrm{w}(p))_K = (\boldsymbol{u}(\boldsymbol{\lambda}) \cdot \boldsymbol{n}, p)_{\partial K} = (\boldsymbol{\lambda}, \mathrm{w}(p)\boldsymbol{t})_{\partial K}, \tag{3.23}$$

$$c_K(p, q) := (\mathrm{w}(p), \mathrm{w}(q))_K = -(q, \mathbf{u}(p) \cdot \boldsymbol{n})_{\partial K}. \tag{3.24}$$

Finally, we have the following identities related to the linear forms ℓ_1 and ℓ_2:

$$(\boldsymbol{f}, \boldsymbol{u}(\boldsymbol{\mu}))_K = -(\mathsf{w}(\boldsymbol{f})\boldsymbol{t}, \boldsymbol{\mu})_{\partial K}, \tag{3.25}$$

$$(w(\boldsymbol{\mu}), g_t)_{\partial K \cap \partial \Omega} = (\boldsymbol{\mu}, w(g_t)\, \boldsymbol{t})_{\partial K}, \tag{3.26}$$

$$(\boldsymbol{f}, \mathbf{u}(q))_K = -(\mathbf{u}(\boldsymbol{f}) \cdot \boldsymbol{n}, q)_{\partial K}, \tag{3.27}$$

$$(\mathrm{w}(q), g_t)_{\partial K \cap \partial \Omega} = (q, \boldsymbol{u}(g_t) \cdot \boldsymbol{n})_{\partial K}. \tag{3.28}$$

Proof. Let us begin by proving the orthogonality identities. Equation (3.18) is obtained by setting $\tau = w(\boldsymbol{\lambda})$ in (3.3) and using (3.2). The proof of (3.19) is analogous. The equations (3.20) and (3.21) follow from similar arguments, as the equations defining the liftings of g_t and $\boldsymbol{\lambda}$ have the same structure.

Next, let us prove the identities associated with the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$. The equation (3.22) is obtained follows. Setting $\tau = w(\boldsymbol{\mu})$ in the definition of the liftings (3.1), we get

$$\begin{aligned}(w(\boldsymbol{\lambda}), \boldsymbol{w}(\boldsymbol{\mu}))_K &= (\boldsymbol{u}(\boldsymbol{\lambda}), \mathbf{curl}\ w(\boldsymbol{\mu}))_K + (\boldsymbol{\lambda}, w(\boldsymbol{\mu})\, \boldsymbol{t})_{\partial K} \\ &= (\boldsymbol{\lambda}, w(\boldsymbol{\mu})\, \boldsymbol{t})_{\partial K},\end{aligned}$$

by (3.2) with $\boldsymbol{\lambda} = \boldsymbol{\mu}$ and $\boldsymbol{v} = \boldsymbol{u}(\boldsymbol{\lambda})$. Let us prove (3.23). Taking $\boldsymbol{v} = \boldsymbol{u}(\boldsymbol{\lambda})$ in (3.4), we get

$$\begin{aligned}(\boldsymbol{u}(\boldsymbol{\lambda}) \cdot \boldsymbol{n}, p)_{\partial K} &= -(\boldsymbol{u}(\boldsymbol{\lambda}), \mathbf{curl}\ \mathrm{w}(p))_K \\ &= (\boldsymbol{\lambda}, \mathrm{w}(p)\, \boldsymbol{t})_{\partial K} - (w(\boldsymbol{\lambda}), \mathrm{w}(p))_K \quad \text{by (3.1) with } \tau = \mathrm{w}(p), \\ &= (\boldsymbol{\lambda}, \mathrm{w}(p)\, \boldsymbol{t})_{\partial K},\end{aligned}$$

by the orthogonality property (3.18). Now let us prove (3.24). We have, by (3.3) with $\tau = \mathrm{w}(q)$,

$$\begin{aligned}(\mathrm{w}(p), \mathrm{w}(q))_K &= (\mathbf{u}(p), \mathbf{curl}\ \mathrm{w}(q))_K\\ &= -(q, \mathbf{u}(p) \cdot \boldsymbol{n})_{\partial K},\end{aligned}$$

by (3.4) with $p = q$ and $\boldsymbol{v} = \mathbf{u}(p)$.

Finally, let us consider the last set of identities. We first prove (3.25). Setting $\boldsymbol{v} = \boldsymbol{u}(\boldsymbol{\mu})$ in (3.8), we get

$$\begin{aligned}(\boldsymbol{f}, \boldsymbol{u}(\boldsymbol{\mu}))_K &= (\boldsymbol{u}(\boldsymbol{\mu}), \mathbf{curl}\ \mathsf{w}(\boldsymbol{f}))_K\\ &= (w(\boldsymbol{\mu}), \mathsf{w}(\boldsymbol{f}))_K - (\boldsymbol{\mu}, \mathsf{w}(\boldsymbol{f})\, \boldsymbol{t})_{\partial K}\end{aligned}$$

by (3.1) with $\boldsymbol{\lambda} = \boldsymbol{\mu}$ and $\tau = \mathsf{w}(\boldsymbol{f})$. The desired equation follows by using the already established orthogonality property (3.19). Next, let us prove equation (3.26). Setting $\tau = w(\boldsymbol{\mu})$ in (3.5) and then using (3.2), we get

$$\begin{aligned}(g_t, w(\boldsymbol{\mu}))_{\partial K \cap \partial \Omega} &= (w(\boldsymbol{\mu}), w(g_t))_K\\ &= (\boldsymbol{\mu}, w(g_t)\boldsymbol{t})_{\partial K} + (\boldsymbol{u}(\boldsymbol{\mu}), \mathbf{curl}\,(w(g_t)))_K,\end{aligned}$$

by (3.1) with $\boldsymbol{\lambda} = \boldsymbol{\mu}$ and $\tau = w(g_t)$. The equation (3.26) follows from (3.6). Equation (3.27) is obtained as follows:

$$\begin{aligned}(\boldsymbol{f}, \mathbf{u}(q))_K &= (\mathbf{u}(q), \mathbf{curl}\ \mathsf{w}(\boldsymbol{f}))_K && \text{by (3.8) with } \boldsymbol{v} = \mathbf{u}(q),\\ &= (\mathrm{w}(q), \mathsf{w}(\boldsymbol{f}))_K && \text{by (3.3) with } p = q \text{ and } \tau = \mathsf{w}(\boldsymbol{f}),\\ &= (\mathbf{u}(\boldsymbol{f}), \mathbf{curl}\ \mathrm{w}(q))_K && \text{by (3.7) with } \tau = \mathrm{w}(q),\\ &= -(q, \mathbf{u}(\boldsymbol{f}))_K),\end{aligned}$$

by (3.4) with $p = q$ and $\boldsymbol{v} = \mathbf{u}(\boldsymbol{f})$. Finally, let us prove equation (3.28). By (3.5) with $\tau = \mathrm{w}(q)$, we have

$$\begin{aligned}(\mathrm{w}(q), g_t)_{\partial K \cup \partial \Omega} &= (w(g_t), \mathrm{w}(q))_K - (\boldsymbol{u}(g_t), \mathbf{curl}\ \mathrm{w}(q))_K,\\ &= -(\boldsymbol{u}(g_t), \mathbf{curl}\ \mathrm{w}(q))_K, && \text{by (3.20)},\\ &= (q, \boldsymbol{u}(g_t) \cdot \boldsymbol{n})_{\partial K}\end{aligned}$$

by (3.4) with $p = q$ and $\boldsymbol{v} = \boldsymbol{u}(g_t)$. This completes the proof. □

Lemma 3.2 (The jump conditions)**.** *For arbitrary $\boldsymbol{\lambda} \in M_h$ and $p \in P_h$ set*

$$\begin{aligned}\widetilde{\omega}_h^{\boldsymbol{\lambda},p} &= w(\boldsymbol{\lambda}) + \mathrm{w}(p) + w(g_t) + \mathsf{w}(\boldsymbol{f}),\\ \widetilde{\boldsymbol{u}}_h^{\boldsymbol{\lambda},p} &= \boldsymbol{u}(\boldsymbol{\lambda}) + \mathbf{u}(p) + \boldsymbol{u}(g_t) + \mathbf{u}(\boldsymbol{f}).\end{aligned}$$

Let $(\omega_h, \boldsymbol{u}_h, \boldsymbol{\lambda}_h, p_h)$ *be the unique solution of* (2.8)–(2.11). *Then the following statements are equivalent:*

$$
\begin{array}{ll}
A. & \textit{For all } \boldsymbol{\mu} \in M_h \textit{ and } q \in P_h, \\
 & \displaystyle\sum_{e\in\mathcal{E}} (\boldsymbol{\mu},\, [\![\widetilde{\omega}_h^{\boldsymbol{\lambda},p} \boldsymbol{t}]\!])_e = 0, \;\; \textit{and} \;\; \sum_{e\in\mathcal{E}} (q,\, [\![\widetilde{\boldsymbol{u}}_h^{\boldsymbol{\lambda},p} \cdot \boldsymbol{n}]\!])_e = (g_n, q)_{\partial\Omega}. \\
B. & \widetilde{\omega}_h^{\boldsymbol{\lambda},p} = \omega_h \;\; \textit{and} \;\; \widetilde{\boldsymbol{u}}_h^{\boldsymbol{\lambda},p} = \boldsymbol{u}_h. \\
C. & \boldsymbol{\lambda} = \boldsymbol{\lambda}_h \;\; \textit{and} \;\; p = p_h. \\
D. & a(\boldsymbol{\lambda}, \boldsymbol{\mu}) + b(\boldsymbol{\mu}, p) = \ell_1(\boldsymbol{\mu}), \quad \textit{for all } \boldsymbol{\mu} \in M_h \textit{ and} \\
 & b(\boldsymbol{\lambda}, q) - c(p, q) = \ell_2(q), \quad \textit{for all } q \in P_h.
\end{array}
$$

Proof. $A \implies B$: By adding the equations defining $(w(\boldsymbol{\lambda}), \boldsymbol{u}(\boldsymbol{\lambda}))$, $(\mathrm{w}(p), \mathbf{u}(p))$, $(\mathsf{w}(\boldsymbol{f}), \mathbf{u}(\boldsymbol{f}))$, and $(w(g_t), \boldsymbol{u}(g_t))$, we find that $\widetilde{\omega}_h^{\boldsymbol{\lambda},p}$ and $\widetilde{\boldsymbol{u}}_h^{\boldsymbol{\lambda},p}$ satisfy the first two equations of our hybridized mixed method, i.e.,

$$
\begin{aligned}
(\widetilde{\omega}_h^{\boldsymbol{\lambda},p}, \tau_h)_\Omega - (\widetilde{\boldsymbol{u}}_h^{\boldsymbol{\lambda},p}, \mathbf{curl}\, \tau_h)_\Omega - \sum_{e\in\mathcal{E}} (\boldsymbol{\lambda},\, [\![\tau_h \boldsymbol{t}]\!])_e &= (g_t, \tau_h)_{\partial\Omega}, \\
(\boldsymbol{v}_h, \mathbf{curl}\, \widetilde{\omega}_h^{\boldsymbol{\lambda},p})_\Omega + \sum_{e\in\mathcal{E}} (p,\, [\![\boldsymbol{v}_h \cdot \boldsymbol{n}]\!])_e &= (\boldsymbol{f}, \boldsymbol{v}_h)_\Omega,
\end{aligned}
$$

for all $\tau_h \in W_h$ and $\boldsymbol{v}_h \in V_h$. Since Statement A holds, they also satisfy the remaining equations of the method. By uniqueness of solutions of the hybridized mixed method (as given by Proposition 2.2) we get Statement B.

$B \implies C$: By linear superposition,

$$
\omega_h = w(\boldsymbol{\lambda}_h) + \mathrm{w}(p_h) + w(g_t) + \mathsf{w}(\boldsymbol{f}), \tag{3.29}
$$

$$
\boldsymbol{u}_h = \boldsymbol{u}(\boldsymbol{\lambda}_h) + \mathbf{u}(p_h) + \boldsymbol{u}(g_t) + \mathbf{u}(\boldsymbol{f}). \tag{3.30}
$$

Comparing with the definitions of $\widetilde{\omega}_h^{\boldsymbol{\lambda},p}$ and $\widetilde{\boldsymbol{u}}_h^{\boldsymbol{\lambda},p}$, we find that Statement B implies

$$
w(\boldsymbol{\lambda}_h) + \mathrm{w}(p_h) = w(\boldsymbol{\lambda}) + \mathrm{w}(p), \tag{3.31}
$$

$$
\boldsymbol{u}(\boldsymbol{\lambda}_h) + \mathbf{u}(p_h) = \boldsymbol{u}(\boldsymbol{\lambda}) + \mathbf{u}(p). \tag{3.32}
$$

In particular,

$$
w(\boldsymbol{\lambda}_h - \boldsymbol{\lambda}) + \mathrm{w}(p_h - p) = 0.
$$

Since the two terms on the left hand side above are $L^2(\Omega)$-orthogonal by (3.18), they both must vanish. Moreover, by the definition of $\mathbf{u}(\cdot)$ (see (3.3)), $w(\boldsymbol{\lambda}_h - \boldsymbol{\lambda}) = 0$ implies

$$
(\mathbf{u}(p_h - p), \mathbf{curl}\, \tau)_K = 0, \quad \text{for all } K \in \mathcal{T}, \tau \in W_h.
$$

Hence $\mathbf{u}(p_h - p) = 0$. By (3.32) we also get $\boldsymbol{u}(\boldsymbol{\lambda}_h - \boldsymbol{\lambda}) = 0$. Thus,

$$
w(\boldsymbol{\lambda}_h - \boldsymbol{\lambda}) = \mathrm{w}(p_h - p) = 0, \quad \boldsymbol{u}(\boldsymbol{\lambda}_h - \boldsymbol{\lambda}) = \mathbf{u}(p_h - p) = \mathbf{0},
$$

so $\boldsymbol{\lambda}_h - \boldsymbol{\lambda} = 0$ and $p_h - p = 0$.

$C \implies D$: We know by (3.29)–(3.30) and the last two equations of the hybridized mixed method, that

$$\Theta := \sum_{e\in\mathcal{E}} \big(\boldsymbol{\mu},\, [\![\big(w(\boldsymbol{\lambda}_h) + \mathrm{w}(p_h) + \mathsf{w}(\boldsymbol{f}) + w(g_t)\big)\boldsymbol{t}]\!]\big)_e = 0,$$

$$\Psi := \sum_{e\in\mathcal{E}} \big(q,\, [\![\big(\boldsymbol{u}(\boldsymbol{\lambda}_h) + \mathbf{u}(p_h) + \mathbf{u}(\boldsymbol{f}) + \boldsymbol{u}(g_t)\big)\cdot \boldsymbol{n}]\!]\big)_e - (g_n, q)_{\partial\Omega} = 0.$$

Hence, it suffices to show that

$$\Theta = a(\boldsymbol{\lambda},\boldsymbol{\mu}) + b(\boldsymbol{\mu}, p) - \ell_1(\boldsymbol{\mu}), \tag{3.33}$$

$$\Psi = b(\boldsymbol{\lambda}, q) - c(p,q) - \ell_2(q). \tag{3.34}$$

To do this, let us split $\Theta =: \theta_1 + \theta_2 + \theta_3 + \theta_4$, where

$$\begin{aligned}
\theta_1 &:= \sum_{e\in\mathcal{E}} \big(\boldsymbol{\mu},\, [\![w(\boldsymbol{\lambda})]\!]\boldsymbol{t}\big)_e = (w(\boldsymbol{\lambda}), w(\boldsymbol{\mu}))_\Omega && \text{by (3.22)},\\
\theta_2 &:= \sum_{e\in\mathcal{E}} \big(\boldsymbol{\mu},\, [\![\mathrm{w}(p)\boldsymbol{t}]\!]\big)_e = -\sum_{K\in\mathcal{T}} (\boldsymbol{u}(\boldsymbol{\mu}), \mathbf{curl}\ \mathrm{w}(p))_K && \text{by (3.23)},\\
\theta_3 &:= \sum_{e\in\mathcal{E}} \big(\boldsymbol{\mu},\, [\![\mathsf{w}(\boldsymbol{f})\boldsymbol{t}]\!]\big)_e = -(\boldsymbol{f}, \boldsymbol{u}(\boldsymbol{\mu}))_\Omega && \text{by (3.25)},\\
\theta_4 &:= \sum_{e\in\mathcal{E}} \big(\boldsymbol{\mu},\, [\![w(g_t)\boldsymbol{t}]\!]\big)_e = (g_t, w(\boldsymbol{\mu}))_{\partial\Omega} && \text{by (3.26)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\theta_1 &= a(\boldsymbol{\lambda},\boldsymbol{\mu}) && \text{by (3.9)},\\
\theta_2 &= b(\boldsymbol{\mu}, p) && \text{by (3.10)},\\
\theta_3 + \theta_4 &= -\ell_1(\boldsymbol{\mu}) && \text{by (3.12)}.
\end{aligned}$$

This proves (3.33).

To prove (3.34), we split $\Psi =: \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5$ where

$$\begin{aligned}
\psi_1 &:= \sum_{e\in\mathcal{E}} \big(q,\, [\![\boldsymbol{u}(\boldsymbol{\lambda})\cdot\boldsymbol{n}]\!]\big)_e = -\sum_{K\in\mathcal{T}} (\boldsymbol{u}(\boldsymbol{\lambda}), \mathbf{curl}\ \mathrm{w}(q))_K && \text{by (3.23)},\\
\psi_2 &:= \sum_{e\in\mathcal{E}} \big(q,\, [\![\mathbf{u}(p)\cdot\boldsymbol{n}]\!]\big)_e = -(w(p), w(q))_\Omega && \text{by (3.24)},\\
\psi_3 &:= \sum_{e\in\mathcal{E}} \big(q,\, [\![\mathbf{u}(\boldsymbol{f})\cdot\boldsymbol{n}]\!]\big)_e = -(\boldsymbol{f}, \mathbf{u}(q))_\Omega && \text{by (3.27)},\\
\psi_4 &:= \sum_{e\in\mathcal{E}} \big(q,\, [\![\boldsymbol{u}(g_t)\cdot\boldsymbol{n}]\!]\big)_e = (g_t, \mathrm{w}(q))_{\partial\Omega} && \text{by (3.28)},\\
\psi_5 &:= -(g_n, q)_{\partial\Omega}.
\end{aligned}$$

Hence

$$\begin{aligned}
\psi_1 &= b(\boldsymbol{\lambda}, q) && \text{by (3.10)},\\
\psi_2 &= -c(p,q) && \text{by (3.11)},\\
\psi_3 + \psi_4 + \psi_5 &= -\ell_2(\boldsymbol{\mu}), && \text{by (3.13)}.
\end{aligned}$$

Adding the above equations we obtain (3.34).

$D \implies A$: If Statement D holds, then, by the previous step, we have

$$\sum_{e\in\mathcal{E}} \big(\boldsymbol{\mu},\, [\![\big(w(\boldsymbol{\lambda}) + \mathsf{w}(p) + \mathsf{w}(\boldsymbol{f}) + w(g_t)\big)\boldsymbol{t}]\!]\big)_e = 0,$$

$$\sum_{e\in\mathcal{E}} \big(q,\, [\![\big(\boldsymbol{u}(\boldsymbol{\lambda}) + \mathbf{u}(p) + \mathbf{u}(\boldsymbol{f}) + \boldsymbol{u}(g_t)\big)\cdot \boldsymbol{n}]\!]\big)_e = (g_n, q)_{\partial\Omega},$$

which is Statement A. □

Proof of Theorem 3.1. The proof of the theorem is immediate from the previous lemmas: The first assertion of the theorem follows from the equivalence of Statements C and D of Lemma 3.2. The second follows from the first by linear superposition. Thus Theorem 3.1 is proved. □

4. Local bases for Lagrange multipliers

For the hybridized method to be of practical use, it is imperative that we develop computable bases of locally supported functions for the multiplier spaces P_h and M_h.

4.1. **The pressure space.** We begin with a characterization of the space of pressure Lagrange multipliers arising from the first hybridization.

Proposition 4.1. *The space P_h defined in* (2.1) *is characterized by*

$$P_h = \left\{ p : p|_e \in P_k(e) \text{ for all } e \in \mathcal{E} \text{ and } \sum_{e\in\mathcal{E}} (p,1)_e = 0 \right\}.$$

Proof. Let Q_h denote the set in the right hand side above. To show that $P_h \subseteq Q_h$, consider any $\boldsymbol{v}_h \in V_h$ and let $p_h = [\![\boldsymbol{v}_h\cdot\boldsymbol{n}]\!]$. Then $[\![\boldsymbol{v}_h\cdot\boldsymbol{n}]\!]_e \in P_k(e)$ and

$$\sum_{e\in\mathcal{E}} (p_h, 1)_e \,\mathrm{d}s = \sum_{K\in\mathcal{T}} (\boldsymbol{v}_h\cdot\boldsymbol{n}, 1)_{\partial K} = \sum_{K\in\mathcal{T}} (\operatorname{div}\boldsymbol{v}_h, 1)_K = 0.$$

Hence $P_h \subseteq Q_h$.

To show the reverse inclusion, consider any $p_h \in Q_h$. Then there is a function $\widetilde{\boldsymbol{v}}_h \in \widetilde{V}_h := \{\boldsymbol{r} : \boldsymbol{r}|_K = \boldsymbol{x} p_k(\boldsymbol{x}) + \boldsymbol{q}_k$ for some $p_k \in P_k(K)$ and $\boldsymbol{q}_k \in P_k(K)^2\}$, such that

$$[\![\widetilde{\boldsymbol{v}}_h\cdot\boldsymbol{n}]\!]_e = p_h|_e \quad \text{for all } e \in \mathcal{E}.$$

Note that $\operatorname{div}(\widetilde{\boldsymbol{v}}_h|_K)$ is not zero in general. Let S_h be the space of functions whose average on Ω is zero and whose restriction to each mesh element $K \in \mathcal{T}$ is in $P_k(K)$. The function $s_h(\boldsymbol{x})$ defined by

$$s_h|_K = \operatorname{div}(\widetilde{\boldsymbol{v}}_h|_K) \quad \text{for all } K \in \mathcal{T},$$

is in S_h because p_h is in Q_h:

$$(s_h, 1)_\Omega = \sum_{K\in\mathcal{T}} (\operatorname{div}\widetilde{\boldsymbol{v}}_h, 1)_K = \sum_{K\in\mathcal{T}} (\widetilde{\boldsymbol{v}}_h\cdot\boldsymbol{n}, 1) = \sum_{e\in\mathcal{E}} (p_h, 1)_e = 0.$$

Now, the space $\widetilde{V}_h \cap H_0(\operatorname{div},\Omega)$ is a standard Raviart-Thomas space, and by its well known properties, $\operatorname{div} : \widetilde{V}_h \cap H_0(\operatorname{div},\Omega) \mapsto S_h$ is a surjection. Hence, there is a $\boldsymbol{z}_h \in \widetilde{V}_h \cap H_0(\operatorname{div},\Omega)$ such that

$$\operatorname{div}\boldsymbol{z}_h = s_h.$$

Then $\boldsymbol{v}_h = \widetilde{\boldsymbol{v}}_h - \boldsymbol{z}_h$ is in V_h and $[\![\boldsymbol{v}_h\cdot\boldsymbol{n}]\!]_e = [\![\widetilde{\boldsymbol{v}}_h\cdot\boldsymbol{n}]\!]_e = p_h|_e$. Hence $Q_h \subseteq P_h$. □

In view of Proposition 4.1, the function q_h belongs to the space P_h if and only if it belongs to

$$\widetilde{P}_h = \{p : p|_e \in P_k(e) \text{ for all } e \in \mathcal{E}\}$$

and satisfies

$$\sum_{e\in\mathcal{E}} (q_h, 1)_e = 0. \tag{4.1}$$

Thus we only need to construct a local basis for $\widetilde{P}_h$ and then enforce the last equation. Obviously, we can construct a basis for $\widetilde{P}_h$ by taking the union of local bases for $P_k(e)$, say Legendre polynomials, on every edge $e \in \mathcal{E}$. In practice, the constraint (4.1) can be handled *a posteriori* in a very simple way, as shown in Section 5.

4.2. **The lowest order tangential velocity space.** In the remainder of this section, we construct a local basis for the space M_h of tangential velocity Lagrange multipliers. In this subsection, we study the lowest order case. In the next, we show how our considerations here generalize to the higher order case.

In order to explicitly give a local basis for M_h, we introduce some more notation. Let K be a mesh triangle and x be one of its vertices. We denote by $\Lambda_{x,K}$ the union of the two edges of K that are connected to the vertex x. Let

$$\hat{\Lambda}_h = \{\Lambda_{x,K} : x \text{ is a vertex of } \mathcal{T} \text{ and } K \in \mathcal{T}\}.$$

For all $\Lambda \in \hat{\Lambda}_h$, we denote by K_Λ the (unique) triangle $K \in \mathcal{T}$ such that $\Lambda \subseteq \partial K$, and by x_Λ the common vertex of Λ and K_Λ. Let ϕ_Λ denote the function (that is discontinuous, in general) which vanishes on all $K \in \mathcal{T}$ except on K_Λ where it equals the linear function that is one on x_Λ and zero on the remaining two vertices of K_Λ. We define a basis for M_h using the functions

$$\boldsymbol{\psi}_\Lambda = [\![\phi_\Lambda \boldsymbol{t}]\!].$$

Obviously $\boldsymbol{\psi}_\Lambda \in M_h$, but not all of $\boldsymbol{\psi}_\Lambda, \Lambda \in \hat{\Lambda}_h$ are linearly independent, e.g., the functions $\boldsymbol{\psi}_\Lambda$ for all Λ connected to one vertex are linked by one equation. Therefore, for every mesh vertex x (including $x \in \partial\Omega$), we arbitrarily pick one element $\Lambda \in \hat{\Lambda}_h$ with vertex $x_\Lambda = x$, denote it by $\triangledown_x$ (see Figure 2), and "omit" it: Define

$$\Lambda_h = \hat{\Lambda}_h \setminus \{\triangledown_x : \text{ for all mesh vertices } x\}.$$

Proposition 4.2. *The set $\mathcal{B} = \{\boldsymbol{\psi}_\Lambda : \Lambda \in \Lambda_h\}$ is a basis for M_h when $k = 0$.*

Proof. Obviously the span of $\mathcal{B}$ is contained in M_h. Hence it suffices to prove that

$$\operatorname{card}\mathcal{B} = \dim M_h, \quad \text{and} \tag{4.2}$$

$$\mathcal{B} \text{ is a linearly independent set.} \tag{4.3}$$

To prove (4.2), let us first count the dimension of M_h. Defining $T_h : W_h \mapsto M_h$ by

$$T_h\tau = [\![\tau\boldsymbol{t}]\!],$$

we note that M_h is the range of T_h. Since the null space of T_h is $\mathcal{W}_h$, by the rank-nullity theorem, we find that

$$\dim(M_h) = \operatorname{rank}(T_h) = \dim(W_h) - \dim(\mathcal{W}_h). \tag{4.4}$$

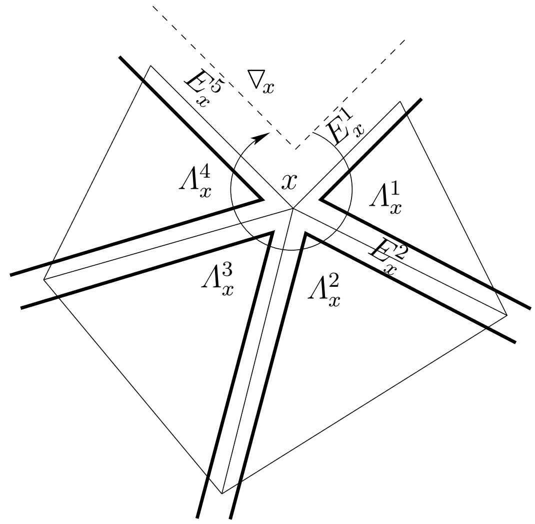

FIGURE 2. Construction of basis functions supported near a mesh vertex x

In the lowest order case, this easily gives

$$\dim(M_h) = 3n_K - n_V,$$

where n_K and n_V are the number of triangles and vertices of the mesh, respectively. Now, since

$$\operatorname{card} \mathcal{B} = \operatorname{card} \Lambda_h = \operatorname{card} \hat{\Lambda}_h - n_V = 3n_K - n_V,$$

we immediately see that (4.2) holds.

To prove (4.3), let $\boldsymbol{\mu}$ be any linear combination of the basis elements:

$$\boldsymbol{\mu} = \sum_{\Lambda \in \Lambda_h} c_\Lambda \boldsymbol{\psi}_\Lambda. \tag{4.5}$$

Then, consider $\boldsymbol{\mu}|_{\nabla_x}$ for any mesh vertex x (including $x \in \partial\Omega$). Enumerate all $\Lambda \in \Lambda_h$ with vertex x as $\Lambda_x^1, \Lambda_x^2, \ldots, \Lambda_x^{N_x}$ and all edges in $\mathcal{E}$ connected to x as $E_x^1, E_x^2, \ldots E_x^{N_x+1}$ as in Figure 2. The enumerations are such that the two edges of Λ_x^j are E_x^j and E_x^{j+1}. Let μ_x^i be the function defined on E_x^i that equals the magnitude of $\boldsymbol{\mu}|_{E_x^i}$. Observe that the limit of $\mu_x^1(y)$ as y approaches x along the edge E_x^1 is $|c_{\Lambda_x^1}|$. Similarly, the limit of $\mu_x^{N_x}(y)$ as y approaches x along the edge $E_x^{N_x+1}$ is $|c_{\Lambda_x^{N_x}}|$. Also note that the limit of $\mu_x^j(y)$ as y approaches x along the edge E_x^j is $|c_{\Lambda_x^j} - c_{\Lambda_x^{j-1}}|$, for all $j = 2, 3, \ldots, N_x - 1$.

Now suppose $\boldsymbol{\mu} \equiv \mathbf{0}$. We have to show that all the coefficients c_Λ in (4.5) are zero. Since $\boldsymbol{\mu}$ vanishes everywhere, in particular, for a mesh vertex x, the function $\mu_x^j(y)$ vanish on the edge E_x^j. Hence its limit as y approaches x along the edge E_x^j equals zero. Thus,

$$\begin{aligned} |c_{\Lambda_x^1}| = |c_{\Lambda_x^{N_x}}| &= 0, && \text{and} \\ |c_{\Lambda_x^j} - c_{\Lambda_x^{j-1}}| &= 0, && \text{for all } j = 2, \ldots, N_x - 1. \end{aligned}$$

This implies that $c_{\Lambda_x^j} = 0$ for all j. The above argument applies to every mesh vertex, so all the coefficients c_Λ in (4.5) are zero. Hence (4.3) follows. $\square$

4.3. **Basis for the space of tangential velocities.** By augmenting the basis $\mathcal{B}$ for the lowest order case constructed above with some locally supported functions, it is possible to construct a basis for M_h of any order. Define $\mathcal{B}_e^{(k+1)}$ to be any basis for the set of polynomials on edge e of degree at most $k+1$ that vanishes at both endpoints of e. Let $\mathcal{E}_0$ denote the set of all *interior* edges of the mesh $\mathcal{T}$. Then we have the following result.

Theorem 4.1. *The set*

$$\mathcal{B}^{(k+1)} = \left(\bigcup_{e \in \mathcal{E}_0} \mathcal{B}_e^{(k+1)} \right) \cup \mathcal{B}$$

is a basis for M_h.

Proof. It is easy to see that each element of $\mathcal{B}_e^{(k+1)}$ can be written as $[\![\phi \boldsymbol{t}]\!]$ for some $\phi \in W_h$. Hence the span of $\mathcal{B}^{(k+1)}$ is contained in M_h. As in the proof of Proposition 4.2, it now suffices to prove that

$$\operatorname{card} \mathcal{B}^{(k+1)} = \dim(M_h), \tag{4.6}$$

and that $\mathcal{B}^{(k+1)}$ is a linearly independent set. Since functions in $\mathcal{B}_e^{(k+1)}$ vanish at endpoints of their edge of support, by a minor modification of the arguments in the proof of Proposition 4.2, the linear independence of $\mathcal{B}^{(k+1)}$ follows.

To prove (4.6), observe that $\operatorname{card} \mathcal{B}_e^{(k+1)} = \dim(P_{k+1}(e)) - 2 = k$. Since

$$\operatorname{card} \mathcal{E}_0 = 3n_K - n_E,$$

where n_E denotes the number of all edges of $\mathcal{T}$ (including boundary edges), we have

$$\begin{aligned} \operatorname{card} \mathcal{B}^{(k+1)} &= \operatorname{card} \mathcal{B} + \sum_{e \in \mathcal{E}_0} \operatorname{card} \mathcal{B}_e^{(k+1)} \\ &= (3n_K - n_V) + (3n_K - n_E)k \\ &= 3n_K(k+1) - n_V - kn_E. \end{aligned} \tag{4.7}$$

Now, let us show that this equals $\dim(M_h)$. Since, by (4.4), $\dim(M_h) = \dim(W_h) - \dim(\mathcal{W}_h)$, we need to compute the dimension of the spaces W_h and $\mathcal{W}_h$. The number of degrees of freedom of $\mathcal{W}_h$ can be computed by splitting them into vertex degrees of freedom (one per vertex), edge degrees of freedom (k per edge), and interior degrees of freedom ($\dim(P^{k-2})$ per triangle):

$$\dim(\mathcal{W}_h) = n_K \left(\frac{1}{2}(k-1)k \right) + kn_E + n_V.$$

Now, since

$$\dim(W_h) = n_K \frac{1}{2}(k+2)(k+3) = n_K \left(3(k+1) + \frac{1}{2}(k-1)k \right),$$

the dimension of M_h can immediately seen to be equal to $\operatorname{card} \mathcal{B}^{(k+1)}$ as calculated in (4.7). Hence (4.6) follows. $\square$

5. Some implementation aspects

In this section, we first point out some general issues in implementing the Lagrange multiplier system. Afterward we specialize to a detailed discussion of the lowest order case. We exhibit explicit expressions for all the local mappings in the lowest order case. We also show how traditional finite element ideas such as matrix assembly through local element stiffness matrices apply for the Lagrange multiplier system provided the local matrices are properly defined.

5.1. **The matrix equations.** In order to solve for $\boldsymbol{\lambda}_h$ and p_h satisfying (3.14)–(3.15), we use the previously introduced local basis. Let $\boldsymbol{\psi}^{(i)}$, $i = 1, 2, \ldots, N_M$ be an enumeration of the basis for M_h introduced in Section 4. Let $p^{(l)}$, $l = 1, 2, \ldots, N_P$ denote any basis for $\widetilde{P}_h$ with the property that a basis function is supported on just one edge. With respect to these bases, let A, B, and C denote the matrices associated to the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$, respectively:

$$
\begin{aligned}
\mathsf{A}_{ij} &= (w(\boldsymbol{\psi}^{(j)}), w(\boldsymbol{\psi}^{(i)}))_\Omega,\\
\mathsf{B}_{lj} &= -(\mathbf{curl}\ \mathrm{w}(p^{(l)}), \boldsymbol{u}(\boldsymbol{\psi}^{(j)}))_\Omega,\\
\mathsf{C}_{lm} &= (\mathrm{w}(p^{(m)}), \mathrm{w}(p^{(l)}))_\Omega.
\end{aligned}
$$

Then the Lagrange multiplier system (3.14)–(3.15) takes the following matrix form:

$$
\begin{bmatrix} \mathsf{A} & \mathsf{B}^t \\ \mathsf{B} & -\mathsf{C} \end{bmatrix} \begin{bmatrix} \mathsf{\Lambda} \\ \mathsf{P} \end{bmatrix} = \begin{bmatrix} \mathsf{L}_1 \\ \mathsf{L}_2 \end{bmatrix}. \tag{5.1}
$$

Here the $\mathsf{\Lambda}$ and P are vectors of coefficients of $\boldsymbol{\lambda}_h$ and p_h, respectively, i.e.,

$$
\boldsymbol{\lambda}_h = \sum_{i=1}^{N_M} \mathsf{\Lambda}_i\, \boldsymbol{\psi}^{(i)} \quad \text{and} \quad p_h = \sum_{l=1}^{N_P} \mathsf{P}_l\, p^{(l)}.
$$

Notice that we have used a basis for the space $\widetilde{P}_h$ and not one for the space P_h. In view of Proposition 4.1, we therefore anticipate the pressure to be given only up to a constant. This, of course, reflects the fact that the pressure in the Stokes system is also defined up to a constant.

To clarify how one can deal with this in practical implementations, let us examine the null space of

$$
\mathsf{M} := \begin{bmatrix} \mathsf{A} & \mathsf{B}^t \\ \mathsf{B} & -\mathsf{C} \end{bmatrix}.
$$

If $\mathsf{M}\left[\begin{smallmatrix} \mathsf{\Lambda} \\ \mathsf{P} \end{smallmatrix}\right] = 0$, then

$$
a(\boldsymbol{\lambda}_h, \boldsymbol{\mu}) + b(\boldsymbol{\mu}, p_h) = 0 \qquad \text{for all } \boldsymbol{\mu} \in M_h \text{ and} \tag{5.2}
$$

$$
b(\boldsymbol{\lambda}_h, q) - c(p_h, q) = 0 \qquad \text{for all } q \in \widetilde{P}_h. \tag{5.3}
$$

Now, an immediate consequence of the definition of the liftings is that for any constant function $\kappa \in \widetilde{P}_h$

$$
\mathrm{w}(\kappa) = 0 \quad \text{and} \quad \mathbf{u}(\kappa) = \mathbf{0}. \tag{5.4}
$$

Any $q \in \widetilde{P}_h$ can be decomposed as $q = \mathring{q} + \bar{q}$ where $\mathring{q} \in P_h$ and $\bar{q}$ is a constant function ($\bar{q}$ equals the global mean of q). Decomposing both p_h and q this way in (5.2)–(5.3), we find that

$$a(\boldsymbol{\lambda}_h, \boldsymbol{\mu}) + b(\boldsymbol{\mu}, \mathring{p}_h) = 0 \qquad \text{for all } \boldsymbol{\mu} \in M_h \text{ and}$$
$$b(\boldsymbol{\lambda}_h, \mathring{q}) - c(\mathring{p}_h, \mathring{q}) = 0 \qquad \text{for all } \mathring{q} \in P_h.$$

By the unique solvability of (3.14)–(3.15) asserted by Theorem 3.1, we conclude that both $\boldsymbol{\lambda}_h$ and $\mathring{p}_h$ vanish. Thus, $\mathsf{M}\left[\begin{smallmatrix}\Lambda\\ \mathsf{P}\end{smallmatrix}\right] = 0$ if and only if $\boldsymbol{\lambda}_h = \mathbf{0}$ and p_h equals a constant function. The null space of M is therefore equal to the span of $\left[\begin{smallmatrix}0\\ 1_{\mathsf{P}}\end{smallmatrix}\right]$ where 1_{P} denotes the vector of coefficients of $\kappa \equiv 1 \in \widetilde{P}_h$. Note that if b denotes the vector $\left[\begin{smallmatrix}\mathsf{L}_1\\ \mathsf{L}_2\end{smallmatrix}\right]$ on the right hand side of (5.1), then by (5.4),

$$\mathsf{b} \cdot \left[\begin{smallmatrix}0\\ 1_{\mathsf{P}}\end{smallmatrix}\right] = \ell_1(0) + \ell_2(\kappa) = 0.$$

Thus (5.1) has a solution, and if $\left[\begin{smallmatrix}\Lambda\\ \mathsf{P}\end{smallmatrix}\right]$ is a solution, then all solutions are of the form $\left[\begin{smallmatrix}\Lambda\\ \mathsf{P}\end{smallmatrix}\right] + \alpha \left[\begin{smallmatrix}0\\ 1_{\mathsf{P}}\end{smallmatrix}\right]$ for some $\alpha \in \mathbb{R}$.

To compute one solution to (5.1), one can now apply variations of standard techniques. For example, if one uses a Krylov space iteration such as MINRES for solving (5.1), then the n-th iterate x_n is in $\mathsf{x}_0 + \operatorname{span}\{\mathsf{r}_0, \mathsf{M}\mathsf{r}_0, \mathsf{M}^2\mathsf{r}_0, \ldots, \mathsf{M}^{n-1}\mathsf{r}_0\}$, where $\mathsf{r}_0 = \mathsf{b} - \mathsf{M}\mathsf{x}_0$ and x_0 is the initial iterate. Since $\left[\begin{smallmatrix}0\\ 1_{\mathsf{P}}\end{smallmatrix}\right] \cdot (\mathsf{M}^j \mathsf{r}_0) = 0$ for all $j \geq 0$, if the initial iterate x_0 satisfies $\mathsf{x}_0 \cdot \left[\begin{smallmatrix}0\\ 1_{\mathsf{P}}\end{smallmatrix}\right] = 0$, then all further iterates x_n satisfy $\mathsf{x}_n \cdot \left[\begin{smallmatrix}0\\ 1_{\mathsf{P}}\end{smallmatrix}\right] = 0$. Hence by adjusting the final pressure iterate by a scalar multiple of 1_{P}, we can obtain the pressure Lagrange multiplier of zero mean. If one uses a direct solver instead, one can convert (5.1) to an invertible system by simply deleting the row and column of M corresponding to one fixed pressure degree of freedom.

5.2. **The Schur complement matrix for the tangential velocity.** Many standard stable choices of mixed finite elements for Stokes equations result in a velocity-pressure discretization of the form (5.1). There is often a preference for solving the discrete system using a positive definite Schur complement system obtained by eliminating the velocity variable (the Schur complement matrix being $\mathsf{C} + \mathsf{B}\mathsf{A}^{-1}\mathsf{B}^t$), because iterative solvers for positive definite systems are well developed. However, this is not feasible for our method, because in contrast to the standard methods, our matrix A in (5.1) is not invertible in general.

But we can obtain an alternate Schur complement system for our discretization by utilizing a feature of our method that is usually not found in standard methods for Stokes equations, namely the invertibility of the other diagonal block (C) on a subspace. More precisely, we have the following result.

Proposition 5.1. *For any $q_h \in \widetilde{P}_h$, $c(q_h, q_h) = 0$ if and only if q_h is constant.*

Proof. It is obvious from (5.4) that if q_h is constant then $c(q_h, q_h) = 0$. To prove the converse, we observe that $c(q_h, q_h) = 0$ implies $\mathrm{w}(q_h) = 0$, so from (3.4) it follows that

$$\sum_{e \in \mathcal{E}} (q_h, [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!])_e = \sum_{e \in \mathcal{E}} (q_h - \bar{q}_h, [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!])_e = 0 \quad \text{for all } \boldsymbol{v} \in V_h, \tag{5.5}$$

where

$$\bar{q}_h = \frac{\sum_{e \in \mathcal{E}} \int_e q_h \, \mathrm{d}s}{\sum_{e \in \mathcal{E}} \int_e \mathrm{d}s}$$

is the global mean of q_h. By Proposition 4.1, there is a $\boldsymbol{v} \in V_h$ such that $q_h - \bar{q}_h = [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!]$. Hence (5.5) implies that $q_h - \bar{q}_h \equiv 0$. $\square$

The above proposition readily implies that the matrix C restricted to the orthogonal complement $1_{\mathsf{P}}^{\perp} := \{\mathsf{Q} : \mathsf{Q} \cdot 1_{\mathsf{P}} = 0\}$ is invertible. Therefore, rewriting (5.1) as

$$\begin{bmatrix} \mathsf{C} & -\mathsf{B} \\ -\mathsf{B}^t & -\mathsf{A} \end{bmatrix} \begin{bmatrix} \mathsf{P} \\ \Lambda \end{bmatrix} = - \begin{bmatrix} \mathsf{L}_2 \\ \mathsf{L}_1 \end{bmatrix} \tag{5.6}$$

and eliminating P, we get an alternate Schur complement system:

$$(\mathsf{B}^t \mathsf{C}^{-1} \mathsf{B} + \mathsf{A})\Lambda = \mathsf{L}_1 + \mathsf{B}^t \mathsf{C}^{-1} \mathsf{L}_2. \tag{5.7}$$

Note that the two applications of C^{-1} above make sense because Range(B) $\subseteq 1_{\mathsf{P}}^{\perp}$ (since $1_{\mathsf{P}} \cdot \mathsf{B}\Lambda = b(\boldsymbol{\lambda}_h, \kappa) = 0$) and $\mathsf{L}_2 \in 1_{\mathsf{P}}^{\perp}$ (since $1_{\mathsf{P}} \cdot \mathsf{L}_2 = \ell_2(\kappa) = 0$). The Schur complement matrix in (5.7) is invertible because (5.6) uniquely determines Λ. Thus (5.7) is a symmetric and positive definite system, well suited to solution by minimization algorithms such as conjugate gradients.

5.3. **The local mappings for lowest order case.** We now give explicit expressions for the local maps which define the Lagrange multiplier bilinear forms in the lowest order case, i.e., $k = 0$. A simple computation gives that, on any triangle T, we have

$$\begin{aligned}
w(\boldsymbol{\lambda}) &= \frac{1}{|T|} \int_{\partial T \setminus \partial\Omega} \boldsymbol{\lambda} \cdot \boldsymbol{t} \, \mathrm{d}s, & \boldsymbol{u}(\boldsymbol{\lambda}) &= \frac{1}{|T|} \int_{\partial T \setminus \partial\Omega} \boldsymbol{\lambda} \cdot \boldsymbol{t} \, (\boldsymbol{x} - \boldsymbol{x}_T)^{\perp} \, \mathrm{d}s, \\
w(g_t) &= \frac{1}{|T|} \int_{\partial T \cap \partial\Omega} g_t \, \mathrm{d}s, & \boldsymbol{u}(g_t) &= \frac{1}{|T|} \int_{\partial T \cap \partial\Omega} g_t \, (\boldsymbol{x} - \boldsymbol{x}_T)^{\perp} \, \mathrm{d}s, \\
\mathrm{w}(p) &= \boldsymbol{w}_p^T \times (\boldsymbol{x} - \boldsymbol{x}_T), & \mathbf{u}(p) &= -\frac{1}{|T|} \int_T (\boldsymbol{x} - \boldsymbol{x}_T)^{\perp} \, \mathrm{w}(p) \, \mathrm{d}x, \\
\mathsf{w}(\boldsymbol{f}) &= \mathbf{w}_{\boldsymbol{f}}^T \times (\boldsymbol{x} - \boldsymbol{x}_T), & \mathbf{u}(\boldsymbol{f}) &= -\frac{1}{|T|} \int_T (\boldsymbol{x} - \boldsymbol{x}_T)^{\perp} \, \mathsf{w}(\boldsymbol{f}) \, \mathrm{d}x,
\end{aligned}$$

where, the point $\boldsymbol{x}_T$ denotes the barycenter of the triangle T,

$$\boldsymbol{w}_p^T = -\frac{1}{|T|} \int_{\partial T} p \, \boldsymbol{n} \, \mathrm{d}s, \qquad \mathbf{w}_{\boldsymbol{f}}^T = \frac{1}{|T|} \int_T \boldsymbol{f} \, \mathrm{d}x.$$

We have used standard notations for vector operations above, e.g., for vectors $\boldsymbol{a} = (a_1, a_2)$ and $\boldsymbol{b} = (b_1, b_2)$,

$$\boldsymbol{a} \times \boldsymbol{b} = a_1 \, b_2 - a_2 \, b_1, \qquad \boldsymbol{a}^{\perp} = (-a_2, a_1).$$

It is easy to simplify the above expressions to obtain the local mappings of our Lagrange multiplier basis functions. We first give the liftings of $\boldsymbol{\psi}_{\Lambda}$ for a basis function $\boldsymbol{\psi}_{\Lambda}$ associated with a $\Lambda \in \Lambda_h$. Let K be the triangle formed by vertices $\boldsymbol{x}_1, \boldsymbol{x}_2$, and $\boldsymbol{x}_3$. Let e_i denote the edge of K opposite to vertex $\boldsymbol{x}_i$, and $\boldsymbol{n}_i$ denotes the outward unit normal of K on edge e_i. These notations when superscripted by L, R, $-$, or $+$ denote the corresponding geometrical parameters of adjacent triangles K^L, K^R, K_e^-, or $K \equiv K_e^+$, respectively, as illustrated in Figure 3.

Consider the basis function associated to $\Lambda \in \Lambda_h$ with vertex $\boldsymbol{x}_3$ and $\Lambda \subseteq \partial K$ as marked in Figure 3. The liftings $w_{\Lambda} := w(\boldsymbol{\psi}_{\Lambda})$ and $\boldsymbol{u}_{\Lambda} := \boldsymbol{u}(\boldsymbol{\psi}_{\Lambda})$ are supported on $K \cup K^R \cup K^L$

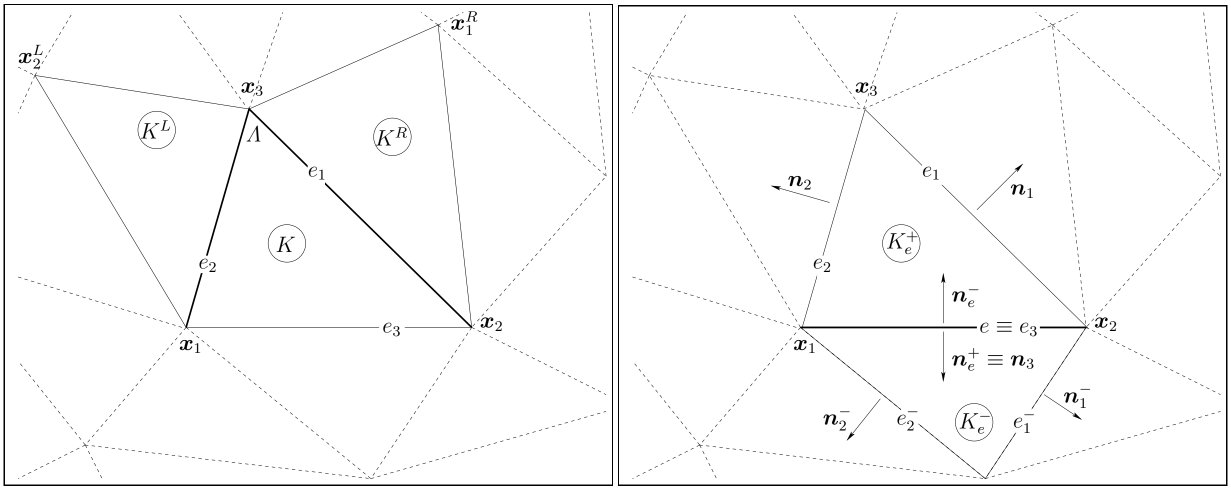

FIGURE 3. Illustration of triangles where liftings associated to a wedge Λ and an edge e are nonzero.

and are given by

$$
\begin{aligned}
w_\Lambda &= \frac{|e_1|+|e_2|}{2|K|}, \quad & \boldsymbol{u}_\Lambda &= \frac{|e_1|}{6|K|}(\boldsymbol{x}_3-\boldsymbol{x}_1)^\perp + \frac{|e_2|}{6|K|}(\boldsymbol{x}_3-\boldsymbol{x}_2)^\perp, \quad && \text{on } K,\\
w_\Lambda &= -\frac{|e_2|}{2|K^L|}, & \boldsymbol{u}_\Lambda &= -\frac{|e_2|}{6|K^L|}(\boldsymbol{x}_3-\boldsymbol{x}_2^L)^\perp, && \text{on } K^L,\\
w_\Lambda &= -\frac{|e_1|}{2|K^R|}, & \boldsymbol{u}_\Lambda &= -\frac{|e_1|}{6|K^R|}(\boldsymbol{x}_3-\boldsymbol{x}_1^R)^\perp, && \text{on } K^R.
\end{aligned}
$$

Here, $|e|$ denotes the length of the edge e and $|K|$ the area of the triangle K. The points $\boldsymbol{x}_1^R$ and $\boldsymbol{x}_2^L$ are shown in Fig 3.

Next, let us display the liftings associated with the pressure. To treat this case, consider an edge e shared by $K \equiv K_e^+$ and another triangle K_e^-. Let p_e denote the indicator function of edge e. The liftings $\mathsf{w}_e := \mathsf{w}(p_e)$ and $\mathbf{u}_e := \mathbf{u}(p_e)$ are supported on $K_e^+ \cup K_e^-$. Using the notation of Figure 3 wherein $e \equiv e_3$, we can express the liftings on $K_e^\pm$ by

$$
\mathsf{w}_e(\boldsymbol{x}) = \boldsymbol{\mathsf{w}}_e^\pm \times (\boldsymbol{x}-\boldsymbol{x}_{K_e^\pm}), \qquad \mathbf{u}_e(\boldsymbol{x}) = \frac{1}{36}\sum_{\ell=1}^{3}(\boldsymbol{\mathsf{w}}_e^\pm \cdot \boldsymbol{E}_\ell^\pm)\boldsymbol{E}_\ell^\pm,
$$

where in accordance with our previous notation $\boldsymbol{x}_{K_e^\pm}$ denotes the barycenter of $K_e^\pm$,

$$
\boldsymbol{\mathsf{w}}_e^\pm = -\frac{1}{|K_e^\pm|}\boldsymbol{n}_e^\pm |e|,
$$

and $\boldsymbol{E}_\ell^\pm = \boldsymbol{n}_\ell^\pm |e_\ell^\pm|$. Here $\boldsymbol{n}_e^\pm$ is as illustrated in Figures 1 and 3.

Finally, we give formulae for the local mappings associated with the body force on the triangle K. If $\boldsymbol{f}$ is supported only on K, then $\mathsf{w}(\boldsymbol{f})$ and $\mathbf{u}(\boldsymbol{f})$ are supported only on K. Their values on K are given by

$$
\mathsf{w}(\boldsymbol{f}) = \mathbf{w} \times (\boldsymbol{x}-\boldsymbol{x}_K), \qquad \mathbf{u}(\boldsymbol{f}) = \frac{1}{36}\sum_{\ell=1}^{3}(\mathbf{w}\cdot\boldsymbol{E}_\ell)\boldsymbol{E}_\ell
$$

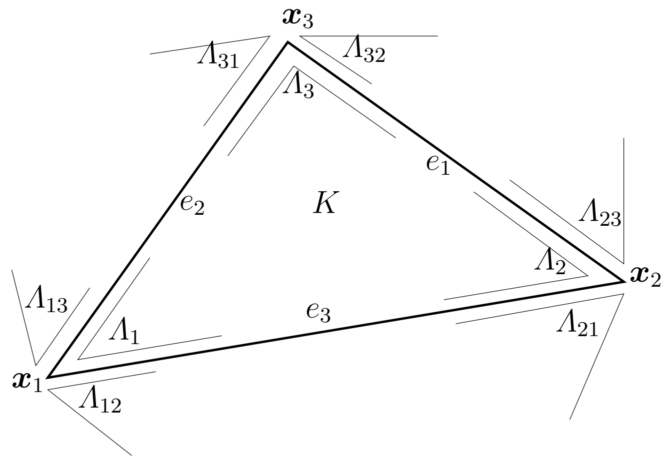

FIGURE 4. Geometry in local element matrix calculations

where as before $\boldsymbol{E}_\ell = |e_\ell|\boldsymbol{n}_\ell$ and

$$\mathbf{w} = \frac{1}{|K|}\int_K \boldsymbol{f}\,\mathrm{d}x.$$

5.4. **The local element matrices for the lowest order case.** It is possible to "assemble" the global stiffness matrix of the Lagrange multiplier equations (3.14)–(3.15) just as one does for traditional finite element methods, provided appropriate local element stiffness matrices are defined for our method. We illustrate this in the lowest order case.

First, we enumerate the degrees of freedom local to an element as in Figure 4. In this enumeration, we include the omitted elements of $\hat{\Lambda}_h$. The omissions can be taken care of during assembly simply by not assembling the rows and columns corresponding to the omitted elements of $\hat{\Lambda}_h$ (just as one would handle zero Dirichlet boundary conditions when solving the Dirichlet problem with standard finite elements). Figure 4 shows nine elements of $\hat{\Lambda}_h$ connected to K, which we have enumerated as Λ_1, Λ_2, Λ_3, Λ_{12}, Λ_{21}, Λ_{13}, Λ_{31}, Λ_{23}, and Λ_{32}, or in short, Λ_I for all I in the index set $\mathfrak{I} := \{1,2,3,12,21,13,31,23,32\}$. The local matrices are made using nine functions in M_h whose local mappings are nonzero on K, namely $\boldsymbol{\psi}_{\Lambda_I}$ for all $I \in \mathfrak{I}$. The local stiffness matrix of an element K has the form

$$\begin{bmatrix} \mathsf{A}^{(K)} & (\mathsf{B}^{(K)})^t \\ \mathsf{B}^{(K)} & -\mathsf{C}^{(K)} \end{bmatrix}$$

where

$$\mathsf{A}^{(K)}_{IJ} = \int_K w(\boldsymbol{\psi}_{\Lambda_I})\, w(\boldsymbol{\psi}_{\Lambda_J})\,\mathrm{d}x, \qquad I, J \in \mathfrak{I},$$

$$\mathsf{B}^{(K)}_{LJ} = -\int_K \mathbf{curl}\, w(p_L)\cdot \boldsymbol{u}(\boldsymbol{\psi}_{\Lambda_J})\,\mathrm{d}x, \qquad J \in \mathfrak{I}, \quad L \in \{1,2,3\},$$

$$\mathsf{C}^{(K)}_{LM} = \int_K w(p_L)\, w(p_M)\,\mathrm{d}x, \qquad L, M \in \{1,2,3\}.$$

Here p_L denotes the characteristic function of the edge e_L in Figure 4. We can calculate the integrals above after substituting the previously given expressions for the liftings of the basis functions in the integrands.

In order to give explicit expressions for $\mathsf{A}^{(K)}, \mathsf{B}^{(K)}$ and $\mathsf{C}^{(K)}$, suppose that $\{i, j, k\}$ is any permutation of $\{1, 2, 3\}$. Let σ_j equal zero if the edge e_j is contained in the boundary $\partial\Omega$ and equal one otherwise. Define

$$W_I = \begin{cases} \sigma_i|e_i| + \sigma_j|e_j|, & \text{if } I = k, \\ -|e_k|, & \text{if } I = ij, \end{cases}$$

$$\boldsymbol{U}_I = \begin{cases} \sigma_i|e_i|(\boldsymbol{x}_k - \boldsymbol{x}_i)^\perp + \sigma_j|e_j|(\boldsymbol{x}_k - \boldsymbol{x}_j)^\perp, & \text{if } I = k, \\ -|e_k|(\boldsymbol{x}_i - \boldsymbol{x}_k)^\perp, & \text{if } I = ij, \end{cases}$$

Then,

$$\mathsf{A}_{IJ}^{(K)} = \frac{1}{4|K|} W_I W_J,$$

$$\mathsf{B}_{LJ}^{(K)} = \frac{1}{6|K|} \boldsymbol{E}_L \cdot \boldsymbol{U}_J,$$

$$\mathsf{C}_{LM}^{(K)} = \frac{1}{36|K|} \sum_{\ell=1}^{3} (\boldsymbol{E}_L \cdot \boldsymbol{E}_\ell)(\boldsymbol{E}_M \cdot \boldsymbol{E}_\ell),$$

where, as before, $\boldsymbol{E}_L = \boldsymbol{n}_L|e_L|$ for all $L \in \{1, 2, 3\}$. With these local matrices, one can assemble all the global matrices of our method as simply as any other finite element method.

6. Conclusion

We have developed new hybridization techniques which when applied to a well known conforming mixed method for the Stokes problem results in a new "tangential velocity-pressure" discretization. The advantages of the new method include fewer globally coupled degrees of freedom and numerical velocity approximations that satisfy the incompressibility condition exactly. Our results are achieved by using the methodology introduced in [8] to study hybridized mixed methods for second-order elliptic problems.

In a sequel, we will discuss the extension of the ideas here to the Stokes problem in three space dimensions, variable degree incompressible finite elements, and other boundary conditions.

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School of Mathematics, University of Minnesota, Vincent Hall, Minneapolis, MN 55455, USA, email: `cockburn@math.umn.edu`.

Department of Mathematics, University of Florida, Gainesville, FL 32611–8105, email: `jayg@math.ufl.edu`.