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# Partial Expansion of a Lipschitz Domain and Some Applications

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# PARTIAL EXPANSION OF A LIPSCHITZ DOMAIN AND SOME APPLICATIONS

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ABSTRACT. We show that a Lipschitz domain can be expanded solely near a part of its boundary, assuming that the part is enclosed by a piecewise $C^1$ curve. The expanded domain as well as the extended part are both Lipschitz. We apply this result to prove a regular decomposition of standard vector Sobolev spaces with vanishing traces only on part of the boundary. Another application in the construction of low-regularity projectors into finite element spaces with partial boundary conditions is also indicated.

## 1. INTRODUCTION

Boundary value problems posed on non-smooth domains, particularly polyhedral domains, are pervasive in computational mathematics. As such, it is central to understand the properties of functions spaces on such domains. Lipschitz regularity of the boundary of a computational domain is often a standard assumption in such studies. In this work we provide a theoretical tool for Lipschitz domains which can be useful when working with function spaces resulting from boundary value problems with mixed boundary conditions, i.e., when part of a Lipschitz boundary is endowed with essential boundary conditions, while the remainder has natural boundary conditions.

Suppose $\Omega$ is a three-dimensional Lipschitz domain. We will show that given a part $\Gamma \subsetneq \partial\Omega$ of the boundary (satisfying certain regularity assumptions), there is a larger Lipschitz domain $\tilde{\Omega}$ which is obtained by extending $\Omega$ only near $\Gamma$. The existence of this domain is proved constructively, by transporting $\Gamma$ using a transversal vector field. A number of technical problems need to be overcome for the proof. We adapt several known techniques [17, 21], such as the construction of a smooth transversal vector field in a neighborhood of Lipschitz domains, and the equivalence between Lischitzness and uniform cone property, to surmount the technicalities.

As an example of how to apply the result, we use the expanded domain to prove a decomposition result for two Sobolev spaces of vector functions whose (tangential or normal) traces vanish only on a part of the boundary. The analogues of these decompositions for the case of no boundary conditions have been known in the literature [4]. They are often called "regular decompositions" [20]. Such decompositions have turned out to be a valuable tool in proving convergence of numerical algorithms. As another application of the

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domain expansion result, we provide a missing detail in the construction of low-regularity bounded cochain projectors (Schöberl projectors) having partial boundary conditions.

We begin by stating our geometrical assumptions in the next section. There we will also state Theorem 2.3 on the existence of the expanded domain. In Section 3, we discuss a few applications. In Section 4 we prove Theorem 2.3.

## 2. Partial expansion of a Lipschitz domain

We consider a three-dimensional domain $\Omega$. In this section, we state our result on how a *part* $\Gamma \subsetneq \partial\Omega$ of a Lipschitz boundary can be transported outward maintaining Lipschitz regularity, under suitable assumptions. The precise assumptions on the domain $\Omega$ and the part of its boundary $\Gamma$ will be detailed below.

2.1. **Geometrical assumptions.** We begin with some standard definitions, restated in an equivalent form convenient for our purposes. These definitions also establish the notations that we will use later in proofs.

*Definition* 2.1. Let $D$ be a nonempty, proper open subset of $\mathbb{R}^3$. Fix $p \in \partial D$. We call $D$ a *locally Lipschitz domain* near $p$ if there exists a open neighborhood $\mathcal{C}_{r,h}$ of $p$ in $\mathbb{R}^3$, and new orthogonal coordinates $(x_1, x_2, x_3)$ such that in the new coordinates, $p = (0,0,0)$, the neighborhood can be represented by

$$\mathcal{C}_{r,h} = \{(x_1, x_2, 0) + t\hat{u} : \ (x_1, x_2) \in (-r, r) \times (-r, r) \text{ and } -h < t < h\},$$

where the vector $\hat{u} = (0,0,1)$ in the local coordinates, and

$$\mathcal{C}_{r,h} \cap D = \mathcal{C}_{r,h} \cap \{(x_1, x_2, 0) + t\hat{u} : \ x_1, x_2 \in (-r, r), \ t > \zeta(x_1, x_2)\}, \tag{1}$$

$$\mathcal{C}_{r,h} \cap \partial D = \mathcal{C}_{r,h} \cap \{(x_1, x_2, 0) + t\hat{u} : \ x_1, x_2 \in (-r, r), \ t = \zeta(x_1, x_2)\}, \tag{2}$$

$$\mathcal{C}_{r,h} \cap \overline{D}^c = \mathcal{C}_{r,h} \cap \{(x_1, x_2, 0) + t\hat{u} : \ x_1, x_2 \in (-r, r), \ t < \zeta(x_1, x_2)\}, \tag{3}$$

for some Lipschitz function $\zeta : [-r, r]^2 \to \mathbb{R}$ satisfying

$$\zeta(p) = 0, \quad \text{and } |\zeta(x_1, x_2)| < h \text{ if } x_1, x_2 \in (-r, r). \tag{4}$$

We call $\mathcal{C}_{r,h}$ a *coordinate box near $p$ in the $\hat{u}$-direction.* The boundary $\partial D$ is then said to be a *Lipschitz hypograph near $p$ in the $\hat{u}$-direction.* A domain $D$ which is locally a Lipschitz hypograph near every point on $\partial D$ is simply called a *locally Lipschitz* domain. We say that $D$ is a *Lipschitz domain* if it is a locally Lipschitz domain and $\partial D$ is compact.

This is a standard definition, e.g., it is equivalent to [17, Definition 1.2.1.1] – see also [23, pp. 89] and [21]. The next definition allows us to talk about parts of the boundary which are regular in a certain sense.

*Definition* 2.2 (Piecewise $C^1$ dissection). Suppose $D$ is a Lipschitz domain in $\mathbb{R}^3$, with the accompanying notations above. Consider a disjoint union

$$\partial D = \Gamma_1 \cup \Pi \cup \Gamma_2, \tag{5}$$

where $\Gamma_1$ and $\Gamma_2$ are disjoint, nonempty, relatively open subsets of $\partial D$, having $\Pi$ as their common boundary in $\partial D$. We call (5) a *piecewise $C^1$ dissection of $\partial D$* if for any $p \in \partial D$, the coordinate box $\mathcal{C}_{r,h}$ near $p$, and the local coordinates $(x_1, x_2, x_3)$, given by the Lipschitz

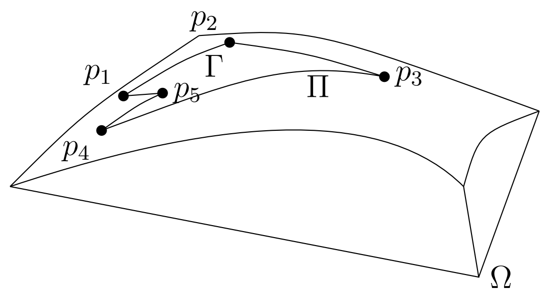


FIGURE 1. A piecewise $C^1$ dissection of the boundary $\partial\Omega$.

regularity (see Definition 2.1) are such that the three sets $\Gamma_1 \cap \mathcal{C}_{r,h}$, $\Pi \cap \mathcal{C}_{r,h}$ and $\Gamma_2 \cap \mathcal{C}_{r,h}$ have the representations

$$\begin{aligned}
\Gamma_1 \cap \mathcal{C}_{r,h} &= \{(x_1, x_2, x_3) \in \mathcal{C}_{r,h} :\ x_3 = \zeta(x_1, \varrho(x_1)),\ x_2 < \varrho(x_1)\},\\
\Pi \cap \mathcal{C}_{r,h} &= \{(x_1, x_2, x_3) \in \mathcal{C}_{r,h} :\ x_3 = \zeta(x_1, \varrho(x_1)),\ x_2 = \varrho(x_1)\},\\
\Gamma_2 \cap \mathcal{C}_{r,h} &= \{(x_1, x_2, x_3) \in \mathcal{C}_{r,h} :\ x_3 = \zeta(x_1, \varrho(x_1)),\ x_2 > \varrho(x_1)\},
\end{aligned}$$

for some Lipschitz function $\varrho : [-r, r] \to \mathbb{R}$, and additionally, the map

$$x_1 \mapsto (x_1, \varrho(x_1), \zeta(x_1, \varrho(x_1)))$$

from $[-r, r]$ into $\mathcal{C}_{r,h}$ is $C^1$ on $[-r, r]$, except finitely many points. These finitely many exceptional points (where $\Pi$ is not $C^1$) will be enumerated as $p_1, \dots, p_m$ (see Figure 1).

With the help of these definitions, we now place the following assumptions on $D$, and the part of the boundary $\Gamma$ used in imposing the mixed boundary condition.

*Assumption* 1. Assume that $\Omega$ is a Lipschitz domain and $\Gamma \cup \Pi \cup (\partial\Omega \setminus \bar{\Gamma})$ is a piecewise $C^1$ dissection of $\partial\Omega$.

A typical case in practical computations occurs when $\Omega$ is a Lipschitz polyhedron and $\Gamma$ is formed by the union of a few faces of the polyhedron. We have in mind boundary value problems where one type of boundary condition is imposed on $\Gamma$, while another boundary condition is imposed on the remainder of the boundary.

The next theorem shows that $\Omega$ can be expanded to a Lipschitz domain in such a way that the expansion occurs only near $\Gamma$.

**Theorem 2.3** (Partial expansion of a Lipschitz domain)**.** *Suppose Assumption 1 holds. Then there exists a Lipschitz domain $\Omega^e$ such that*

$$\Omega \cap \Omega^e = \emptyset \qquad \textit{and} \qquad \partial\Omega \cap \partial\Omega^e = \Gamma \cup \Pi.$$

*Furthermore, $\tilde{\Omega} = \Omega \cup \Gamma \cup \Omega^e$ is also Lipschitz.*

The proof of this result is technical mainly because we cannot assume more than Lipschitz regularity for $\partial\Omega$. The proof, together with all the lemmas needed, are gathered in Section 4.

## 3. Applications

In this section, we give some applications of Theorem 2.3 to questions in computational mathematics. In § 3.1, we use the first conclusion of Theorem 2.3, namely that $\Omega^e$ is Lipschitz, while in § 3.2, we use the second conclusion, namely that $\tilde{\Omega}$ is also Lipschitz.

Let us first establish notations for Sobolev spaces. The set of functions from $\Omega$ into $\mathbb{X}$ whose components are square (Lebesgue) integrable will be denoted by $L^2(\Omega, \mathbb{X})$, when $\mathbb{X}$ is $\mathbb{R}, \mathbb{R}^3, \mathbb{R}^{3\times 3}$, etc. Let $H^1(\Omega, \mathbb{R}^3) = \{v \in L^2(\Omega, \mathbb{R}^3) : \operatorname{grad} v \in L^2(\Omega, \mathbb{R}^{3\times 3})\}$, and $H^1(\Omega, \mathbb{R})$ is similarly defined. Also, let

$$H(\operatorname{div}, \Omega, \mathbb{R}^3) = \{v \in L^2(\Omega, \mathbb{R}^3) : \ \operatorname{div} v \in L^2(\Omega, \mathbb{R})\},$$
$$H(\operatorname{curl}, \Omega, \mathbb{R}^3) = \{v \in L^2(\Omega, \mathbb{R}^3) : \ \operatorname{curl} v \in L^2(\Omega, \mathbb{R}^3)\}.$$

Higher order analogues of this space are defined by

$$H^k(\operatorname{curl}, \Omega, \mathbb{R}^3) = \{v \in H^k(\Omega, \mathbb{R}^3) : \operatorname{curl} v \in H^k(\Omega, \mathbb{R}^3)\},$$

for $k \geq 0$ (for the $k = 0$ case, we obtain the previous space). The space $H^k(\operatorname{div}, \Omega, \mathbb{R}^3)$ is defined similarly. Before we proceed to the applications, let us review an interesting result for the above defined space, recently obtained in [20].

**Theorem 3.1** (Hiptmair, Li, and Zhou [20])**.** *Let $\Omega$ be a Lipschitz domain and $k \geq 0$ be an integer. Then, there are extension operators $\mathcal{E}^{\operatorname{curl}} : H^k(\operatorname{curl}, \Omega, \mathbb{R}^3) \mapsto H^k(\operatorname{curl}, \mathbb{R}^3, \mathbb{R}^3)$ and $\mathcal{E}^{\operatorname{div}} : H^k(\operatorname{div}, \Omega, \mathbb{R}^3) \mapsto H^k(\operatorname{div}, \mathbb{R}^3, \mathbb{R}^3)$ and a $C > 0$ (dependent on $k$) such that*

$$\left\|\mathcal{E}^{\operatorname{curl}} v\right\|_{H^k(\operatorname{curl}, \mathbb{R}^3)} \leq C \|v\|_{H^k(\operatorname{curl}, \Omega)}, \qquad \text{for all } v \in H^k(\operatorname{curl}, \Omega, \mathbb{R}^3),$$
$$\left\|\mathcal{E}^{\operatorname{div}} u\right\|_{H^k(\operatorname{div}, \mathbb{R}^3)} \leq C \|u\|_{H^k(\operatorname{div}, \Omega)}, \qquad \text{for all } u \in H^k(\operatorname{div}, \Omega, \mathbb{R}^3),$$

*where $\|w\|_{H^k(\operatorname{curl}, D)} = (\|w\|^2_{H^k(D)} + \|\operatorname{curl} w\|^2_{H^k(D)})^{1/2}$, and $\|w\|_{H^k(\operatorname{div}, D)}$ is similarly defined.*

This is a generalization of the Stein extension [30] (and we will use it in § 3.1 below). The construction of the above extensions are based on generalizing an integral formula of Stein that extends functions on Lipschitz hypographs, in such a way that a target commutativity property is satisfied. For any given $k$, a simpler such extension based on Hestenes' generalized reflections [18] can be constructed, as in [12, § 2.1]. However the result of [20] is stronger and gives a universal extension for all $k$, as stated above.

### 3.1. **A decomposition of spaces.**

As an application of Theorem 2.3 (and Theorem 3.1) we now prove a decomposition of Sobolev spaces that finds utility in analyses of certain computational algorithms.

By way of preliminaries, recall [14] that the trace operator and the normal trace operator, namely,

$$\operatorname{trc}(v) = v|_{\partial\Omega}, \quad \text{and} \quad \operatorname{trc}_n(v) = v \cdot n|_{\partial\Omega},$$

resp., can be continuously extended from smooth vector functions to $H^1(\Omega, \mathbb{R}^3)$ and $H(\operatorname{div}, \Omega, \mathbb{R}^3)$, resp. Throughout, $n$ denotes the outward unit normal on $\partial\Omega$. Let

$$H^1_{0,\Gamma}(\Omega, \mathbb{R}^3) = \{v \in H^1(\Omega, \mathbb{R}^3) : \ \operatorname{trc}(v)|_\Gamma = 0\}.$$

Note that all components of a vector function in this space vanish on $\Gamma$. Here, the range of $\operatorname{trc}(\cdot)$ is $H^{1/2}(\partial\Omega)$, so the restriction $\operatorname{trc}(v)|_\Gamma$ is obviously well-defined. We can also give meaning to a similar statement on the normal trace as follows. The range of $\operatorname{trc}_n$ as a map from $H(\operatorname{div}, \Omega, \mathbb{R}^3)$ equals $H^{-1/2}(\partial\Omega)$. Let $\langle \cdot, \cdot \rangle_{H^{1/2}}$ be the duality pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$, and let $H^1_{0,\partial\Omega\setminus\Gamma}(\Omega) = \{z \in L^2(\Omega,\mathbb{R}) : \operatorname{grad} z \in L^2(\Omega,\mathbb{R}^3), \text{ and } z|_{\partial\Omega\setminus\Gamma} = 0\}$. For vector functions $v \in H(\operatorname{div},\Omega,\mathbb{R}^3)$, we say that

$$\operatorname{trc}_n(v)|_\Gamma = 0, \tag{6}$$

if $\langle \operatorname{trc}_n(v), \phi\rangle_{H^{1/2}} = 0$ for all $\phi \in H^1_{0,\partial\Omega\setminus\Gamma}(\Omega,\mathbb{R})$. Define

$$H_{0,\Gamma}(\operatorname{div},\Omega) \stackrel{\text{def}}{=} \{v \in H(\operatorname{div},\Omega,\mathbb{R}^3) : \operatorname{trc}_n(v)|_\Gamma = 0\}.$$

Similarly we define

$$H_{0,\Gamma}(\operatorname{curl},\Omega) \stackrel{\text{def}}{=} \{v \in H(\operatorname{curl},\Omega,\mathbb{R}^3) : \langle n\times v, \operatorname{trc}_\tau(\phi)\rangle_{H^{1/2}} = 0 \text{ for all } \phi \in H^1_{0,\partial\Omega\setminus\Gamma}(\Omega;\mathbb{R}^3)\},$$

where the tangential trace operator is defined by

$$\operatorname{trc}_\tau(v) = (v - (v\cdot n)n)\,|_{\partial\Omega}$$

for smooth functions $v$. It is well known [14] that $\operatorname{trc}_\tau$ can also be extended as a continuous map from $H(\operatorname{curl},\Omega,\mathbb{R}^3)$ into $H^{-1/2}(\partial\Omega,\mathbb{T})$, where $\mathbb{T}$ is the tangent space (homeomorphic to $\mathbb{R}^2$) and so we interpret $\operatorname{trc}_\tau(v)|_\Gamma$ just as we did $\operatorname{trc}_n(v)|_\Gamma$ in (6).

**Theorem 3.2.** *Suppose $\Omega$ is contractible and Assumption 1 holds. Then, any $v \in H_{0,\Gamma}(\operatorname{div},\Omega)$ can be decomposed into*

$$v = \operatorname{curl}\varphi + \phi, \qquad \textit{with} \quad \varphi \in H^1_{0,\Gamma}(\Omega,\mathbb{R}^3) \ \textit{ and } \phi \in H^1_{0,\Gamma}(\Omega,\mathbb{R}^3), \tag{7}$$

*and any $u$ in $H_{0,\Gamma}(\operatorname{curl},\Omega)$ can be decomposed into*

$$u = \operatorname{grad}\xi + \zeta, \qquad \textit{with} \quad \xi \in H^1_{0,\Gamma}(\Omega,\mathbb{R}) \ \textit{ and } \zeta \in H^1_{0,\Gamma}(\Omega,\mathbb{R}^3). \tag{8}$$

*Moreover, both decompositions are stable, i.e., $\varphi$ and $\phi$ depend continuously on $v$ (in their respective norms), and similarly $\xi$ and $\zeta$ depend continuously on $u$.*

*Proof.* Let us prove the first decomposition for $v \in H_{0,\Gamma}(\operatorname{div},\Omega)$. Let $\Omega^e$ and $\tilde\Omega$ be as given by Theorem 2.3 and consider the trivial extension

$$\tilde v = \begin{cases} v, \text{ on } \Omega, \\ 0, \text{ on } \tilde\Omega\setminus\Omega, \end{cases} \tag{9}$$

Clearly, $\tilde v$ is in $H(\operatorname{div},\tilde\Omega,\mathbb{R}^3)$.

First, we claim that there are functions $\bar\varphi$ and $\tilde\theta$, both in $H^1(\tilde\Omega,\mathbb{R}^3)$, such that $\tilde v$ can be decomposed as

$$\tilde v = \operatorname{curl}\bar\varphi + \tilde\theta, \qquad \text{on } \tilde\Omega. \tag{10}$$

and the component functions $\bar\varphi, \tilde\theta$ continuously depend on $\tilde v$. To see this, we first use a well known regular right inverse of divergence (see [24], [14, Corollary 2.4], or more recently [6]) to obtain a $\tilde\theta \in H^1(\tilde\Omega,\mathbb{R}^3)$ satisfying $\operatorname{div}\tilde\theta = \operatorname{div}\tilde v$ and

$$\|\tilde\theta\|_{H^1(\tilde\Omega)} \le C\|\operatorname{div}\tilde v\|_{L^2(\tilde\Omega)} = C\|\operatorname{div} v\|_{L^2(\Omega)}. \tag{11}$$

Next, since $\operatorname{div}(\tilde\theta - \tilde v) = 0$, by [1, Lemma 3.5], there is a $\bar\varphi \in H^1(\tilde\Omega,\mathbb{R}^3)$ such that $\tilde\theta - \tilde v = \operatorname{curl}\bar\varphi$ (where we have used the contractibility of $\Omega$) and

$$\|\bar\varphi\|_{H^1(\tilde\Omega)} \le C\|\tilde\theta - \tilde v\|_{L^2(\tilde\Omega)} \le C\|v\|_{H(\operatorname{div},\Omega)}. \tag{12}$$

This proves (10).

Next, observe that when (10) is restricted to $\Omega^e$, since $\tilde{v}$ vanishes on $\Omega^e$, we have

$$(\operatorname{curl}\bar{\varphi})\big|_{\Omega^e} = -\tilde{\theta}\big|_{\Omega^e} \in H^1(\Omega^e, \mathbb{R}^3). \tag{13}$$

Hence

$$\bar{\varphi}|_{\Omega^e} \in H^1(\operatorname{curl}, \Omega^e, \mathbb{R}^3).$$

By Theorem 2.3, $\Omega^e$ is Lipschitz, so we can apply the universal extension of Theorem 3.1 to $\bar{\varphi}|_{\Omega^e}$, yielding $\hat{\varphi} = \mathcal{E}^{\operatorname{curl}}\bar{\varphi}$ in $H^1(\operatorname{curl}, \mathbb{R}^3, \mathbb{R}^3)$. Adding and subtracting $\operatorname{curl}\hat{\varphi}$ in (10), we thus have $\tilde{v} = \operatorname{curl}(\bar{\varphi} - \hat{\varphi}) + (\tilde{\theta} + \operatorname{curl}\hat{\varphi})$. In other words,

$$\tilde{v} = \operatorname{curl}\tilde{\varphi} + \tilde{\phi}, \qquad \text{on } \tilde{\Omega}, \tag{14}$$

with $\tilde{\varphi} = \bar{\varphi} - \hat{\varphi}$, and $\tilde{\phi} = \tilde{\theta} + \operatorname{curl}\hat{\varphi}$.

To finish the proof of (7), we now only need to restrict the functions in (14) to $\Omega$. Indeed, with $\varphi = \tilde{\varphi}|_\Omega$ and $\phi = \tilde{\phi}|_\Omega$, we have $v = \operatorname{curl}\tilde{\varphi} + \tilde{\phi}$. We need to verify the boundary conditions of $\varphi$ and $\phi$. To show that $\varphi|_\Gamma = 0$, we only need to observe that $\tilde{\varphi}|_{\Omega^e} = (\bar{\varphi} - \hat{\varphi})|_{\Omega^e} = 0$ because $\hat{\varphi}$ is the extension of $\tilde{\phi}$ from $\Omega^e$. We note that *all* components of $\varphi$ vanish on $\Gamma$.

To verify that all components of $\phi$ also vanish on $\Gamma$, recall that $\tilde{v}|_{\Omega^e} = 0$. Since we observed above that $\tilde{\varphi}|_{\Omega^e} = 0$, all the terms other than $\tilde{\phi}$ in (14) vanish on $\Omega^e$, so $\tilde{\phi}$ must vanish on $\Omega^e$ too, and consequently, $\phi|_\Gamma = 0$.

It only remains to prove that the decomposition is stable. For this,

$$\|\phi\|_{H^1(\Omega)} \le \|\tilde{\theta}\|_{H^1(\Omega)} + \|\operatorname{curl}(\mathcal{E}^{\operatorname{curl}}\tilde{\varphi})\|_{H^1(\Omega)} \le \|v\|_{H(\operatorname{div},\Omega)}$$

by (11), Theorem 3.1, and (12). Similarly,

$$\begin{aligned}
\|\varphi\|_{H^1(\Omega)} &\le \|\bar{\varphi}\|_{H^1(\tilde{\Omega})} + \|\mathcal{E}^{\operatorname{curl}}\bar{\varphi}\|_{H^1(\tilde{\Omega})} \\
&\le C(\|\bar{\varphi}\|_{H^1(\tilde{\Omega})} + \|\operatorname{curl}\bar{\varphi}\|_{H^1(\Omega^e)}) && \text{by Theorem 3.1} \\
&\le C(\|\bar{\varphi}\|_{H^1(\tilde{\Omega})} + \|\tilde{\theta}\|_{H^1(\Omega^e)}) && \text{by (13)} \\
&\le C\|v\|_{H(\operatorname{div},\Omega)} && \text{by (11) and (12).}
\end{aligned}$$

Thus $\varphi$ and $\phi$ satisfy all the properties stated in the theorem.

The proof of the other decomposition (8) is similar. □

The topological assumption that $\Omega$ is contractible is used only to convey the simplicity of the idea of the proof. It is possible to prove a more general version of the theorem, accounting for nontrivial harmonic forms. We conclude with the following remarks on the applications of the above decomposition.

*Remark* 3.3 (*Overlapping Schwarz preconditioner).* In [25] we find a decomposition similar to (8) but for the case of boundary conditions on the entire boundary (i.e., the case $\Gamma = \partial\Omega$). This is a critical ingredient in their proof that additive and multiplicative overlapping Schwarz algorithms give uniform preconditioners for the inner product in $H_0(\operatorname{curl}, \Omega)$, even on non-convex domains. Other related works that paved the way for [25] include [2, 19, 31]. In particular [2, 19] proved the uniformity of the preconditioner in the convex domain case. These results were used in [16] to prove that the overlapping Schwarz algorithms give a uniform preconditioner for the indefinite time-harmonic Maxwell equations by a perturbation argument. In view of Theorem 3.2, one can now extend the results of [25] and [16] to the case of *Maxwell equations with mixed boundary conditions* on general domains.

*Remark* 3.4. We note that decompositions similar to (8) were also used in the analysis of the *singular field method* in [5, Proposition 5.1], but again only for the case of boundary conditions on the entire boundary.

*Remark* 3.5. Another application of Theorem 3.2 is in the *characterization of traces* on Lipschitz boundaries. A Hodge decomposition of the space of tangential traces, namely $\operatorname{trc}_\tau(H(\operatorname{curl}, \Omega, \mathbb{R}^3))$, is already known [8] (and such results are useful in the analysis of boundary element methods for Maxwell equations). Theorem 3.2 gives a decomposition (different from the Hodge decomposition) of the traces into a regular and a singular part. Specifically, taking the tangential trace of (8), any $v_\tau \in \operatorname{trc}_\tau(H_{0,\Gamma}(\operatorname{curl}, \Omega))$ can be decomposed into

$$v_\tau = \operatorname{grad}_\tau \xi + \zeta_\tau \tag{15}$$

where $\xi \in H^1_{0,\Gamma}(\Omega, \mathbb{R})$ and $\zeta \in H^1_{0,\Gamma}(\Omega, \mathbb{R}^3)$. The $\zeta_\tau$ part is regular, while the singular part is entirely a surface gradient. Moreover, both components of the decomposition vanish on $\Gamma$. Such decompositions were used (albeit on the surface of a tetrahedron) in [11, 12]. A decomposition of normal traces analogous to (15), but using a surface curl, also follows from Theorem 3.2 (using (7)).

*Remark* 3.6. A *right inverse of the divergence* operator, with mixed boundary conditions, is provided by Theorem 3.2. To see this, first note that given any $z \in L^2(\Omega, \mathbb{R})$, it is easy to see that there exists a $v$ in $H(\operatorname{div}, \Omega, \mathbb{R}^3)$ satisfying

$$\operatorname{div} v = z, \quad \operatorname{trc}_n(v)|_\Gamma = 0, \quad \text{and} \quad \|v\|_{L^2(\Omega)} \le C\|z\|_{L^2(\Omega)}$$

(consider the solution of $\Delta\psi = z$ with mixed boundary conditions $(\partial\psi/\partial n)|_\Gamma = 0$ and $\psi|_{\partial\Omega\setminus\Gamma} = 0$ and set $v = \operatorname{grad}\psi$). When this $v$ is decomposed using (7), the resulting $\phi$ has all its components in $H^1_{0,\Gamma}(\Omega)$ and satisfies

$$\operatorname{div} \phi = z, \quad \phi|_\Gamma = 0, \quad \text{and} \quad \|\phi\|_{H^1(\Omega)} \le C\|z\|_{L^2(\Omega)}. \tag{16}$$

Thus, the map $z \mapsto \phi$ is a regular right inverse of divergence. (Note that (16) can also be proved by other methods.) Right inverses of divergence are fundamental in the study of *Stokes flow* [22, 29]. The above result implies that the Stokes system with no-slip conditions only on $\Gamma$ is well posed. Another application of (16) is in proving the well-posedness of mixed formulations of *linear elasticity with weakly imposed symmetry*. Under purely traction boundary conditions or purely kinematic boundary conditions, a proof of well-posedness can be found in [13]. The same applies almost verbatim for mixed boundary conditions, once (16) is used, in place of the right inverse of divergence used there.

3.2. **Schöberl projectors with partial boundary conditions.** Projectors from Sobolev spaces into finite element spaces with optimal approximation properties find many applications in finite elements. It is well known that every finite element has a canonical projector defined by its degrees of freedom, but this projection is often unbounded in the natural Sobolev space where the solution is sought. This problem was first overcome by the Clément interpolant [10]. Although Clément interpolation yielded operators bounded just in the $L^2$-norm, it had neither the projection property, nor the commutativity with the exterior derivative important in analyses of mixed methods. Clément's idea was substantially generalized by Schöberl in [26, 27, 28] to obtain similar projectors with the additional commutativity properties. These developments are reviewed in [15] where Schöberl's ideas were generalized to weighted norms. We refer to the operators obtained by his method as Schöberl projectors. The importance of these projectors have also been highlighted in a recent review [3] of finite element exterior calculus. They called the projectors bounded cochain projectors, because the spaces formed a cochain complex. Another recent work which refined Schöberl's ideas is [9], where the operators were called smoothed projectors.

However, all these recent works dealt either with the case of no boundary conditions or the case of homogeneous boundary conditions on the entire boundary. With the help of Theorem 2.3, it is easy to generalize their arguments to obtain a Schöberl projector with partial boundary conditions (only on $\Gamma$). (Actually in [28], the partial boundary condition case is considered under the tacit unverified assumption that a result like Theorem 2.3 holds.) We now, very briefly, discuss the case of the projectors with vanishing traces on $\Gamma \subsetneq \partial\Omega$.

Let $\Omega$ be a polyhedron satisfying Assumption 1, meshed by a geometrically conforming tetrahedral finite element mesh $\mathcal{T}_h$. Apply Theorem 2.3 to obtain the associated $\Omega^e$ and $\tilde{\Omega}$. Assume that $\mathcal{T}_h$ is quasiuniform of mesh size $h$. Corresponding to each mesh vertex $x$, we associate a ball $\omega_x$ of radius $h\delta$, where $\delta > 0$ is a global parameter to be chosen shortly. For vertices $x$ in $\bar{\Gamma}$, we choose $\omega_x$ to be centered at some $\tilde{x}$ satisfying $|\tilde{x} - x| \le ch\delta$ (where $c$ is another globally fixed constant) and

$$\omega_x \subset \Omega^e.$$

For all other vertices $x$, the ball $\omega_x$ is centered at $x$.

Now, given $u \in H^1_{0,\Gamma}(\Omega, \mathbb{R})$, $v \in H_{0,\Gamma}(\mathrm{curl}\,, \Omega)$, and $w \in H_{0,\Gamma}(\mathrm{div}, \Omega)$, we extend each by zero to $\Omega^e$ to obtain $\tilde{u}, \tilde{v}$, and $\tilde{w}$ in $\tilde{\Omega}$. By Theorem 2.3, $\tilde{\Omega}$ is a Lipschitz domain. Therefore, we can use the universal extension of Theorem 3.1 to extend these functions to all $\mathbb{R}^3$. Let us denote these extended functions on $\mathbb{R}^3$ by $\hat{u}, \hat{v}$ and $\hat{w}$, resp. We will also consider a function $z \in L^2(\Omega)$ and its trivial extension (by zero) $\hat{z}$ to $L^2(\mathbb{R}^3)$.

Following [28], we now define the smoothing operators. Let $K \in \mathcal{T}_h$ and $x \in K$. Denote by $a_i$ the vertices of $K$. and by $\lambda_i(x)$ the barycentric coordinates at $x$. Define $\tilde{x} \equiv \tilde{x}(x, y_1, y_2, y_3, y_4) \equiv \sum_{i=1}^4 \lambda_i(x) y_i$. Let $\omega = \omega_{a_1} \times \omega_{a_2} \times \omega_{a_3} \times \omega_{a_4}$ and abbreviate the (12 dimensional) measure on this product domain to $dy = \ dy_4 dy_3 dy_2 dy_1$. Let $f_i$ denote a function in $L^\infty(\omega_{a_i})$ such that $\int_{\omega_{a_i}} f_i(y_i) p(y_i)\, dy_i = p(a_i)$ for all polynomials of some fixed degree. Then, setting $\kappa \equiv \kappa(y_1, y_2, y_3, y_4) \equiv f_1(y_1) f_2(y_2) f_3(y_3) f_4(y_4)$, define

$$\begin{aligned}
S^g u(x) &= \int_\omega \kappa\ \hat{u}(\tilde{x})\ dy \\
S^c v(x) &= \int_\omega \kappa\ \left(\frac{d\tilde{x}}{dx}\right)^T \hat{v}(\tilde{x})\ dy, \\
S^d w(x) &= \int_\omega \kappa\ \det\left(\frac{d\tilde{x}}{dx}\right) \left(\frac{d\tilde{x}}{dx}\right)^{-T} \hat{w}(\tilde{x})\ dy \\
S^o z(x) &= \int_\omega \kappa\ \det\left(\frac{d\tilde{x}}{dx}\right) \hat{z}(\tilde{x})\ dy
\end{aligned}$$

for all $x \in K$ and for each $K \in \mathcal{T}_h$.

Next, let $I_h^g, I_h^c, I_h^d$, and $I_h^o$ denote the canonical interpolation operators of the lowest order Lagrange ($U_h$), Nédélec ($V_h$), Raviart-Thomas ($W_h$), and $L^2$-conforming ($Z_h$) finite element spaces. The Schöberl quasi-interpolation operators are now defined by

$$R_h^i = I_h^i \circ S^i, \qquad \text{for } i \in \{g, c, d, o\}.$$

One can then prove, as indicated in [27] (or see more details in [15, Lemma 4.2]), that the operators norms $\|I - R_h^i\|_{L^2(\Omega)} = O(\delta)$. Hence, choosing $\delta$ sufficiently small, the operator $R_h^i$ restricted to the finite element subspace is invertible. Let the inverse be $J_h^i$. The Schöberl projectors are defined by

$$\varPi_h^i = J_h^i \circ R_h^i.$$

As in [27, 28] (or cf. [15, Theorem 5.1]), one can then continue on to prove that these projectors are continuous in the $L^2(\Omega)$-norm, satisfy the commuting diagram

$$\begin{array}{ccccccc}
H^1_{0,\Gamma}(\Omega) & \xrightarrow{\text{grad}} & H_{0,\Gamma}(\text{curl}\,,\Omega) & \xrightarrow{\text{curl}} & H_{0,\Gamma}(\text{div},\Omega) & \xrightarrow{\text{div}} & L^2(\Omega) \\
\Big\downarrow \varPi_h^g & & \Big\downarrow \varPi_h^c & & \Big\downarrow \varPi_h^d & & \Big\downarrow \varPi_h^o \\
U_h & \xrightarrow{\text{grad}} & V_h & \xrightarrow{\text{curl}} & W_h & \xrightarrow{\text{div}} & Z_h,
\end{array}$$

and yield optimal approximation error estimates. This completes our brief sketch of the construction of Schöberl projectors for the partial boundary condition case.

## 4. Proof of Theorem 2.3

This section is devoted to proving Theorem 2.3. The idea is to construct the extended domain by "transporting" the boundary $\Gamma$ outward along a continuous transversal vector field. Technicalities arise when one makes change of variables and exhibits coordinate directions with respect to which the protruded boundary is a Lipschitz hypograph.

Let $\partial D$ be a Lipschitz hypograph near $p$ in the $\hat{u}$-direction, and let $\mathcal{C}_{r,h}$, $\zeta$ and $r$ be as in Definition 2.1. We say that $M$ is the *Lipschitz constant* of the Lipschitz hypograph $\partial D$ if

$$|\zeta(x_1, x_2) - \zeta(y_1, y_2)| \le M\|(x_1, x_2) - (y_1, y_2)\|_2 \tag{17}$$

for all $(x_1, x_2)$ and $(y_1, y_2)$ in $(-r, r)$, i.e., $M$ is the Lipschitz constant of $\zeta$. (Above and throughout $\|\cdot\|_2$ denotes the Euclidean distance.) Let $\gamma_M$ denote the acute angle such that

$$\tan \gamma_M = M \qquad (\gamma_M < \frac{\pi}{2}). \tag{18}$$

Suppose $A$ and $B$ are any two points on $\partial D \cap \mathcal{C}_{r,h}$ (see Figure 2). Then, the line segment $AB$ connecting them has slope bounded by $M$ (for all such $A$ and $B$) if and only if (17) holds.

**Lemma 4.1** (Perturbed coordinate direction)**.** *Suppose $\partial D$ is a Lipschitz hypograph near $p \in \partial D$ in the $\hat{u}$-direction (and let $\gamma_M$ be as above). Let $\hat{v}$ be a unit vector such that*

$$\sin \gamma_M < \hat{u} \cdot \hat{v}. \tag{19}$$

*Then $\partial D$ is a Lipschitz hypograph near $p$ in the $\hat{v}$-direction.*

*Proof.* Let $\theta$ and $\theta_1$ denote the acute angles such that $\sin\theta = \hat{u}\cdot\hat{v}$ and $\tan\theta_1$ equals the slope of $AB$, respectively (considering any two points $A$ and $B$ as mentioned above – see also Figure 2). Let $\theta_2$ denote the smaller of the angles that $AB$ makes with $\hat{v}$. Then (19) implies $\gamma_M < \theta \le \theta_1 + \theta_2$ while (17) implies $\theta_1 \le \gamma_M$. Hence $\theta_2 > \gamma_M - \theta_1 \ge 0$, i.e.,

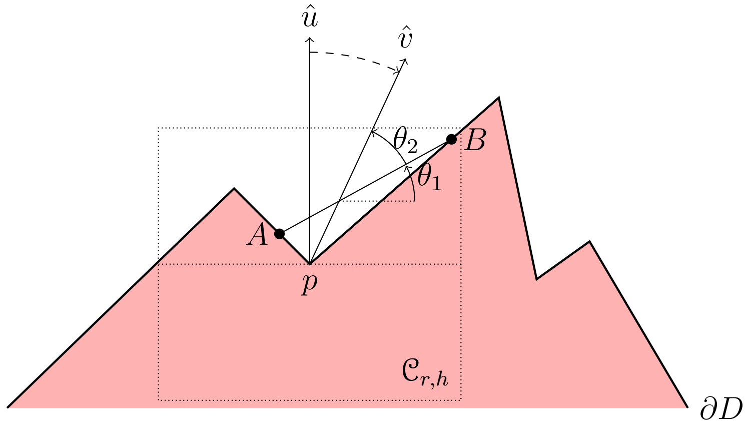


FIGURE 2. A 2D illustration. Note that (17) is equivalent to $\theta_1 \le \gamma_M$.

there is a $\underline{\theta} > 0$ such that $\theta_2 \ge \underline{\theta}$ for all $A$ and $B$ in a neighborhood of $p$. This implies that $\partial D$ is still a Lipschitz hypograph near $p$ in the direction of $\hat{v}$ with Lipschitz constant $\tan(\pi/2 - \underline{\theta})$. □

Next, let us recall the well known fact that a domain is Lipschitz if and only if it satisfies the uniform cone property, which we now state.

*Definition* 4.1 (Cone property). Let $D$ be an open subset of $\mathbb{R}^3$. We say that $D$ has the *cone property at $p \in \partial D$ in the direction $\hat{u}$* if there are (i) new coordinates $(y_1, y_2, y_3)$, where the $y_3$-direction is $\hat{u}$, (ii) a hypercube

$$V = \{(y_1, y_2, y_3) : -a_j \le y_j \le a_j, 1 \le j \le 3\},$$

and (iii) constants $\theta \in (0, \pi/2)$ and $h > 0$, and a corresponding open cone $K_{\theta,h,\hat{u}} = \{y : (y_1^2 + y_2^2)^{1/2} < y_3(\tan\theta) < h(\tan\theta)\}$, such that

$$y - z \in D \text{ whenever } y \in \overline{D} \cap V \text{ and } z \in K_{\theta,h,\hat{u}}.$$

We say that $D$ has the *uniform cone property* if every point $p$ on $\partial D$ satisfies the cone property (in some direction) with the same $\theta$ and $h$.

This definition can be found in [17, Definition 1.2.2.1], and so can the following theorem. The statement of the theorem in [17, Theorem 1.2.2.2] assumes boundedness of $D$. Following that proof we however find that boundedness of $D$ is unnecessary. We only need to assume that $\partial D$ is compact.

**Theorem 4.2** (see Theorem 1.2.2.2 in [17]). *Let $D$ be an open subset of $\mathbb{R}^3$ with compact boundary $\partial D$. It has the cone property in $\hat{u}$-direction at some $p \in \partial D$ if and only if $\partial D$ is a Lipschitz hypograph in the $\hat{u}$-direction near $p$. Moreover, $D$ is Lipschitz if and only if $D$ has the uniform cone property.*

**Lemma 4.2** (Combination of directions). *Let $\nu$ denote the unit outward normal on $\partial D$ at $p \in \partial D$. If $\partial D$ is a Lipschitz hypograph near some $p \in \partial D$ in both the directions $\hat{u}$ and $\hat{v}$, and*

$$(20) \qquad \hat{u} \cdot \nu > \kappa \quad \text{and} \quad \hat{v} \cdot \nu > \kappa, \qquad \text{for some } \kappa > 0,$$

*a.e. on a neighborhood of $p$ on $\partial D$, then $\partial D$ is also a Lipschitz hypograph near $p$ in the direction $\hat{w} = a\hat{u} + b\hat{v}$ for any $a, b > 0$.*

*Proof.* We use Theorem 4.2. By reducing, if necessary, the neighborhoods given by the cone property in $\hat{u}$ and $\hat{v}$ directions, we can find a single hypercube $V$ containing $p$ such that for any $q \in \overline{D} \cap V$, we have $q - K_{\theta,h,\hat{u}} \subset D$ and $q - K_{\theta,h,\hat{v}} \subset D$. Let $\mathcal{C}_{r,h}$ be the coordinate box at $p$ in the $\hat{u}$-direction. We choose $V$ so small that the sets $q - K_{\theta,h,\hat{u}}$, $q - K_{\theta,h,\hat{v}}$, and $q - K_{\theta,h,\hat{w}}$ are all contained in $\mathcal{C}_{r,h}$ for all $q \in \bar{D} \cap V$.

We now claim that $D$ satisfies the cone property in the $\hat{w}$-direction at $p$. If not, then by Definition 4.1, for every neighborhood $V_n$, cone angle $\theta_n = \theta/n$, and cone height $h_n = h/n$, there is a point $y_n \in V_n \cap \overline{D}$ such that $y_n - K_{\theta_n,h_n,\hat{w}} \not\subseteq D$. Here we choose $V_n$ to be a hypercube of side-lengths $1/n$ centered at $p$. Thus, there exists $q_n$ in $y_n - K_{\theta_n,h_n,\hat{w}}$ such that

$$q_n \notin D, \tag{21}$$

and $q_n$ converges to $p$.

Now, by (20), $\hat{u}$ and $\hat{v}$ point above the hypograph in $\mathcal{C}_{r,h}$, and since $a, b > 0$, so does $\hat{w}$. Since the heights and angles of the cones $y_n - K_{\theta_n,h_n,\hat{w}}$ approach 0 and since their vertices approach $p$, we find that for sufficiently large $n$, the cone $y_n - K_{\theta_n,h_n,\hat{w}} \subseteq \mathcal{C}_{r,h} \cap D$ and $q_n \in D$. But this is in contradiction with (21). □

**Lemma 4.3** (Separation of transported points)**.** *Let $u = (u_1, u_2, u_3)$. Suppose $u_j$ and $\zeta$ are Lipschitz functions on $(-r, r)^2$ for some $r > 0$, $u_1(0) = u_2(0) = 0$, and $u_3(0) \neq 0$. Let $(y_1, y_2)$ be mapped to*

$$L(y_1, y_2, s) = (y_1, y_2, \zeta(y_1, y_2)) + s\, u(y_1, y_2).$$

*Then there exists $0 < r_0 < r$ and $C > 0$ (depending on $\zeta$ and $u$) such that*

$$\|L(y_1, y_2, s) - L(z_1, z_2, t)\|_2 \geq C\|(y_1, y_2, s) - (z_1, z_2, t)\|_2$$

*for all $(y_1, y_2, s)$ and $(z_1, z_2, t)$ in the cube $(-r_0, r_0)^3$.*

*Proof.* Let $M$ be the maximum of the Lipschitz constants of $u_j$ and $\zeta$. Then, denoting by $[\cdot]_j$ the $j$th component, we have, for small enough $|s|$,

$$\big|\, [L(y_1, y_2, s) - L(z_1, z_2, s)]_3 \,\big| \leq\ 2M\|(y_1, y_2) - (z_1, z_2)\|_2, \tag{22a}$$

$$\|\ell(y_1, y_2, s) - \ell(z_1, z_2, s)\|_2 \geq (3/4)\|(y_1, y_2) - (z_1, z_2)\|_2, \tag{22b}$$

where $l(y_1, y_2, s) = ([L(y_1, y_2, s)]_1, [L(y_1, y_2, s)]_2)$. Let $c_0 = |u_3(0)|/16M > 0$. Since

$$\|u(z_1, z_2) - (0, 0, u_3(0))\|_2^2 \leq 3M^2\|(z_1, z_2)\|_2^2,$$

for small enough $\|(z_1, z_2)\|_2$, we have

$$|(s - t)\, u_3(z_1, z_2)| \geq \frac{3}{4}|u_3(0)|\, |s - t|, \tag{22c}$$

$$\big|(s - t)\, u_j(z_1, z_2)\big| \leq c_0|s - t|, \text{ for } j = 1, \text{ and } 2, \tag{22d}$$

for any $s, t \in (-r, r)$. In view of these, there exists an $0 < r_0 < r$ such that all the inequalities of (22) hold for any $(y_1, y_2, s)$ and $(z_1, z_2, t)$ in $(-r_0, r_0)^3$. The remainder of the proof splits into two cases:

*Case 1:* $|s-t| \le \|(y_1,y_2)-(z_1,z_2)\|_2/(2\sqrt{2}c_0)$. In this case, (22d) implies

$$|(s-t)\,u_j(z_1,z_2)| \le \frac{1}{2\sqrt{2}}\|(y_1,y_2)-(z_1,z_2)\|_2. \tag{23}$$

Then, we have

$$\begin{aligned}\|L(y_1,y_2,s)-L(z_1,z_2,t)\|_2 &\ge \|l(y_1,y_2,s)-l(z_1,z_2,t)\|_2\\ &\ge \|l(y_1,y_2,s)-l(z_1,z_2,s)\|_2 - \|l(z_1,z_2,s)-l(z_1,z_2,t)\|_2\\ &\ge (\frac{3}{4}-\frac{1}{2})\|(y_1,y_2)-(z_1,z_2)\|_2,\end{aligned}$$

by (22b) and (23). This proves the result in Case 1.

*Case 2:* $|s-t| \ge \|(y_1,y_2)-(z_1,z_2)\|_2/(2\sqrt{2}c_0)$. We now estimate using the last component of $L$, namely

$$\begin{aligned}\|L(y_1,y_2,s)-L(z_1,z_2,t)\|_2 &\ge \big|[L(z_1,z_2,t)-L(y_1,y_2,s)]_3\big|\\ &\ge \big|[L(z_1,z_2,t)-L(z_1,z_2,s)]_3\big| - \big|[L(z_1,z_2,s)-L(y_1,y_2,s)]_3\big|\\ &= |(s-t)u_3(z_1,z_2)| - \big|[L(z_1,z_2,s)-L(y_1,y_2,s)]_3\big|\end{aligned}$$

Now, by (22a), and the inequality of Case 2, we have

$$\big|[L(z_1,z_2,s)-L(y_1,y_2,s)]_3\big| \le 2M\|(y_1,y_2)-(z_1,z_2)\|_2 \le 4\sqrt{2}Mc_0|s-t|.$$

Hence, using (22c) and the definition of $c_0$, we obtain

$$\|L(y_1,y_2,s)-L(z_1,z_2,t)\|_2 \ge (\frac{3}{4}-\frac{\sqrt{2}}{4})|u_3(0)|\;|s-t|.$$

Using the inequality of Case 2 to bound $|s-t|$ from below again, we finish the proof. $\square$

*Remark* 4.3. Note that Lemma 4.3 and its proof in fact holds more generally in $n+1$ space dimension, for any $n \ge 1$, for maps

$$L(y_1,\dots,y_n,s) = (y_1,\dots,y_n,\zeta(y_1,\dots,y_n)) + s\,u(y_1,\dots,y_n)$$

satisfying $u_j(0)=0$ for all $j=1,\dots,n$, but $u_{n+1}(0)\ne 0$. We only described it above for the $n=2$ case for simplicity. In the remainder of the paper, we will use it with $n=1$ and 2.

Now, let $D$ be a Lipschitz domain in $\mathbb{R}^3$. Then there exist finitely many coordinate boxes $\{\mathcal{C}_{r_j,h_j}\}_{j=1}^N$ such that

$$\partial D \subset \bigcup_{j=1}^N \mathcal{C}_{r_j,h_j}. \tag{24}$$

Moreover, there exists [23] a partition of unity $\{\psi_j\}_{j=1}^N$ subordinate to $\{\mathcal{C}_{r_j,h_j}\}_{j=1}^N$ with $\sum_{j=1}^N \psi_j = 1$ on $\partial D$. We denote by $\nu_j$ the direction of $\mathcal{C}_{r_j,h_j}$ for any $1 \le j \le N$ (where the "direction" of the box is as in Definition 2.1). We define a vector field

$$\hat{v}' := \sum_{j=1}^N \psi_j \nu_j. \tag{25a}$$

Then $\hat{v}'$ is nonzero on $\partial D$. Hence, we may normalize and define

$$\hat{v} := \frac{\hat{v}'}{|\hat{v}'|}. \tag{25b}$$

The following lemma proves that this yields a continuous transversal vector field with an additional property we shall need later. The arguments are standard (see e.g., [17, 21]) and we give the proof only for completeness.

**Lemma 4.4** (Transversal vector field)**.** *Let $D$ be Lipschitz. The unit vector field $\hat{v}$ defined by* (25b) *satisfies the following properties:*

(1) *It is tranversal, i.e., if $\nu$ is the outward unit normal on $\partial D$, there is a constant $\kappa > 0$ such that $\hat{v} \cdot \nu > \kappa$ a.e. on $\partial D$.*
(2) *If $p_i$'s are the exceptional points in the piecewise $C^1$ dissection (see Definition 2.2), then $\hat{v}$ is constant on a neighborhood of each $p_i$.*
(3) *In a neighborhood of any point $p \in \partial D$, the boundary is a Lipschitz hypograph in the $\hat{v}(p)$-direction.*

*Proof.* In each of the coordinate boxes in (24), there is a $\kappa_j > 0$ such that $\nu_j \cdot \nu > \kappa_j$. Set $\kappa = \min_j \kappa_j$. Then

$$\hat{v}' \cdot \nu = \sum_{j=1}^{N} \psi_j \nu_j \cdot \nu \geq \kappa \sum_{j=1}^{N} \psi_j = \kappa$$

a.e. on $\partial D$.

To ensure that $\hat{v}$ is constant in the neighborhood of each $p_i$, we choose the covering of boxes in (24) as follows. We first select the coordinate boxes around each $p_i$. Next, we construct an open cover for the remainder such that the distances between $p_i$ and the open sets of this covering are bounded away from 0. A covering of $\partial D$ is obtained by the union of this cover and the coordinate boxes of $p_i$. Then we use a partition of unity subordinate to this union to define (24). Then, only one of the summands in (25a) is nonzero (say the $j$th) in a neighborhood of $p_i$ and $\psi_j$ is constant there.

The final statement, item (3), immediately follows from Lemma 4.2. □

*Remark* 4.4. Although $\hat{v}'$ is obviously a smooth vector field in a three-dimensional neighborhood of $\partial D$, below we will need to use the two-dimensional restriction $\breve{v} \equiv \hat{v}|_{\partial D}$, which is *not* smooth, in general. Indeed, in each coordinate box where $\partial D$ takes the form $(x_1, x_2, \zeta(x_1, x_2))$, this vector field on $\partial D$ is

$$\breve{v}(x_1, x_2) = \hat{v}(x_1, x_2, \zeta(x_1, x_2)),$$

which only has Lipschitz regularity. To avoid proliferation of notations, we will avoid using $\breve{v}$ and continue to denote various restrictions of $\hat{v}$ by $\hat{v}$ itself.

Using the above defined $\hat{v}$, we can transport the entire boundary $\partial D$ to create a new domain. Namely, define

$$\Sigma_{t^-,t^+} := \{p + s\hat{v}(p) : p \in \partial D,\ t^- < s < t^+\}, \qquad \text{for any } t^- < t^+. \tag{26}$$

**Lemma 4.5** (Expansion of the entire boundary)**.** *Let $D$ be a Lipschitz domain and $\hat{v}$ be as above. Then there exists $t_0 > 0$ such that:*

(1) *For any $p, q \in \partial D$ and any $s_1, s_2 \in [-t_0, t_0]$,*

$$p + s_1\hat{v}(p) \neq q + s_2\hat{v}(q) \qquad \textit{whenever } (p, s_1) \neq (q, s_2).$$

(2) *Let $-t_0 \le t^- < t^+ \le t_0$. For any $q \in \partial\Sigma_{t^-,t^+}$, the boundary $\partial\Sigma_{t^-,t^+}$ is a Lipschitz hypograph near $q$ in the direction $\hat{v}(p)$. Moreover, $\partial\Sigma_{t^-,t^+}$ is compact. Consequently $\Sigma_{t^-,t^+}$ is a Lipschitz domain.*

*Proof.* Let $p \in \partial D$ and let $\mathcal{C}_{r,h}$ be the coordinate box in the $\hat{v}(p)$-direction near $p$. There is a Lipschitz function $\zeta$ such that $\partial D$ around $p$ can be parameterized by the mapping $(x_1, x_2) \to (x_1, x_2, \zeta(x_1, x_2))$ in orthogonal coordinates $(x_1, x_2, x_3)$.

*Case 1:* We first show that the item (1) holds, *assuming* that $q \in \partial D$ is in the same coordinate box $\mathcal{C}_{r,h}$ as $p$. Then, since $\zeta$ and $\hat{v}$ are Lipschitz functions of $x_1, x_2$, we can apply Lemma 4.3 (with $u$ there set to $\hat{v}$) to get that the distance between $p + s_1\hat{v}(p)$ and $q + s_2\hat{v}(q)$ is bounded below by $C\|(p, s_1) - (q, s_2)\|_2$. Hence this distance cannot be zero.

*Case 2:* Now we prove the item (1) in general. If the result is not true, then for any $t^- < t^+$, there exists $p, q \in \partial D$ and $s, t \in [t^-, t^+]$ such that $p + s\hat{v}(p) = q + t\hat{v}(q)$ and $p \ne q$. (If $p = q$, then we fall into Case 1 and the proof is done.) This implies (choosing $t_\pm = \pm 1/i$), that there are $\{p_i\}_{i=1}^\infty, \{q_i\}_{i=1}^\infty \in \partial D$ and $s_i, t_i \in [-1/i, 1/i]$ such that $p_i \ne q_i$ for any $i$ and $p_i + s_i\hat{v}(p_i) = q_i + t_i\hat{v}(q_i)$. Since $\partial D$ is compact, a subsequence of $\{(p_i, q_i)\}_{i=1}^\infty$ converges. This implies (since $s_i, t_i \to 0$) that there exists an $i$, large enough, such that $p_i$ and $q_i$ are in the same coordinate box. Then, by Case 1, it is impossible that $p_i + s_i\hat{v}(p_i) = q_i + t_i\hat{v}(q_i)$, a contradiction.

To prove the second item, we start by noting that by virtue of the first item, any point on $\partial\Sigma_{t^-,t^+}$ can be written (uniquely) as $p + t_0\hat{v}(p)$ for some $p \in \partial D$. With $\mathcal{C}_{r,h}$ and $(x_1, x_2, \zeta(x_1, x_2))$ as in the beginning of this proof, define new coordinates

$$X(x_1, x_2) = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \equiv \begin{bmatrix} x_1 + t_0\hat{v}_1(x_1, x_2) \\ x_2 + t_0\hat{v}_2(x_1, x_2) \end{bmatrix}$$

Clearly, by choosing $t_0$ small enough, we can ensure that

$$\|X(x_1, x_2) - X(y_1, y_2)\|_2 \ge C\|(x_1, x_2) - (y_1, y_2)\|_2, \tag{27}$$

so the inverse map $T = X^{-1}$ exists, i.e., $T(X_1, X_2) = (x_1, x_2)$.

To prove that $\partial\Sigma_{t^-,t^+}$ is a Lipschitz hypograph near $p + t_0\hat{v}(p)$, we now only need to show that the third component $Z = \zeta(x_1, x_2) + t_0\hat{v}_3(x_1, x_2)$ is a Lipschitz function of the new coordinates $X_1$ and $X_2$. In these variables,

$$Z = \zeta \circ T + t_0\hat{v}_3 \circ T. \tag{28}$$

Since $\zeta$ and $\hat{v}_3$ are Lipschitz, and since $T$ is Lipschitz by (27), we conclude from (28) that $Z$ is also Lipschitz, hence $\partial\Sigma_{t^-,t^+}$ is a Lipschitz hypograph in the $\hat{v}$-direction. Obviously, $\partial\Sigma_{t_-,t_+}$ is also compact. $\square$

Next, given a $C^1$ dissection $\partial D = \Gamma_1 \cup \Pi \cup \Gamma_2$ of a Lipschitz boundary $\partial D$, we define, for $t \in \mathbb{R}$, the following sets of transported points

$$S_t := \{p + t\hat{v}(p) : \forall p \in \partial D\}, \qquad S_{1,t} := \{p + t\hat{v}(p) : \forall p \in \Gamma_1\},$$
$$\Pi_t := \{p + t\hat{v}(p) : \forall p \in \Pi\}, \qquad S_{2,t} := \{p + t\hat{v}(p) : \forall p \in \Gamma_2\}.$$

**Lemma 4.6** (Transported dissection)**.** *Let $\partial D = \Gamma_1 \cup \Pi \cup \Gamma_2$ be a piecewise $C^1$ dissection of a Lipschitz $\partial D$ and let $\hat{v}$ be transversal field of Lemma 4.4. Then, there exists $t_0 > 0$ such that for any $-t_0 \le t \le t_0$, $S_t = S_{1,t} \cup \Pi_t \cup S_{2,t}$ is a piecewise $C^1$ dissection of $S_t$. The only points where $\Pi_t$ is not $C^1$ are $\{p_i + t\hat{v}(p_i)\}_{i=1}^m$.*

*Proof.* By Lemma 4.5, there is a $t_0$ such that for all $t \in [-t_0, t_0]$, the surface $S_t$ is a Lipschitz hypograph at $p + t\hat{v}(p)$ in the $\hat{v}(p)$-direction for any $p \in \partial D$.

To show that $\Pi_t$ is piecewise $C^1$ dissection, we first consider the exceptional points $p_i$ of $\Pi$. The coordinate boxes near $p_i$ may simply be translated to form coordinate boxes around $p_i + t\hat{v}(p_i)$ because $\hat{v}$ is a constant vector field near $p_i$ (by Lemma 4.4). Since $\Pi_t$ is a merely a translation of $\Pi$, there is nothing to prove at these points.

Next, consider the remaining $p \in \Pi$. In a neighborhood of $p + t\hat{v}(p)$, the transported curve $\Pi_t$ is $C^1$. This is because $\hat{v}|_\Pi$ is $C^1$ (and $\hat{v}$ is globally smooth), so $\Pi_t$ is locally the image of a $C^1$ curve under a $C^1$ map. Next, let $w$ denote the tangent vector of $\Pi_t$ at $p+t\hat{v}(p)$. We construct a coordinate box around the $p+t\hat{v}(p)$ as follows: The $x_3$-direction is provided by $\hat{v}(p)$. The $x_1$-direction is provided by the projection $\hat{u} = w - (w \cdot \hat{v}(p))\hat{v}(p)$ (and the $x_2$-direction is then determined). Then (because $\Pi_t$ can be locally parametrized using its tangent $w$) it is easy to see that $\Pi_t$ can be parametrized using a $C^1$ function of the new $x_1$ coordinate. □

Now, we are ready to define the partial expansion we need. Let $\partial D = \Gamma_1 \cup \Pi \cup \Gamma_2$ be a piecewise $C^1$ dissection of a Lipschitz $\partial D$ and $\hat{v}$ be the vector field of Lemma 4.4, define

$$D_t^e = \{p + s\hat{v}(p) : p \in \Gamma_1, s \in (0, t)\}. \tag{29}$$

Additionally let

$$\Psi_t = \{p + s\hat{v}(p) : p \in \Pi, 0 < s < t\}, \qquad \Gamma_{1,t} = \{p + t\hat{v}(p) : p \in \Gamma_1\}.$$

Clearly, for any $0 < t \le t_0$, $D_t^e$ is an open subset of $\mathbb{R}^3$ and $\partial D_t^e = \Gamma_1 \cup \Gamma_{1,t} \cup \Psi_t \cup \Pi \cup \Pi_t$.

**Lemma 4.7** (The protrusion is Lipschitz)**.** *Let $D$ be a Lipschitz domain and let $\partial D = \Gamma_1 \cup \Pi \cup \Gamma_2$ be a piecewise $C^1$ dissection. Then, there exists $t_0 > 0$ such that for any $0 < t \le t_0$, the domain $D_t^e$ in* (29) *is a Lipschitz domain.*

*Proof.* We prove the Lipschtizness of each of the components in the decomposition $\partial D_t^e = \Gamma_1 \cup \Gamma_{1,t} \cup \Psi_t \cup \Pi \cup \Pi_t$.

Obviously $\Gamma_1$, being part of $\partial D$, is Lipschitz (since a Lipschitz function remains Lipschitz when the $x_3$-direction is reversed). That the surface $\Gamma_{1,t}$ is locally Lipschitz at all of its (interior) points follows from Lemma 4.5. Hence it suffices to consider points in the remaining components $\Psi_t, \Pi$, and $\Pi_t$.

To prove that $\Psi_t$ is a Lipschitz hypograph near a point $q \in \Psi_t$, we first note that $q \in \Pi_{t'}$ for some $0 < t' < t$. By Lemma 4.6, the curve $\Pi_{t'}$ is a piecewise $C^1$-dissection of $S_{t'}$. Then, reviewing the proof of Lemma 4.6, we find that there is a coordinate box $\mathcal{C}_{r,h}$ near $q$ in the $\hat{v}(q)$-direction, where $\hat{v}$ is same transversal vector field given by Lemma 4.4, such that $\Pi_{t'} \cap \mathcal{C}_{r,h}$ in the local coordinates $(x_1, x_2, x_3)$ takes the form $(x_1, \rho(x_1), \zeta(x_1, \rho(x_1))$ (see Definition 2.2). This means that near $q$, the surface $\Psi_t$ can be parametrized by

$$(x_1, s) \mapsto X(x_1, s) \stackrel{\text{def}}{=} (x_1 + s\hat{v}_1, \rho(x_1) + s\hat{v}_2, \zeta(x_1, \rho(x_1)) + s\hat{v}_3) \tag{30}$$

where $\hat{v}_j$ is the $j$th component of $\hat{v}(x_1, \rho(x_1), \zeta(x_1, \rho(x_1))$. Clearly, each $\hat{v}_j$ is Lipschitz. We apply Lemma 4.3 in $\operatorname{span}(\hat{v}_1, \hat{v}_3)$ with $L = (X_1, X_3) \equiv X_{13}(x_1, s) \equiv (x_1, \zeta(x_1, \rho(x_1))) + s(\hat{v}_1, \hat{v}_3)$. (Note that the lemma is applicable in two-dimensions also – see Remark 4.3.) Thus we obtain

$$\|X_{13}(x_1, s) - X_{13}(x_1', s')\|_2 \ge C\|(x_1, s) - (x_1', s')\|_2. \tag{31}$$

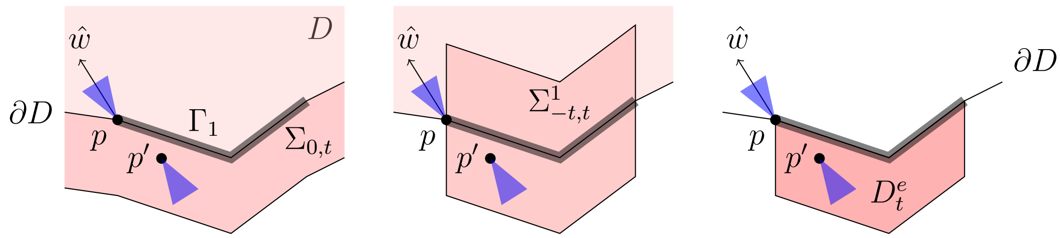


FIGURE 3. The uniform cone property in the $\hat{w}$-direction (near $p \in \Pi$) holds for the domains $\Sigma_{0,t}$ (left) and $\Sigma^1_{-t,t}$ (center), so it holds for the intersection $D^e_t$ (right).

Hence there is a Lipschitz inverse map $T = X_{13}^{-1}$ such that $T_1(X_1, X_3) = x_1$ and $T_2(X_1, X_3) = s$. We may therefore write $X_2$ in terms of $X_1$ and $X_3$, i.e., $X_2 = \rho(x_1) + s\hat{v}_2 = \rho \circ T_1 + T_2 \hat{v}_2(T_1, \rho \circ T_1, \zeta(T_1, \rho \circ T_1)) \equiv Z(X_1, X_3)$. Clearly $Z$ is a Lipschitz function of $X_1$ and $X_3$. Consequently, we may rewrite the surface representation in (30) using new independent variables $X_1$ and $X_3$ (keeping the same coordinate directions) as

$$(X_1, X_3) \mapsto (X_1, Z(X_1, X_3), X_3).$$

This proves that $\Psi_t$ is a Lipschitz hypograph near $q$ *in the $x_2$-direction.* (Without loss of generality, we choose the sign so that the $x_2$-direction points outward of $\Psi_t$.)

It now only remains to consider points on $\Pi$ and $\Pi_t$. We will only consider $p \in \Pi$ as the other case is similar. We will use the fact that

$$D^e_t = \Sigma_{0,t} \cap \Sigma^1_{-t,t} \tag{32}$$

where

$$\Sigma^1_{-t,t} = \{q + s\hat{v}(q) : q \in \Gamma_1, s \in (-t, t)\},$$
$$\Sigma_{0,t} = \{q + s\hat{v}(q) : q \in \partial D, s \in (0, t)\}.$$

Clearly $p \in \partial\Sigma^1_{-t,t} \cap \partial\Sigma_{0,t}$. We will prove that $\partial D^e_t$ is a Lipschitz hypograph near $p \in \Pi$ in four steps:

*Step 1.* We claim that if there is a direction $\hat{w}$ such that *both* $\partial\Sigma^1_{-t,t}$ and $\partial\Sigma_{0,t}$ are Lipschitz hypographs near $p \in \Pi$ in the $\hat{w}$-direction, then the $D^e_t$ is also a Lipschitz hypograph near $p$ in the $\hat{w}$-direction.

To prove this claim, we use the uniform cone property. By Theorem 4.2, there is a neighborhood $V_1$ of $p$ and a cone $K_{\theta_1,h_1,\hat{w}}$ such that $p' - K_{\theta_1,h_1,\hat{w}} \subseteq \Sigma^1_{-t,t}$ for all $p' \in V_1 \cap \bar{\Sigma}^1_{-t,t}$ (see Figure 3). Similarly there is a neighborhood $V_2$ and a cone $K_{\theta_2,h_2,\hat{w}}$ such that $p' - K_{\theta_2,h_2,\hat{w}} \subseteq \Sigma_{0,t}$ for all $p' \in V_2 \cap \bar{\Sigma}_{0,t}$. Hence considering a smaller hypercube $V$ in the intersection $V_1 \cap V_2$, in view of (32), we find that for all $p' \in V \cap D^e_t$, we have $p' - K \subset D^e_t$, where $K$ is the smaller of the two cones. Thus $D^e_t$ satisfies the uniform cone property in the $\hat{w}$-direction and the claim follows.

*Step 2.* We now find a $\hat{w}$ such that $\partial\Sigma_{0,t}$ is a Lipschitz hypograph near $p$ in the $\hat{w}$-direction.

We know that $\partial D$ is a Lipschitz hypograph near $p$ in the $-\hat{v}(p)$ direction, which we now take to be our local $x_3$-direction – see Figure 4. Let $\tan\gamma_M$ be the corresponding

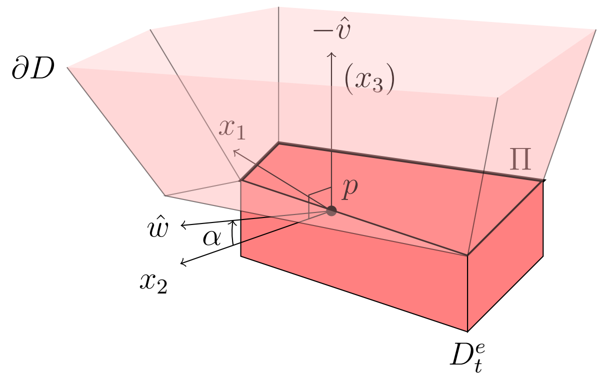


FIGURE 4. Determining the direction vector $(\hat{w})$ for the Lipschitz hypograph near $p \in \Pi$.

Lipschitz constant (as in (18)). Define

$$\hat{w} = (0, \cos\alpha, \sin\alpha), \qquad \text{where} \qquad \alpha = \frac{1}{2}\left(\frac{\pi}{2} + \gamma_M\right). \tag{33}$$

Clearly, by construction, $0 < \gamma_M < \alpha < \pi/2$. Hence, (19) is satisfied since $\sin\gamma_M < \sin\alpha$. By Lemma 4.1, we conclude that $\partial\Sigma_{0,t}$ is a Lipschitz hypograph near $p$ in the $\hat{w}$-direction.

*Step 3.* Suppose $p$ is not one of the exceptional points $p_i$. Then we claim that $\partial\Sigma^1_{-t,t}$ is a Lipschitz hypograph near $p$ in the $\hat{w}$-direction (with the same $\hat{w}$ as in (33)).

To prove this, we recall that $\partial\Sigma^1_{-t,t}$ can locally be represented by the same map as in (30). However, this time, the components are not merely Lipschitz: they are in fact $C^1$. This is because, near $p$, the curve $(x_1, \rho(x_1), \zeta(x_1, \rho(x_1))$ is $C^1$, so each of the components must be $C^1$. Moreover, since $\hat{v}_j$ is the $j$th component of the globally smooth $\hat{v}$ along the $C^1$ curve, it is also $C^1$. Thus the map (30) defines a $C^1$ surface near $p$, so in particular, $\partial\Sigma^1_{-t,t}$ is Lipschitz.

It only remains to verify that $\partial\Sigma^1_{-t,t}$ is a Lipschitz hypograph *in the $\hat{w}$-direction*. Standard geometrical arguments can be used to show this. But to be self-contained, we give a proof: The vector $\hat{v}(p)$, which in local coordinates is $(0,0,1)$, together with the tangent to the curve $\Pi$ at $p$, which we denote by $(a,b,c)$, span the tangent plane of the $C^1$ surface $\partial\Sigma^1_{-t,t}$ at $p$. Moreover, due to the representation (30), $a \neq 0$. Hence the normal vector $n$ to $\partial\Sigma^1_{-t,t}$ is in the direction $(0,0,1)\times(a,b,c)$. In particular, $|n\cdot\hat{w}| = |a\cos\alpha| \neq 0$. Now, since $\partial\Sigma^1_{-t,t}$ is $C^1$, in the coordinate system $(z_1, z_2, z_3)$, with $p$ as the origin, and with $z_3$-direction equal to the $n$-direction, the surface $\partial\Sigma^1_{-t,t}$ can be parametrized a $(z_1, z_2, \eta(z_1, z_2))$ for a $C^1$ function $\eta$ such that $\eta(z_1,z_2) = \lambda_1 z_1^2 + \lambda_2 z_2^2 + o(\|(z_1,z_2)\|_2^2)$. This implies that the Lipschitz constant $M$ in these coordinates can be made arbitrarily small by considering a small enough neighborhood. Therefore we can apply Lemma 4.1 (with $\hat{u} = n$ and $\hat{v} = \hat{w}$). Since $|n\cdot\hat{w}| > 0$, by choosing a small enough neighborhood, condition (19) can be satisfied. We conclude that $\partial\Sigma^1_{-t,t}$ is a Lipschitz hypograph in the $\hat{w}$-direction.

*Step 4.* If $p$ coincides with one of the exceptional points $p_i$, then $\partial\Sigma^1_{-t,t}$ is a Lipschitz hypograph near $p$ in the $\hat{w}$-direction.

To prove this, we recall that in a neighborhood of the exceptional points, the vector field $\hat{v}$ is designed to be constant (Lemma 4.4). In the local coordinates, this constant vector is $\hat{v} = (0,0,1)$. Hence the parametric representation of the surface $\partial\Sigma^1_{-t,t}$ in (30) takes the form

$$X(x_1, s) = (X_1, X_2, X_3) = (x_1, \rho(x_1), \zeta(x_1, \rho(x_1)) + s).$$

Applying Lemma 4.3 to $X_{13}(x_1, s) = (x_1, \zeta(x_1, \rho(x_1))) + s(0, 1)$, we find, as before, that (31) holds. Hence, as before, $X_{13}$ is invertible and the change of variable $(x_1, s) \mapsto (X_1, X_3)$ is locally one-one. Reparametrizing in new $(X_1, X_2, X_3)$-variables (but keeping the old coordinate directions) the surface takes the form

$$(X_1, \rho(X_1), X_3). \tag{34}$$

Since $\rho$ is Lipschitz, this means that $\partial\Sigma^1_{-t,t}$ is a Lipschitz hypograph near $p$ in the $x_2$-direction.

This fact can be used to show that it is also a Lipschitz hypograph near $p$ in the $\hat{w}$-direction. Denote the unit vectors in the $x_i$-coordinate direction by $e_i$. We rotate the coordinate directions in the $e_2$-$e_3$ plane, about the $e_1$-axis, by the angle $\alpha$, to get new orthogonal coordinate directions $e_i'$. Clearly, $e_2' = \hat{w}$ due to (33). If the coordinates $x_i'$ are such that $x_1'e_1' + x_2'e_2' + x_3'e_3' = X_1e_1 + X_2e_2 + X_3e_3$, then

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}.$$

Using this to map (34), we conclude that the points on surface $\partial\Sigma^1_{-t,t}$ near $p$ takes the form $x_1'e_1' + x_2'e_2' + x_3'e_3'$ with

$$x_1' = X_1, \qquad x_2' = \rho(X_1)\cos\alpha + X_3\sin\alpha, \quad x_3' = -\rho(X_1)\sin\alpha + X_3\cos\alpha.$$

We will now make one more change of variables: We can apply Lemma 4.3 to the two-dimensional map

$$(X_1, X_3) \mapsto (X_1', X_3') \equiv (X_1, -\rho(X_1)\sin\alpha) + X_3(0, \cos\alpha),$$

because $\cos\alpha \neq 0$. As a result, this change of variable is locally one-one, so $X_3$ can be expressed as a Lipschitz function of $X_1'$ and $X_3'$, namely $X_3 \equiv X_3(X_1', X_3')$. This means that in the $\{e_1', e_2', e_3'\}$-coordinate system, the surface $\partial\Sigma^1_{-t,t}$ near $p$ takes the form

$$(X_1',\ \rho(X_1')\cos\alpha + X_3(X_1', X_3')\sin\alpha,\ X_3').$$

Since $e_2' = \hat{w}$, the Lipschtizness of the second component above shows that the surface is a Lipschitz hypograph in the $\hat{w}$-direction. This completes Step 4.

*Finally*, we conclude the proof of the lemma by noting that we have verified the condition in Step 1, so $D_t^e$ is a Lipschitz hypograph near all the points $p \in \Pi$. □

*Proof of Theorem 2.3.* Define $\Omega_t^e = \{p + s\hat{v}(p) :\ p \in \Gamma,\ s \in (0, t)\}$, cf. (29). Let $t_0$ be as given by Lemma 4.7. Set $\Omega^e = \Omega_t^e$ for some $0 < t < t_0$. Then by Lemma 4.7, $\Omega^e$ is Lipschitz.

That $\tilde{\Omega}$ is also Lipschitz follows by the same techniques. The only points that require any further explanation are the points $p \in \Pi$. Consider the domain obtained by transporting $\Gamma_2 = \partial\Omega \setminus \Gamma$, namely $\Omega_{2,t}^e = \{q + s\hat{v}(q) :\ q \in \Gamma_2,\ s \in (0, t)\}$. By Lemma 4.7, there exists a $t_0$ such that for all $0 < t < t_0$, its boundary $\partial\Omega_{2,t}^e$ is Lipschitz. Now, observe that the surface $\partial\tilde{\Omega}$ coincides with the boundary of $\partial\Omega_{2,t}^e$ (for some value of $t$) on a small neighborhood of $p \in \Pi$. Hence $\partial\tilde{\Omega}$ is a Lipschitz hypograph near such $p$. □

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