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Uniform Materials and the Multiplicative Decomposition of the Deformation Gradient in Finite Elasto-Plasticity

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Abstract

In this work we analyze the relation between the multiplicative decomposition $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$ of the deformation gradient as a product of the elastic and plastic factors and the theory of uniform materials. We prove that postulating such a decomposition is equivalent to having a uniform material model with two configurations – total ϕ and the inelastic ϕ_1.

We introduce strain tensors characterizing different types of evolutions of the material and discuss the form of the internal energy and that of the dissipative potential. The evolution equations are obtained for the configurations (ϕ, ϕ_1) and the material metric $\mathbf{g}$.

Finally, the dissipative inequality for the materials of this type is presented. It is shown that the conditions of positivity of the internal dissipation terms related to the processes of plastic and metric evolution provide the anisotropic yield criteria.

1. Introduction

The objective of this work is to investigate the relation between the geometrical theory of uniform materials and the multiplicative elasto-plastic decomposition of the deformation gradient of Bilby–Kröner–Lee (BKL decomposition) and Nemat–Nasser (see [1–4]).

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Such a relation was first studied in [5]. In particular, the relation between the inhomogeneity velocity gradient $\mathbf{L}_P$ (see below) and the plastic distortion rate $\bar{\mathbf{L}} = \bar{\mathbf{F}}^p \cdot (\bar{\mathbf{F}}^p)^{-1}$ was introduced. In this paper we study the geometrical form of the relation introduced in [5].

In Section 2 we introduce the basic concepts and review properties of uniform materials. In Section 3 a bijective correspondence between the BKL decompositions of the gradient of a configuration ϕ of an elasto-plastic solid and the triple (ϕ, ϕ_1, P) is established. Here P represents the **uniform material structure** and ϕ and ϕ_1 are, respectively, total and **inelastic** (intermediate) material configurations.

In Section 4 we introduce the natural strain tensors measuring the relations between the Cauchy–Green deformation tensors $C(\phi)$ and $C(\phi_1)$ and the material metric $\mathbf{g}$ induced by the uniform structure P. In the same section, the combinations of these tensors, material metric and its curvature characteristic independent of the decomposition of plastic deformation gradient $F^p = \phi_{1*} \circ D$ are determined and the strain rate tensors are introduced.

In Section 5 the form of the internal energy u depending on variables $(\phi, \phi_1, \mathbf{g})$ and their derivatives is postulated and the dissipative potential $\mathcal{D}$ is introduced. We also formulate the system of equations describing evolution of dynamical variables $(\phi, \phi_1, \mathbf{g})$. In the same section, different stress tensors present in our scheme are defined and relations between them are discussed.

In Section 6 we write down the dissipative inequality for the suggested scheme and separate the terms corresponding to the internal dissipation related to the processes of integrable inelastic and uniform structure evolutions. We show that the conditions of positivity of the corresponding terms in dissipative inequality provide the anisotropic yield criteria for initiating the corresponding processes.

Another form of a relation between the finite elasto-plasticity based on the multiplicative decomposition and the uniformity structures using the second-order connection was suggested by Cleja-Tigoui, see [6].

2. Uniform materials: material connections and material metrics

Uniform materials entered the scene of material science in about 1952 when Kondo introduced the material connection and the material metric as the tools to model properties of materials. Later developments in the works by Kondo, Bilby and his collaborators, Kroner, Noll ([7]) and Wang (see [8, 9]) establish the basis of this theory. In the works from 1980 to present by

Elzanowski, Epstein, de Leon, Maugin, different aspects of this theory were further developed: models of higher grade uniform materials, dynamics of material properties, thermodynamical properties of such materials, role of Eshelby stress tensor, geometry of functionally graduate material, etc.

In this section we present the basic geometrical structures of the theory of uniform materials that will be used in later parts of the paper. Our presentation is based on [5,10–13].

2.1. Material and physical spaces

A material body (**material manifold**) is usually represented by a connected 3-dimensional smooth oriented manifold M with a piecewise smooth boundary ∂M. Constructions of this paper are local, so it is sufficient to consider M as a connected open domain in $\mathbb{R}^3$ with local coordinates X^I, $I = 1, 2, 3$.

As the physical space our body is placed in, we consider the 3-dimensional Euclidean vector space $(E^3, \mathbf{h})$, $\mathbf{h}$ being the (flat) Euclidean metric. We introduce global Cartesian coordinates x^i in $\mathbb{R}^3$. In these coordinates the metric $\mathbf{h}$ takes the form $\mathbf{h} = h_{ij}dx^i dx^j$.

We will also use the concept of "archetype" (see [10] or [11], Sec. 1.2.), a 3-dimensional vector space V endowed with a standard Euclidean metric and the orthonormal basis $\mathbf{e}_0 = \{e_i,\ i = 1, 2, 3\}$. For convenience, we identify the "archetype" space V with the tangent space at the origin O of the physical space: $V = T_O(\mathbb{R}^3)$ and its metric with the metric $\mathbf{h}$ at the origin.

2.2. Configurations and the Cauchy metric

Configuration of the body M is a (diffeomorphic) embedding $\phi : M \to E^3$ into the physical space E^3; see [15], Ch. 1. To each configuration ϕ, there corresponds the **deformation gradient** – the mapping from the tangent space $T_X(M)$ at the point $X \in M$ to the tangent space $T_{\phi(X)}(E^3)$ at the point $\phi(X) \in E^3$,

$$\mathbf{F}(X) = \phi_{*X} : T_X(M) \to T_{\phi(X)}(E^3),$$

given, in coordinates X^A, x^i, by the matrix of partial derivatives

$$\mathbf{F}(X)^i_I = \phi^i_{,I}.$$

Here and below we will use the notation $\phi^i_{,I} = \frac{\partial \phi^i}{\partial X^I}(X)$ for the partial derivatives of configuration components $\phi^i(X)$.

To a configuration $\phi(X)$ there corresponds the right Cauchy–Green deformation tensor – the flat metric $\mathbf{C}(\phi) = \phi^*\mathbf{h}$ in M obtained as the pullback of Euclidian metric h in physical space by the configuration mapping ϕ. In coordinates (X^I) tensor $C(\phi)$ has the form

$$C(\phi)_{IJ} = h_{ij}\phi^i_I\phi^j_J. \tag{1}$$

We will fix a specific configuration ϕ_o and call it the **reference configuration**. Usually it presents the state of the material body that is free from loads and stresses (see [14], p. 15 or [16], p. 48), although it might happen that such a configuration does not exist and one has to choose a reference configuration differently. The body M is often identified with its image under the embedding ϕ_o.

To the reference configuration ϕ_o there corresponds its Cauchy–Green tensor called the *reference metric* in M,

$$\mathbf{g}_o = \mathbf{C}(\phi_o),\ g_{oIJ} = h_{ij}\phi^i_{o,I}\phi^j_{o,J}, \tag{2}$$

and the corresponding *reference volume form* $v_o(X) = \sqrt{|g_o|}dX^1 \wedge \ldots \wedge dX^n$.

Using the mapping inverse to the reference configuration $\phi : M \to E^3$, one can define the frame $\mathbf{p}_o$ in M by the rule[1]

$$\mathbf{p}_o(X) = \phi^{-1}_{o,*X}(\mathbf{e}_0),\ (p_o)_i = \frac{\partial\phi_o^{-1}{}^I}{\partial x^i}\frac{\partial}{\partial X^I},\ i = 1, 2, 3.$$

From now on we assume that the coordinates X^I are introduced in the material manifold M using the reference configuration, i.e., $X^I(X) = \phi^I_o(X)$. Then the vectors of the frame $\mathbf{p}_o$ take the form $(p_o)_I = \frac{\partial}{\partial X^I},\ I = 1, 2, 3$.

Finally we define **a history of deformation** as a time parameterized family of smooth configurations: $\phi(t, X) : M \times \mathbb{R} \to E^3$.

2.3. Uniform materials, I

Recall [7,8] that a material is called **hyperelastic** if its constitutive response (to a loading condition) at any configuration ϕ is completely characterized by two scalar functions:

[1] Here and below for a differentiable mapping $\psi : M \to N$ between manifolds M and N we denote by $\psi_{*X} : T_X(M) \to T_{\psi(X)}(N)$ the linear mapping of tangent spaces at a point $X \in M$. In coordinates (X^I, x^i) mapping ψ_{*X} is given by the matrix $F^i_I = \phi^i_{,I}$. Corresponding mapping of the tangent bundles will be denoted by ψ_*: $\psi_* : T(M) \to T(N)$; see [15], Ch. 1.

1. The **elastic energy density function** (per unit of reference volume v_o) $W(X, \mathbf{F}(X))$ depending on a material point $X \in M$ and the deformation gradient $\mathbf{F}(X)$ at this point; and

2. The mass density function $\rho_{ref}(X) > 0$ in the reference configuration ϕ_o.

Next, we introduce the basic notion of a **uniform material (body)**. Intuitively speaking, a **uniform body** is one that is made of the *same material at all its points*. The property of *uniformity* is characterized in terms of a *parallelism* K_X^Y in the body M [8,10,16]. More specifically, a hyperelastic material body (M, W) is called **uniform** if for any two material points X, Y there exists a linear isomorphism $K_X^Y : T_X(M) \to T_Y(M)$ between tangent spaces at these points such that

$$K_X^{Y*}(W(Y, \mathbf{F}(Y))dv_0(Y)) = W(X, F(Y) \circ K_X^Y)dv_0(X) \tag{3}$$

for all values of deformation gradients $\mathbf{F}(Y)$ at Y. Here K_X^{Y*} is the pullback of the n-form of energy density by the mapping K_X^Y.

We now introduce the scalar factor λ_X^Y, characterizing the behavior of the reference volume form under the parallelism K_X^Y: $K_X^{Y*}v_o(Y) = \lambda_X^Y v_o(X)$. Then, in terms of the *energy density function* W, condition (3) takes the form

$$\lambda_X^Y W(Y, \mathbf{F}(Y)) = W(X, \mathbf{F}(Y) \circ K_X^Y) \tag{4}$$

for all points X, Y in M and for all values of deformation gradient $\mathbf{F}(Y)$ at the point Y.

2.4. Material connections

The localization of the definition of uniform materials given above leads to the introduction of a linear connection (**material connection**) ω in M having vanishing curvature (an absolute parallelism; see [17], Ch. 3, Sec. 2. Having such a connection available, the mappings K_X^Y are defined by the parallel translation defined by connection ω from the point X to the point Y along any curve connecting X and Y (result of such translation is independent of the choice of a curve due to the vanishing of the curvature). The torsion tensor T of connection ω provides the measure of *nonhomogeneity* of the material; see [10,18].

It is known (see [17], Ch. 2) that in a simply connected body M, which admits a global tangent frame, a zero curvature connection is determined by a choice of a *global tangent frame parallel with respect to the connection* ω:

$$\mathbf{p}(X) = \{\mathbf{p}_k = p_k^I(X)\partial_{X^I},\ k = 1, \ldots, 3,\ \nabla^\omega \mathbf{p}_k = 0\}.$$

Remark 1 *A choice of such a frame is unique up to the (natural) right action of the group GL(n, R) on the tangent frames and the left action of the* **symmetry gauge group** G^M *of the connection* ω *(see [11,13]).*

A global frame **p** may also be defined by the **uniformity mapping** smoothly depending on the point X:

$$P_X : V \to T_X(M),\ P_X(\mathbf{e}_i) = (P_X)_i^I \partial_{X^I},\ i = 1, 2, 3. \tag{5}$$

Mapping P_X defines the linear isomorphism of the archetype space V with the tangent space at each point $X \in M$. Section **p** and the uniformity map P are related by

$$\mathbf{p}(X) = P_X(\mathbf{e}_0) \Leftrightarrow p_I(X) = P_I^J \frac{\partial}{\partial X^J}. \tag{6}$$

Parallel translation K_X^Y defined by the connection ω can be written in terms of the uniformity mapping as the composition

$$K_X^Y = P_Y \circ P_X^{-1}.$$

Using the reference frame $\mathbf{p}_o$ (see above) and the frame $\{\mathbf{e}_i\}$ in the space V, one can associate to a material frame **p** two other geometrical objects:

1. A smooth mapping $k : M \to GL(V), X \to k(X)$ (an element of the gauge group $GL(V)^M$) such that for all $X \in M$

   $$\mathbf{p}_J(X) = \mathbf{p}_{o\,J}(X) \cdot k(X) \Leftrightarrow p_J^I(X) = p_{0\,J}^L k(X)_L^I,\ I, J = 1, 2, 3,$$

   here $GL(V)$ is the group of invertible linear transformations of the archetype space V;

2. A nondegenerate (1,1)-tensor field $D_J^I(X)$ such that

   $$\mathbf{D}(X)\mathbf{p}_o(X) = \mathbf{p}(X), \quad \text{i.e.}$$
   $$p_i^I(X) = D(X)_J^I (p_0)_i^J(X) = D_i^I(X),\ i, I = 1, \ldots, 3,$$

   last equality being true due to $(p_0)_i^I(X) = \delta_i^I$.

Nondegeneracy of the (1,1)-tensor $\mathbf{D}(X)$ means that $\mathbf{D}(X) \in GL(T_X(M))$.

Using the relation between the frame **p** and the corresponding gauge mapping $k : M \to GL(V)$, we get the relation between k and the uniformity mapping

P corresponding to the frame $\mathbf{p}$, namely, $\mathbf{p}_i(X) = P_X(\mathbf{e}_i) = (\mathbf{p}_0)_i k(X) = P_{o,X}(\mathbf{e}_i)k(X)$, so that

$$P_X = P_{o,X} \circ k(X).$$

These considerations are summarized in the following:

Proposition 1 *Let M be a simply connected parallelizable (i.e., admitting a global frame) manifold. With a choice of a reference configuration ϕ_o and a frame $\boldsymbol{e}_i$ in the archetype space V, there is a bijection between the following objects:*

1. *Global frames* $\mathbf{p}$ *in M (global smooth sections of the frame bundle $F(M)$);*
2. *Smooth uniformity mappings $P_X : V \to T_X(M)$;*
3. *Smooth mappings $\mathbf{k} : M \to GL(V), X \to \mathbf{k}(X)$ (elements of the gauge group $GL(V)^M$) such that for all $X \in M$*

$$\mathbf{p}(X) = \mathbf{p}_o(X)k(X);$$

4. *Nondegenerate smooth (1,1)-tensor fields $D^I_J(X)$ in M such that*

$$\mathbf{D}(X)\mathbf{p}_o(X) = \mathbf{p}(X),$$

or, in terms of uniformity mappings P and P_o,

$$\mathbf{D}(X) = P_X \circ P_o^{-1}.$$

Remark 2 *It is the bijection between the first two and the last types of geometrical objects (nondegenerate (1,1)-tensor fields) that will be primarily used in this paper.*

2.5. Uniform materials, II

A uniformity mapping P determines its own volume form by translating to the material the Euclidian volume element from the archetype: $v_P(X) = P_X^{-1\,*}(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3)$. Denote by $J_P(X)$ the factor relating two volume forms v_o and v_P,

$$v_P(X) = J_P(X)v_0(X)$$

namely the Jacobian of the mapping P^{-1}.

Comparing the definition of the factor λ_X^Y in Eq. (4) with the definition of the factor $J_P(X)$, we get, for a uniform material, the following relation between these factors:

$$\lambda_X^Y = \frac{J_P(X)}{J_P(Y)}. \tag{7}$$

In terms of the volume factor μ_P uniformity condition (4) takes the form

$$J_P^{-1}(Y)W(Y, F(Y)) = J_P^{-1}(X)W(X, F(Y) \circ P_Y \circ P_X^{-1}). \tag{8}$$

Combining the deformation gradient $\mathbf{F}(X)$ and the uniformity mapping P_X, one gets the linear automorphism of the archetype space $A_X = \mathbf{F}(X) \circ P_X \in GL(V)$. Comparing Eq. (4) with Eq. (8), we rewrite the condition (8) as follows:

$$\begin{aligned} &J_P^{-1}(X)W(X, \mathbf{F}(Y) \circ P_Y \circ P_X^{-1}) \\ &\qquad = J_P^{-1}(Y)W(Y, \mathbf{F}(Y)) = J_P^{-1}(Y)W(Y, \mathbf{F}(Y) \circ P(Y) \circ P(Y)^{-1}) \end{aligned}$$

for arbitrary points $X, Y \in M$ and an arbitrary value of the deformation gradient $\mathbf{F}(Y)$ at the point Y.

Define a function $\hat{W}$ of a point $X \in M$ and a linear mapping $A \in GL(V)$ by setting

$$\hat{W}(X, A) = J_P^{-1}(X)W(X, A \circ P_X^{-1}). \tag{9}$$

In terms of the function $\hat{W}$, definition (8) of uniform material takes the very simple form

$$\hat{W}(X, A) = \hat{W}(Y, A). \tag{10}$$

Thus, **the uniformity condition (4) for the strain energy function W is equivalent to the statement that the function $\hat{W}(X, A))$, $X \in M, A \in GL(V)$ does not depend on the point $X \in M$.** As a result, $\hat{W}(X, A)$ is a function on the linear group $GL(V)$ only. This result is the central point of the theory of (first-grade) uniform hyperelastic materials. It reduces the study of material properties of body M and the evolution of those to the study of the uniformity mapping P_X and the function $\hat{W}$ on the linear group $GL(V)$.

Additional physical requirements (e.g., material frame indifference, presence of a nontrivial material symmetry group, etc.) lead to additional restrictions on the form of the energy function W. For instance, material frame indifference requirement leads to the conclusion that $\hat{W}(A)$ is a function of invariants of

matrix A. If a uniform material is isotropic, function $W(A)$ is left invariant with respect to the multiplication by elements of $SO(3)$ [11,13].

Returning to the the strain energy density function $W(X, F(X))$, we see that for a uniform material with the uniformity mapping P the strain function W takes the form [10,12]

$$W(X, F(X)) = J_P(X)\hat{W}(F(X) \circ P(X)). \tag{11}$$

2.6. Material metric of a uniform structure

As was already known to Cartan (see [19]), to a zero curvature linear connection ω (absolute parallelism) determined by a frame **p** (or by the corresponding uniformity map P) there corresponds the **material metric g** defined as the pullback of Euclidian metric h by the mapping P_X^{-1}:

$$\mathbf{g}(X) = P_{X*}^{-1}\mathbf{h}. \tag{12}$$

This definition is equivalent to declaring the frame **p g-orthonormal** at each point $X \in M$. In local coordinates X^I, the metric **g** has the form

$$g_{IJ}(X) = (P_X^{-1})_I^i (P_X^{-1})_J^j h_{ij} = (D(X)^{-1})_I^M (D(X)^{-1})_J^N g_{0\,MN},$$

the first expression being given in terms of the uniformity mapping P while the second is in terms of the corresponding (1,1)-tensor field **D**.

The curvature of the metric **g** is then defined by the torsion of the connection ω (see [16], eq. (34.19)).

2.7. Examples

Elastic strain tensor of a body in a configuration ϕ is defined by

$$\mathbf{E}_c^{el} = \frac{1}{2} ln(\mathbf{g}_0^{-1}\mathbf{C}(\phi)) \approx \frac{1}{2}(g_0^{-1}C(\phi) - I),$$

where the second expression is the linear approximation of the first one [15, 16]. Recall that the strain energy function of an isotropic material in linear elasticity has the form

$$W(\phi) = \lambda[Tr(\mathbf{E}^{el})]^2 + \mu\, Tr[(\mathbf{E}^{el\,2})],$$

where λ, μ are Lamé coefficients (see [15], Sec. 4.3).

Using the same function $\hat{W} = \lambda[Tr(A)]^2 + \mu Tr[A^2]$ on the linear group $GL(V)$ but a nontrivial uniformity mapping P, we come to the model of a **quasi-isotropic material**. Uniformity mapping P defined the material metric g as above. This allows us to *redefine* the elastic strain tensor using metric g instead of the reference metric g_0:

$$\mathbf{E}^{el} = \frac{1}{2} ln(\mathbf{g}^{-1}\mathbf{C}(\phi)) \approx \frac{1}{2}(g_0^{-1}C(\phi) - I). \tag{13}$$

Strain energy of a quasi-isotropic material in linear elasticity is defied as follows:

$$W_P(X, \mathbf{F}(X)) = \mu_P(X)[\lambda(Tr(\mathbf{E}^{el}))^2 + \mu Tr(\mathbf{E}^{el\ 2})]. \tag{14}$$

It is easy to see that the strain energy is the quadratic function of the conventional elastic strain tensor E_c^{el} with the *tensor of elastic moduli depending on the material point* X.

Another example is provided by a **quasi-Hookean** material (see [15], p. 11), i.e., the uniform analog of the **neo-Hookean** material with

$$W(\phi) = \alpha[Tr(\mathbf{E}_c^{el\ 2}) - 3].$$

The quasi-Hookean material corresponding to a uniformity structure P is defined by the same strain energy function but with the *redefined strain tensor*

$$\mathbf{E}^{el} = \frac{1}{2} ln(\mathbf{g}^{-1}\mathbf{C}(\phi)),$$

where material metric g is used instead of the reference metric g_0:

$$W_P(X, F(X)) = \alpha(Tr(\mathbf{E}^{el\ 2}) - 3)]. \tag{15}$$

In the case of a homogeneous uniformity structure, the last expression reduces to the strain energy of standard neo-Hookean material.

2.8. Evolution of the uniform structure

Evolution of the properties of a uniform material is characterized by the time-dependence of the uniformity mapping P and that of the function $\hat{W}$. An appropriate characteristic of the evolution of the uniform structure P is the material velocity $\mathbf{L}(X)$, which has been studied by different authors, see for instance [5,12,20].

The **material velocity** of the uniformity structure P is defined as the material point and time-dependent linear mapping,

$$\mathbf{L}_t(X) = P_X^{-1} \circ \frac{\partial P_X}{\partial t} : V \to V.$$

Under a loading, both the uniform structure P and the deformation mapping ϕ are evolving. As a result, the couple $(P_X(t), \phi(t, X))$ (or $(g(t, X), \phi(t, X))$ describes both the (total) deformation of a material and the evolution of its properties (elastic moduli, reference density, etc.). The rate of change of this couple is given by $(\mathbf{L}_t(X), \mathbf{V}(t, X))$, where $\mathbf{V}(t, X) = \frac{\partial \phi}{\partial t}$ is the physical velocity.

3. Elasto-plastic multiplicative decompositions of the deformation gradient

At the end of 1950s, Bilby, Kröner ([2]) and later on Lee ([3]) proposed the following **multiplicative decomposition of the deformation gradient (BKL decomposition)**,

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p, \tag{16}$$

as the product of two **smooth (1,1)-tensor fields** of **elastic and plastic deformations**, respectively. To provide a geometrical illustration of this decomposition, an **intermediate configuration** C_t^* was introduced between the material body M and the current configuration $C_t = \phi_t(M)$.

The decomposition $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$ is used to study the behavior exemplified by an elasto-plastic behavior of a material that undergoes deformation under a slowly applied load beyond the elastic range and then, after unloading, preserves some "permanent" strain (deformation). We refer to the works [4,21] for more examples and references concerning multiplicative decompositions of the deformation gradient F and their applications.

3.1. Relation between the BKL decomposition and the theory of uniform materials

Recall that the deformation gradient $\mathbf{F}_t(X)$ of a configuration $\phi : M \to E^3$ is the two-point (1,1)-tensor field in M defined by the linear isomorphism of the tangent spaces $\phi_* : T(M) \to T(\phi_t(M))$ at $X \in M$. Here $C_t = \phi_t(M)$ is the configuration of the body at the time t.

The decomposition (16) can be hardly interpreted other than as the composition of tangent bundle mappings over some mappings of corresponding base manifolds:

$$\begin{array}{ccccc} T(M) & \longrightarrow & T(C_t^*) & \longrightarrow & T(\phi_t(M)) \\ \pi \downarrow & & \pi \downarrow & & \pi \downarrow \\ M & \longrightarrow & C_t^* & \longrightarrow & \phi_t(M), \end{array}$$

since the tensor fields $\mathbf{F}^e$ and $\mathbf{F}^p$ should be strictly anchored at some manifolds (domain and target of each). Moreover, the first mapping $\mathbf{F}^p$ should define a mapping from the tangent space $T_X(M)$ at a point $X \in M$ to the tangent space at some point Y_X of intermediate configuration C_t^*. The correspondence $X \to Y_X$ should be one-to-one, otherwise the composition of mappings (16) cannot be an isomorphism of the tangent bundles. Therefore, there exists a unique one-to-one mapping $\phi_1 : M \to C_t^*$ underlying the tangent bundle mapping $\mathbf{F}^p$. Mapping ϕ_1 can be assumed to be differentiable.

In the same way, $\mathbf{F}^e$ can be viewed as a mapping of tangent bundles $T(C_t^*) \to T(C_t)$ over the differentiable mapping $\phi_2 : C_t^* \to T(C_t)$ of basis manifolds.

We obviously have $\phi = \phi_2 \circ \phi_1$. Therefore, ϕ_2 is onto. Restricting, if necessary, the intermediate configuration manifold, one may assume, without losing generality, that ϕ_1 is onto and ϕ_2 is one to one. Thus, both ϕ_1 and ϕ_2 can be considered as diffeomorphisms.

Remark 3 *Defining the decomposition (16), some authors presume that the mappings* $\mathbf{F}^e$ *and* $\mathbf{F}^p$ *are nonsmooth or even noncontinuous, reflecting microdefect densities in the manifold* M*. Translating this into the language of tensor fields and using the derivatives of these tensor fields, one should however assume some smoothness. Usually it is done by considering these tensor fields as smooth* averaged *characteristics of the structural state of the material.*

Remark 4 *Mapping* ϕ_1 *presents the* intermediate configuration *introduced in 1960 by a variety of researchers; see [3,22,23]. It was used for the construction of plastic deformation gradient* F^p *and the elasto-plastic decompositions of total deformation gradient* F*, but as far as we know, was not considered previously as an independent dynamical variable.*

Now we are ready to take the next step. Consider the tangent mapping $\phi_{1t*} : T(M) \to T(C_t^*)$ and compare it with the mapping $\mathbf{F}^p(t, X) : T(M) \to T(C_t^*)$. Since mapping ϕ_{1t*} is linear isomorphism at each point $X \in M$, one can write, for all tangent vectors $\xi \in T_X(M)$,

$$\mathbf{F}^p(t; (X, \xi)) = \phi_{1t*X} \circ \mathbf{D}_t(X) \cdot \xi, \qquad \star$$

where $\mathbf{D}_t(X)$ is **uniquely defined smooth (1,1)-tensor field in** M.

In exactly the same way, one can present

$$\mathbf{F}^e(t, Y, \eta) = \phi_{2t*Y} \circ \mathbf{F}^{e*}(t, Y) \cdot \xi$$

for the **uniquely defined smooth (1,1)-tensor field** $\mathbf{F}^{e*}(t, Y)$ **in** C_t^*.

If we pull back the (spatial index of) tensor field $\mathbf{F}^{e*}(t, Y)$ from C_t^* onto M by the differential ϕ_{1t*} of the point mapping ϕ_1, we get another (1,1)-tensor $\mathbf{D}^e$ on M.

Since $\phi(t, X) = \phi_{t2} \circ \phi_{t1}$ for all t and, therefore, $\phi(t, X)_{*X} = \phi_{t2\ *\phi_{t1}(\mathbf{X})} \circ \phi_{t1\ *X}$, combining this with the decomposition ($\bigstar$) we get

$$\begin{aligned}\phi_* &= \mathbf{F}^e \circ \mathbf{F}^p = (\phi_{2*} \circ \mathbf{D}^e) \circ (\phi_{1*} \circ \mathbf{D}) = (\phi_{2*} \circ \phi_{1*}) \circ (\phi_{1*}^{-1} \circ \mathbf{D}^e \circ \phi_{1*}) \circ \mathbf{D} \\ &= (\phi_2 \circ \phi_1)_* \circ (\mathbf{D}^e \circ \mathbf{D}) = \phi_* \circ (\mathbf{D}^e \circ \mathbf{D}),\end{aligned}$$

so that

$$\mathbf{D}^e(t, X) \cdot \mathbf{D}(t, X) = id_{T(M)}.$$

As a result, **being transferred to the material manifold** M, the (1,1)-tensor fields connecting integrable mappings ϕ_{i*} $(i = 1, 2)$ to the tangent bundle mappings $\mathbf{F}^p$ and $\mathbf{F}^e$ are **inverse to one other**. This is hardly a surprise since in the physical literature only one of these tensors was considered an independent dynamical variable; see [4,24].

In the same way, $\phi_2 = \phi \circ \phi_1^{-1}$ would also be redundant. As a result, the only independent dynamical variables in this scheme are diffeomorphic embeddings ϕ, ϕ_1 and the material (1,1)-tensor field $\mathbf{D}$.

Remark 5 *One can of course choose another triple of variables as independent dynamical quantities, for instance one may use* $(\phi_1, \phi_2, \mathbf{D})$ *if it is preferable to deal with the elastic deformation* ϕ_2 *explicitly.*

Remark 6 *We consider here only the decomposition* $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$*, but the same arguments would produce a geometrical representation of the reverse* $\mathbf{F} = \mathbf{F}^p\mathbf{F}^e$ *decomposition as well.*

Remark 7 *Notice that the choice of an intermediate configuration* $(C_t^*, \phi_{1\ t})$ *participating in the decomposition* ($\bigstar$) *is far from being unique. In particular, let us show that we may formally choose the image* $C_t^* = \phi_o(M)$ *as the intermediate configuration with* $\phi_{1\ t} = \phi_o$ *being* time independent. *To do this, denote by* $\psi : C_t^* \to C_o$ *the diffeomorphism* $\psi = \phi_o \circ \phi_1^{-1}$. *Transfer the tensor* $\mathbf{F}^e$ *to* C_o *as follows:* $\mathbf{F}^{e\ 0} = \psi_*(\mathbf{F}^{e\ *})$, *where we are using the diffeomorphism* ψ *together with its inverse to push forward the (1,1)-tensor* $\mathbf{F}^{e\ *}$. *Thus we get the mapping of tangent bundles*

$$\hat{\mathbf{F}}\mathbf{F}^e = \chi_{t*} \circ \mathbf{F}^{e\ o} = \chi_{t*} \circ \psi_{t*} \circ \mathbf{F}^{e\ *} \circ \psi_{t*}^{-1}.$$

Define also the diffeomorphism $\chi_t : C_o \to C_t$ *as* $\chi_t = \phi \circ \phi_o^{-1}$. *and define the* **mapping of the tangent bundles** *by setting*

$$\hat{\mathbf{F}}^p = \phi_{o*} \circ \mathbf{F}^{p\,M}.$$

Then we have $\phi = \chi \circ \phi_o$, *and, as is easy to check,* $\hat{\mathbf{F}}^e \circ \hat{\mathbf{F}}^p = \phi_*$ *as required.*

As a result, we get a simplified scheme of the elastic-plastic $\mathbf{F}^e\mathbf{F}^p$ *decomposition of the deformation gradient* $F = \phi_*$ *of a uniform material. Notice that the integrable part* $\phi_{1\,t}$ *of the plastic deformation gradient* $\mathbf{F}^p$ *is lost in this simplified scheme. That is why it is preferable to work with the previous scheme where the intermediate configuration is different from the image of the reference embedding* ϕ_o.

Remark 8 *Notice that the couple* $(\mathbf{D}, \phi_1)$ *represents* another model of evolution of the material *of the same type* **with the same uniformity structure**. *This model of pure inelastic evolution is related to the model* (ϕ, ϕ_1, D) **by the elastic deformation** ϕ_2.

If we start with a time-dependent uniformity mapping P_t and two configurations $\phi, \phi_1 : M \to R^3$, then one can (reversing the arguments above) construct the "elastic deformation" $\phi_2 = \phi \circ \phi_1^{-1}$ and the mappings of tangent bundles $\mathbf{F}^p : T(M) \to T(C^* = Im(\phi_{1,t}),\ \mathbf{F}^e : T(C_t^* = Im(\phi_{1,t}) \to T(C_t = Im(\phi_t))$, such that the construction above returns us to the triple (P_t, ϕ, ϕ_1).

Finally, there is freedom in the choice of the decomposition $\mathbf{F}^p = \phi_{1*} \circ \mathbf{D}$ given by an arbitrary diffeomorphism $\psi \in Diff(M)$:

$$\mathbf{F}^p = \phi_{1*} \circ \mathbf{D} = (\phi_1 \circ \psi^{-1})_* \circ (\psi_* \circ \mathbf{D}). \tag{17}$$

Thus, we can introduce the following equivalence relation between the pairs $(\phi_1, \mathbf{D})$ of the (time-dependent) mappings $\phi_1 : M \to R^n$ and nondegenerate (1,1)-tensor fields $\mathbf{D}$ in M. We say that two pairs $(\phi_1, \mathbf{D})$, (χ_1, K) are equivalent if there is a diffeomorphism $\psi \in Diff(M)$ such that

$$\chi_1 = \phi_1 \circ \psi^{-1}, K = \psi_* \circ \mathbf{D}. \tag{18}$$

Collecting the considerations presented in this section, we come to the following conclusions:

1. The BKL decomposition $F = F^pF^e$ of the deformation gradient $F = \phi_*$ of a (total) configuration $\phi_t : M \to E^3$ **presupposes** the existence of (intermediate) inelastic configuration $\phi_{1\,t} : M \to E^3$ and of the nondegenerate (1,1)-tensor field D_t in the material space M such that

$$\mathbf{F}^p = \phi_{1*} \circ \mathbf{D}, \tag{19}$$

$$\mathbf{F}^e = \phi_* \circ \mathbf{D}^{-1} \circ \phi_{1*}^{-1}. \tag{20}$$

2. Configuration ϕ_1 is the mapping $\phi_{1\ t} : M \to E^3$ defining the integrable part of inelastic (plastic!) deformation gradient $\mathbf{F}^p$.

3. (1,1)-tensor $\mathbf{D}$ is equal to

$$\mathbf{D}_t = \phi_{1*}^{-1} \circ \mathbf{F}^p(t, X).$$

4. Vice versa, to any triple $(\phi_t, \mathbf{F}_t^p, \mathbf{F}_t^e)$ consisting of two configurations and a nondegenerate (1,1)-tensor field D_t in M there corresponds the multiplicative decomposition

$$\phi_* = F = \mathbf{F}^e \circ \mathbf{F}^p$$

of the deformation gradient F of the total deformation history ϕ_t.

5. Tensor field D_t defines the time-dependent uniform structure $P_t = D_t \circ P_0$ (see Proposition 1) in the material body M.

6. Uniform structure D determines the time-dependent Riemannian metric $\mathbf{g}_t$ in the material manifold M (see Section 2.6) by the formula:

$$g_{t\ IJ} = h_{ij} D_I^i D_J^j. \tag{21}$$

Remark 9 *Notice that the decomposition* $\mathbf{F} = \phi_* = \mathbf{F}^e \circ \mathbf{F}^p$ *determines the tensor field* $\mathbf{D}$ *and the plastic integrable deformation* ϕ_1 *up to an action of a diffeomorphism* χ.

On the other hand, there are arguments showing that the inelastic configuration ϕ_1 *is defined uniquely by the history of deformation. If total deformation* $\phi_0 \to \phi_t$ *is subject to certain conditions of loading, heating, etc., the unloading or turning of the heat at some moment* t_1 *produces certain configuration* $\phi_1 : M \to E^3$. *Often the unloading happens fast and is not accompanied by an essential change in the material structure. See [25] where relaxation of a material to the intermediate configuration* ϕ_1 *during unloading is discussed. As a result, we may associate with the moment* t_1 *the final configuration* $\phi_{1\ t_1}$ *taken by the body M after unloading. As a result,* under physically reasonable assumptions on the evolution process*, the intermediate configuration* ϕ_1*, and therefore the tensor field* $\mathbf{D}$*, are* determined uniquely *(up to a composition of* ϕ_1 *with the Euclidean motion of the physical space* $\mathbf{E}^3$*).*

*Using the terminology of Maugin and Muschik [26], the configuration ϕ_1 is "*observable but not controllable*" (or, citing the same work, "*partly controllable*" through the loading conditions).*

In this analysis of the kinematics of the BKL decomposition, we identified three variables: the uniform structure P_t, the integrable part of the "plastic deformation gradient" ϕ_1 (intermediate configuration), and the total deformation ϕ, which we will consider as **physically independent**. Thus, the full dynamical/thermodynamical picture with these kinematics should include all three components. Below, discussing the dynamical structure of the presented model, we will be using material metric g instead of the tensor field D as the material dynamical variable. Metric g contains the essential information about the uniformity structure P_t and is more convenient to use when describing the evolution of material properties than the tensor D or uniformity mapping P; see [11], Ch. 11.

4. Strain tensors: elastic, inelastic, and metric

4.1. Strain tensors

Nonlinear elasticity theory has, as its geometrical keystone, the question of the comparison of two metrics: the reference metric $\mathbf{g}_o$ and the Cauchy metric $\mathbf{C}(\phi_t)$ of a configuration ϕ_t, [15].

As Theorem 1 shows, the multiplicative decomposition leads, in its geometrical form, to the presence of four metrics in the material manifold M: the reference metric $\mathbf{g}_o$, the material metric $\mathbf{g}$ generated by the (1,1)-tensor field $\mathbf{D} = \mathbf{F}^{p\,M}$, the Cauchy metric of the integrable part of plastic deformation $\mathbf{C}(\phi_1)$, and, finally, the Cauchy metric of the total deformation $\mathbf{C}(\phi)$. It seems natural to define appropriate strain tensors as measures of comparison between pairs of these metrics and use these tensors for description of different processes developing in the material.

We will introduce six strain tensors as suited to describe the state of our solid and characterize specific processes undergoing in the body.

1. **Elastic strain tensor:**

$$\mathbf{E}^{el} = \frac{1}{2} ln[\mathbf{C}(\phi_1)^{-1}\mathbf{C}(\phi)] \approx \frac{1}{2}\mathbf{C}(\phi_1)^{-1}[\mathbf{C}(\phi) - \mathbf{C}(\phi_1)] \approx \frac{1}{2}\mathbf{g}^{-1}[\mathbf{C}(\phi) - \mathbf{C}(\phi_1)].$$

Elastic strain tensor measures the elastic part of the deformation at each instant of time and vanishes under unloading. Tensor E^{el} and its linearized

variant are $C(\phi_1)$-symmetrical (i.e., $E^{el}_{IJ} = C(\phi_1)_{IK}E^{el\ K}_{J}$ is symmetrical (0,2)-tensor).

2. **Inelastic strain tensor:**

$$\mathbf{E}^{in} = \frac{1}{2}ln[\mathbf{g}^{-1}\mathbf{C}(\phi_1)] \approx \frac{1}{2}\mathbf{g}^{-1}[\mathbf{C}(\phi) - \mathbf{g}] \approx \frac{1}{2}\mathbf{g}_o^{-1}[\mathbf{C}(\phi) - \mathbf{g}].$$

Inelastic (plastic) strain tensor measures the plastic but still Euclidean **deformation** of the body, i.e., permanent after unloading but not leading to any **residual stresses** in the material. Tensor E^{in} and its linearized version E^{in}_{lin} are g-symmetrical, i.e., (0,2)-tensors $g_{IK}E^{in\ K}_{J}$, $g_{IK}E^{in\ K}_{lin\ J}$ are symmetrical.

3. **Material strain tensor:**

$$\mathbf{E}^{m} = \frac{1}{2}ln(\mathbf{g}_o^{-1}\mathbf{g}) \approx \frac{1}{2}\mathbf{g}_o^{-1}(\mathbf{g} - \mathbf{g}_o).$$

Material strain tensor measures the pure metrical evolution of the material, not leading to any deformation (material points displacement). Tensor E^{m} and its linearized version E^{m}_{lin} are g_0-symmetrical.

4. **Euclidian strain tensor:**

$$\mathbf{E}^{eucl} := \frac{1}{2}ln[\mathbf{g}^{-1}\mathbf{C}(\phi)] \approx \frac{1}{2}\mathbf{g}^{-1}[\mathbf{C}(\phi) - \mathbf{g}].$$

Euclidian strain tensor measures the integrable part of the total deformation.

5. **Total strain tensor:**

$$\mathbf{E}^{tot} = \frac{1}{2}ln(\mathbf{g}_o^{-1}\mathbf{C}(\phi)) \approx \frac{1}{2}\mathbf{g}_o^{-1}(\mathbf{C}(\phi) - \mathbf{g}_o).$$

Total strain tensor measures the decline of the Cauchy metric of total deformation ϕ from the reference (euclidian) metric $\mathbf{g}_o$. It is observable. Tensor E^{tot} and its linearized version E^{tot}_{lin} are g_0-symmetrical.

6. **Total inelastic strain tensor:**

$$\mathbf{E}^{tin} = \frac{1}{2}ln(\mathbf{g}_o^{-1}\mathbf{C}(\phi_1)) \approx \frac{1}{2}\mathbf{g}_o^{-1}(\mathbf{C}(\phi_1) - \mathbf{g}_o).$$

Total inelastic strain tensor measures the decline of the Cauchy metric of inelastic deformation ϕ_1 from the reference (euclidian) metric $\mathbf{g}_o$. It is observable (after unload).

In each case we provide the linear approximation form(s) of the strain tensors suited for small deviation of the former metric from the latter one.

Remark 10 *In some simple cases, say when the (1,1)-tensors* $\mathbf{g}^{-1}\mathbf{C}(\phi_1)$ *and* $\mathbf{C}(\phi_1)^{-1}\mathbf{C}(\phi)$ *commute, from the relation* $\mathbf{g}^{-1}\mathbf{C}(\phi) = \mathbf{g}^{-1}\mathbf{C}(\phi_1)\cdot\mathbf{C}(\phi_1)^{-1}\mathbf{C}(\phi)$ *we conclude that the linearized* **Euclidean strain tensor** $\mathbf{E}^{eucl}$ *splits as follows:*

$$\begin{aligned}\mathbf{E}^{eucl} &\approx \frac{1}{2}\mathbf{g}^{-1}[\mathbf{C}(\phi) - \mathbf{g}] = \frac{1}{2}\mathbf{g}^{-1}[\mathbf{C}(\phi) - \mathbf{C}(\phi_1)] + \frac{1}{2}\mathbf{g}^{-1}[\mathbf{C}(\phi_1) - \mathbf{g}] \\ &\approx \mathbf{g}^{-1}\mathbf{C}(\phi_1)\mathbf{E}^{el} + \mathbf{E}^{m} \approx \mathbf{E}^{el} + \mathbf{E}^{in}. \end{aligned} \tag{22}$$

If all three (1,1)-tensors $\mathbf{g}_o^{-1}\mathbf{g}$, $\mathbf{g}^{-1}\mathbf{C}(\phi_1)$, *and* $\mathbf{C}(\phi_1)^{-1}\mathbf{C}(\phi)$ *commute, we have an additive decomposition of the* **linearized total strain tensor** $\mathbf{E}^{tot}$*:*

$$\begin{aligned}\mathbf{E}^{tot} &\approx \frac{1}{2}\mathbf{g}_0^{-1}[\mathbf{C}(\phi) - \mathbf{g}_o] = \frac{1}{2}\mathbf{g}_0^{-1}[\mathbf{C}(\phi) - \mathbf{C}(\phi_1) + \mathbf{C}(\phi_1) - \mathbf{g} + \mathbf{g} - \mathbf{g}_o] \\ &= \frac{1}{2}\mathbf{g}_0^{-1}\mathbf{C}(\phi_1)\mathbf{C}(\phi_1)^{-1}[\mathbf{C}(\phi) - \mathbf{C}(\phi_1)] + \frac{1}{2}\mathbf{g}_0^{-1}\mathbf{g}\mathbf{g}^{-1}[\mathbf{C}(\phi_1) - \mathbf{g}] + \frac{1}{2}\mathbf{g}_0^{-1}(\mathbf{g} - \mathbf{g}_o) \\ &= \mathbf{E}^{m} + \mathbf{g}_0^{-1}\mathbf{g}\mathbf{E}^{in} + \mathbf{g}_0^{-1}\mathbf{C}(\phi_1)\mathbf{E}^{el} \approx \mathbf{E}^{m} + \mathbf{E}^{in} + \mathbf{E}^{el}. \end{aligned} \tag{23}$$

4.2. Choice of the dynamical variables

In order to determine which combinations of dynamical variables (ϕ, ϕ_1, D) and their derivatives might enter the internal or free energy, dissipative potential, entropy, and other dynamical and thermodynamical quantities, we have to take into account requirements of invariance or covariance of these quantities with respect to the appropriate material and spatial transformations. For instance, the frame indifference requirement [27,28] leads to the conclusion that the deformation gradient $F(X)$ of the total deformation ϕ enters these quantities only in combinations $C(\phi_1)^{-1}C(\phi)$, $g_0^{-1}C(\phi)$, or $g^{-1}C(\phi)$.

Material metric $\mathbf{g}$ and the Cauchy–Green tensor $\mathbf{C}(\phi_1)$ depend on the choice of the plastic decomposition (19). Thus, it is important to determine which tensors or combinations of tensors constructed from the dynamical fields $(\phi, \phi_1, \mathbf{D})$ are independent of the choice of the plastic decomposition (19).

We consider several such combinations.

1. The total strain tensor

$$\mathbf{E}^{tot} = \frac{1}{2}ln(\mathbf{g}_o^{-1}\phi^{*}\mathbf{h}) \approx \frac{1}{2}\mathbf{g}_o^{-1}[\mathbf{C}(\phi) - \mathbf{g}_o] \tag{24}$$

is independent of the choice of a plastic decomposition (19). Since it is a (1,1)-tensor, its invariants are independent of the choice of a decomposition (19) and, moreover, one can combine it with other similar tensors to produce new invariant combinations.

2. The plastic deformation gradient $\mathbf{F}^p = \phi_{1*} \circ \mathbf{D}$ does not depend on the choice of a decomposition (19) but it is convenient to transform it into a material tensor. For instance, one can use the following variant of the Cauchy–Green tensor:

$$\mathbf{F}^{p*}\mathbf{h} \equiv \mathbf{D}^*(\phi_1^*\mathbf{h}) = \mathbf{D}^*\mathbf{C}(\phi_1). \tag{25}$$

Lifting one index in this tensor by means of the reference metric $\mathbf{g}_0$ we get the material (1,1)-tensor $\mathbf{g}_o^{-1} {}^*\mathbf{F}^{p*}\mathbf{h}$. This tensor carries information on both integrable and nonintegrable parts of the plastic deformations.

3. The material metric $\mathbf{g} = P^{-1*}\mathbf{h}$ can be written in a number of different ways. For instance, by using the reference metric $\mathbf{g}_o = P_o^{-1*}\mathbf{h}$, we can get

$$\mathbf{g} = P^{-1*}\mathbf{h} = P^{-1*}(P_o^*\mathbf{g}_o) = (P_o \circ P^{-1})^*\mathbf{g}_o = (P \circ P_o^{-1})^{-1\,*}\mathbf{g}_o = \mathbf{D}^{-1\,*}\mathbf{g}_o. \tag{26}$$

In local coordinates we have

$$g_{AB} = (D^{-1})_A^M (D^{-1})_B^N g_{o\,MN},\ g^{-1\,AB} = D_M^A D_N^B g_0^{MN}. \tag{27}$$

Under the change of decomposition (19), the material metric $\mathbf{g}$ is transformed by a diffeomorphism ψ into a new metric $\mathbf{g}^\psi$ as follows:

$$\mathbf{g}^\psi = (\psi_* \circ \mathbf{D})^{-1\,*}\mathbf{g}_0 = (\mathbf{D}^{-1} \circ \psi_*)^*\mathbf{g}_0 = \psi^{-1\,*} \circ \mathbf{D}^{-1\,*}\mathbf{g}_0 = \psi^*\mathbf{g}. \tag{28}$$

In coordinates,

$$(g^\psi)_{AB} = (\psi^{-1})_A^M (\psi^{-1})_B^N g_{MN},\ (g^{\psi\,-1})^{CD} = \psi_M^C \psi_N^D g^{-1\,MN}.$$

Therefore, the curvature tensor R^I_{JKL} (as well as the corresponding Ricci tensor $R_{IJ}(\mathbf{g})$) is transformed tensorially by ψ^* or ψ_*. Thus, invariants of these curvature tensors (in particular the scalar curvature $R(\mathbf{g})$), do not depend on a choice of plastic decomposition (19).

4. At the same time the Cauchy metric of the inelastic deformation $\mathbf{C}(\phi_1)$ is transformed by ψ^{-1} tensorially as well:

$$\begin{aligned} C(\phi_1 \circ \psi^{-1})_{AB} &= (\psi^{-1\,*} \circ \phi_1^* h)_{AB} = [\psi^{-1\,*} C(\phi_1)]_{AB} \\ &= (\psi^{-1})_A^M (\psi^{-1})_B^N C(\phi_1)_{MN}. \end{aligned} \tag{29}$$

Combining the material metric and the Cauchy tensor of the inelastic deformation ϕ_1, we finally get the inelastic strain tensor $\mathbf{E}^{in}$. For a different decomposition (19), we have

$$\begin{aligned} (g^{\psi\,-1})^{AC} C(\phi_1 \circ \psi^{-1})_{CB} &= \psi_M^A \psi_N^C g^{-1\,MN} (\psi^{-1})_C^K (\psi^{-1})_B^L C(\phi_1)_{KL} \\ &= \psi_M^A g^{-1\,MN} \delta_N^K (\psi^{-1})_B^L C(\phi_1)_{KL} = \psi_M^A (\psi^{-1})_B^L g^{-1\,MK} C(\phi_1)_{KL} \\ &= \psi_M^A (\psi^{-1})_B^L (g^{-1} C(\phi_1))_L^M. \end{aligned} \tag{30}$$

Thus, $\mathbf{E}^{in}$ transforms tensorially under a change of plastic decomposition (19) and its invariants are independent of this decomposition.

The presence of the two quantities $\mathbf{F}^p$ and $\mathbf{E}^{in}$, whose invariants are independent of the choice of plastic decomposition, makes it important to compare these two quantities. We have

$$\begin{aligned}\mathbf{g}_o^{-1}(\mathbf{F}^{p*})\mathbf{h}) &= \mathbf{g}_o^{-1}(\mathbf{D}^*\mathbf{C}(\phi_1)) = g_o^{-AC} D_C^M D_B^N C(\phi_1)_{MN} \\ &= D_L^{-1\,A}(g_o^{-1\,KC} D_K^L D_C^M) C(\phi_1)_{MN} D_B^N = D_L^{-1\,A}(g^{-1\,ML}) C(\phi_1)_{MN} D_B^N \\ &= D_L^{-1\,A}(g^{-1} C(\phi_1))_N^L D_B^N = D_L^{-1\,A}(exp(2E^{in}))_N^L D_B^N,\end{aligned}$$

i.e.,

$$\mathbf{g}_o^{-1}(\mathbf{F}^{p*}\mathbf{h}) = D_L^{-1\,A}[exp(2E^{in})]_N^L D_B^N, \tag{31}$$

and therefore invariants of the tensor $\mathbf{g}_o^{-1}(\mathbf{F}^{p*}\mathbf{h})$ (i.e., tensorial characteristics of the plastic gradient deformation $\mathbf{F}^p$) contain the same information as the invariants of $\mathbf{E}^{in}$.

Thus, in the restricted case, it seems natural to choose an internal energy u as a function of the invariants of the two strain tensors $\mathbf{E}^{tot}$ and $\mathbf{E}^{in}$, of the invariants of the curvature tensor $R^i_{jkl}(\mathbf{g})$ (of the Ricci Tensor $Ric(\mathbf{g})^I_J$ in the 3-dim case, see [17], Ch. 6, Sec. 5), temperature, and its $\mathbf{g}_o$-gradient: $u = u([\mathbf{E}^{tot}(\phi), \mathbf{E}^{in}(\phi_1, \mathbf{g}), Ric(\mathbf{g}), \theta, \nabla^{\mathbf{g}_o}\theta])$.

If we adopt the assumptions of Remark 5 (i.e., removing the restriction to use only tensors that do not depend on a choice of plastic decomposition (19)), we may consider all three strain tensors $\mathbf{E}^{tot}$, $\mathbf{E}^{in}$, $\mathbf{E}^m$ (one can replace $\mathbf{E}^{tot}$ in this list by $\mathbf{E}^{el}$ if it is preferable) as independent dynamical variables and, together with the Ricci tensor of the material metric $\mathbf{g}$, include them as arguments in the internal energy:

$$u = u[\mathbf{E}^{el}, \mathbf{E}^{in}, \mathbf{E}^m, Ric(\mathbf{g}), \theta, \nabla^{\mathbf{g}_o}\theta]. \tag{32}$$

In this approach, the effects of different types of processes are directly separated.

4.3. Additional strain decompositions

Between the strain tensors introduced above, the conventional strain tensors, and both deformation gradients F^e, F^p of the multiplicative decomposition, there exist different relations that may be in some partial cases more convenient than those presented above. Below are two examples of such relations, the first being valid in the linear case, the second in a nonlinear situation.

1. Linear case:

Since $F = \phi_* = \mathbf{F}^e \circ \mathbf{F}^p$, we have

$$\mathbf{g}_o^{-1}\phi^*\mathbf{h} = \mathbf{g}_o^{-1}\mathbf{F}^{p*}(\mathbf{F}^{e*}\mathbf{h} - \mathbf{h} + \mathbf{h}) = \mathbf{g}_o^{-1}\mathbf{F}^{p*}\mathbf{h} + \mathbf{g}_o^{-1}\mathbf{F}^{p*}(\mathbf{F}^{e*}\mathbf{h} - \mathbf{h})$$

and therefore

$$\begin{aligned}\mathbf{E}_{lin}^{tot} &= \frac{1}{2}(\mathbf{g}_o^{-1}\phi^*\mathbf{h} - \boldsymbol{\delta}) = \frac{1}{2}(\mathbf{g}_o^{-1}\mathbf{F}^{p*}\mathbf{h} - \boldsymbol{\delta}) + \mathbf{g}_o^{-1}\mathbf{F}^{p*}\frac{1}{2}(\mathbf{F}^{e*}\mathbf{h} - \mathbf{h}) \\ &\approx \mathbf{D}^{-1}\mathbf{E}_{lin}^{in}\mathbf{D} + \mathbf{g}_o^{-1}\mathbf{F}^{p*}\mathbf{E}_{lin}^{el}, \end{aligned} \tag{33}$$

where we defined

$$\mathbf{E}_{lin}^{el} = \frac{1}{2}(\mathbf{F}^{e*}\mathbf{h} - \mathbf{h}).$$

This decomposition can be compared with those in Section 4.1.

2. Nonlinear case:

$$\begin{aligned}\mathbf{E}^{tot} &= \frac{1}{2}ln(\mathbf{g}_o^{-1}\phi^*\mathbf{h}) = \frac{1}{2}ln(\mathbf{g}_o^{-1}(\mathbf{F}^{p*}\mathbf{h})\mathbf{h}^{-1}(\mathbf{F}^{e*}\mathbf{h})) \\ &= \frac{1}{2}ln(\mathbf{g}_o^{-1}(\mathbf{F}^{p*}\mathbf{h}) \cdot exp(2\mathbf{E}_{old}^{el})) \\ &= \frac{1}{2}ln(\mathbf{g}_o^{-1}(\mathbf{F}^{p*}\mathbf{h}) \cdot (\mathbf{F}^{p*}exp(2\mathbf{E}_{old}^{el}))) \\ &\approx \frac{1}{2}ln(\mathbf{g}_o^{-1}\mathbf{F}^{p*}\mathbf{h}) + \mathbf{g}_o^{-1}\mathbf{F}^{p*}(\mathbf{E}_{old}^{el}), \end{aligned} \tag{34}$$

here $\mathbf{E}_{old}^{el} = \frac{1}{2}ln(\mathbf{h}^{-1}\mathbf{F}^{el*}\mathbf{h})$ as in conventional finite elasticity.

4.4. Strain rate tensors

We define the strain rate tensors as time derivatives of strain tensors. As a result we get the *strain rate tensors* $\dot{\mathbf{E}}^{tot}$, $\dot{\mathbf{E}}^{in}$, $\dot{\mathbf{E}}^{el}$, and $\dot{\mathbf{E}}^{m}$.

On the other hand, there are other rate characteristics for each of the three participating structures, i.e.:

1. The material velocity:

$$\mathbf{L}_D(t, X) = \mathbf{D}(t, X)^{-1} \circ \frac{\partial}{\partial t}\mathbf{D}(t, X).$$

This velocity is related to the velocity $\mathbf{L}_P = P^{-1} \circ P_{,t}$ introduced in Sec. 2.8 by the relation $\mathbf{L}_D = P \circ \mathbf{L}_P \circ P^{-1}$. We see now that $\dot{\mathbf{D}} = \mathbf{D}\mathbf{L}_D$ and therefore we get the relation between the material strain rate tensor and the material velocity $\mathbf{L}_D$ used in [5,14,20] and other papers.

$$\begin{aligned}\dot{\mathbf{E}}^m &= \frac{1}{2}\mathbf{g}_o^{-1}\dot{\mathbf{g}} = \frac{1}{2}\mathbf{g}_o^{-1}(\dot{D}_I^M D_J^N g_{0\,MN} + D_I^M \dot{D}_J^N g_{o\,MN}) \\ &= \frac{1}{2}\mathbf{g}_o^{-1}(D_K^M L_{D\,I}^K D_J^N g_{o\,MN} + D_I^M D_S^N L_{D\,J}^S g_{o\,MN}) \\ &= \frac{1}{2}\mathbf{g}_o^{-1}\left(L_{D\,I}^K g_{P\,KJ} + L_{D\,J}^S g_{IS}\right) = \mathbf{g}_o^{-1}(L_D)_{(I}^K \cdot g_{|K|J)}, \end{aligned} \tag{35}$$

where in the last formula there is symmetrization by indices IJ.

2. The total velocity is defined as

$$\mathbf{V}(X,t) = \frac{\partial}{\partial t}\phi(X,t),$$

and its gradient is related to the linearized total strain rate tensor $\mathbf{E}_{lin}^{tot} = \frac{1}{2}g_0^{-1}(C(\phi) - g_0)$ by the relation

$$(g_0 \dot{\mathbf{E}}_{lin}^{tot})_{MN} = h_{ij} V_{(,M}^i \phi_{N)}^j.$$

Being written in Euler (spatial) coordinates, this relation reduces to the standard one [16].

3. Finally, the velocity of the inelastic deformation,

$$\mathbf{V}_1(X,t) = \frac{\partial}{\partial t}\phi_1(X,t),$$

is related, in a linear approximation, to the (linearized) inelastic strain rate tensor $\dot{\mathbf{E}}_{lin}^{in}$ by a relation containing the symmetrized velocity gradient and the material velocity $\mathbf{L}_D$. In the calculation that follows, we are using the formula $\frac{\partial}{\partial t}C(\phi_1) = 2h_{ij}V_{1,(M}^i \phi_{1,N)}^j$ for time derivative of Cauchy–Green tensor of configuration ϕ_1. We have

$$\begin{aligned}\dot{\mathbf{E}}_{lin}^{in} &= \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{g}^{-1}(\mathbf{C}(\phi_1) - \mathbf{g})\right) = \frac{1}{2}\frac{\partial}{\partial t}\left(\mathbf{g}^{-1} \cdot \mathbf{C}(\phi_1)\right) \\ &= \frac{1}{2}\mathbf{g}^{-1}\frac{\partial}{\partial t}\mathbf{C}(\phi_1) + \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{g}^{-1}) \cdot \mathbf{C}(\phi_1) \\ &= \mathbf{g}^{-1}[h_{ij}V_{1,(M}^i \phi_{1,N)}^j] + \frac{1}{2}\mathbf{g}^{-1}\dot{\mathbf{g}}\mathbf{g}^{-1} \cdot \mathbf{C}(\phi_1) \\ &= \mathbf{g}^{-1}[h_{ij}V_{1,(M}^i \phi_{1,N)}^j] + \mathbf{g}^{-1}(\mathbf{g}_o \dot{\mathbf{E}}^m)\mathbf{g}^{-1}\mathbf{C}(\phi_1) \\ &\approx (\nabla^{\mathbf{g}_o}\mathbf{v}_1)_{sym} + \dot{\mathbf{E}}^m. \end{aligned} \tag{36}$$

Here $\mathbf{v_1}_M = \phi^i_{,M} h_{ij} \mathbf{V}^j_1$ is the covariant form of the *convective velocity* of inelastic configuration ϕ_1; see [29]. In the last approximation, we replaced $\mathbf{g} \approx \mathbf{g}_o$ in the first and the second terms and $\mathbf{C}(\phi_1) \approx \mathbf{g}$ in the second term.

5. Lagrangian, free energy, dissipative potential, and the stress tensors

Dynamical equations describing the evolution of the system characterized by the variables $(\phi, \phi_1, \mathbf{g})$ are obtained by combining the canonical (Lagrangian) component and the the dissipative forces.

The Lagrangian in our model is the combination of the kinetic, potential, and internal energy terms:

$$L = K - \rho_{ref} u - U(\phi), \tag{37}$$

where $K = \frac{\rho}{2}|\mathbf{V}|_h$ is the density of kinetic energy, $U(\phi)$ is the potential of the volume forces, and u is the internal energy per unit of mass (see Section 4.2).

5.1. Lagrangian and internal energy

It is traditional to define the free energy density ψ as a function of the elastic deformation gradient, the temperature ϑ, a material point X, and additional internal parameters $\boldsymbol{\alpha}$ (see [26,30]).

Dissipative pseudo-potential is, in this approach, the function of rates of deformation gradients and time derivatives of internal variables $\mathcal{D} = \bar{D}(\dot{\mathbf{F}}^e, \dot{\mathbf{F}}^p, \dot{\boldsymbol{\alpha}}, \nabla\vartheta, \nabla\boldsymbol{\alpha}, \vartheta)$ [26,30]. This allows one to define the total, elastic, and plastic stress tensors and the thermodynamical forces conjugate to the parameters $\boldsymbol{\alpha}$, thus separating different factors in the dissipation inequality (see eq. (10.21.) in [14]).

Comparing the expression for the internal energy (32) with these in [26,30], one sees that the metric $\mathbf{g}$ entering the free energy through the tensor $\mathbf{E}^m$ plays here the role of an internal variable $\boldsymbol{\alpha}$ and its Ricci tensor $Ric(\mathbf{g})$ takes the place of the space gradient $\nabla\boldsymbol{\alpha}$ [26,30]. The elastic $\mathbf{E}^{el}$ (respectively, inelastic $\mathbf{E}^{in}$) strain tensors are direct material analogs of $\boldsymbol{\epsilon}_e$ (respectively, of $\boldsymbol{\epsilon}_p$). Thus it is conceivable to adopt the internal variables approach in searching for the form of the equations governing the behavior of our system.

We take the Lagrangian in the form

$$\begin{aligned} L &= L(\rho_{ref}, \mathbf{V}, \mathbf{E}^{el}, \mathbf{E}^{in}, \mathbf{E}^{m}, Ric(\mathbf{g}), \vartheta, \nabla^{g_o}\vartheta) = K - \rho_{ref}u - U(\phi) \\ &= \frac{\rho_{ref}}{2}|\mathbf{V}|_h^2 + \rho_{ref}[\boldsymbol{\gamma} \cdot Ric(\mathbf{g}) + \mu\|\nabla^{g_o}\vartheta\|_g^2 + f_0(\mathbf{E}^{in}, \mathbf{E}^{m}, \vartheta) \\ &\quad + f(\mathbf{E}^{in}, \mathbf{E}^{m}, \vartheta; \mathbf{E}^{el})] - U(\phi). \end{aligned} \tag{38}$$

In this expression $\boldsymbol{\gamma}$ is a constitutive tensor, U is the potential of body forces, f_0 is the *"basic inelastic energy"*, and f is the strain energy of linear thermoelasticity, i.e., a quadratic function of the elastic strain tensor with coefficients depending on the temperature and the remaining inelastic strains:

$$f(\mathbf{E}^{in}, \mathbf{E}^{m}, \vartheta; \mathbf{E}^{el}) = [\mathbf{c_0} + (\vartheta - \vartheta_0)\mathbf{c_1}] : \mathbf{E}^{el} + (\mathbf{e} : \mathbf{E}^{el} : \mathbf{E}^{el}). \tag{39}$$

Here $\mathbf{c_0}$ and $\mathbf{c_1}$ are tensors characterizing the interaction of the elastic processes with the inelastic ones and temperature, respectively (for instance, $\mathbf{c_1}$ is the thermal expansion tensor). The tensor $\mathbf{e}$ is the elasticity tensor.

Assuming that the decomposition (23) is valid, substitution of the total strain tensor $\mathbf{E}^{tot}$ instead of the elastic strain tensor $\mathbf{E}^{el}$ into the expressions for internal energy and dissipative potential (below) does not change the form of function (39):

$$\begin{aligned} f(\mathbf{E}^{in}, \mathbf{E}^{m}, \vartheta; \mathbf{E}^{tot}) &= f(\mathbf{E}^{in}, \mathbf{E}^{m}, \vartheta; \mathbf{E}^{tot} - \mathbf{E}^{in} - \mathbf{E}^{m}) \\ &= [\mathbf{c_0} + (\vartheta - \vartheta_0)\mathbf{c_1}] : (\mathbf{E}^{tot} - \mathbf{E}^{in} - \mathbf{E}^{m})) \\ &\quad + (\mathbf{e} : (\mathbf{E}^{tot} - \mathbf{E}^{in} - \mathbf{E}^{m}) : (\mathbf{E}^{tot} - \mathbf{E}^{in} - \mathbf{E}^{m})) \\ &= + (\mathbf{e} : (\mathbf{E}^{in} + \mathbf{E}^{m}) : (\mathbf{E}^{in} + \mathbf{E}^{m})) + [(\mathbf{c_0} - 2\mathbf{e} : (\mathbf{E}^{in} + \mathbf{E}^{m}) \\ &\quad + (\vartheta - \vartheta_0)\mathbf{c_1}] : \mathbf{E}^{tot} + \mathbf{e} : \mathbf{E}^{tot} : \mathbf{E}^{tot}, \end{aligned} \tag{40}$$

but changes the tensor $\mathbf{c}_0$ and adds a term to the inelastic energy f_0.

This allows us to replace $\mathbf{E}^{el}$ by $\mathbf{E}^{tot}$ in the internal energy, so that we can equivalently use Lagrangian in the form

$$\begin{aligned} L &= L[\rho_{ref}, \mathbf{V}, \mathbf{E}^{tot}, \mathbf{E}^{in}, \mathbf{E}^{m}, Ric(\mathbf{g}), \vartheta, \nabla^{g_o}\vartheta] \\ &= \frac{\rho_{ref}}{2}|\mathbf{V}|_h^2 + \rho_{ref}[\gamma \cdot Ric(\mathbf{g}) + \mu\|\nabla^{g_o}\vartheta\|_g^2 + f_0(\mathbf{E}^{in}, \mathbf{E}^{m}, \vartheta) \\ &\quad + f(\mathbf{E}^{in}, \mathbf{E}^{m}, \vartheta; \mathbf{E}^{tot})] - U(\phi). \end{aligned} \tag{41}$$

The strain energy here has the form (40) where $\mathbf{E}^{el}$ is replaced by $\mathbf{E}^{tot}$.

In the dynamic equations, it is more convenient to use $\mathbf{E}^{tot}$ since ϕ is a geometrically explicit and observable quantity while in the dissipative inequality it is more convenient to use $\mathbf{E}^{el}$ because it allows one to separate inputs of different processes into the entropy production.

5.2. Free energy

The **free energy** is defined, as usual, by the equality

$$\psi = u - s\theta,$$

where s is the **specific entropy**.

5.3. Dissipative potential

The **dissipative (pseudo) potential** is chosen to be a function of the following variables:

$$\mathcal{D} = \mathcal{D}(\dot{\mathbf{E}}^{in}, \dot{\mathbf{E}}^{m}; \mathbf{E}^{m}, \vartheta). \tag{42}$$

We include $\dot{\mathbf{E}}^{in}$ together with $\dot{\mathbf{g}}$ to emphasize the difference between the kinetic energy related with ϕ_t^{tot} and the inelastic strain rate $\dot{\mathbf{E}}^{in}$ participating in the process of viscous dissipation; see [31].

5.4. Evolution equations

Introducing the action for the material in a domain $G \subset M$, corresponding to the Lagrangian L,

$$A(\phi, \phi_1, \mathbf{g}) = \int_G L dv_g,$$

we will write down the equations of evolution for the system characterized by the dynamical variables $\phi, \phi_1, \mathbf{g}$:

1. Equilibrium equation:

$$\frac{\delta A}{\delta \phi} = 0. \tag{43}$$

Since the total deformation ϕ enters only through the elastic strain tensor $\mathbf{E}^{el}$, this equation is, essentially, the equilibrium equation of elasticity

theory. If the strain energy is chosen in the form (40), this equation takes the conventional form of the elasticity equilibrium (Euler) equation with the *elastic moduli depending on the material point and temperature* (see Eq. (40)):

$$\frac{1}{\sqrt{|g|}}\frac{\partial(\rho_{ref}\sqrt{|g|}h_{ij}V^j)}{\partial t} - \mathrm{P}^I_{i\ ;I} = v_i(\phi(X)). \tag{44}$$

Here $\boldsymbol{v} = -dU(\phi)$ is the 1-form of the body forces and P^I_i is the elastic first Piola–Kirchhoff stress tensor (see Section 5.5 below). Covariant derivative is taken with respect to the material metric g.

2. Equation of plastic deformation:

$$\frac{\delta A}{\delta \phi^i_1} = -\frac{\delta \mathbf{D}}{\delta \dot{\phi}^i_1}. \tag{45}$$

Notice that in difference to the usual form of this equation [26,30], we take the variation of dissipative potential by the derivative $\dot{\phi}_1$ of *internal variable* ϕ_1 rather than the partial derivative. This is necessary due to the fact that ϕ_1 enters Lagrangian through its spatial gradient.

To clarify the form of this equation, we notice that

$$\begin{aligned}\frac{\delta A}{\delta \phi^i_1} &= -\frac{1}{\sqrt{|g|}}\partial_{X^I}\left[\frac{\partial u}{\partial \phi^i_{1,I}}\sqrt{|g|}\right] = -div_g(\mathrm{P}^I_{1\ i}) = -\left[T^{in\ N}_M \frac{\partial E^{in\ M}_N}{\partial \phi^i_{1,I}}\right]_{;I} \\ &= -\left[T^{in\ N}_M g^{MK} h_{ij}(\phi^j_{,N}\delta^I_K + \phi^j_K \delta^I_N)\right]_{;I}. \end{aligned} \tag{46}$$

Here $\mathrm{P}^I_{1\ i}$ is the first Piola–Kirchhoff stress tensor density of inelastic configuration ϕ_1; see below. Last equality is due to the fact that $\frac{\partial E^{in\ M}_{lin\ N}}{\partial \phi^i_{1,I}} = g^{MK}h_{ij}(\phi^j_{,N}\delta^I_K + \phi^j_K\delta^I_N)$.

On the other hand,

$$\begin{aligned}\frac{\delta \mathbf{D}}{\delta \dot{\phi}^i_1} &= -\partial_K\left[\frac{\partial \mathbf{D}}{\partial \dot{E}^{in\ M}_N}\frac{\partial \dot{E}^{in\ M}_N}{\partial \dot{\phi}^i_{1,K}}\right] \\ &= -\partial_K\left[\frac{\partial \mathbf{D}}{\partial \dot{E}^{in\ M}_N}\cdot\frac{1}{2}h_{ij}(g^{MK}\phi^i_{1,N} + g^{MS}\phi^j_{1,S}\delta^K_N)\right],\end{aligned}$$

where we have used Eq. (36). As a result, Eq. (45) has the form

$$div_g(\mathrm{P}^I_{1\ i}) + \partial_K \left[\frac{\partial \mathbf{D}}{\partial \dot{E}^{in\ M}_N} \cdot \frac{1}{2} h_{ij} (g^{MK} \phi^i_{1,N} + g^{MS} \phi^j_{1,S} \delta^K_N) \right] = 0. \quad (47)$$

3. Equation of metric evolution:

$$\frac{\delta A}{\delta \mathbf{g}} = -\frac{\partial \mathbf{D}}{\partial \dot{\mathbf{g}}}. \quad (48)$$

Here we have used the partial derivatives in the right side of the equation since $\mathbf{E}^m = \frac{1}{2} ln(\mathbf{g}_0^{-1}\mathbf{g})$ depends on $\mathbf{g}$ but not on its derivatives.

If the free energy depends on the scalar curvature $R(\mathbf{g})$ instead, the full Ricci tensor, Eq. (48), has the form

$$c \cdot \mathcal{E}(\mathbf{g})^{IJ} = -S^{m\ IJ} - \frac{\partial \mathbf{D}}{\partial \dot{\mathbf{E}}^m}, \quad (49)$$

where

$$S^{m\ IJ} = \frac{1}{\sqrt{|\mathbf{g}|}} \frac{\delta}{\delta g_{IJ}} \left([\frac{\rho_{ref}}{2} |\mathbf{V}|^2_h + \rho_{ref} [\mu \| \nabla^{g_o} \vartheta \|^2_{g_o} + f_0 + f] - U(\phi)] \sqrt{|\mathbf{g}|} \right)$$

is the canonical energy-momentum tensor including elastic effects, effects of inelastic deformation, and some thermal effects, and $\mathcal{E}(\mathbf{g})^{IJ}$ is the Einstein tensor of metric g [32]. If $\mathbf{g} = \mathbf{g}_o$ is the reference metric, then this equation is absent ($\mathbf{g}_0$ is fixed).

Remark 11 *In the 2-dim elasticity, any metric g in M is Einstein metric, i.e, $Ric_{IJ} = \frac{R(g)}{2} g_{IJ}$. In this case using the scalar curvature $R(g)$ instead of the Ricci tensor in (38, 41) does not place any restrictions on the material metric g.*

5.5. Stress tensors

Stress tensors characterizing the material's response to the deformations, heating and other physical processes play a crucial role in the formulation of the evolution equations and dissipative inequalities. In the material (Lagrangian) formulation, there are several stress tensors playing different roles in the dynamical picture. They are related to one another and, through the deformation ϕ, to the only stress tensor that is usually present in the Euler picture – the Cauchy tensor $\boldsymbol{\sigma}$. Such a plurality of material stress tensors is related to the presence of two material metrics, i.e., $\mathbf{g}_o$, $\mathbf{g}$, used to raise and lower the indices in tensors, and the two different Cauchy metrics – $\mathbf{C}(\phi)$, $\mathbf{C}(\phi_1)$.

Here we recall the definitions of the most useful stress tensors through the total internal energy u or the strain energy f and provide the formulas relating them to one another [14,15,33].

For the total deformation ϕ, we introduce three stress tensors defined by the differentiation of internal energy by the deformation gradient $F = \phi_*$, Cauchy metric $C(\phi)$, and the strain tensor E^{tot}.

Table 1 Stress tensors defined by total deformation ϕ.

Type	I Piola–Kirchhoff	II Piola–Kirchhoff	Strain dual
Tensor	$\mathrm{P}_i^I = \rho_{ref}\frac{\partial u}{\partial \phi^i_{,I}}$	$S^{IJ} = 2\rho_{ref}\frac{\partial u}{\partial C(\phi)_{IJ}}$	$T_J^{el\ I} = \rho_{ref}\frac{\partial u}{\partial E^{el\ J}_I}$
Relations	$\mathrm{P}_i^I = J(\phi)\sigma_i^j\phi_j^{-1\ I}$	$\mathrm{P}^{Ii} = S^{IK}\phi^i_{,K}$	$T_J^{el\ I} = S^{IK}C(\phi_1)_{KJ}$

The formula relating S and T^{el} is obtained in the assumption of linear approximation $C(\phi) = C(\phi_1) + 2C(\phi_1)E^{el}$; see Section 4.1. $J(\phi)$ here is the Jacobian of the total deformation ϕ calculated with respect to the metrics $\mathbf{h}$ and $\mathbf{g}_o$; see [15], Sec. 2.2. Expression for the tensor $S^{tot\ IJ}$ in Table 1 is the material Doyle–Erickson formula (see [33]).

For the inelastic deformation ϕ_1, we introduce three stress tensors defined by the differentiation of internal energy by the deformation gradient $F = \phi_{1*}$, Cauchy metric $C(\phi_1)$, and the strain tensor E^{in} (see Section 4.1) (Table 2).

Table 2 Stress tensors defined by inelastic deformation ϕ_1.

Type	I Piola–Kirchhoff	II Piola–Kirchhoff	Strain dual
Tensor	$\mathrm{P}^I_{1\ i} = \rho_{ref}\frac{\partial u}{\partial \phi^i_{1,I}}$	$S_1^{IJ} = 2\rho_{ref}\frac{\partial u}{\partial C(\phi)_{1\ IJ}}$	$T_J^{in\ I} = \rho_{ref}\frac{\partial u}{\partial E^{in\ J}_I}$
Relations	$\mathrm{P}^I_{1\ i} = J(\phi_1)\sigma_i^j\phi_j^{-1\ I}$	$\mathrm{P}_1^{Ii} = S_1^{IK}\phi^i_{1,K}$	$T_J^{in\ I} = S_1^{IK}g_{KJ}$

The relation between S_1 and T^{in} is obtained in the assumption of linear approximation $E^{in} = \frac{1}{2}g^{-1}(C(\phi_1) - g)$.

For the deformation (evolution) of material metric $g_0 \to g_t$, there are defined the stress tensors as shown in Table 3.

Here $P^I_{X\ i}$ is the uniformity mapping $P : V \to T_X(X)$ and the internal energy is referered to the reference volume $d_{g_0}V$, [11], Ch. 5, Sec. 5.5. Formula relating S^m and T^m is obtained in the assumption of linear approximation $E^m = \frac{1}{2}g_0^{-1}(g - g_0)$.

Table 3 Stress tensors defined by the material metric.

Type	Eshelby stress	Canonical	Strain dual
Tensor	$b^i_I = -\rho_{ref}\frac{\partial u}{\partial P^I_i}$	$S^{m\,IJ} = 2\rho_{ref}\frac{\partial u}{\partial g_{IJ}}$	$T^{m\,I}_J = \rho_{ref}\frac{\partial u}{\partial E^{m\,J}_I}$
Relations		$b^i_I = S^{m\,MN}P^{-1\;i}_M g_{NI}$	$T^{m\,I}_J = S^{m\,IK}g_{0\,KJ}$

One can define the variant of the Eshelby stress [11,12,14] by $\tilde{b}^I_J = -\rho_0\frac{\partial u}{\partial D^J_I}$ using the material (1,1)-tensor D. Its relation to the tensor b^i_I is given by $\tilde{b}^I_J = b^i_J P^{-1\;I}_{0\;i}$.

Notice also that the canonical stress tensor S^m is the direct analog of the spatial part of the energy-momentum tensor of the gravity theory [32].

Remark 12 *It is instructive to compare our definition of the* ***elastic*** *first Piola–Kirchhoff tensor with its definition as the difference (cf. [4], Ch. 10)*

$$T^{elI}_i = T^I_i - \phi^j_{2,i}T^{in\,I}_j, \tag{50}$$

where $\phi_2 = \phi \circ \phi_1^{-1}$ *is the elastic part of total deformation.*

6. Dissipation inequality

In this section we present the dissipative inequality for the (ϕ, ϕ_1, g) model.

Below, ∇ means ∇^{g_o}. We will adopt here the expression (32) for the internal energy but assume, for simplicity, the quasi-static behavior of the material (i.e., velocity $\mathbf{V}$ is negligible), potential energy U is absent, and internal energy u depends on the scalar curvature $R(\mathbf{g})$ only, instead of on the full Ricci tensor:

$$\psi = u(\mathbf{E}^{el}, \mathbf{E}^{in}, \mathbf{E}^m, R(\mathbf{g}), \vartheta, \nabla^{g_o}\vartheta) - s\theta, \tag{51}$$

with $\mathbf{E}^m = \frac{1}{2}ln(\mathbf{g}_o^{-1}\mathbf{g})$ and $\mathbf{g}$ playing the role of an internal parameter $\boldsymbol{\alpha}$ (cf. [30]).

We will be using the notations

$$\tilde{s} = -\frac{\partial\psi}{\partial\vartheta},\ \mathbf{A} = \frac{\partial\psi}{\partial\nabla\vartheta}, \tag{52}$$

and the formula for the time derivative of the scalar curvature

$$\frac{\partial}{\partial t}R(\mathbf{g}) = lim_{\Delta t\to 0}[R(\mathbf{g})(t+\Delta t) - R(\mathbf{g})(t)] = \boldsymbol{\mathcal{E}}(\mathbf{g})\cdot\dot{\mathbf{g}}, \tag{53}$$

where $\boldsymbol{\mathcal{E}}(\mathbf{g}) = \frac{\delta R(\mathbf{g})}{\delta \mathbf{g}}$ is the Einstein tensor of the material metric $\mathbf{g}$ (see [32], eq. (8.49)).

Calculate now the derivative of the free energy:

$$\dot{\psi} = \mathbf{T}^{el} \cdot \dot{\mathbf{E}}^{el} + \mathbf{T}^{in} \cdot \dot{\mathbf{E}}^{in} + \frac{1}{2}\mathbf{S}^{m} \cdot \dot{\mathbf{g}} + \boldsymbol{\mathcal{E}}(\mathbf{g}) \cdot \dot{\mathbf{g}}$$
$$-\tilde{s} \cdot \dot{\theta} + \nabla \cdot (\mathbf{A}\dot{\vartheta}) - (\nabla \cdot \mathbf{A})\dot{\vartheta}, \tag{54}$$

where the vectorial identity $\nabla \cdot (\mathbf{A}\dot{\vartheta}) = (\nabla \cdot \mathbf{A})\dot{\vartheta} + \mathbf{A} \cdot \nabla\dot{\vartheta}$ was used and where tensors T^{in}, T^{el}, S^{m} are as in the Section 5.

Recalling the formula for the variation

$$\frac{\delta \psi}{\delta \mathbf{g}} = \frac{\partial \psi}{\partial \mathbf{g}} - \nabla \cdot \frac{\partial \psi}{\partial \nabla \mathbf{g}_{o} \mathbf{g}}$$

and using the notation $\mathcal{A} = \frac{\delta \psi}{\delta \mathbf{g}}$, we find

$$[\frac{1}{2}\mathbf{S}^{m} + \boldsymbol{\mathcal{E}}(\mathbf{g})] \cdot \dot{\mathbf{g}} = \frac{\delta \psi}{\delta \mathbf{g}} \cdot \dot{\mathbf{g}} = \mathcal{A}(\mathbf{g}) \cdot \dot{\mathbf{g}}. \tag{55}$$

Then the time derivative of the free energy takes the form

$$\dot{\psi} = \mathbf{T}^{el} \cdot \dot{\mathbf{E}}^{el} + \mathbf{T}^{in} \cdot \dot{\mathbf{E}}^{in} + \mathcal{A}(\mathbf{g}) \cdot \dot{\mathbf{g}} - \tilde{s} \cdot \dot{\vartheta} - \nabla \cdot (\mathbf{A})\dot{\vartheta} + \nabla \cdot (\mathbf{A}\dot{\vartheta}). \tag{56}$$

Recall now the Gibbs inequality for a thermodynamical system with internal parameter $\boldsymbol{\alpha}$ (here $\boldsymbol{\alpha} = \mathbf{g}$); see [30]:

$$-(\dot{\psi} + s\dot{\theta}) + p_i + \nabla \cdot (\theta \mathbf{k}) - (s \cdot \nabla)\theta \geq 0. \tag{57}$$

Here

$$p_i = \mathbf{T} \cdot \dot{\mathbf{E}}^{tot}$$

is the power of the internal work, stress tensor $\mathbf{T}$ will be specified below, and $\mathbf{k}$ is the extra entropy flux density assumed to include contributions from the flux of the internal variables.

Substituting the expression (56) for $\dot{\psi}$ into the Gibbs inequality (57), we get

$$-\mathbf{T}^{el} \cdot \dot{\mathbf{E}}^{el} - \mathbf{T}^{in} \cdot \dot{\mathbf{E}}^{in} - \mathcal{A}(g) \cdot \dot{\mathbf{g}} + \tilde{s} \cdot \dot{\vartheta} + \nabla \cdot (\mathbf{A}\dot{\vartheta}) -, \tag{58}$$
$$\nabla \cdot (\mathbf{A}\dot{\vartheta} - s\dot{\theta}) + p_i + \nabla \cdot (\vartheta \mathbf{k}) - (s \cdot \nabla)\vartheta \geq 0. \tag{59}$$

In the special case of when one uses the linearized definitions of strain tensors (see Section 4.1) and the commutativity condition that allows us to write the

total strain rate $\dot{E}^{tot}$ in the form (23) is fulfilled, the previous inequality takes the form

$$\begin{aligned}&(\mathbf{T}(1+2E^m)(1+2E^{in})-\mathbf{T}^{el})\cdot\dot{\mathbf{E}}^{el}\\&\quad+(\mathbf{T}(1+2E^m)(1+2E^{el})-\mathbf{T}^{in})\cdot\dot{\mathbf{E}}^{in}\\&\quad+\mathbf{T}(1+2E^{el})(1+2E^{in})\cdot\dot{\mathbf{E}}^m-\mathcal{A}(\mathbf{g})\cdot\dot{\mathbf{g}}\\&\quad+(s-\tilde{s}+\nabla\cdot\mathbf{A})\dot{\vartheta}+\nabla\cdot(\theta\mathbf{k}-\mathbf{A}\dot{\theta})-(s\cdot\nabla)\theta\geq 0.\end{aligned}\tag{60}$$

Now we use the fact that the derivatives $\dot{E}^{el}$, $\dot{\vartheta}$ are controllable variables and can take arbitrary positive and negative values and, therefore, their coefficients should be equal to zero [30]. Thus we obtain the relations

$$\mathbf{T}=\mathbf{T}^{el}(1+2E^{in})^{-1}(1+2E^m)^{-1}\tag{61}$$

and

$$s=-(\tilde{s}-\nabla\cdot\mathbf{A})=-\Big(\frac{\partial\psi}{\partial\vartheta}-\nabla\cdot\frac{\partial\psi}{\partial\nabla\vartheta}\Big)=-\frac{\delta\psi}{\delta\vartheta}.\tag{62}$$

Assuming for $\mathbf{k}$ the prescription

$$\mathbf{k}=\vartheta^{-1}\mathbf{A}\dot{\vartheta}=\vartheta^{-1}\frac{\partial\psi}{\partial\nabla\vartheta}\dot{\vartheta},\tag{63}$$

the reduced dissipation inequality is obtained in the form

$$\begin{aligned}&[\mathbf{T}^{el}(1+2E^{in})^{-1}(1+2E^{el})-\mathbf{T}^{in}]\cdot\dot{\mathbf{E}}^{in}\\&\quad+[\mathbf{T}^{el}(1+2E^{in})^{-1}(1+2E^m)^{-1}(1+2E^{el})(1+2E^{in})\\&\quad-2\mathcal{A}(\mathbf{g})g_0]\cdot\dot{\mathbf{E}}^m-(s\cdot\nabla)\vartheta\geq 0,\end{aligned}\tag{64}$$

where we have used the expression $\dot{\mathbf{E}}^m=\frac{1}{2}(g_0^{-1}\dot{g}-I)=\frac{1}{2}g_0^{-1}\dot{g}$ for the linearized metric strain tensor $E^m=\frac{1}{2}(g_0^{-1}g-I)$.

Dissipation inequality (64) is satisfied if one requests the independent fulfillment of the stronger conditions – two intrinsic dissipation inequalities:

$$\begin{cases}[\mathbf{T}^{el}(1+2E^{in})^{-1}(1+2E^{el})-\mathbf{T}^{in}]\cdot\dot{\mathbf{E}}^{in}\geq 0,\\ [T^{el}(1+2E^{in})^{-1}(1+2E^m)^{-1}(1+2E^{el})(1+2E^{in})-2\mathcal{A}(\mathbf{g})g_0]\cdot\dot{\mathbf{E}}^m\geq 0,\end{cases}\tag{65}$$

and the thermal dissipation inequality:

$$-(s\cdot\nabla)\theta\geq 0.\tag{66}$$

Using the relation between the tensor S^m and T^m presented in Table 3, we can rewrite second inequality in the form

$$[T^{el}(1+2E^{in})^{-1}(1+2E^m)^{-1}(1+2E^{el})(1+2E^{in}) - T^m - 2\mathcal{E}(g)g_0]\cdot\dot{\mathbf{E}}^m \geq 0. \tag{67}$$

Comparing inequalities (65–67) with similar dissipative inequalities in [30, 31], we see that the coefficient of $\dot{E}^{in}$ (respectively, $\dot{E}^m$) can be interpreted as the *effective stress tensor* for integrable inelastic deformation (respectively, for evolution of the uniform structure). Such modifications of the stress tensors are customary in studying the entropy production by a combination of interrelated elastic and inelastic processes; cf. [14], Ch. 10.

6.1. Yield condition from dissipative inequality

If all three strain tensors in Eq. (67) are small (in comparison with the unit tensor), the inequalities (67) take the (approximate) form

$$\begin{cases} [\mathbf{T}^{el} - \mathbf{T}^{in}]\cdot\dot{\mathbf{E}}^{in} \geqq 0, \\ [T^{el} - T^m - 2\mathcal{E}(g)g_0]\cdot\dot{\mathbf{E}}^m \geqq 0. \end{cases} \tag{68}$$

These inequalities can be interpreted as the yield conditions determining when the corresponding type of inelastic evolution (plastic integrable: $\dot{\phi}_1 \neq 0$ and/or material metric $\dot{g}_t \neq 0$, respectively) may proceed. In each case, the elastic stress $\mathbf{T}^{el}$ should be large enough to overcome the barrier necessary for initiation of the corresponding process.

This form of yield condition is similar to the condition for the plastic deformation to proceed obtained from the Drucker postulate; see [34], Sec. 8.11, inequality (8.85).

Solutions of evolution equations (45) and (48) describe also the evolution of stress tensors T^{in}, T^m. Therefore, the conditions (68) for elastic stress T^{el} evolves in time. This evolution can be related to the *hardening processes* during an elasto-plastic deformation of materials.

Consider, for instance, a homogeneous isotropic case. Let Q_{IJ} be a symmetric (0,2)-tensor. The evolution in the direction of this tensor, i.e., the evolution for which $\dot{E}^{in}_{IJ} = \lambda(t)Q_{IJ}$, $\lambda(t) > 0$, may proceed only if the difference $(T^{el}_{IJ} - T^{in}_{IJ})$ is such that $Tr_g((T^{el}_{IJ} - T^{in}_{IJ})Q^{IJ}) \geqq 0$, i.e., if this difference is positive in the direction of tensor Q_{IJ}.

Leaving further study and comparison of these conditions with the usual yield criteria [34,35] for future work, we notice only that the conditions (68) are

anisotropic by nature and might possibly provide a useful supplement to the usual criteria in essentially anisotropic situations.

7. Conclusions

In this work we analyzed the relation between the Bilby–Kröner–Lee multiplicative decomposition $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$ of the total deformation gradient into elastic and plastic factors [2–4,12] and the theory of uniform materials [7–9]. We prove that the Bilby–Kröner–Lee multiplicative decomposition is equivalent to the uniform material model with two deformation mappings, i.e., the total ϕ and the inelastic ϕ_1 deformations together with the uniformity structure. Uniformity enters through the (1,1)-tensor field $\mathbf{D}$ in the material manifold M or through the material metric $\mathbf{g}$. We introduced the total, the elastic, and the inelastic strain tensors characterizing different types of the geometrical evolution of the material. After discussing the relations between these strain tensors and the deformation gradients $\mathbf{F}^e$ and $\mathbf{F}^p$, we chose the form of the internal energy (38) and of the dissipative potential (42) for the materials modeled by the triple $(\phi, \phi_1, \mathbf{g})$. The evolution equations were written down for all dynamical variables $(\phi, \phi_1, \mathbf{g})$. We discussed different types of stress tensors that naturally enter the scheme of our work. Finally, we wrote down the dissipative inequalities for the materials of $(\phi, \phi_1, \mathbf{g})$-type, where the terms corresponding to the different types of dissipative processes are separated.

Further research along the lines indicated in this paper seems to be in order. First, one should compare our results with those obtained by Maugin in a different framework [14,30]. Second, in the continuation of this work we will study the evolution equations (43), (45), (48), obtain the energy balance law, and the heat equation that follows from it along the lines of [14]. Third, some special cases and examples will be considered.

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8. Appendix

In this Appendix we present the calculation of the total strain rate $\dot{E}^{tot}$ that was used in Section 6.

From the formula (23) for the linearized definition of strain tensors, we get

$$E^{tot} = g_0^{-1} C(\phi_1) E^{el} + g_0^{-1} g E^{in} + E^m.$$

Taking derivative, we get

$$\dot{E}^{tot} = g_0^{-1} C(\dot{\phi}_1) E^{el} + g_0^{-1} C(\phi_1) \dot{E}^{el} + g_0^{-1} \dot{g} E^{in} + g_0^{-1} g \dot{E}^{in} + \dot{E}^m. \quad (69)$$

From the definition of linearized $E^{in} = \frac{1}{2} g^{-1}(C(\phi_1) - g)$, we get $C(\phi_1) = g + 2gE^{in}$ and, therefore, $\dot{C}(\phi_1) = \dot{g} + 2\dot{g}E^{in} + 2g\dot{E}^{in}$. As a result,

$$g_0^{-1} C(\dot{\phi}_1) = g_0^{-1}(\dot{g} + 2\dot{g}E^{in} + 2g\dot{E}^{in}) = 2\dot{E}^m + 4\dot{E}^m E^{in} + 2(1 + 2E^m)\dot{E}^{in},$$

where we have used $g_0^{-1} g = 1 + 2E^m$.

In the second term in Eq. (69), $g_0^{-1} C(\phi_1) = g_0^{-1} g g^{-1} C(\phi_1) = (1 + 2E^m)(1 + 2E^{in})$, in the third one, $g_0^{-1}\dot{g} = 2\dot{E}^m$. Substituting these expressions into Eq. (69) and collecting coefficients of strain rate tensors, we get

$$\dot{E}^{tot} = (1+2E^m)(1+2E^{in})\dot{E}^{el} + (1+2E^m)\dot{E}^{in}(1+2E^{el}) + \dot{E}^m(1+2E^{el})(1+2E^{in}). \quad (70)$$

In a case where strain tensors participating in the second and third terms of the last formula commute with the corresponding strain rate tensor, we get

$$\dot{E}^{tot} = (1+2E^m)(1+2E^{in})\dot{E}^{el} + (1+2E^m)(1+2E^{el})\dot{E}^{in} + (1+2E^{el})(1+2E^{in})\dot{E}^m. \quad (71)$$

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