2002

Multiple-Valued Quantum Logic Synthesis

Marek Perkowski
Portland State University, marek.perkowski@pdx.edu

Anas Al-Rabadi
Portland State University

Pawel Kerttopf
Warsaw University of Technology

Multiple-Valued Quantum Logic Synthesis

2002 International Symposium on New Paradigm VLSI Computing

December 12, 2002, Sendai, Japan,

- **Marek A. Perkowski*, Anas Al-Rabadi^ and Pawel Kerntopf+**
- *Department of Electrical Engineering and Computer Science, Korea Advanced Institute of Science and Technology (**KAIST**), 373-1 Gusong-dong, Yusong-gu, Daejeon 305-701, KOREA, *mperkows@ee.kaist.ac.kr*
- ^ Department of Electrical and Computer Engineering, Portland State University, Portland, Oregon, 97207-0751, 1900 S.W. Fourth Ave. *alrabadi@ee.pdx.edu*
- +Institute of Computer Science, Warsaw University of Technology, Nowowiejska 15/19, 00-665 Warsaw, POLAND, *pke@ii.pw.edu.pl*

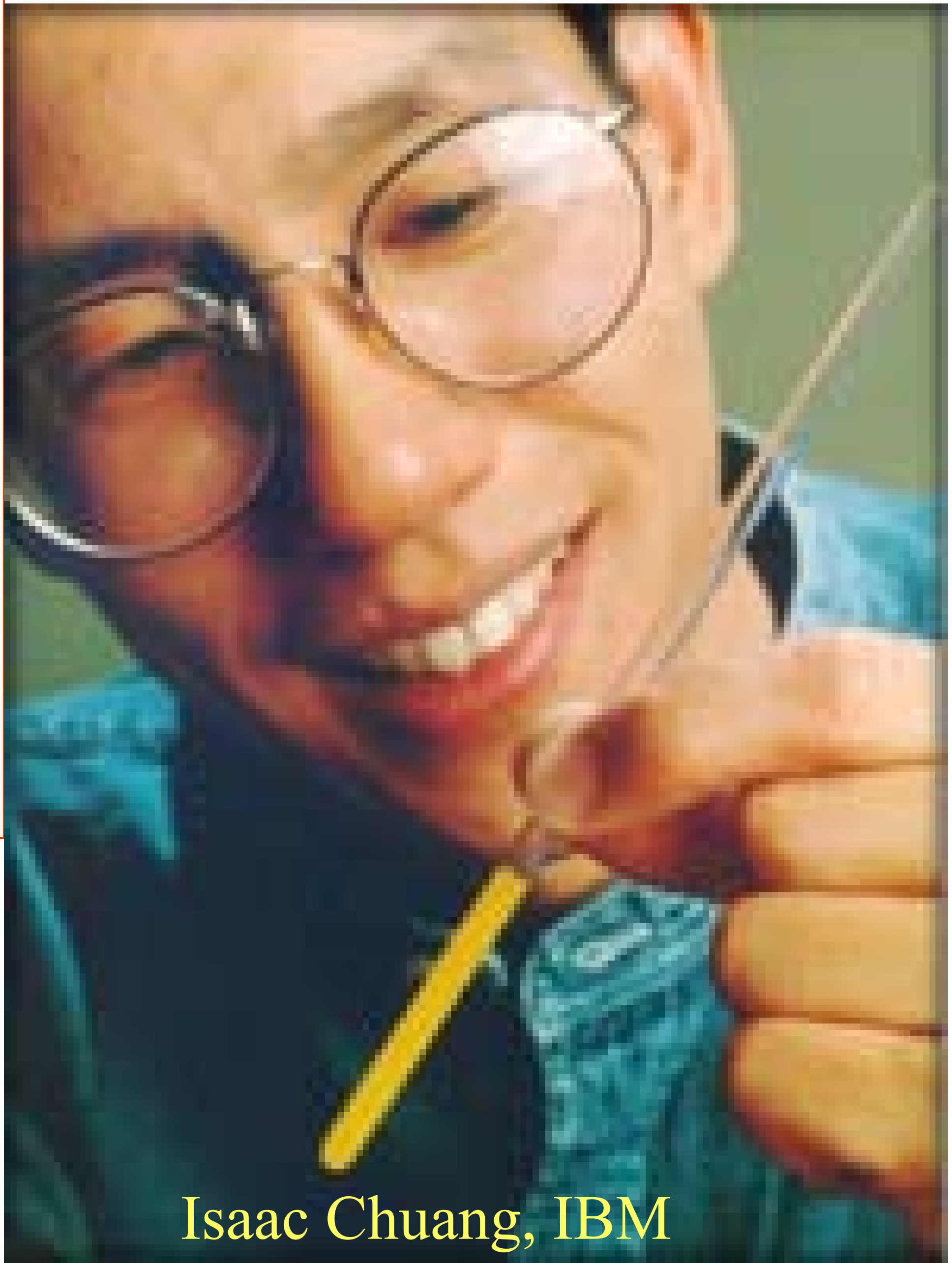
What size of (binary) Quantum Computers can be build in year 2002?

- *7 bits*

Is logic synthesis for quantum computers a practical research subject?

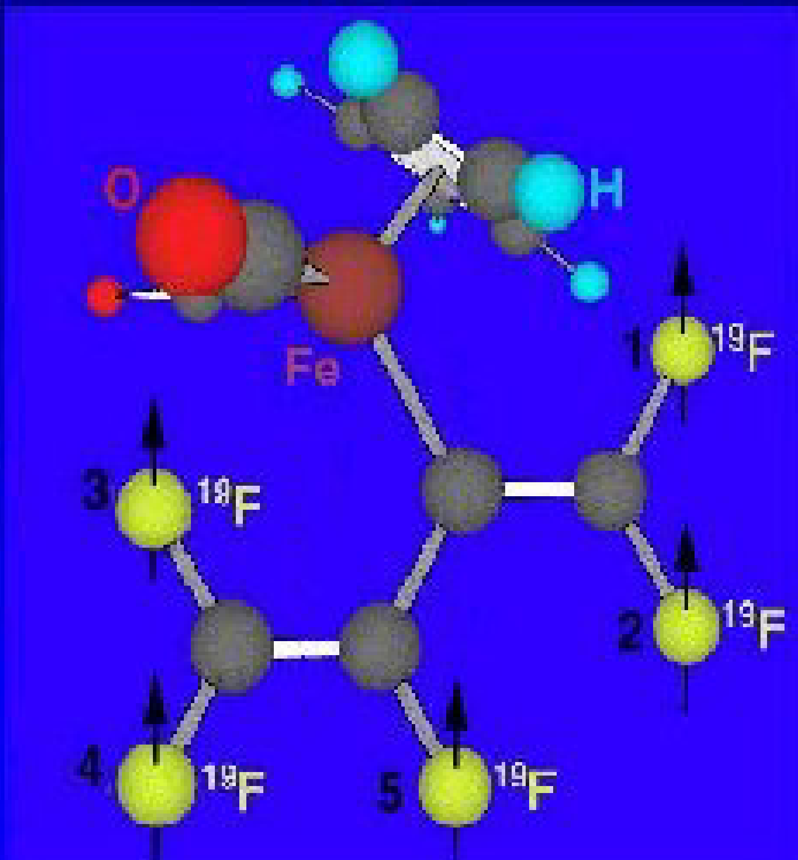
Yes, it is a useful technique for physicists who are mapping logic operations to NMR computers.

CAD for physicists.

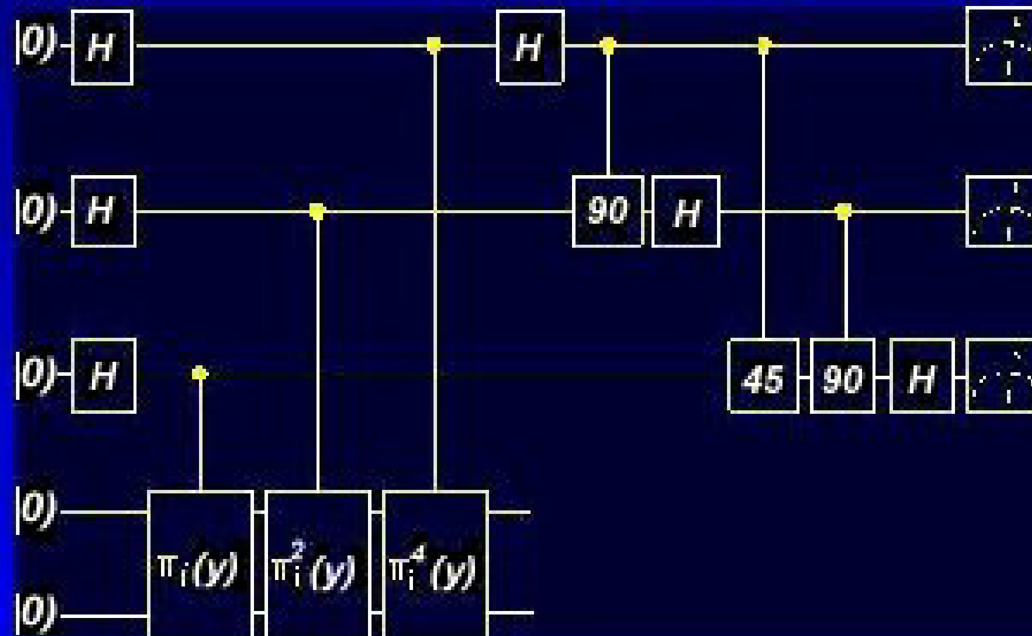


5 qubit 215 Hz Q. Processor

(Vandersypen, Steffen, Breyta Yannoni, Cleve, and Chuang, 2000)



- The Molecule

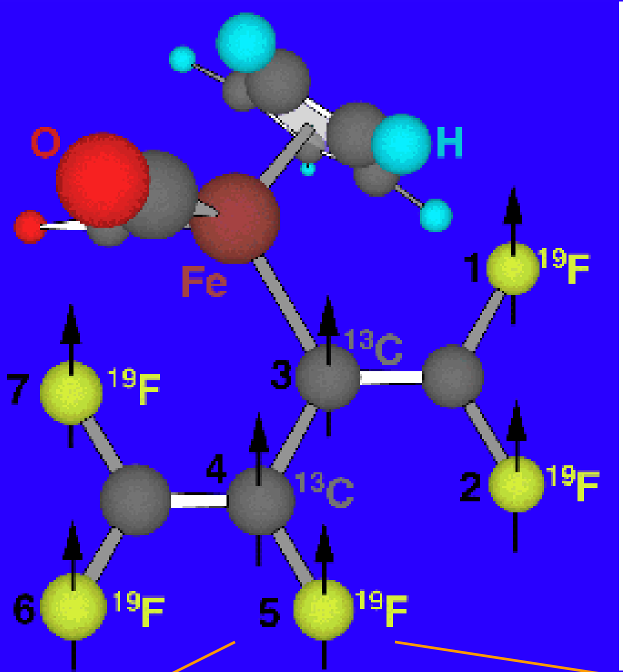


- Quantum Circuit

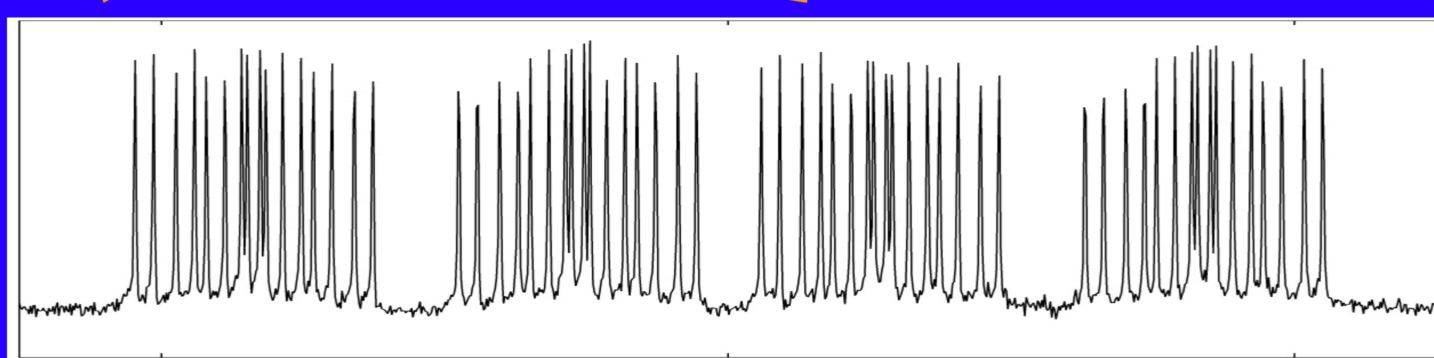
T_2 > 0.3 sec ; ~ 200 gates

Source: IBM, Hot Chips Conference, 2000

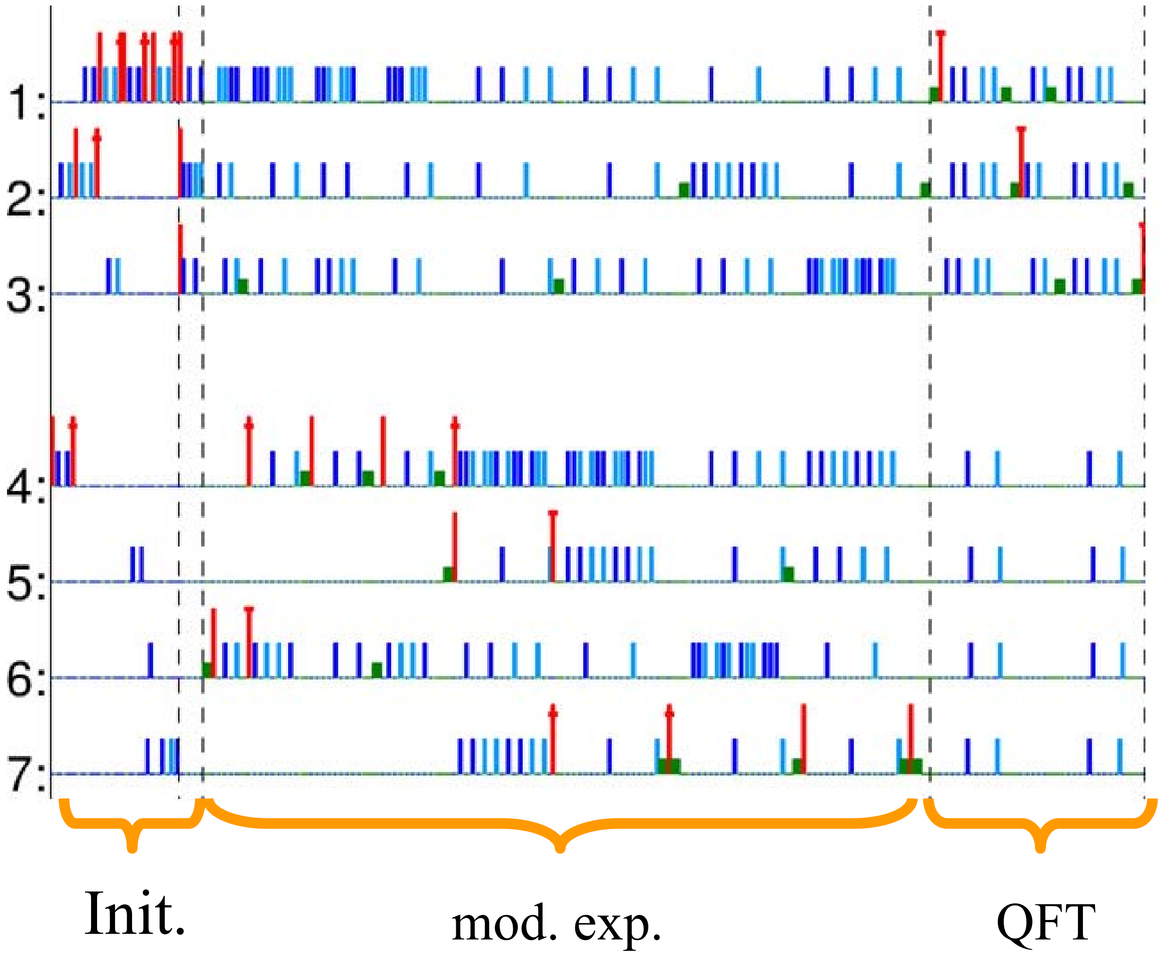
The molecule



i	$\omega_i/2$	$T_{1,i}$	$T_{2,i}$	J_{7i}	J_{6i}	J_{5i}	J_{4i}	J_{3i}	J_{2i}
1	−22052.0	5.0	1.3	−221.0	37.7	6.6	−114.3	14.5	25.16
2	489.5	13.7	1.8	18.6	−3.9	2.5	79.9	3.9	
3	25088.3	3.0	2.5	1.0	−13.5	41.6	12.9		
4	−4918.7	10.0	1.7	54.1	−5.7	2.1			
5	15186.6	2.8	1.8	19.4	59.5				
6	−4519.1	45.4	2.0	68.9					
7	4244.3	31.6	2.0						

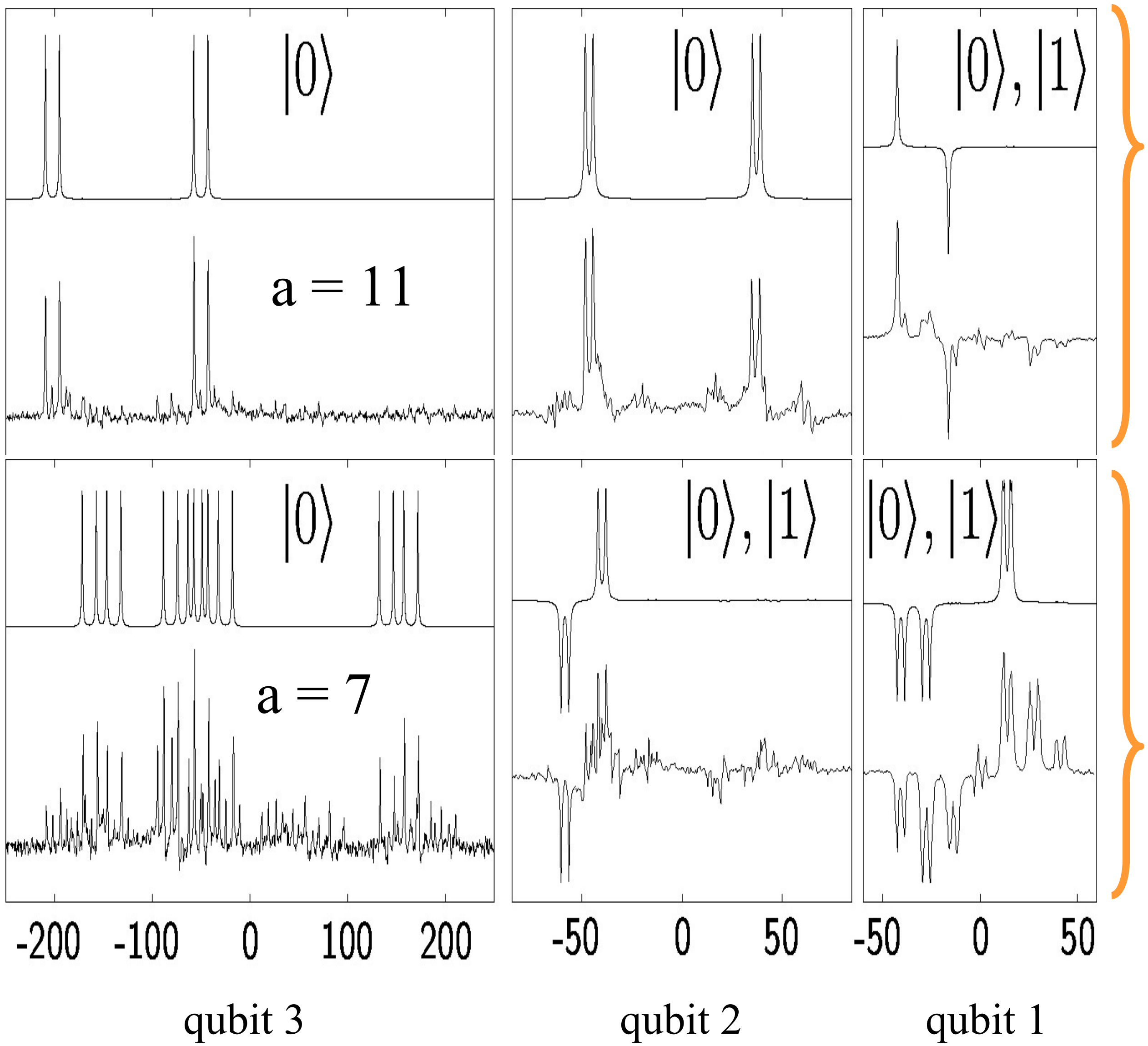


Pulse Sequence



~ 300 RF pulses || ~ 750 ms duration

Results: Spectra



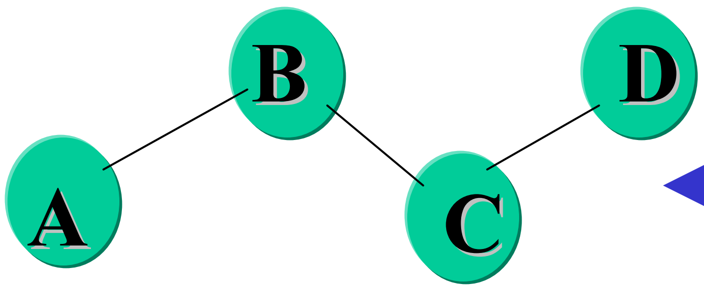
Mixture of $|0\rangle, |4\rangle$
$2^3/4 = r = 2$
$\gcd(11^{2/2} \pm 1, 15) = 3, 5$

$15 = 3 \cdot 5$

Mixture of $|0\rangle, |2\rangle, |4\rangle, |6\rangle$
$2^3/2 = r = 4$
$\gcd(7^{4/2} \pm 1, 15) = 3, 5$

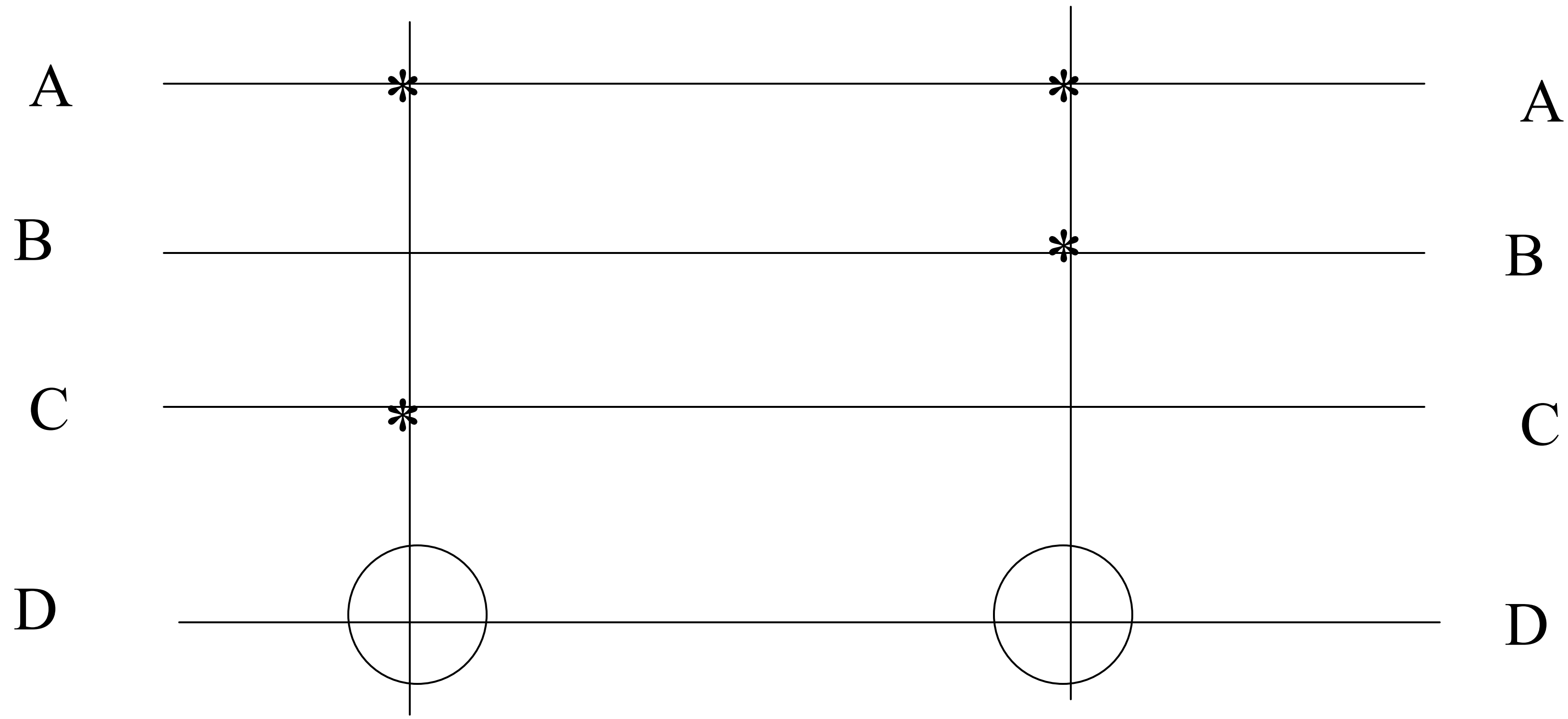
Problem

- We would like to assume that any two quantum wires can interact, but we are limited by the realization constraints
- Structure of atomic bonds in the molecule determines neighborhoods in the circuit.
- This is similar to restricted routing in FPGA layout - *link between logic and layout synthesis* known from CMOS design **now appears in quantum**.
- Below we are interested only in the so-called **"permutation circuits"** - their unitary quantum matrices are permutation matrices



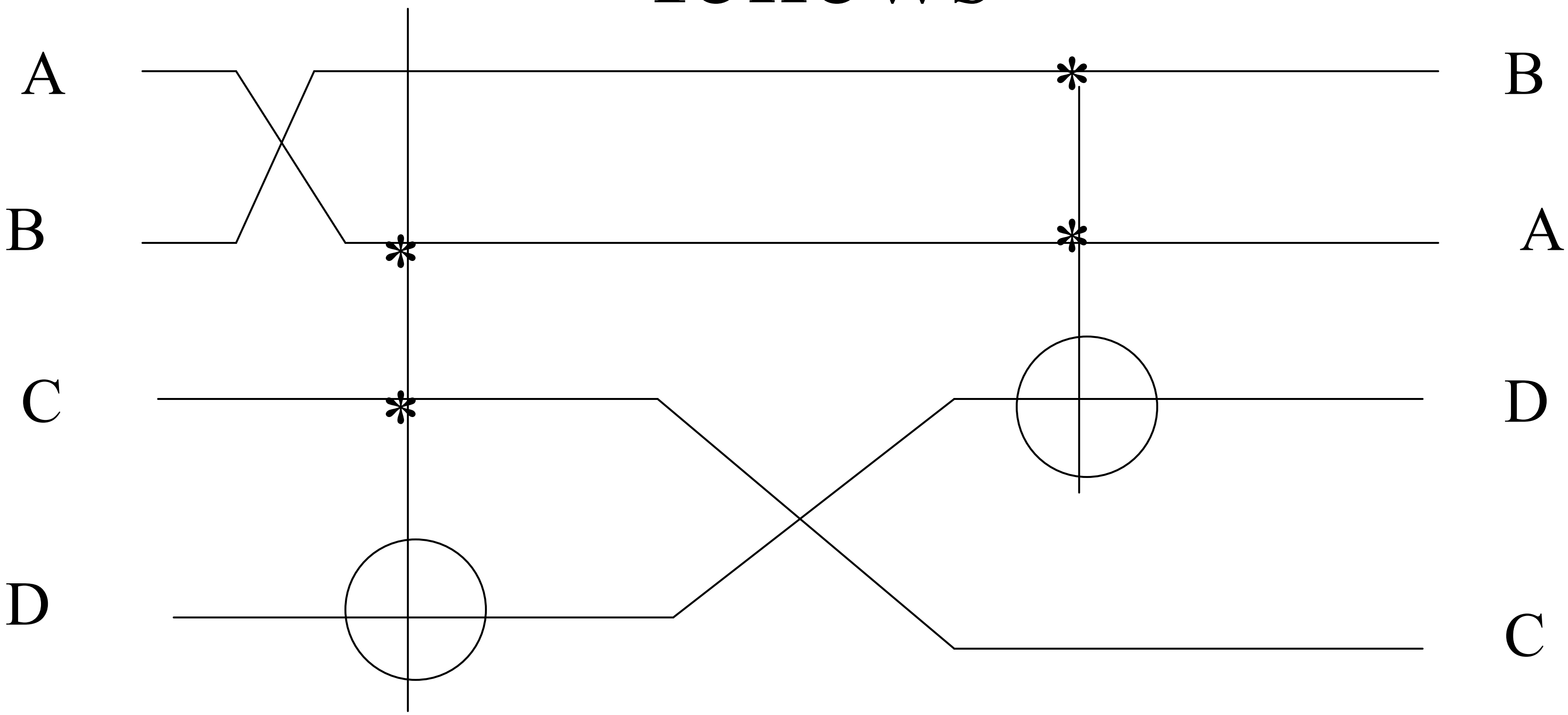
Quantum wires A and C are not neighbors

A schematics with two binary Toffoli gates

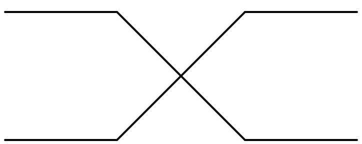


This is a result of our ESOP minimizer program, but this is not realizable in NMR for the above molecule

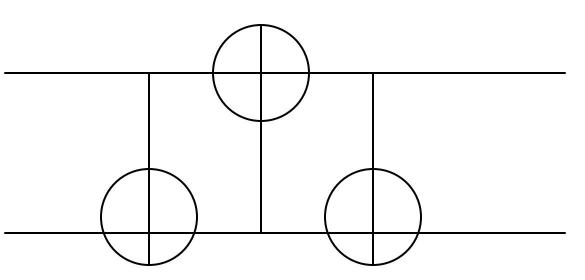
So I modify the schematics as follows



But this costs me two swap gates



Costs 3
Feynmans



Solution

- One solution to connection problem in VLSI has been to increase the number of values in wires.
- Have a **“quantum wire”** have **more than two** eigenstates.
- Increase from superpositions of 2^n to **superpositions of 3^n**
- Basic gate in quantum logic is a **2*2** (2-qubit gate). We have to build from such gates.

Can we build multiple-valued Quantum Computers in year 2002?

- *In principle, yes*

Has one tried?

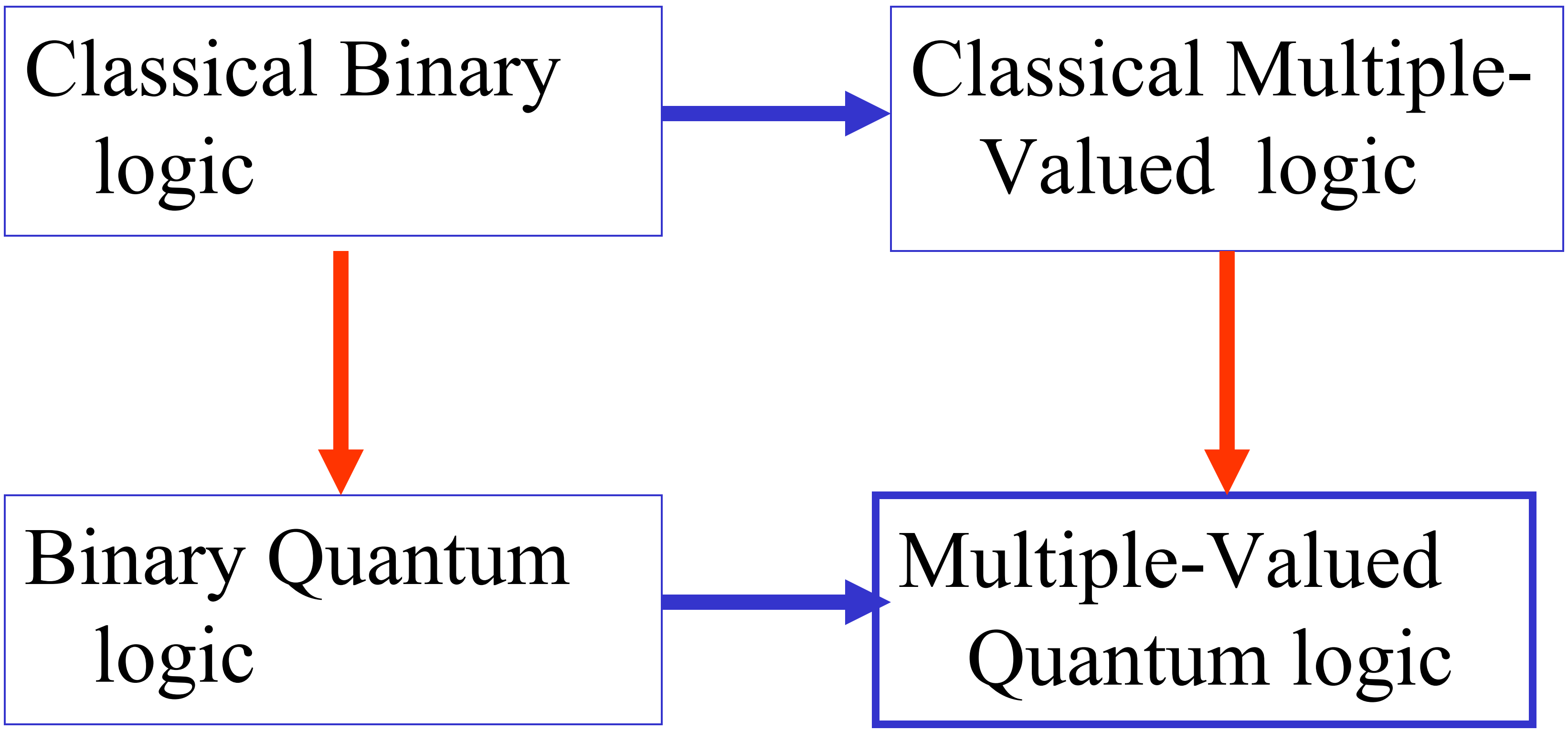
No.

Gates, yes

Qudits not qubits

- In ternary logic, the notation for the superposition is **$\alpha|0\rangle + \beta|1\rangle + \gamma\,|2\rangle$.**
- These intermediate states cannot be distinguished, rather a **measurement** will yield that the qudit is in one of the basis states, |0>, |1> or |2>.
- The probability that a measurement of a qudit yields state |0> is $|\alpha|^2$, the probability is $|\beta|^2$ for state |1> and $|\gamma|^2$ for state γ . The sum of these probabilites is one. The absolute values are required, since in general, α β and γ are

The concept of Multiple-Valued Quantum Logic



What is known?

- **Mattle** 1996 - *Trit |0>, |1>, |2>*
- Chau 1997 - *qudit, error correcting quantum codes*
- Ashikhmin and Knill 1999, *MV codes.*
- Gottesman, Aharonov and Ben-Or 1999 - *MV fault tolerant gates.*
- Burlakov 1999 - *correlated photon realization of ternary qubit.*
- Muthukrishnan and Stroud 2000 - *multi-valued universal quantum logic for linear ion trapped devices.*
- Picton 2000 - *Multi-valued reversible PLA.*
- *Perkowski, Al-Rabadi, Kerntopf and Portland Quantum Logic Group 2001 - Galois Field quantum logic synthesis*

What is known?

- Al-Rabadi, 2002 - *ternary EPR and Chrestenson Gate*
- De Vos 2002 - *Two ternary 1*1 gates and two ternary 2*2 gates for reversible logic*.
- Zilic and Radecka 2002 - *Super-Fast Fourier Transform*
- Bartlett et al, 2002 - *Quantum Encoding in Spin Systems*
- Brassard, Braunstein and Cleve, 2002 - *Teleportation*
- Rungta, Munro et al *Qudit Entanglement*.

Ternary Galois Field (GF3) operations.

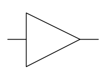
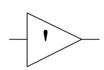
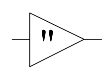
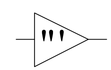
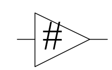
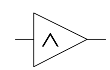
+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

(a) Addition

•	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

(b) Multiplication

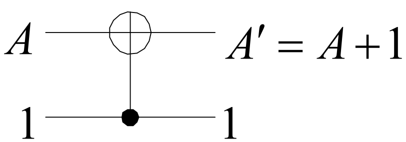
Reversible ternary shift operations.

Operator Name		Buffer	Single-Shift	Dual-Shift	Self-Shift	Self-Single-Shift	Self-Dual-Shift
Operator symbol & equation		A	$A' = A + 1$	$A'' = A + 2$	$A''' = A + A = 2A$	$A^{\#} = 2A + 1$	$A^{\wedge} = 2A + 2$
Gate symbol							
A	0	0	1	2	0	1	2
	1	1	2	0	2	0	1
	2	2	0	1	1	2	0

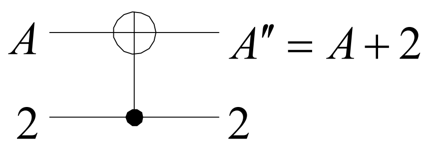
Conversion of one shift form to another shift form using ternary shift gates

Input	Output					
	A	A'	A''	A'''	$A^{\#}$	$A^{\wedge}$
A		'	''	'''	#	^
A'	''		'	#	^	'''
A''	'	''		^	'''	#
A'''	'''	#	^		'	''
$A^{\#}$	#	^	'''	''		'
$A^{\wedge}$	^	'''	#	'	''	

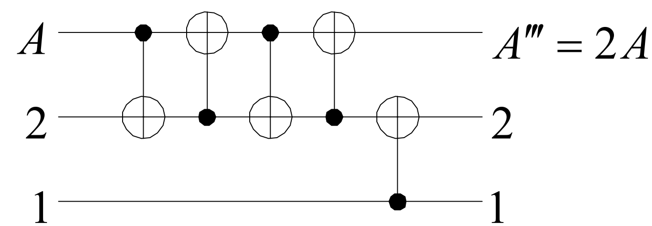
Quantum realization of ternary shift gates.



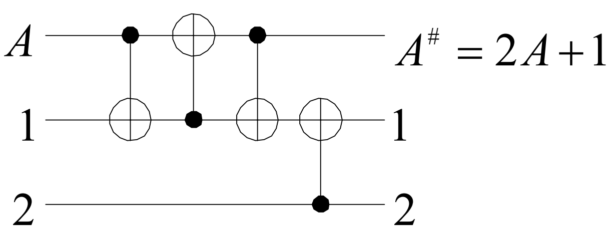
(a) Single-Shift



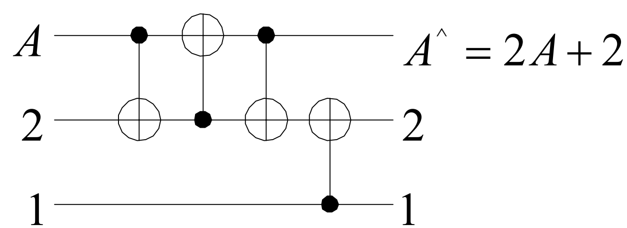
(b) Dual-Shift



(c) Self-Shift

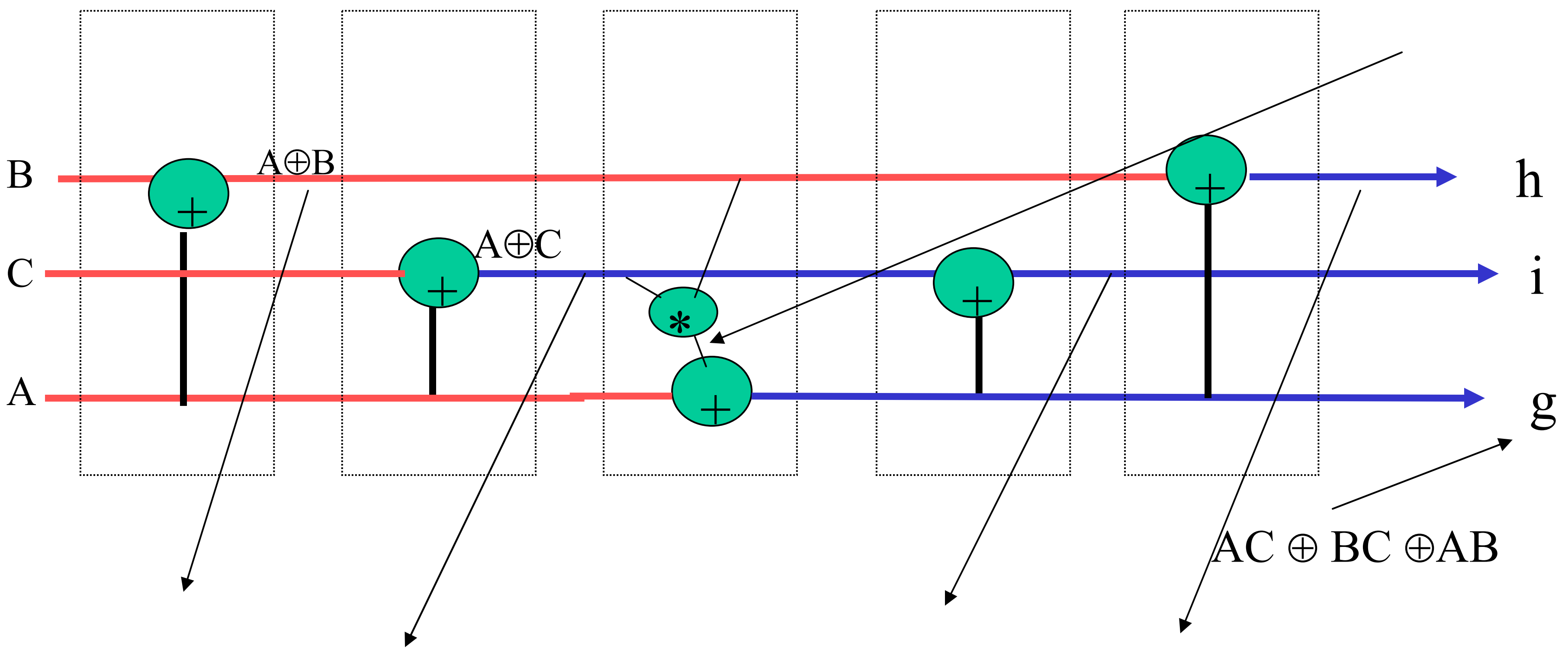


(d) Self-Single-Shift



(e) Self-Dual-Shift

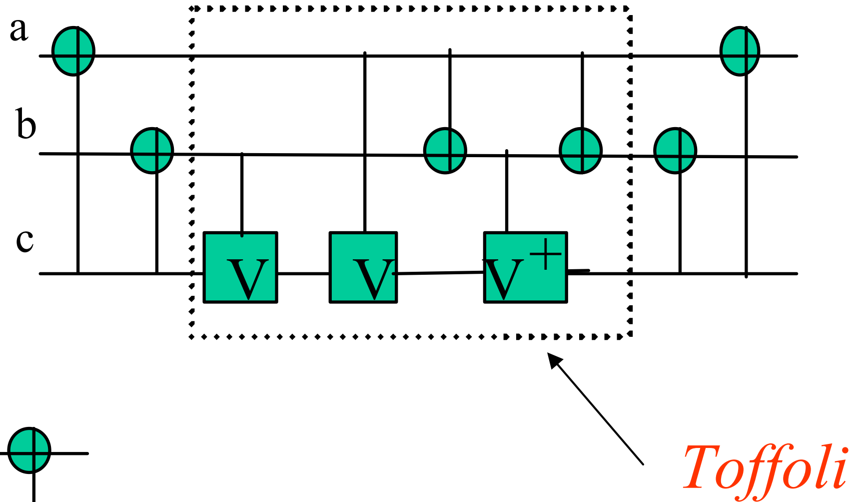
Optimal Solution to Ternary Miller Function



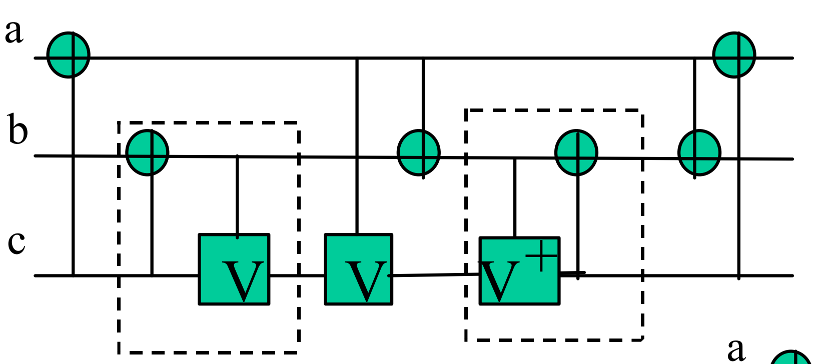
Check ternary maps

2-qubit quantum realization of Miller Gate

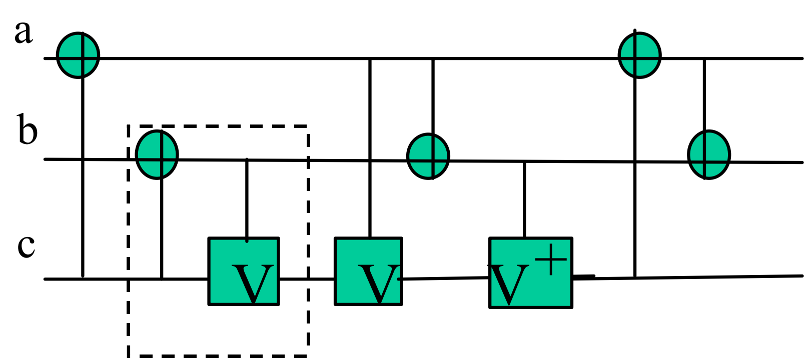
a)



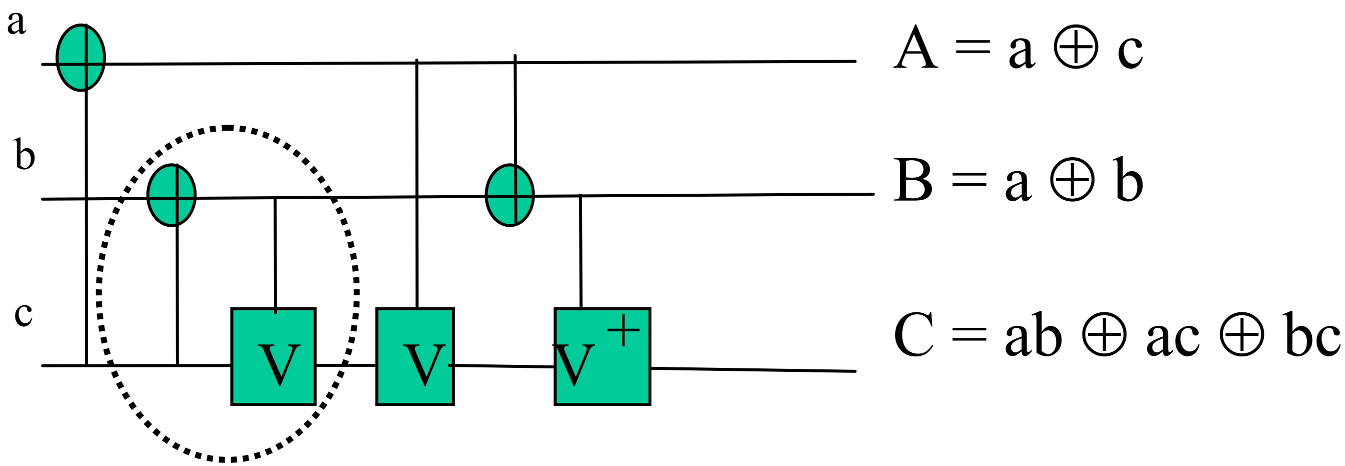
b)



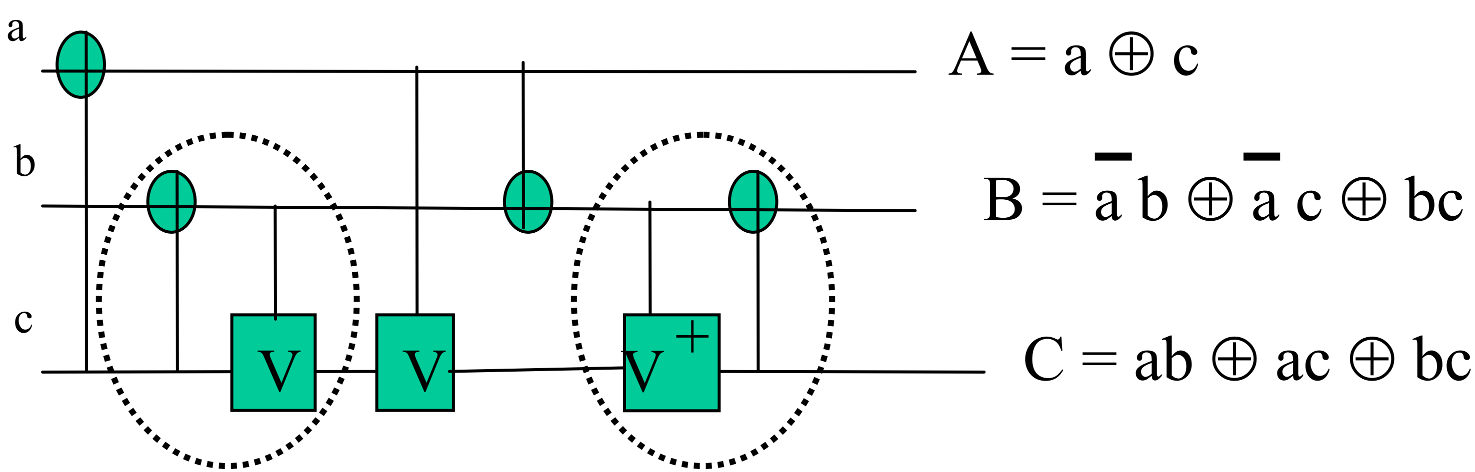
c)



These techniques can be also applied for Multiple-Valued Quantum Logic a)



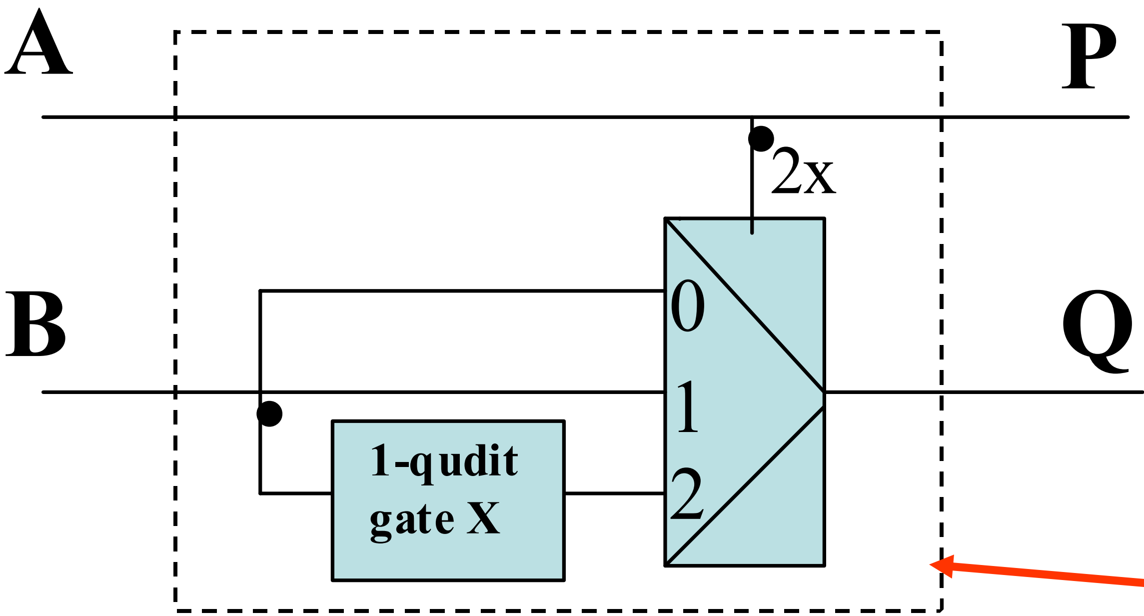
b)



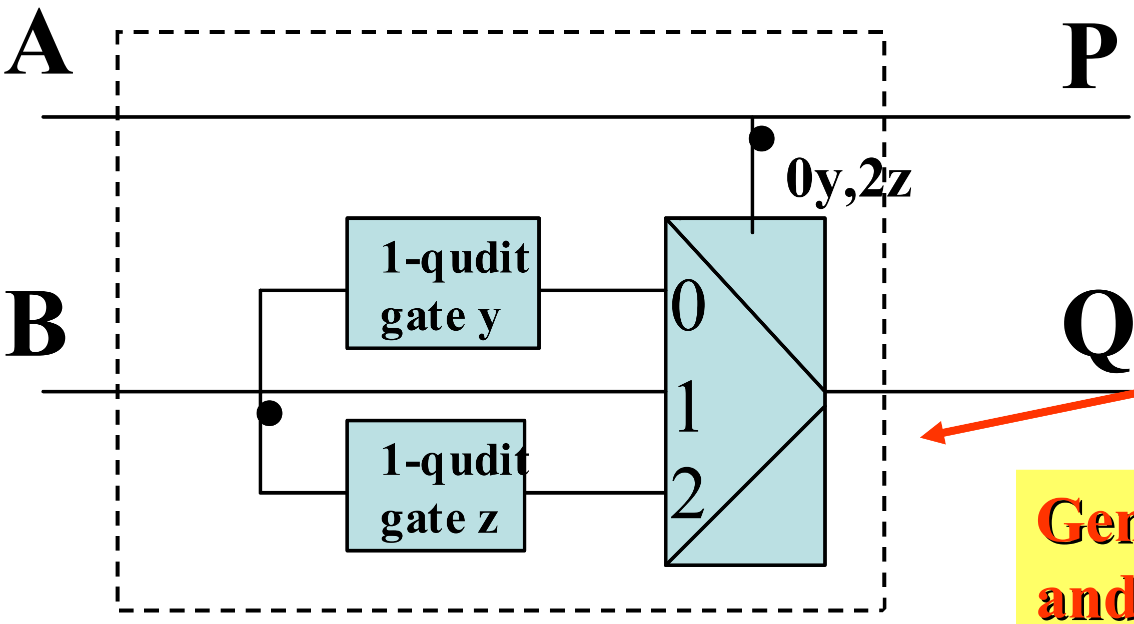
Design a Ternary Toffoli Gate from 2-qubit quantum primitives

Ternary controlled gates

First task is to demonstrate that a universal 3-qubit gate can be built from MV quantum primitives

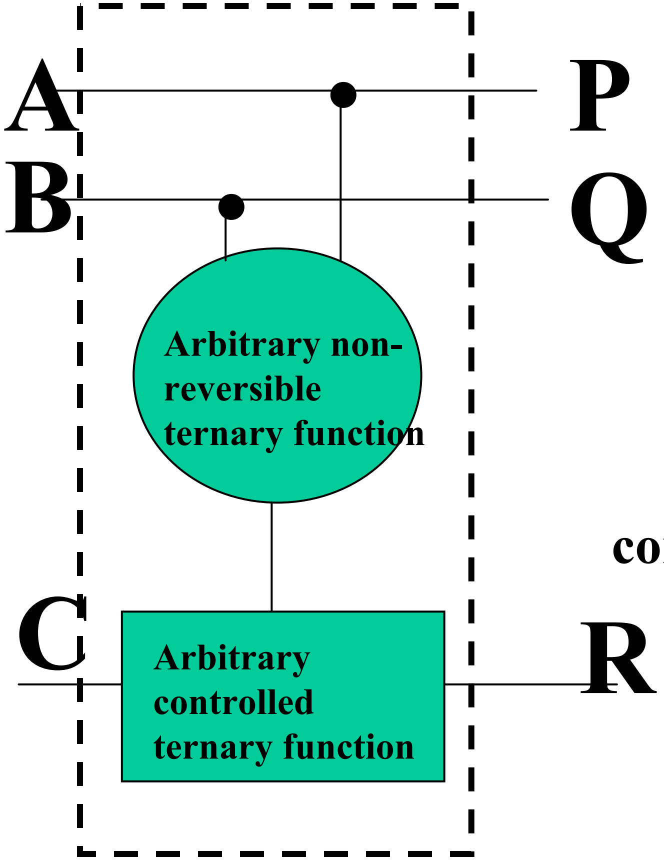


2-qubit controlled gate with controlling value d-1 = 2

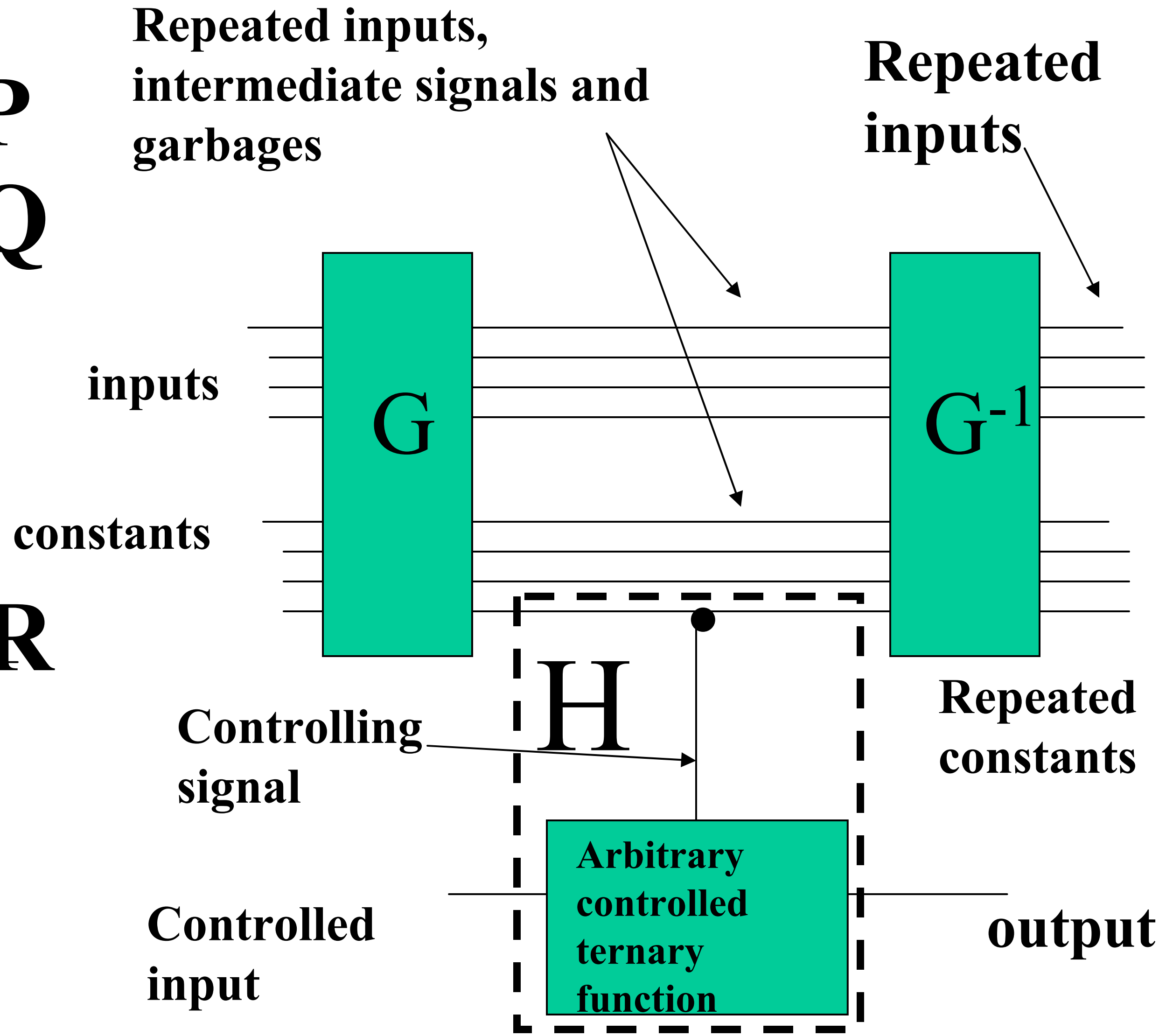


2-qubit controlled gate with controlling values 0 and 2

Generalization of Stroud and De Vos gates

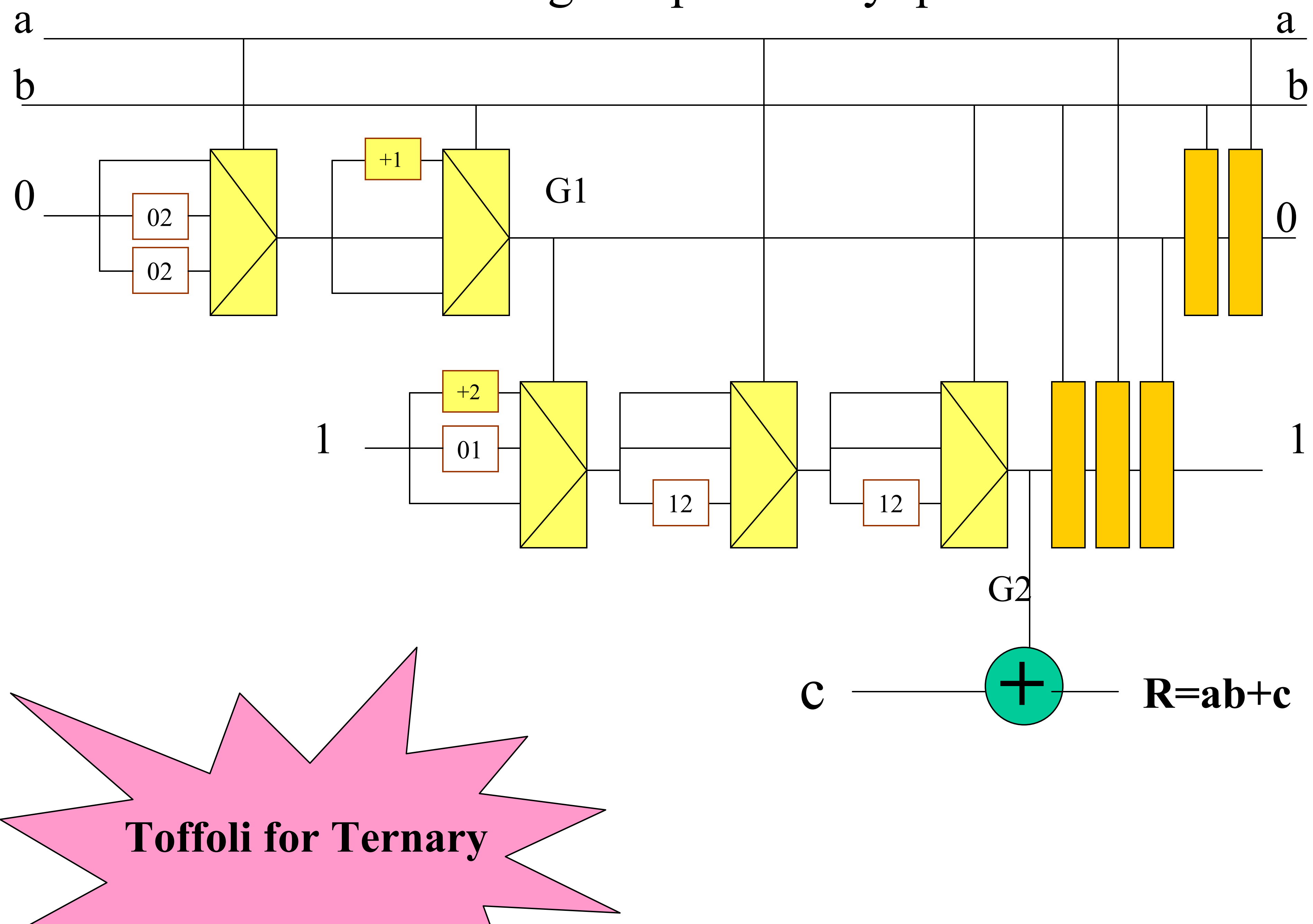


Ternary generalized Toffoli

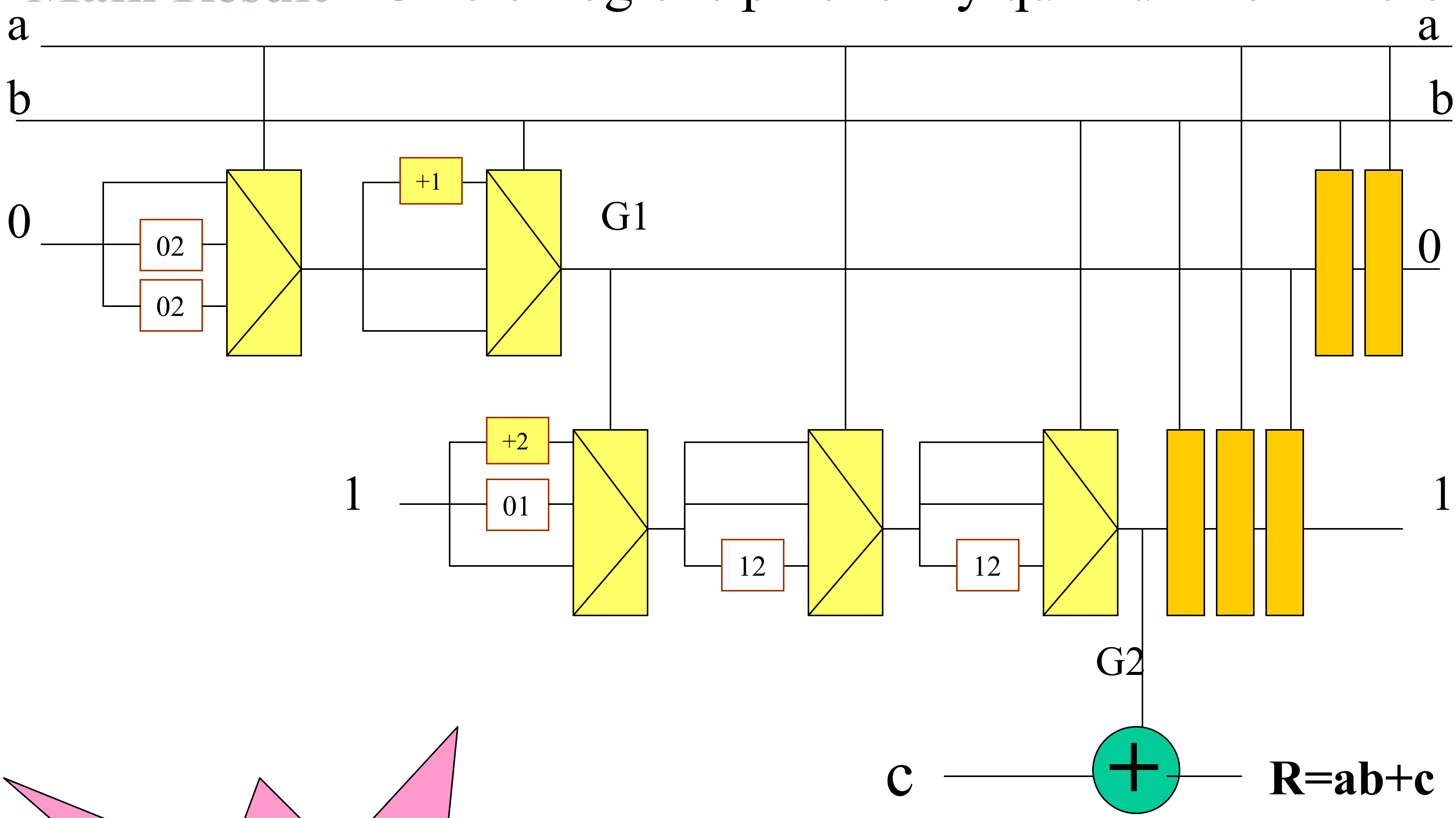


Principle of creating arbitrary reversible gates

Main Result - Galois Logic is practically quantum-realizable



Main Result - Galois Logic is practically quantum-realizable



But is it worthy?

This structure realizes also a very huge family of ternary Toffoli-like gates

Complete ternary systems

- System 1. Post literal, min, max
- System 2. Power of variable, shifts of variable (two of them for ternary – these are optional), Galois ADD, Galois MUL
- System 3. Post Literals, MIN, MODSUM.
- These three are most popular, but there are many other.

Are they good for quantum?

Ternary Operator Kmaps

A \ B	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Galois Multiplication. Also has latin square for non-zero columns and rows

A \ B	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

MODSUM which for primary number 3 is the same as Galois Addition. Observe latin square property, very important

Example : Ternary Kmaps of ternary adder

C:

A \ B	0	1	2
0	0	0	0
1	0	0	1
2	0	1	1

S:

A \ B	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$C = {}^{1}A * {}^{2}B + {}^{2}A * {}^{1}B + {}^{2}A * {}^{2}B$

Step1: write from Kmap the formula for mv minterms

This is modsum3 by inspection, so $S = A +_3 B$. But you can also calculate is with much formula writing the same as I show for C

Step 2. Algebraic Simplifications using rules of ternary Galois Field Algebra

$$C = {}^{1}A * {}^{2}B + {}^{2}A * {}^{1}B + {}^{2}A * {}^{2}B = (2A^2+2A) * (2B^2+B) + (2A^2+A) * (2B^2+2B) + (2A^2+A) * (2B^2+B)$$

$$= 2(A^2+A) * (2B^2+B) + 2*2A^2B^2 + 2*2A^2B + 2AB^2 + 2AB + 2*2A^2B^2 + 2A^2B + 2AB^2 + AB = \mathbf{2A^2B + 2AB^2 + 2AB}$$

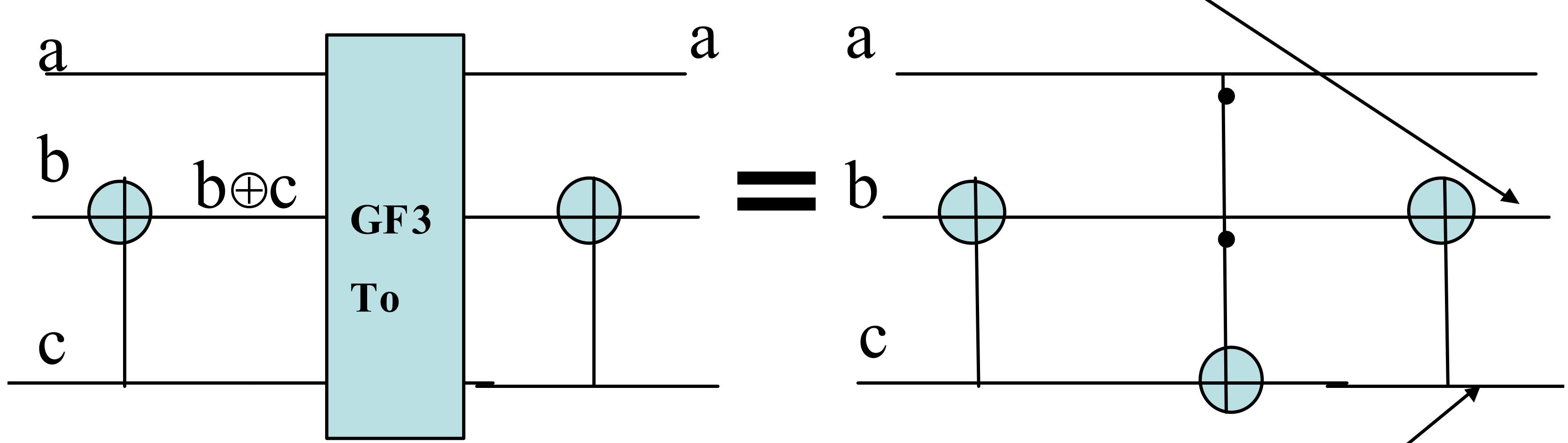
Example of Post literal, it has value 1 for argument value 1 and 0 otherwise

Here Post literals are next replaced by tautological polynomials in Galois Field

Complex Ternary Quantum Gates

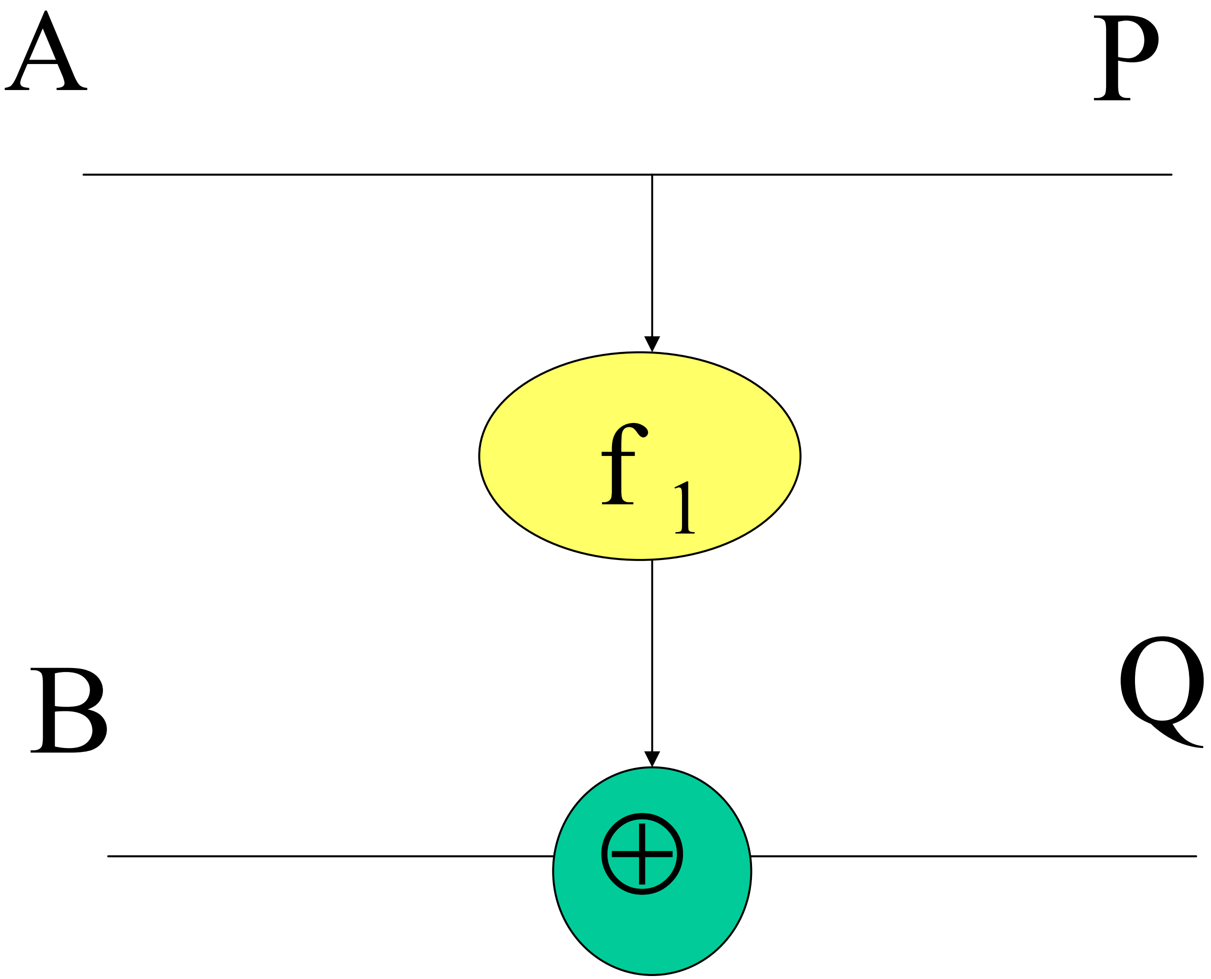
Ternary Fredkin Gate build from Ternary Toffoli and Ternary Feynman gates

$b \oplus c \oplus ab \oplus a'c = ac \oplus ba''$



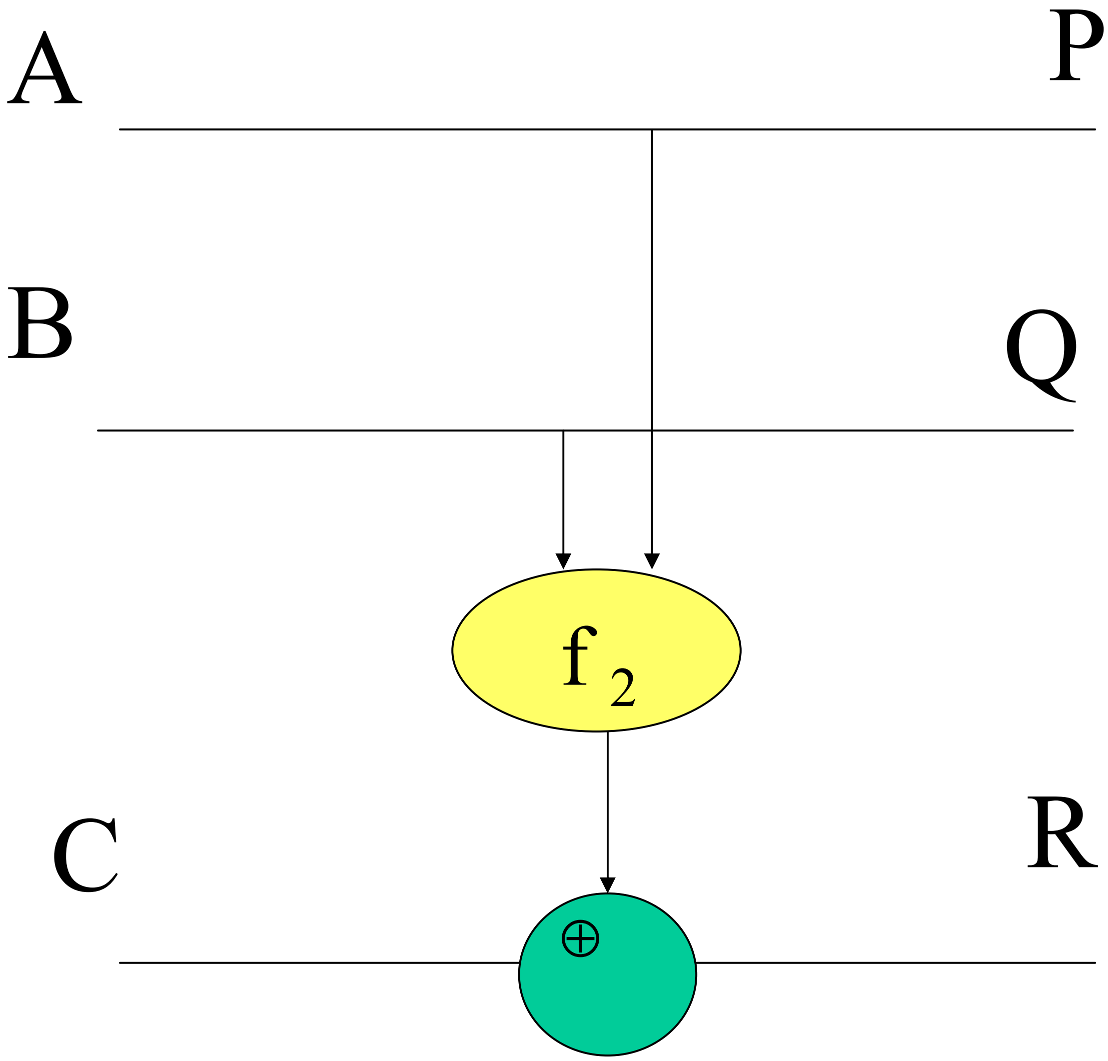
$c \oplus a(b \oplus c) = c \oplus ab \oplus ac = ca' \oplus ab$

Generalized Ternary Feynman Gate



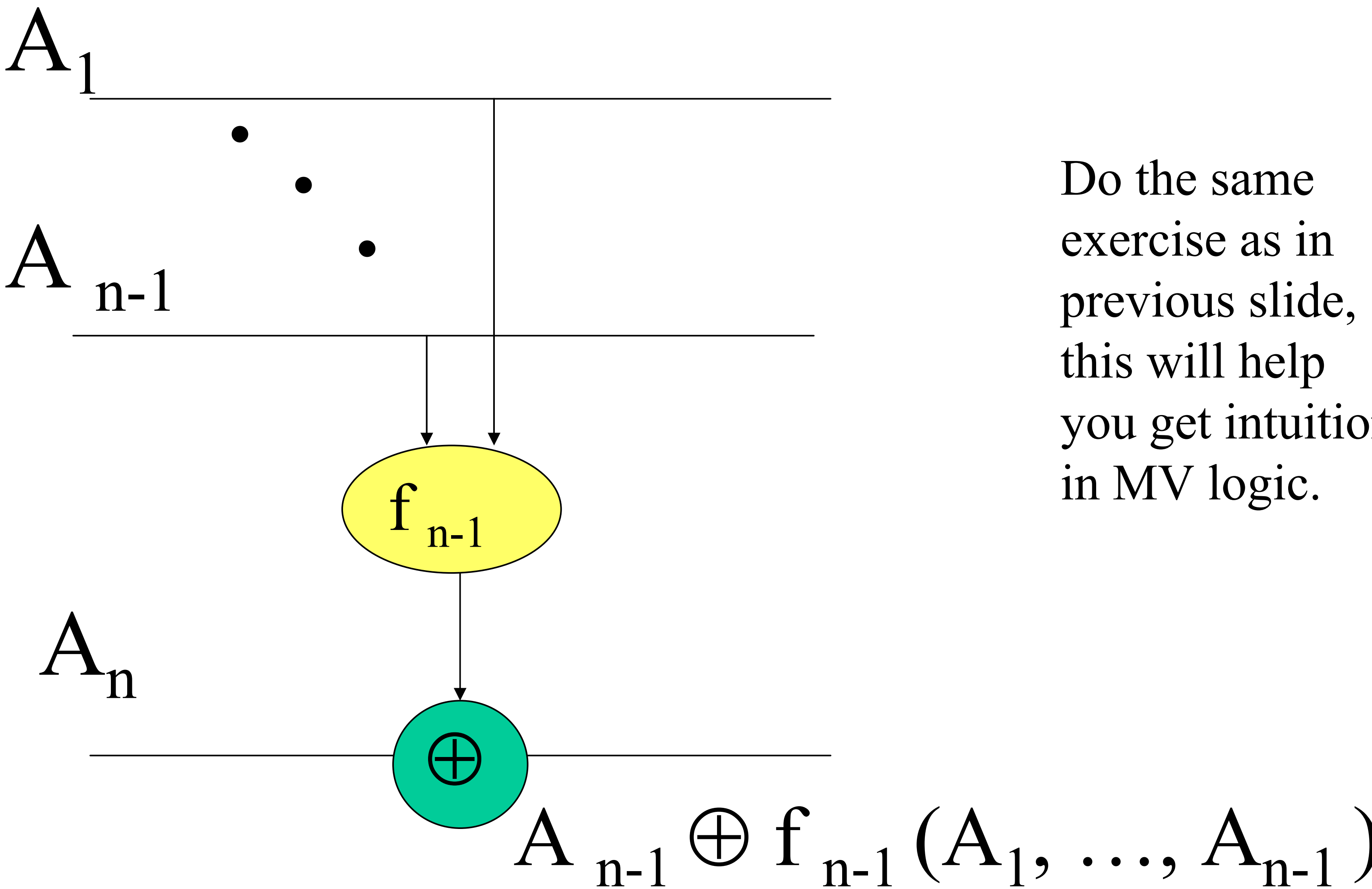
If f1 is reversible , gate is correct, what about non-reversible f1, please check if the gate is still reversible

Generalized Ternary 3*3 Toffoli Gate

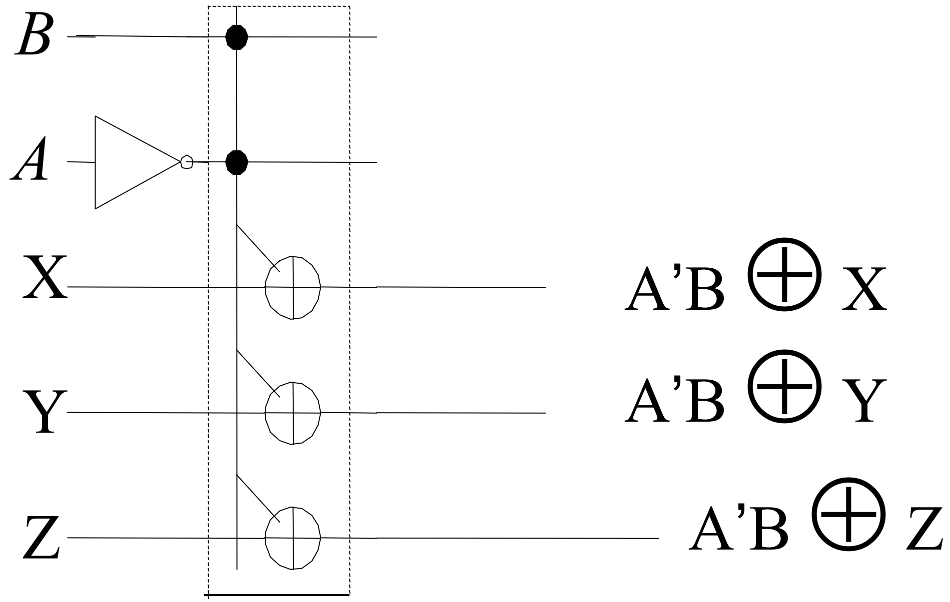


Do the same exercise as in previous slide, this will help you get intuition in MV logic.

Generalized Ternary n*n Toffoli Gate



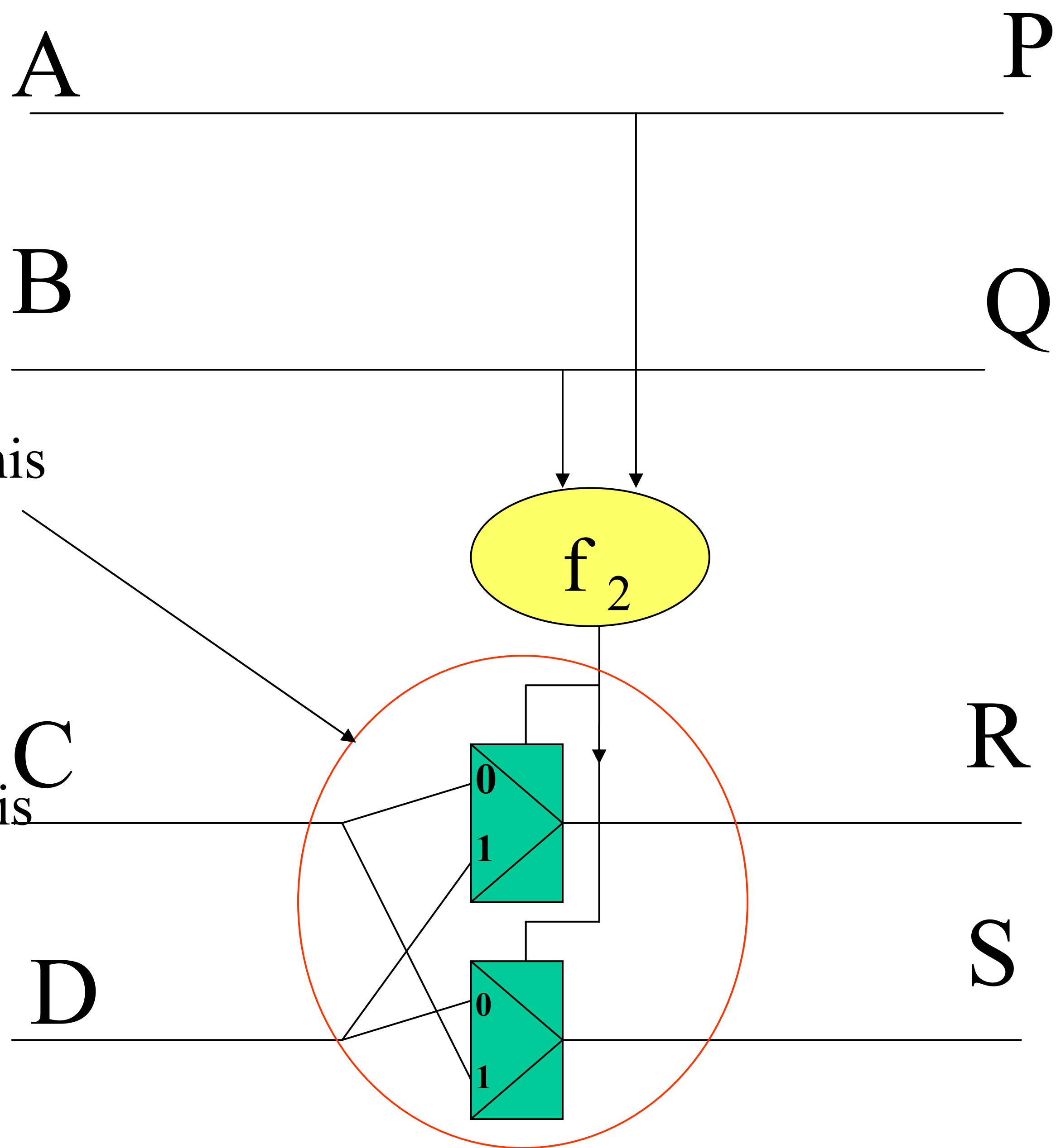
Do the same exercise as in previous slide, this will help you get intuition in MV logic.



Is this a realizable quantum gate ? -yes

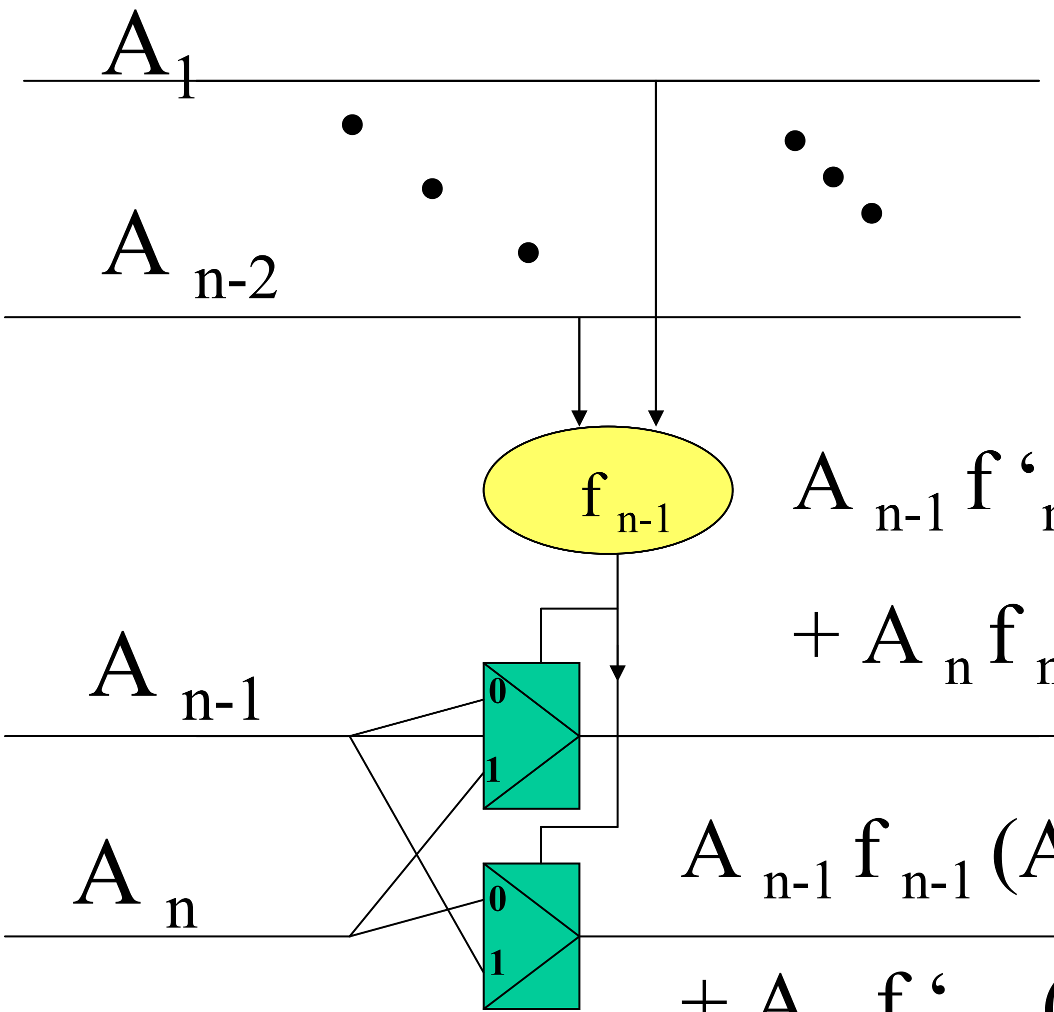
Generalized Ternary 4*4 Fredkin Gate

Rewrite this part to quantum ternary notation with Galois Logic



Generalized Ternary n*n Fredkin Gate

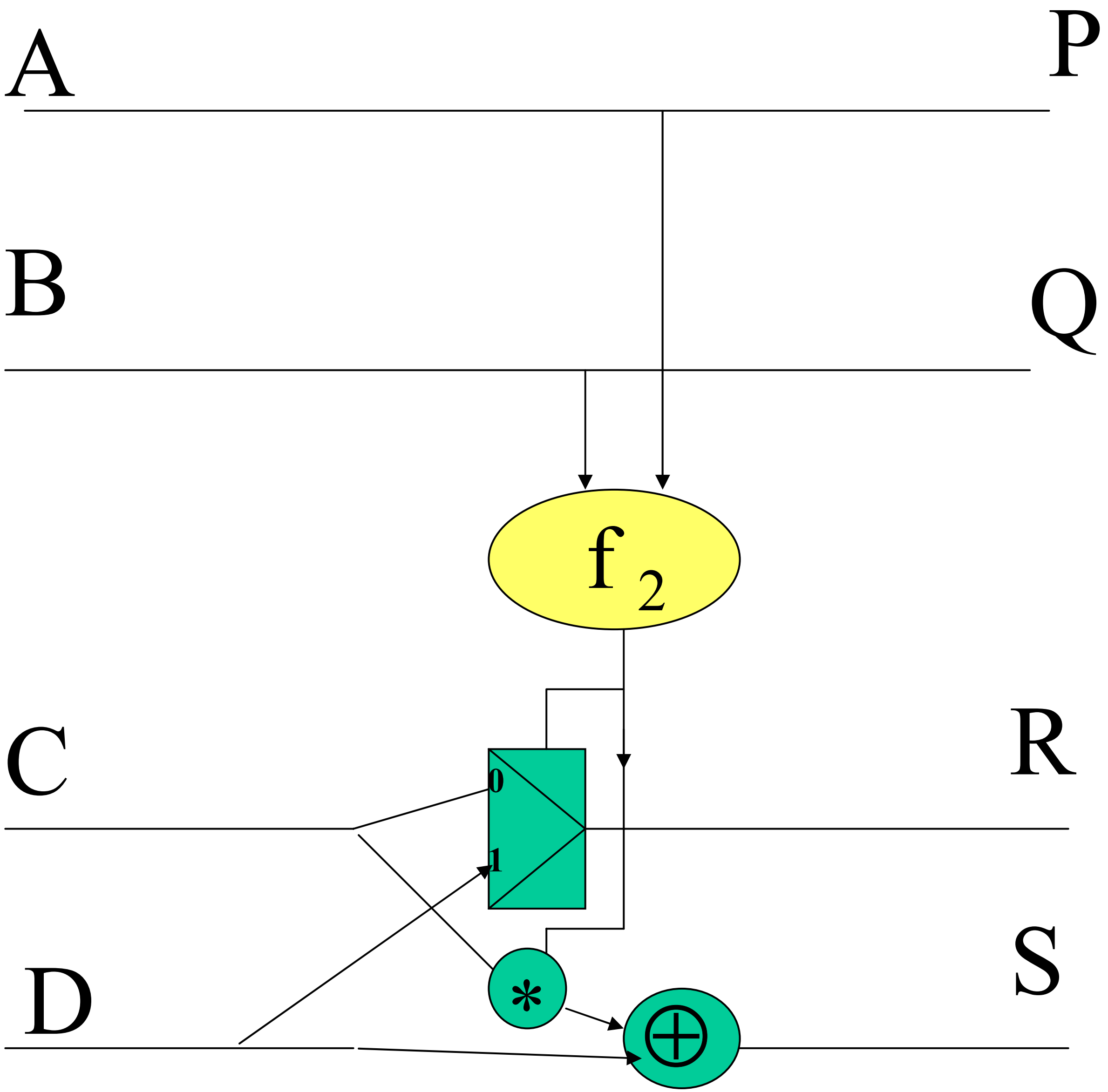
Do the same exercise as in previous slide, this will help you get intuition in MV logic.



$$A_{n-1}\, f'_{n-1}(A_1, \ldots, A_{n-1}) + A_n\, f_{n-1}(A_1, \ldots, A_{n-1})$$

$$A_{n-1}\, f_{n-1}(A_1, \ldots, A_{n-1}) + A_n\, f'_{n-1}(A_1, \ldots, A_{n-1})$$

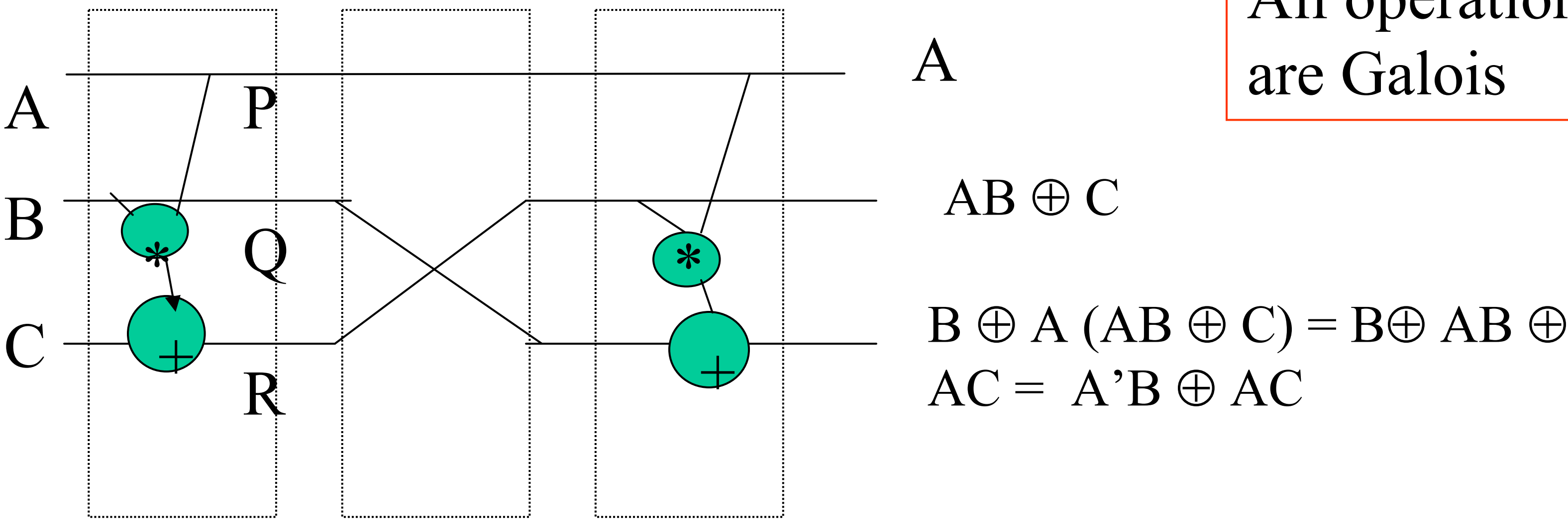
Generalized Ternary 4*4 Kerntopf Gate



Do the same exercise as in previous slide, this will help you get intuition in MV logic.

Ternary Quantum Circuits

Ternary GFSOP Cascade (non-optimal)



All operations are Galois

<u>Notation for EACH gate:</u>

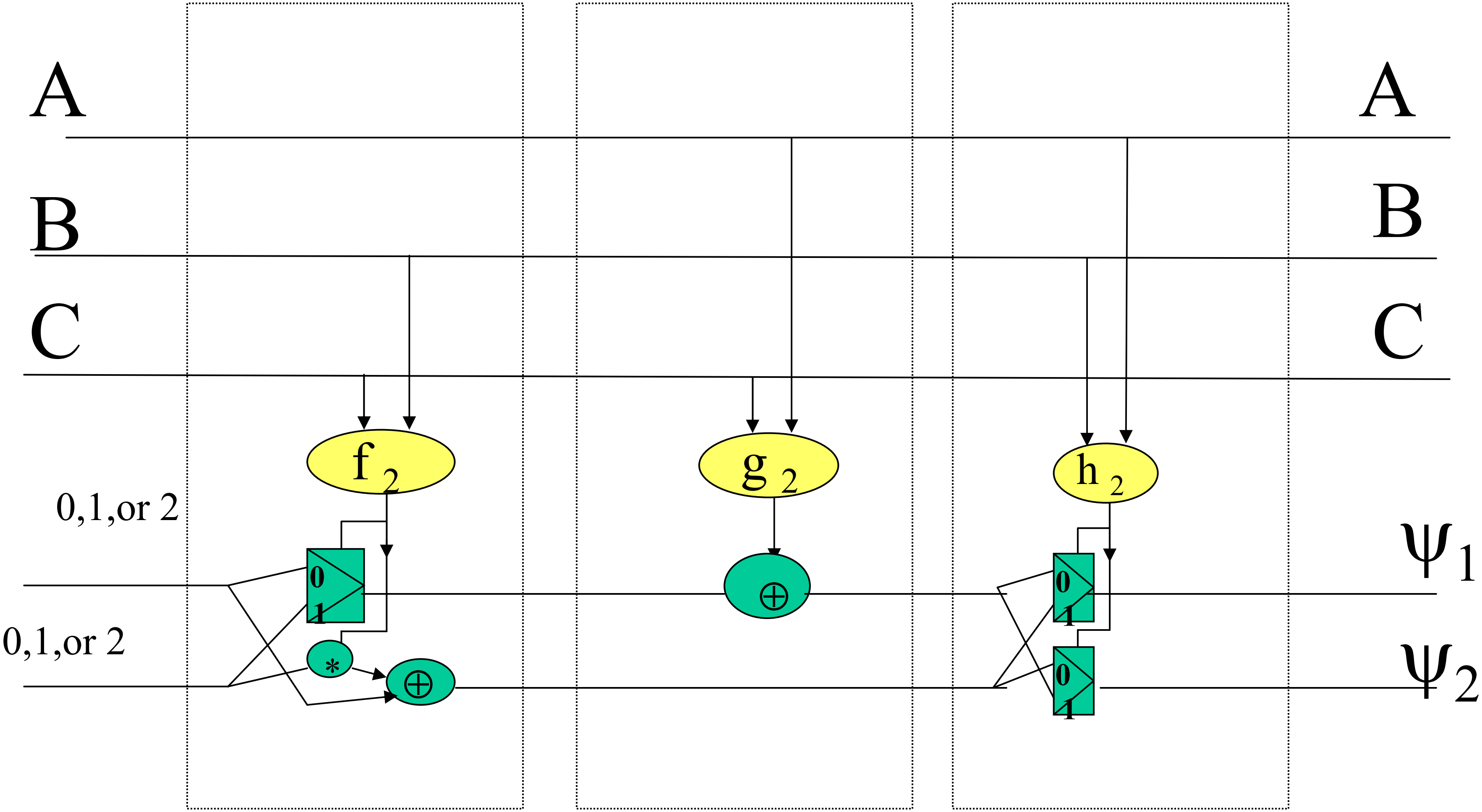
Inputs: A,B,C

Outputs: P,Q,R

How to realize ternary swap gate?

In any case, this is very costly!

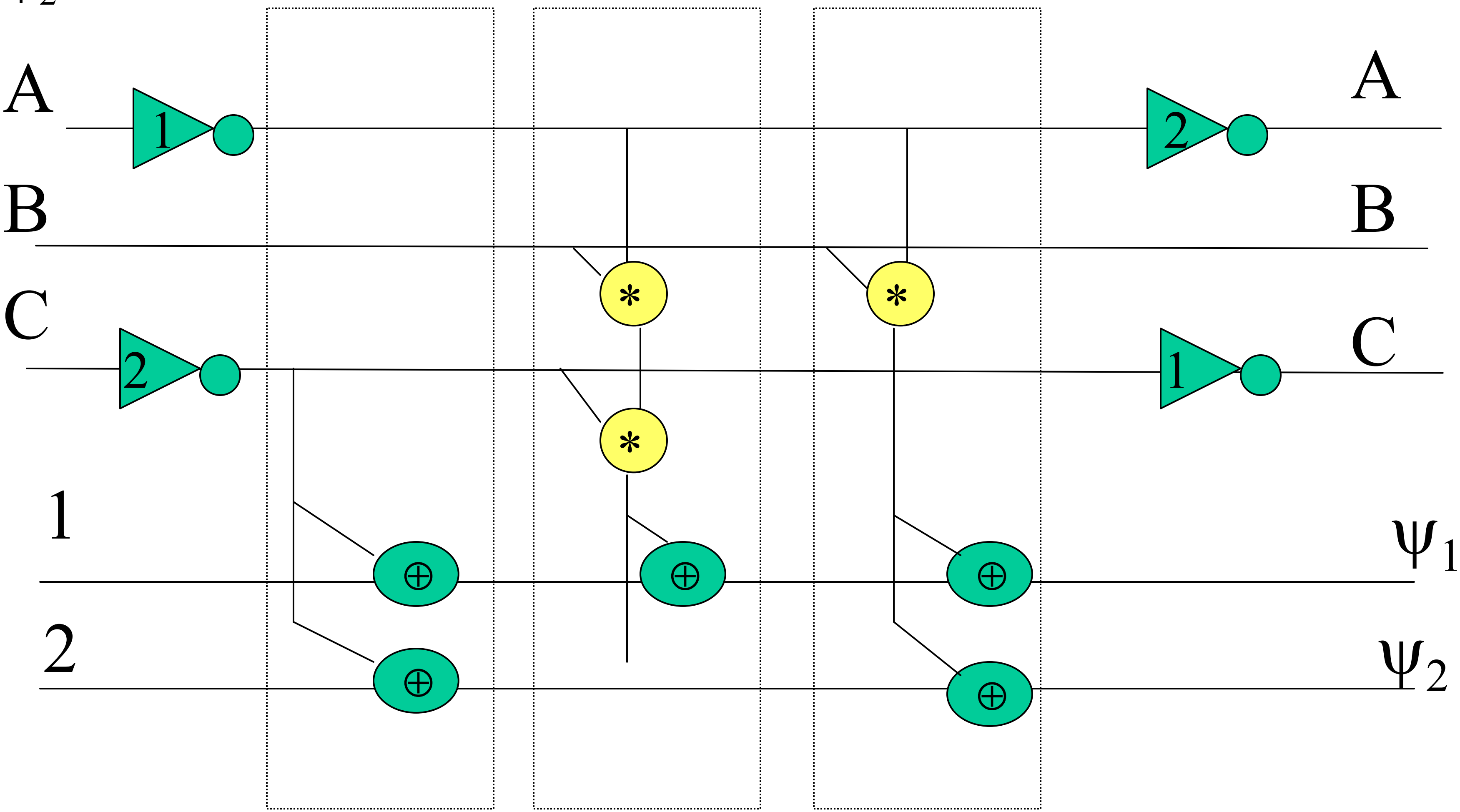
General Ternary Cascade of Kerntopf, Toffoli and Fredkin Family Gates



Example of multi-output FPRM-like GFSOP cascade of Toffoli family gates

$\psi_1 = 1 \oplus C'' \oplus A'BC \oplus A'\,B$

$\psi_2 = 2 \oplus C'' \oplus A'\,B$



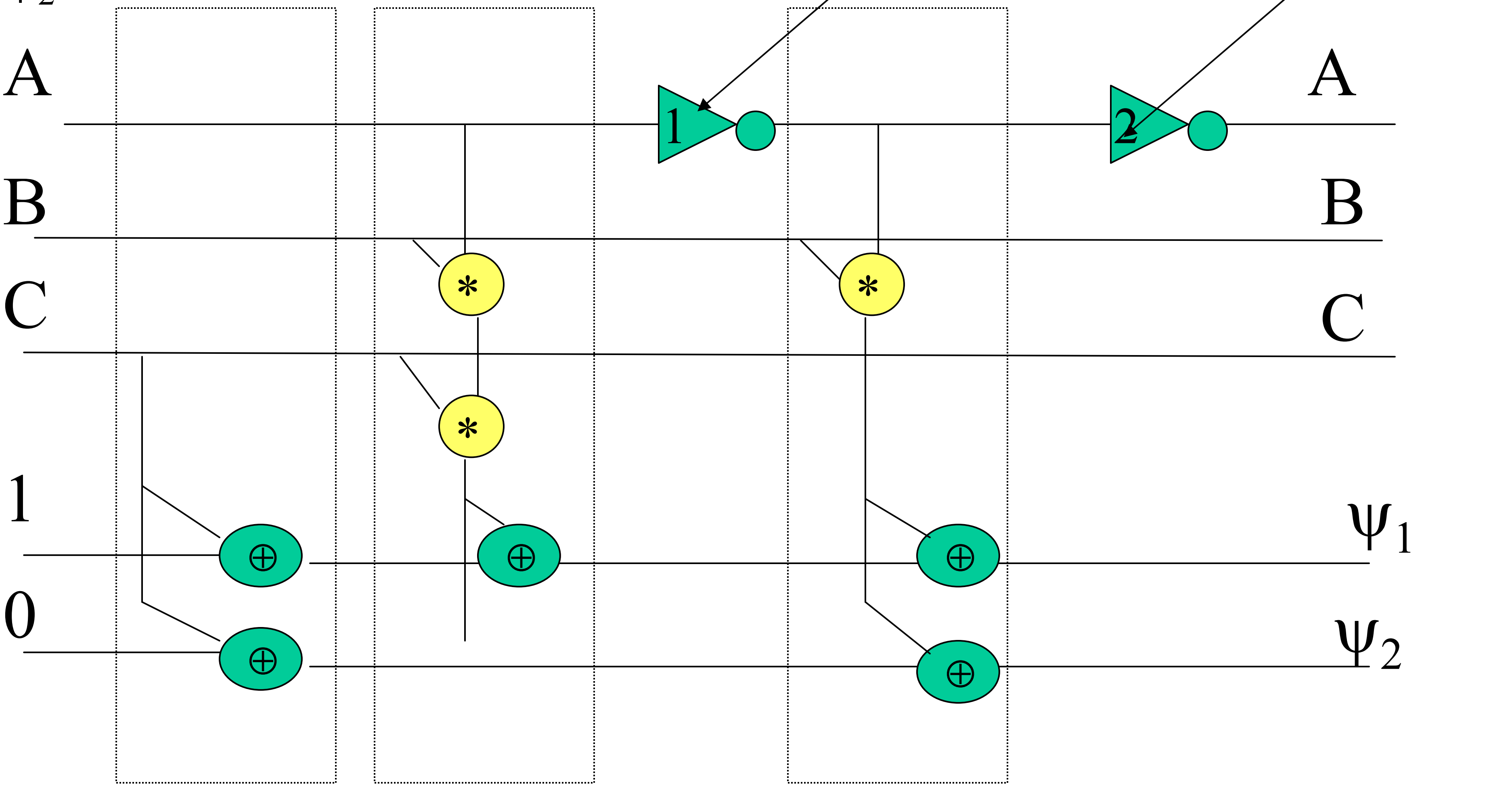
Example of ternary multi-output GFSOP cascade of Toffoli family gates

$\psi_1 = 1 \oplus C \oplus ABC \oplus A' B$

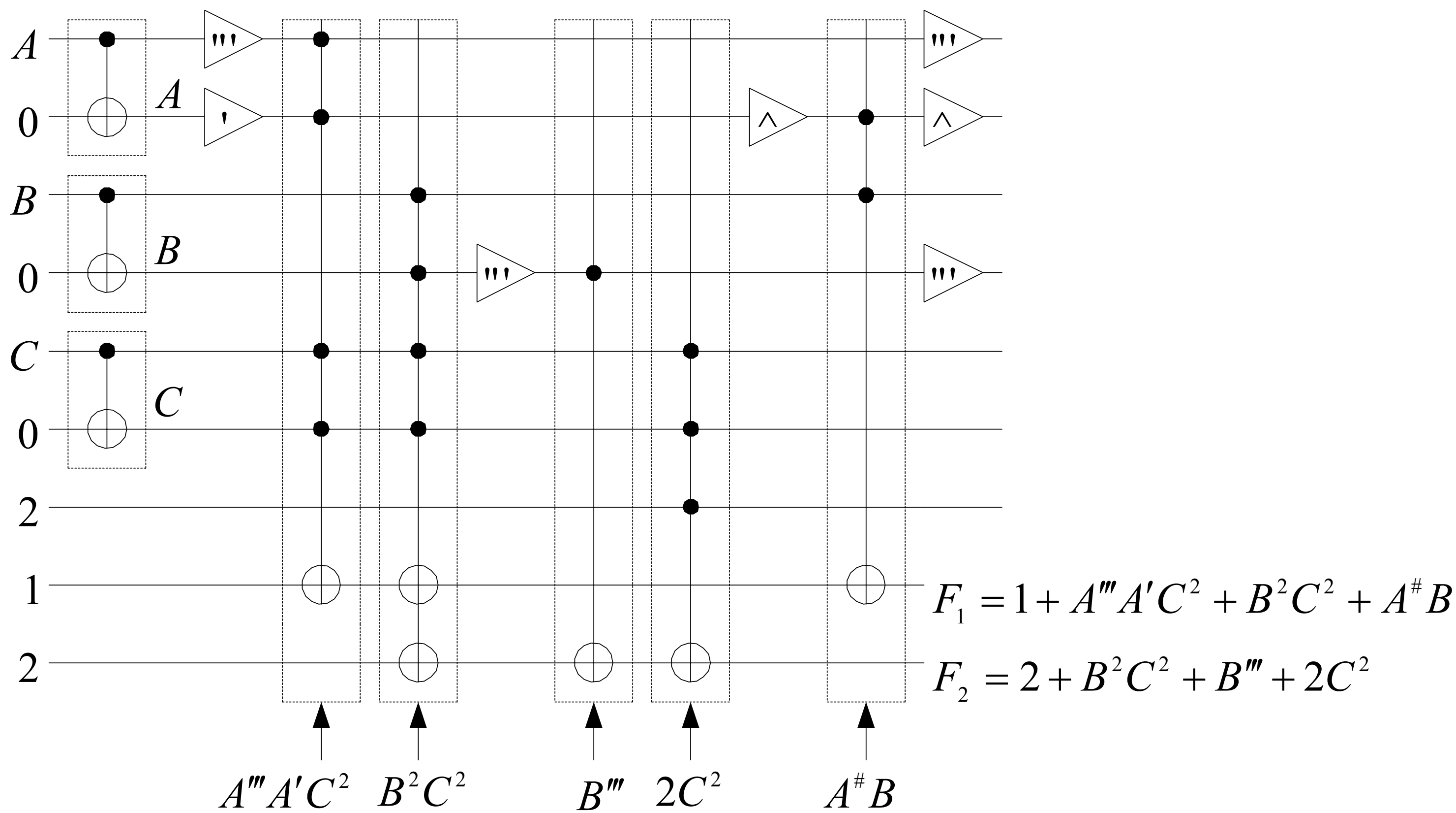
$\psi_2 = 0 \oplus C \oplus A' B$

This is notation for single shift

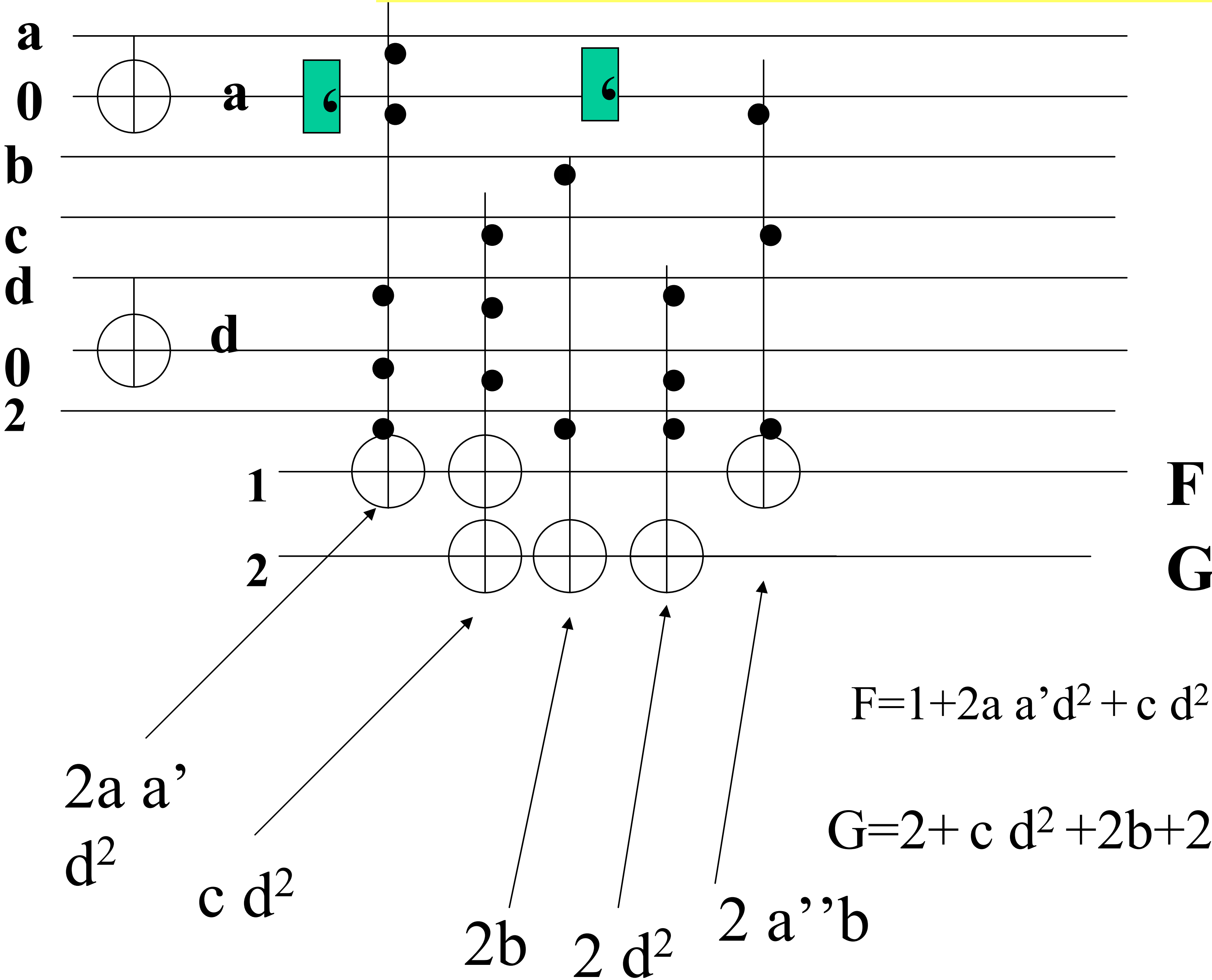
This is notation for dual shift



The general pattern of a cascade to implement any ternary function using ternary Toffoli gates

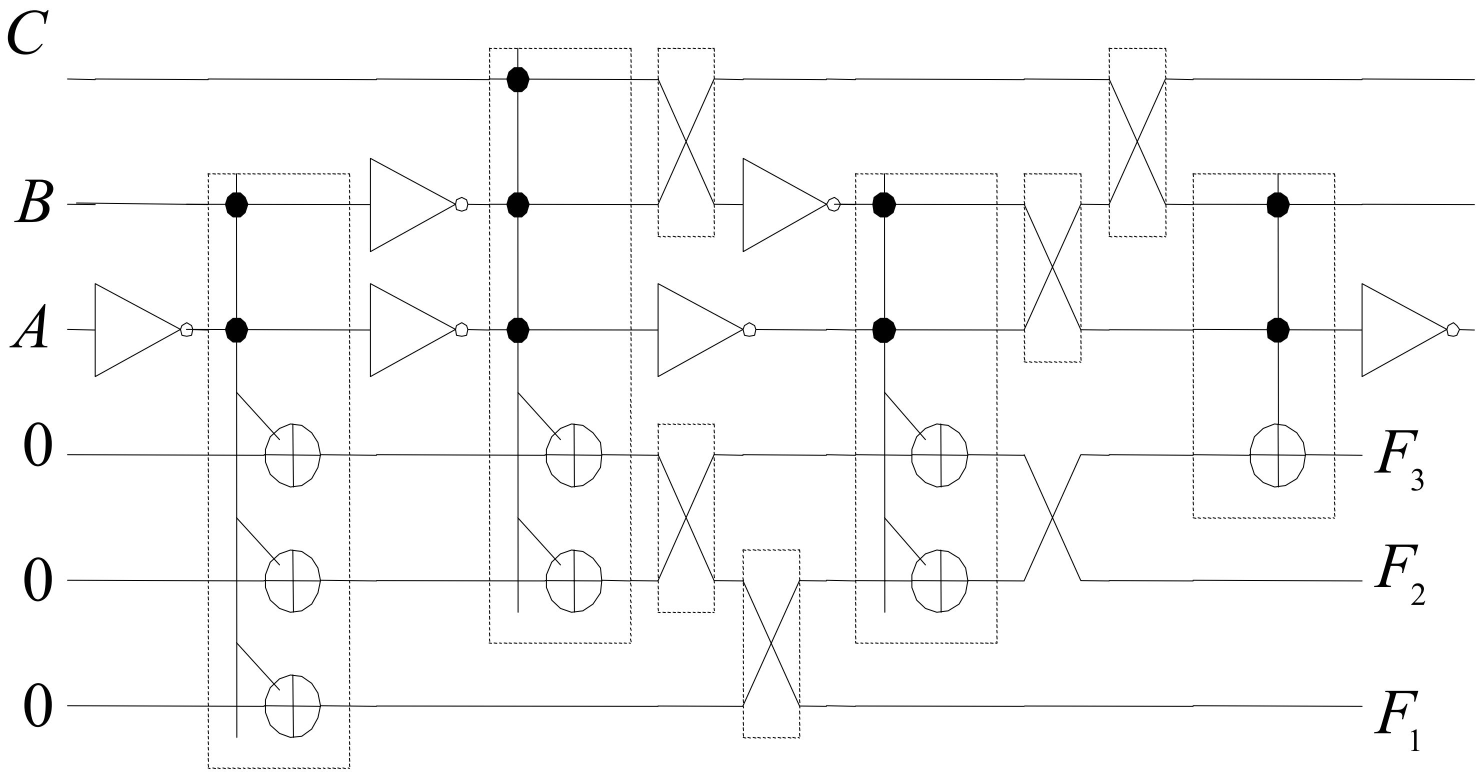


Simplified GFSOP array when powers are not used for some variables. Function of four variables



$F = 1 + 2a\,a'd^2 + c\,d^2 + 2\,a''c$

$G = 2 + c\,d^2 + 2b + 2d^2$



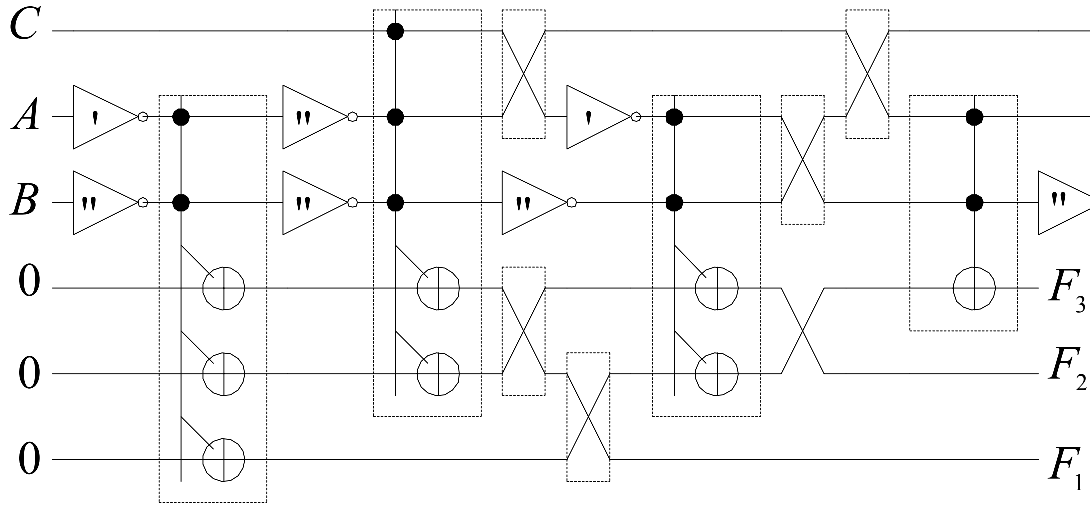
$$F_1 = A'B \oplus A B' C$$

$$F_2 = A'B \oplus A'C \oplus B'C'$$

$$F_3 = A'B \oplus B'C \oplus AC'$$

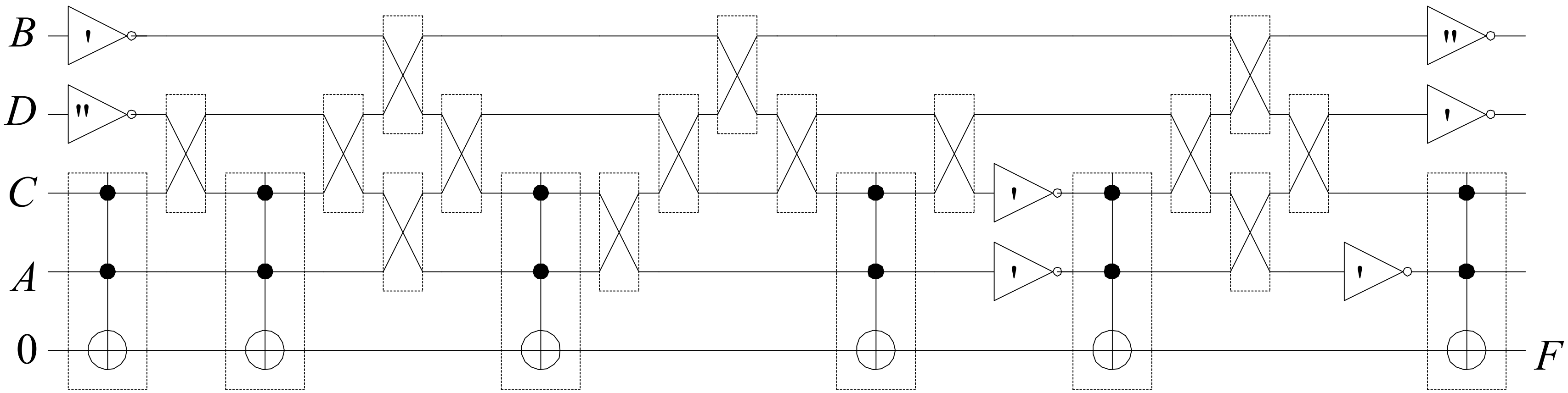
(a) Realization of multi-output ESOP

Macrogeneration introduces many Feynman gates that originate from swaps



$$F_1 = A'B'' \oplus AB'C$$
$$F_2 = A'B'' \oplus AB'C \oplus BC'$$
$$F_3 = A'B'' \oplus BC' \oplus AC'$$

(a) Realization of multi-output GFSOP



$$F = AC \oplus AD'' \oplus B'C \oplus B'D'' \oplus CD \oplus A'B''$$

(b) Realization of single-output GFSOP

MV Quantum Design Structures and Approaches

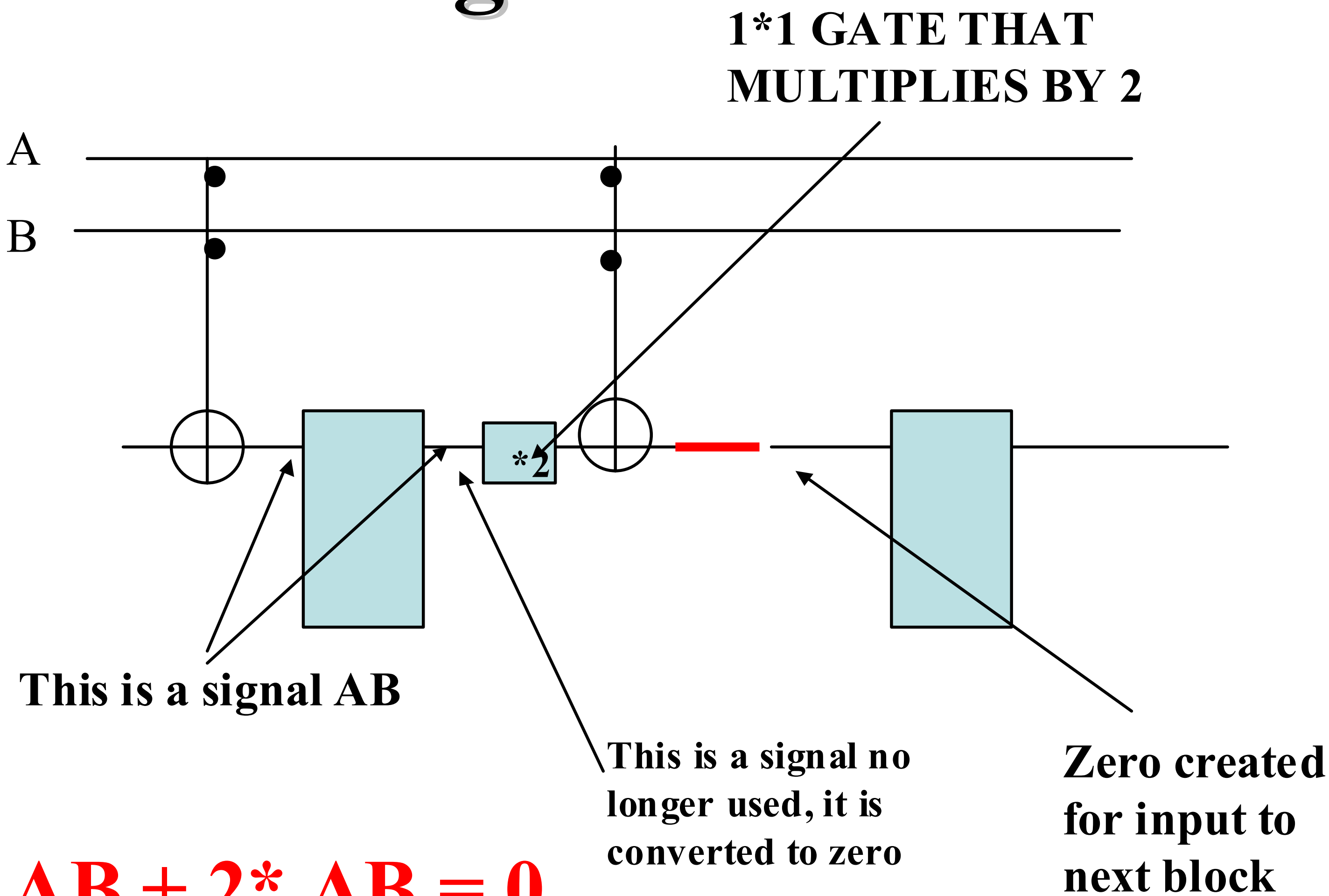
- **1.** GFSOP
- **2.** Multiple-Valued Reed-Muller
- **3.** Canonical Forms over Galois Logic (equivalents of PPRM, FPRM, GRM, etc)
- **4.** Multiple-Valued Maitra Cascades and Wave Cascades.
- **5.** Other cascades of specific type of elements
- **6.** Cascades of general gates

Design Issues

- **1.** Local mirroring
- **2.** Variable ordering versus gate ordering
- **3.** Return to zero and folding
- **4.** Realization of complex multiple-valued reversible gates (permutation gates) using directly 1-qubit and 2-qubit quantum primitives

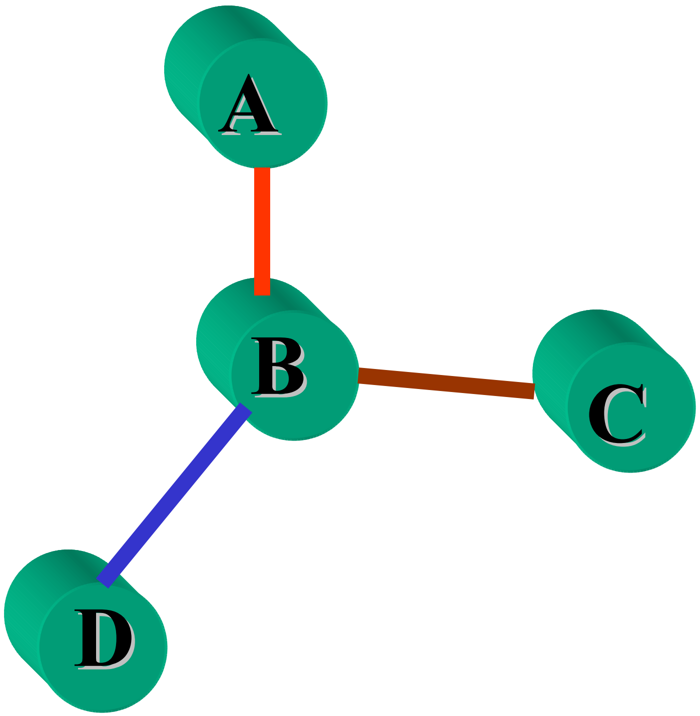
"Return to Zero" and Folding

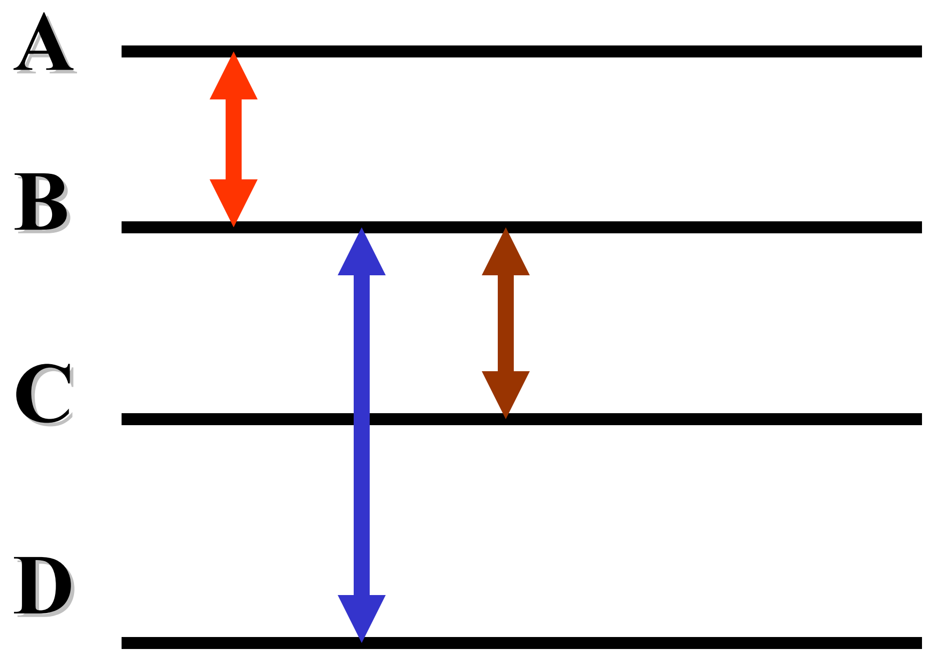
Technique of local mirror can improve your ternary circuits, reduce the number of zeros in inputs. Here is the explanation for **ternary** logic



$$AB + 2^* AB = 0$$

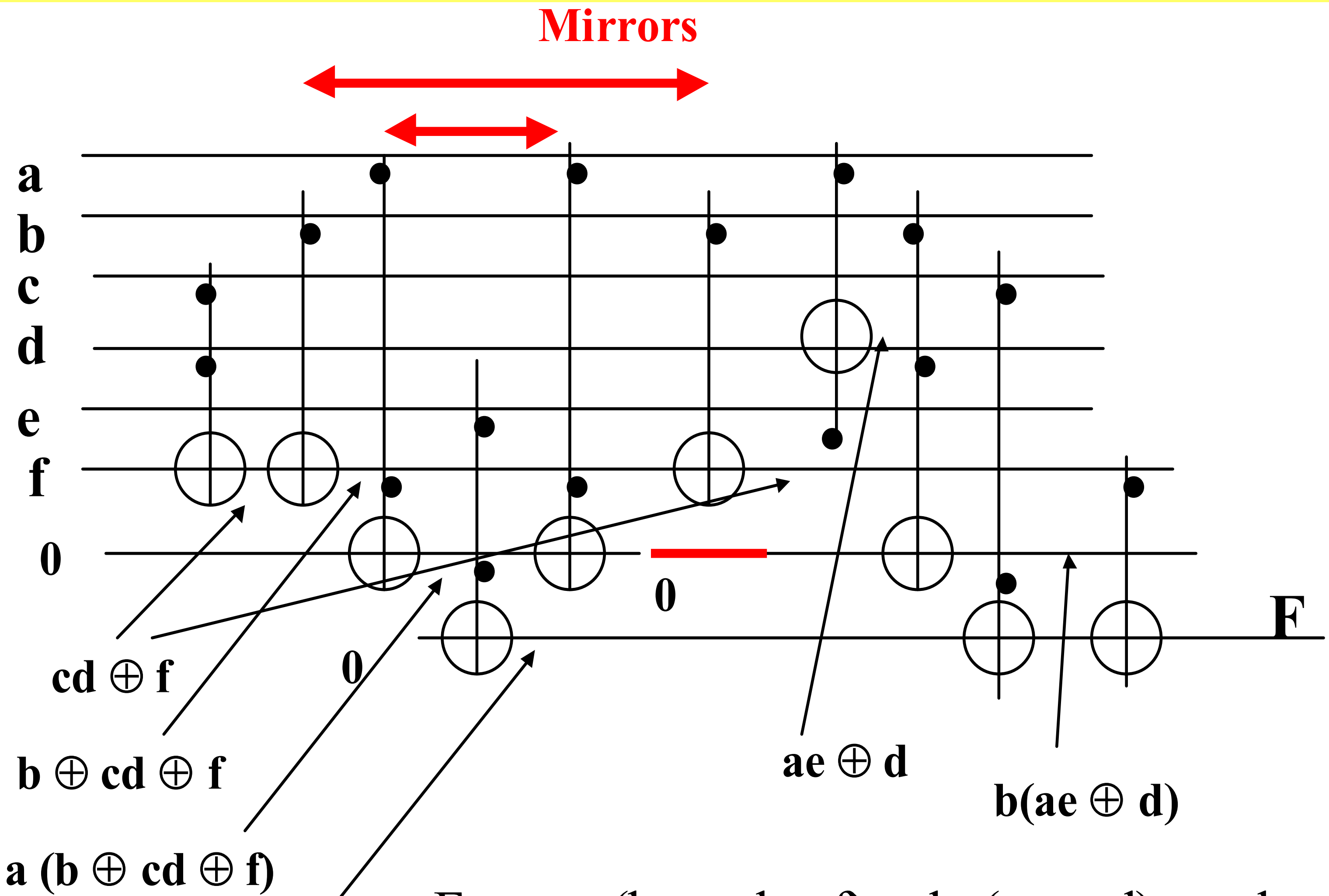
Molecule - Driven Layout and Logic Synthesis



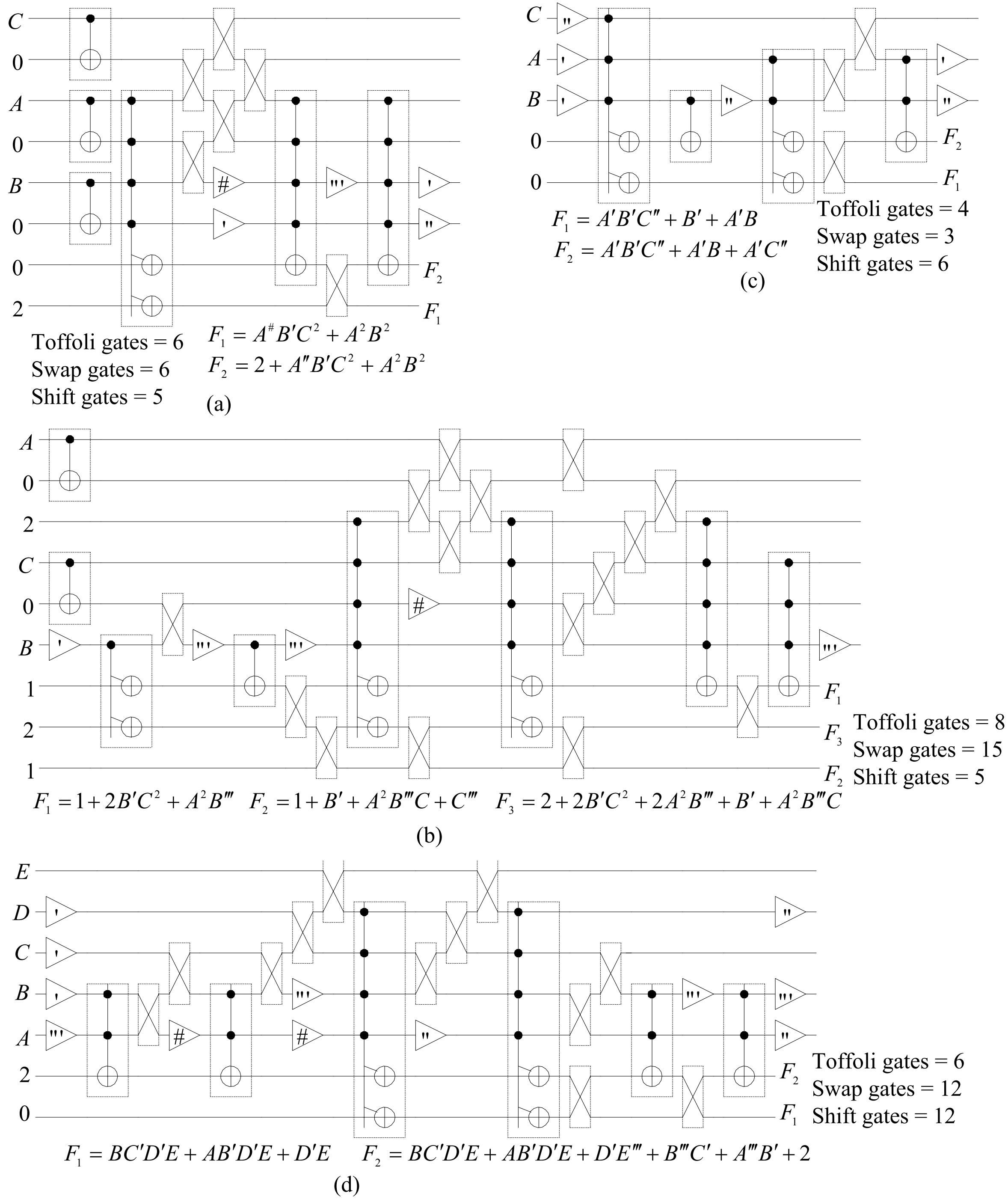


Allowed gate neighborhood for 2 qubit gates

Using Local Mirrors and Return-to-zero factorization

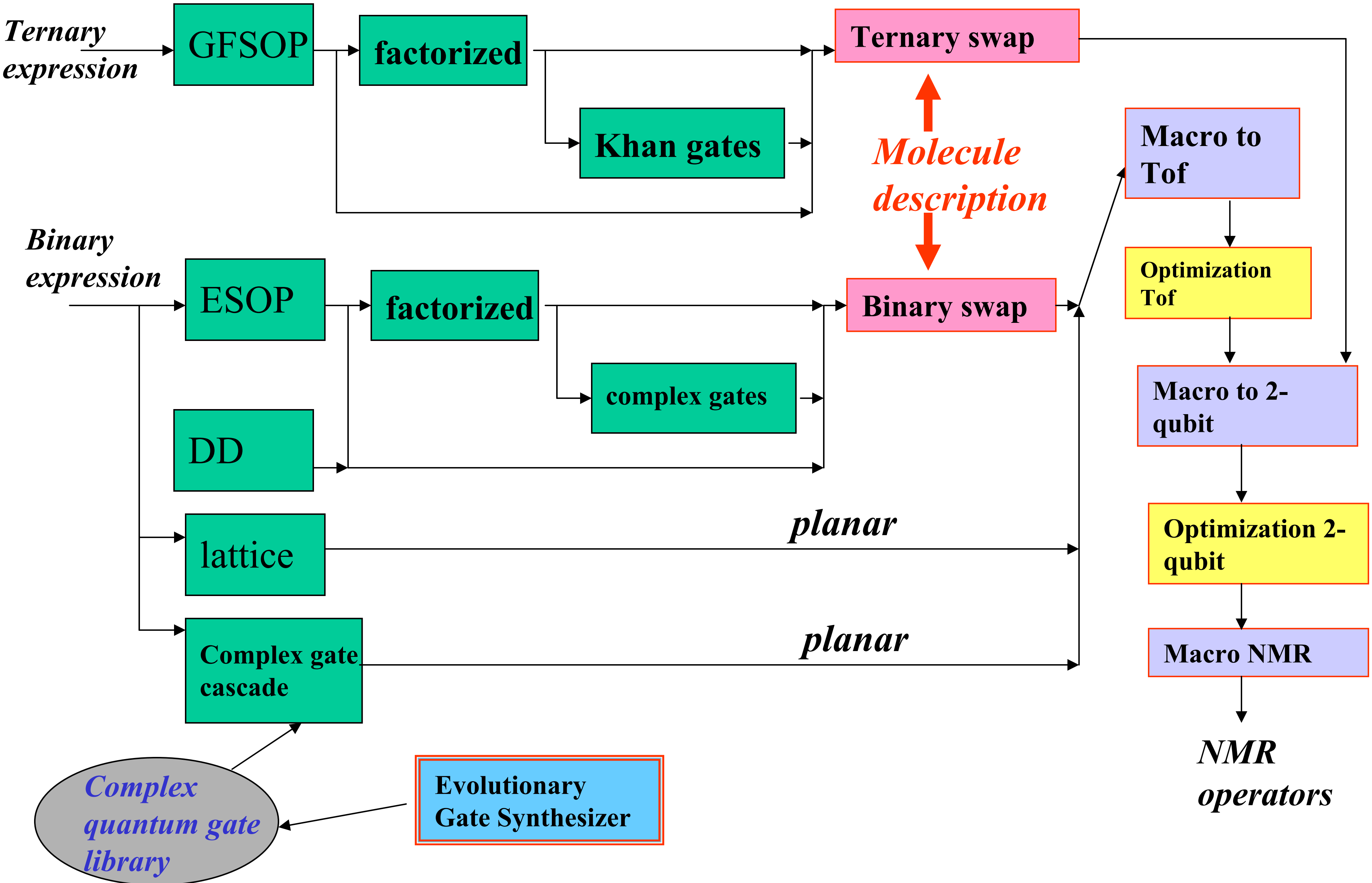


F = a e (b ⊕ cd ⊕ f) ⊕ bc(ae ⊕ d) ⊕ cd ⊕ f



C
0
A
0
B
0
0
2
F2
F1
Toffoli gates = 6
Swap gates = 6
Shift gates = 5
$F_1 = A^{\#}B'C^2 + A^2B^2$
$F_2 = 2 + A''B'C^2 + A^2B^2$
(a)
C
A
B
0
0
F2
F1
$F_1 = A'B'C'' + B' + A'B$
$F_2 = A'B'C'' + A'B + A'C''$
(c)
Toffoli gates = 4
Swap gates = 3
Shift gates = 6
A
0
2
C
0
B
1
2
1
F1
F3
F2
Toffoli gates = 8
Swap gates = 15
Shift gates = 5
$F_1 = 1 + 2B'C^2 + A^2B'''$
$F_2 = 1 + B' + A^2B'''C + C'''$
$F_3 = 2 + 2B'C^2 + 2A^2B''' + B' + A^2B'''C$
(b)
E
D
C
B
A
2
0
F2
F1
Toffoli gates = 6
Swap gates = 12
Shift gates = 12
$F_1 = BC'D'E + AB'D'E + D'E$
$F_2 = BC'D'E + AB'D'E + D'E''' + B'''C' + A'''B' + 2$
(d)

System for mixed quantum logic NMR



Open Problems

1. How to select the best gates for permutation circuit synthesis.

2. Simplest practical realization of a ternary Toffoli-like gate

3. Best realization, in quantum circuit sense (simplicity and ease of realization), of other Galois gates and non-Galois standard MV operators such as minimum, maximum, truncated sum and others.

4. Synthesis algorithms for MV reversible circuit families:
   - GFSOP ,
   - nets,
   - lattices,
   - PLAs
   - MV counterparts of Maitra cascades and wave cascades
   - other reversible cascades

Conclusion

- Practical algorithms for MV quantum circuits. Quantum permutation circuits design (for NMR) is **not** the same as standard reversible logic.
- **CAD Tools** for quantum physicists: *link levels of design*.
- Evolutionary Approaches versus GFSOP-like approaches
- MV Quantum Simulation
- MV Quantum Circuits Verification
- Designing MV counterparts of Deutch, Shorr, Grover and other original MV quantum algorithms
- Generalization to MV Of efficient Garbage-less quantum gates by Barenco, DiVincenzo, etc.
- NMR realization of ternary logic.
- MV Quantum Computational Intelligence