Analysis of the secular problem for triple star systems

J. P. S. Carvalho,[1] **R. Vilhena de Moraes,**[1] **A. F. B. A. Prado,**[2] **and O. C. Winter,**[3]

[1]UNIFESP-ICT, Universidade Federal de São Paulo, São José dos Campos - SP, Brazil.
[2] Division of Space Mechanics and Control - INPE, São José dos Campos - SP, Brazil
[3] UNESP- Univ Estadual Paulista, Guaratinguetá - SP, Brazil.

E-mail: `jeanfeg@gmail.com`

Abstract. The long-term dynamics of the three-body problem is studied. The goal is to study the motion of a planet (m_1) around a star (m_0) that is perturbed by a third-body (m_2) (a planet or a brown dwarf star). The gravitational potential is developed in closed form up to the fourth order. Taking into account the triple system, it is shown here the evolution of some orbital parameters of the planet (m_1). A comparison considering models with different orders for the disturbing potential is presented. We show that the behavior of the orbit of the inner planet can flip from prograde to retrograde trajectories. This is due to the third-order term, which strongly affects the eccentricity and inclination. We show that the effect of the fourth order term is to change the times when the phenomenon occurs.

1. Introduction

In the literature, several authors considered the three-body problem to study the dynamics of triple systems, such as [1]-[7]. Here, a triple system is presented as a binary system perturbed by a distant star. In a triple system like this, the distance between the two stars (or planets) is very small when compared to their distances to the third-body (here either a planet or a brown dwarf). Reference [6] presented a study where the authors took into account the octupole term in the disturbing potential. They showed that the inclination of the inner planet varies from prograde ($i < 90°$) to retrograde ($i > 90°$) for a specific problem, where i is the mutual inclination between the two orbits. The authors showed that, considering only the quadrupole term in the potential, the inclination varies according to the Kozai mechanism [8] (see fig. 1 in [6]), i.e., the inclination oscillates with large amplitude when the initial inclination has a value larger than the critical inclination (39.23°) and small amplitudes when the initial inclination has a value smaller than the critical inclination. However, the orbit always remains prograde. When they considered the octupole term in the potential, the inclination grows a lot and can flip from prograde to retrograde trajectories. This mechanism is defined by [9] as eccentric Kozai mechanism (EKM).

In this work we develop the disturbing potential in closed form up to the fourth order in a small parameter ($\alpha = a_1/a_2$), where a_1 is the semi-major axis of the planet and a_2 is the semi-major axis of the disturbing body, to analyze the effects caused by the terms: quadrupole

Journal of Physics: Conference Series **465** (2013) 012010 doi:10.1088/1742-6596/465/1/012010

(equivalent to the second order term, the Legendre polynomial P_2), octupole (equivalent to the third order term, the Legendre polynomial P_3) and the P_4 term (order fourth in the Legendre polynomial) on the orbital elements of the planet. The equations of motion are developed in closed form to avoid expansions in power series of the eccentricity and inclination. We compared the models mentioned here in different orders and we also compared with the results obtained by [6]. We investigate what happens in the dynamics when the term of the Legendre polynomial P_4 is considered, and we make various applications to analyze the motion of the planet around a star that is perturbed by a third-body. The approach for the development of the equations is based on [10]-[14]. In order to develop the long-period disturbing potential, the double-averaged method is applied. We analyze the disturbing potential effects on some orbital elements, such as eccentricity, inclination and argument of the periapsis. The subscripts 1 and 2 will refer to the inner and outer orbits, respectively.

2. Equations of motion

The triple system under study is characterized by a body m_1 in an elliptical inner orbit around the body m_0 and a third-body (m_2) moving in an outer elliptical orbit around the center of mass of the system m_0-m_1, but in a very distant trajectory that also has large eccentricity, as shown in Fig. 1. The vector $\mathbf{r_1}$ represents the position of m_1 with respect to the center of mass of the system and the vector $\mathbf{r_2}$ is the position of the body m_2 to the center of mass of the inner orbit. Φ is the angle between $\mathbf{r_1}$ and $\mathbf{r_2}$.

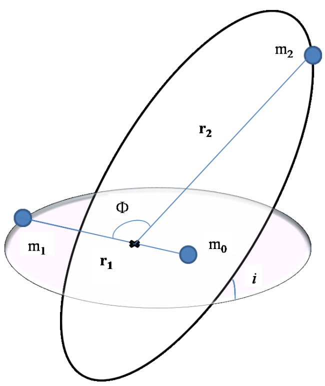

Figure 1. Coordinate system used to describe the triple star system.

The Hamiltonian of the triple system can be written as follows [2, 3, 15]

$$F = \frac{Gm_0m_1}{2a_1} + \frac{G(m_0+m_1)m_2}{2a_2} + \frac{G}{a_2}\sum_{j=2}^{\infty}\alpha^j M_j\left(\frac{r_1}{a_1}\right)^j\left(\frac{a_2}{r_2}\right)^{j+1}P_j(\cos\Phi) \tag{1}$$

where G is the gravitational constant, P_j are the Legendre polynomials and $M_j = m_0m_1m_2\frac{m_0^{j-1}-(-m_1)^{j-1}}{(m_0+m_1)^j}$. We shall deal with the expansion up to the fourth-order in α. The $\cos\Phi$ term is given by [8]

$$\cos\Phi = \cos(f_1+g_1)\cos(f_2+g_2) + c\sin(f_1+g_1)\sin(f_2+g_2) \tag{2}$$

where we will use the shortcut $s = \sin i$, and $c = \cos i$. Here i is the mutual inclination of the inner orbit with respect to the outer orbit. Let g_1, h_1, f_1 and g_2, h_2 and f_2 is the argument of periastron, longitude ascending node and true anomaly of the inner and outer orbits, respectively. The longitude of the ascending node (h) does not appear in Eq (2) because the h_1 and h_2 terms appear as a combination of $h_1 - h_2$. The variables h_1 and h_2 can be eliminated by the theorem of elimination of the nodes [16].

We developed the disturbing potential using a relation given by [9], where g_2 can be written as $g_2 = \pi - h_1$. The disturbing potential is a function of g_1, h_1, so we have

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the equation of motion to study the behavior of the inclination of the orbit of the planet, as is proved in [9]. For the model considered in the present paper, it is necessary to calculate the terms R_2, R_3 and R_4 of the disturbing function due to P_2, P_3 and P_4 terms, respectively. We get, $R_2 = \frac{G}{a_2}\alpha^2 M_2(\frac{r_1}{a_1})^2(\frac{a_2}{r_2})^3 P_2(\cos\Phi)$, $R_3 = \frac{G}{a_2}\alpha^3 M_3(\frac{r_1}{a_1})^3(\frac{a_2}{r_2})^4 P_3(\cos\Phi)$ and $R_4 = \frac{G}{a_2}\alpha^4 M_4(\frac{r_1}{a_1})^4(\frac{a_2}{r_2})^5 P_4(\cos\Phi)$. The disturbing potential given by Eq. (1) can be written as $F = R_0 + R_2 + R_3 + R_4$, where $R_0 = \frac{Gm_0m_1}{2a_1} + \frac{G(m_0+m_1)m_2}{2a_2}$. To eliminate the short-period terms of the potential the double-averaged method is applied. Thus, we obtain the disturbing potential expanded up to the fourth order in a small parameter. The long-period disturbing potential (R_2, R_3 and R_4) can be written as

$$R_2 = -\frac{{a_1}^2\mu}{16{a_2}^3(1-{e_2}^2)^{3/2}}[2 + 15\,c^2{e_1}^2\cos(2\,g_1) - 9\,{e_1}^2c^2 - 15\,{e_1}^2\cos(2\,g_1) + 3\,{e_1}^2 - 6\,c^2] \tag{3}$$

$$\begin{aligned} R_3 = {} & \frac{675}{512(1-{e_2}^2)^{\frac{5}{2}}{a_2}^4} e_1\,{a_1}^3\sigma\,e_2\Big[\tfrac{7}{9}\,{e_1}^2(c+1)(-1+c)^2\cos(3\,g_1 - h_1) - (c^2 + \tfrac{2}{3}\,c - \tfrac{1}{15})\times \\ & (-1+c)({e_1}^2 + \tfrac{4}{3})\cos(g_1 - h_1) + (c+1)((-\tfrac{7}{9}\,{e_1}^2c^2 + \tfrac{7}{9}\,{e_1}^2)\cos(3\,g_1 + h_1) + \\ & \cos(g_1 + h_1)({e_1}^2 + \tfrac{4}{3})(-\tfrac{2}{3}\,c + c^2 - \tfrac{1}{15}))\Big] \end{aligned} \tag{4}$$

$$\begin{aligned} R_4 = {} & -\frac{\rho\,{a_1}^4}{(1-{e_2}^2)^{\frac{7}{2}}{a_2}^5}\frac{6615}{8192}\Big[-\tfrac{4}{3}\,(c^2 + c + \tfrac{1}{7})({e_1}^2 + 2)(c-1)^2{e_2}^2{e_1}^2\cos(2\,g_1 - 2\,h_1) - \\ & \tfrac{4}{3}\,({e_1}^2 + 2)(c^2 - c + \tfrac{1}{7}){e_2}^2{e_1}^2(c+1)^2\cos(2\,g_1 + 2\,h_1) + {e_2}^2{e_1}^4(c-1)(c+1)^3\times \\ & \cos(4\,g_1 + 2\,h_1) + {e_2}^2{e_1}^4(c+1)(c-1)^3\cos(4\,g_1 - 2\,h_1) + 4\,(\tfrac{2}{3} + {e_2}^2)({e_1}^2 + 2)(c- \\ & 1)(c^2 - \tfrac{1}{7}){e_1}^2(c+1)\cos(2\,g_1) + \tfrac{10}{7}\,(c-1){e_2}^2(c^2 - \tfrac{1}{7})(\tfrac{8}{15} + \tfrac{8}{3}\,{e_1}^2 + {e_1}^4)(c+1)\cos(2\,h_1) - \\ & 3\,({e_1}^4(c-1)^2(c+1)^2\cos(4\,g_1) + \tfrac{5}{7}\,(c^4 - \tfrac{6}{7}\,c^2 + \tfrac{3}{35})(\tfrac{8}{15} + \tfrac{8}{3}\,{e_1}^2 + {e_1}^4))(\tfrac{2}{3} + {e_2}^2)\Big] \end{aligned} \tag{5}$$

where $\mu = \frac{Gm_0\,m_1\,m_2}{m_0+m_1}$, $\sigma = \frac{Gm_0\,m_1\,m_2\,(m_0-m_1)}{(m_0+m_1)^2}$ and $\rho = \frac{Gm_0\,m_1\,m_2\,({m_0}^3+{m_1}^3)}{(m_0+m_1)^4}$. After some algebraic manipulations we can show that the potential given by Eq. (3) is in agreement with the ones shown in references [4, 9, 13, 17]; the potential given by Eq. (4) is in agreement with the references [9] and [17] and the potential given by Eq. (5) is in agreement with the reference [17]. Therefore, the long-period disturbing potential is written as

$$< F >= R_0 + R_2 + R_3 + R_4 \tag{6}$$

3. Analysis of the long-period disturbing potential

Here we replaced Eq. (6) in the Lagrange planetary equations ([18]) and numerically integrated the set of nonlinear differential equations using the software Maple to analyze the orbital behavior of the body m_1 for some particular cases. We considered the following initial conditions. The star has mass $1M_{\odot}$, the planet has mass $1M_J$ and the outer brown dwarf has mass $40M_J$. The $M_{\odot}$ and M_J terms represent the mass of the Sun and Jupiter, respectively. The inner orbit has $a_1 = 6$ AU, $e_1 = 0.001$ and the initial value for the relative inclination is $i = 65°$. These initial conditions were obtained from [6] and are used for all the simulations made here. When we consider the $R_2 + R_3$ terms, the perturbation of the third-body causes the effect of making the inclination of the planet to oscillate between a retrograde and a prograde orbit, as shown in Fig. 2, which confirms the result shown in [6]. Fig. 3 shows the effect when the R_4 term

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is considered in the disturbing potential. Note that the angular frequency is changed, however, the behavior of the orbit is maintained. In this situation the effects of R_4 is time dependent.

Figs. 4 and 5 show the behavior of the inclination as a function of the time, taking into account the disturbing potential expanded up to the fourth order, and considering different initial values for the argument of the periapsis and for the longitude of the ascending node. It is easy to check the characteristics of the trajectories. As shown in Fig. 4, the inclination oscillates in a prograde orbit for a time longer than the one shown in Fig. 3, and then it starts to oscillate in a retrograde orbit. It also remains in orbit for a time longer than the ones shown in Fig. 2. This difference in terms of orbital behavior is due to the h_1 term that has been changed to another initial condition, here $h_1 = 90°$. Fig. 5 shows that the inclination remains in a prograde orbit for a longer time compared to Figs. 2 and 3, but shorter when compared to Fig. 4.

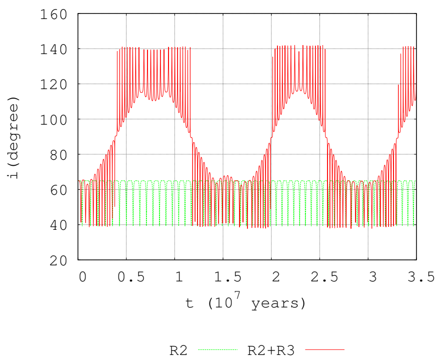

Figure 2. i vs t. Initial conditions: $a_2 = 100$ AU, $e_2 = 0.6$, $g_1 = 45°$ and $h_1 = 43°$.

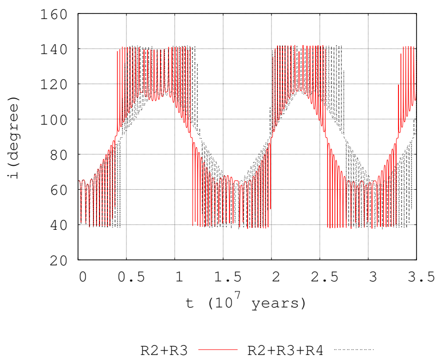

Figure 3. i vs t. Initial conditions: $a_2 = 100$ AU, $e_2 = 0.6$, $g_1 = 45°$ and $h_1 = 43°$.

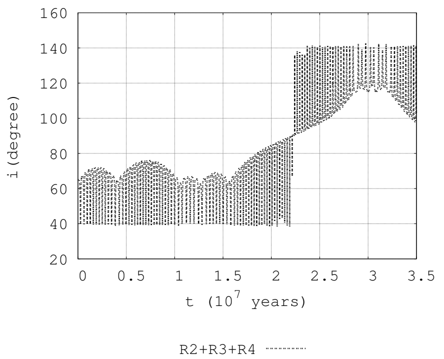

Figure 4. i vs t. Initial conditions: $a_2 = 100$ AU, $e_2 = 0.6$, $g_1 = 45°$ and $h_1 = 90°$.

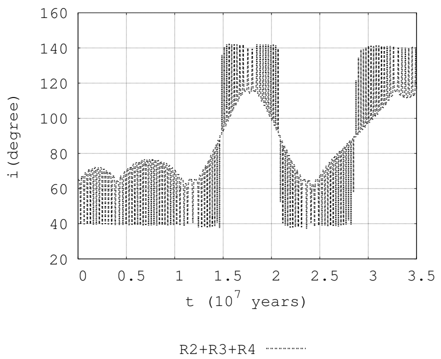

Figure 5. i vs t. Initial conditions: $a_2 = 100$ AU, $e_2 = 0.6$, $g_1 = 65°$ and $h_1 = 90°$.

The diagram e_1 vs g_1 (see Figs. 6 and 7) shows the regions of libration for various values of the eccentricity. For large eccentricities the orbits librates around an equilibrium point for $g_1 = 90°$ with small amplitude, especially for eccentricities $e_1 = 0.6$, 0.7 and 0.8. Fig. 6 were generated taking into account the octupole potential and Fig. 7 were generated taking into account in the potential the R_2, R_3 and R_4 terms. Note that the behavior of the eccentricity in the diagram e_1 vs g_1 (Fig. 7) is a little bit different from the one shown in Fig. 6. The term R_4

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has the effect of reducing the amplitude of the librations. These regions increase when g_1 moves away from 90 degrees. This is a consequence of the R_3 and R_4 terms included in the disturbing potential. Note that we found orbits that librate around the equilibrium point for $g_1 = 90°$. This value of g_1 satisfies the condition of frozen orbit ([19]). The search for frozen orbits ([19]) is made looking for inclinations where the eccentricity versus argument of the periapsis diagram has orbits that librate around the equilibrium point. It is interesting to note that Figs. 6 and 7 do not show lines, as Fig. 3 and 4 in [2], but show small regions of libration. This is due to the fact that the inclinations are oscillating between prograde and retrograde orbits.

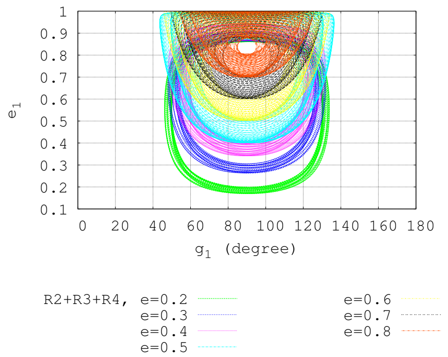

Figure 6. e_1 vs g_1. Initial conditions: $a_2 = 100$ AU, $e_2 = 0.6$, $g_1 = 90°$ and $h_1 = 270°$.

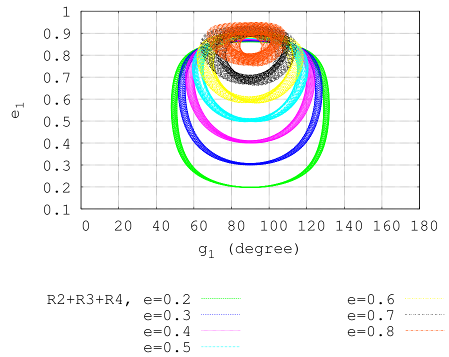

Figure 7. e_1 vs g_1. Initial conditions: $a_2 = 100$ AU, $e_2 = 0.6$, $g_1 = 90°$ and $h_1 = 270°$.

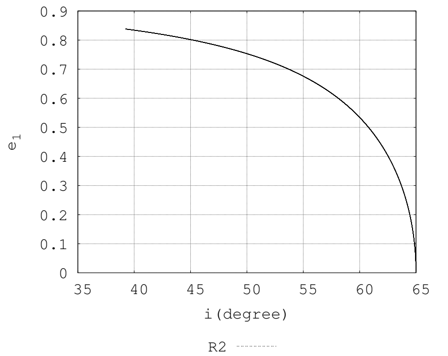

Figure 8. e_1 vs i. Initial conditions: $a_2 = 100$ AU, $e_2 = 0.6$, $g_1 = 45°$ and $h_1 = 43°$.

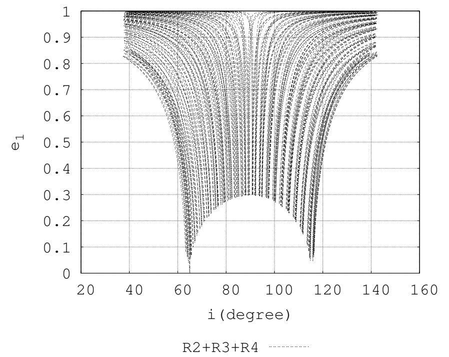

Figure 9. e_1 vs i. Initial conditions: $a_2 = 100$ AU, $e_2 = 0.6$, $g_1 = 45°$ and $h_1 = 43°$.

Fig. 8 shows the behavior of the eccentricity as a function of the inclination. As shown in Fig. 8, when the eccentricity increases the inclination decreases. This is consistent with the Kozai mechanism [8] when taking into account the quadrupole potential. However, including the potential up to the fourth-order in a small parameter (α) (Fig. 9), the inclination shows a different behavior, in comparison with Fig. 8. At the time $t = 0$, the initial mutual inclination is 65°. After a certain time period, the inclination begins to oscillate and the eccentricity reaches a maximum around 0.8, but, when the eccentricity reaches its maximum value the orbit slides to the right, starting to oscillate again. This motion is repeated until near the inclination of 90°. When the inclination reaches this value, the orbit changes its behavior and starts to oscillate in a retrograde orbit. Note also that, in the region near the polar inclination, the eccentricity has a minimum ($e_1 = 0.3$) that is different in comparison with other values of the inclination, as shown in Fig. 9. When the orbit slides right into the polar inclination, the eccentricity reaches a maximum value that is close to unity. At this point, the trajectory approaches an escape orbit and thus the planet may crash onto the star. Because the eccentricity has a large amplitude in its variation, the orbit has a different characteristic when compared to the ones obtained by the Kozai mechanism.

4. CONCLUSIONS

We investigate the dynamics of the three-body problem considering a triple system. We developed the disturbing potential in closed form up to the fourth-order in a small parameter. We analyzed the effects caused by the potential of the third and fourth order, where the dynamics is strongly modified when compared with the Kozai classic problem, where only the potential up to the second-order is considered. We show that the potential developed up to the fourth-order contributes effectively to the inversion of the orbit of the planet from prograde to retrograde. This is due to the strong growth of the eccentricity, inclination and argument of the periapsis caused by the R_3 and R_4 terms.

We show that the behavior of the inclination is sensitive to the initial conditions and that the behavior of the orbit is completely changed when the initial conditions are modified. We show that the effect of the fourth order term preserves the characteristic of oscillating the inclination between a prograde and a retrograde orbit, however, the angular frequency is changed. We found that the R_4 term changes the time when the phenomenon occurs. Taking into account the potential up to the fourth-order, we show a diagram e_1 vs g_1 where our results do not show lines (see Figs. 6 and 7), as Fig. 3 and 4 in [2], but show small regions of libration. This is due to the fact that the inclinations are oscillating between prograde and retrograde orbits. These regions increase when g_1 moves away from 90 degrees.

4.1. Acknowledgments

This work was accomplished with support of FAPESP under the contracts N° 2011/05671-5 and 2012/21023-6 SP-Brazil, CAPES, and CNPQ (300952/2008-2).

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