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Title:
Exploiting Vagueness for Multi-Agent Consensus

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Exploiting Vagueness for Multi-Agent Consensus

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This dissertation is submitted for the degree of Doctor of Philosophy in Engineering Mathematics.

Abstract

The ultimate objective of artificial intelligence is to develop intelligent agents that can think and act rationally. In intelligent systems, agents rarely exist in isolation, but instead form part of a larger group of agents all sharing the same (or similar) goals. As such, a population of agents needs to be able to reach an agreement about the state of the world efficiently and accurately, and in a distributed manner, so that they can then make collective decisions.

In this thesis we attempt to exploit vagueness in natural language so as to allow agents to be more effective in forming consensus. In classical logic, a proposition can be either true or false, which inevitably leads to situations in which agents that disagree about the truth of a proposition cannot resolve their inconsistencies in an intuitive manner. By adopting an intermediate truth state in cases where there is direct conflict between the beliefs of agents (i.e. where one believes the proposition to be true, and the other believes it to be false), we can combine the beliefs of agents in order to form consensus. We can then repeat this process across the population by forming consensus between agents in an iterative manner, until the population converges to a single, shared belief. This forms the basis of our initial model. We then extend this model of consensus for vague beliefs to take account of epistemic uncertainty. After demonstrating strong convergence properties of both models, we apply our work to a swarm of 400 Kilobot robots, and study the resulting convergence in such a setting. Finally, we propose a model of consensus in which agents attempt to reach an agreement about a set of compound sentences, rather than just a set of propositional variables.

Author's Declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

SIGNED: .... DATE: ....

Michael Crosscombe

Supervisor: Jonathan Lawry
1st December, 2017

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CHAPTER 1

Introduction

1.1 Overview

> "*I think vagueness is very much more important in the theory of knowledge than you would judge it to be from the writings of most people. Everything is vague to a degree you do not realize till you have tried to make it precise, and everything precise is so remote from everything that we normally think, that you cannot for a moment suppose that is what we really mean when we say what we think.*"
>
> — Bertrand Russell, *The Philosophy of Logical Atomism (1918-19)*

In the same lecture series from which the quote above is taken, Bertrand Russell states that he wishes he would have been able to spend the time to learn more about the concept of vagueness[1]. One of the oldest and most well-known studies of vagueness relates to the sôritês paradox, and is attributed to the Greek philosopher Eubulides [77, pp. 11] in 4^{th} century BCE. The paradox is as follows: you are presented with what you would define to be a heap of sand, and agree to the proposed assumption that the removal of a single grain of sand does not cause the heap of sand to no longer be a heap. In other words, that the difference between a heap of sand and a non-heap cannot be a single grain of sand. Then, by this admission, the repeated process of removing sand from the heap, a single grain at a time, could never render the heap to become a non-heap. Of course, it cannot be the case that a single grain of sand corresponds to be a heap, nor can two grains of sand. Then, at what point would the removal of a grain of sand change the heap to a non-heap? This fundamental question remains as relevant today as it was over 2 000 years ago, as vagueness is inherent in natural language, and is prevalent in the way in which we communicate with one another. Indeed, vagueness appears to be an inextricable part of conversation, writing, and even science and mathematics. Vagueness is so pervasive in language that 'even when artificial languages such as Esperanto are created and taught, little is done to clarify the meaning of vague expressions' [77, pp. 9]. One question, therefore, is why? Why is this seemingly suboptimal form of language so unavoidable when we communicate with one another? Shouldn't society have advanced so as to eliminate this apparent inefficiency? The

[1] Bertrand Russell, The Philosophy of Logical Atomism, Lecture 1: 'Facts and Propositions'. *The Monist*, 495-509. Oct 1918. `https://users.drew.edu/jlenz/br-logical-atomism1.html`

answer seems to be ‘perhaps not’, but the reasons for this are much more complex than might first be expected. In fact, [53] proposes that vagueness is actually an unfortunate side-effect of the empirical way in which we learn language. Specifically, O’Connor suggests that vagueness is a side-effect of our ability to generalise learnt behaviour to similar situations. This is indirectly supported by the unpublished work of Lipman [47] who argues that vagueness is sub-optimal if considered in the context of signalling games [46].

The ubiquity of vagueness might suggest, however, that it is useful in some way, and that it has a beneficial role to play in language. Indeed, there are several arguments to support this. In contrast to Lipman’s claim that vagueness is sub-optimal for *single-sender* signalling games, Lawry and James [43] show that vagueness is in fact advantageous when aggregating signals from *multiple* senders in the same context. Similarly, van Deemter [76] supports the more general argument that vagueness plays a positive role in a number of communication scenarios where the meaning of an expression cannot be assumed to be agreed upon. That is to say, definitions vary between individuals and so the use of a vague expression can enable others to understand its meaning without a precise definition being agreed upon. Another possible use of vagueness in language is in risk management. Lawry and Tang [44] suggest that the use of vague expressions are particularly helpful in mitigating the risk of making a promise which cannot be kept or a forecast which turns out to be wrong (e.g. in weather forecasts). More specifically, the authors demonstrate how vagueness can be exploited to minimise risk and maximise gain in multi-agent dialogues. The intuition is that the presence of borderline cases allows agents to make more flexible assertions in the presence of uncertainty. This point is further supported in [76] where it is argued that the risk of incurring costs (e.g. loss of trust in weather forecaster, programme, broadcaster etc.) is significantly reduced by making a vague assertion over making a more precise assertion, should such an assertion prove to be incorrect.

In this thesis, we argue that vagueness is useful in another way, by allowing agents to reach consensus when they are in conflict with one another. When two agents attempt to reach an agreement, they must find a way of resolving inconsistencies between their beliefs in order to achieve *consensus*. Here we define consensus to mean an agreement between both agents regarding a shared set of propositions, such that they both adopt the same resulting belief. This belief is formed by merging the beliefs of both agents and resolving any inconsistencies by adopting an intermediate truth state. In classical logic, where propositions are Boolean such that they are either true or false, there is no obvious way to resolve conflict between two opposing viewpoints. However, a feature of vague concepts is that they admit borderline cases, and so we can utilise vagueness as a means of resolving conflicting beliefs by having the agents weaken their beliefs by adopting a more vague interpretation of the underlying concept. Borderline cases are neither true nor false, but explicitly *borderline* meaning that they do not fully satisfy a proposition nor its negation. We use Kleene’s strong three-valued logic to model these borderline cases, where the third truth value represents ‘borderline true/false’. While there are different possible theories of vagueness, from supervaluationism [25, 68] to many-valued logics (including fuzzy logic [82]), we adopt a three-valued approach to model explicitly borderline cases only, and do not consider ‘higher-order’ vagueness or multiple truth values as in fuzzy logic. The primary reason being that an intermediate truth value representing ‘borderline’ is intuitive from a representational standpoint, and it provides a natural way in which conflicting agents can 'meet-in-the-middle'. Further discussions on the practicality of valuations as a representation of borderline cases, as well as arguments in opposition of representing higher-order statements as part of the underlying language, can be found in [44].

Consensus formation is an important part of distributed decision-making and negotiation scenarios. In human societies, for example, beliefs do not exist in isolation but inform and influence our decisions and actions. In this context is is unreasonable for individuals to only base their beliefs on first-hand experience. Instead, individuals rely on others to share and disseminate their beliefs so that they can make the most informed decisions given limited time and resources. This results in a dependent population, sharing beliefs between one another and, in doing so, also requiring the ability to differentiate inaccurate or erroneous information from trustworthy sources.

In artificial intelligence (AI) and robotics, we encounter scenarios in which systems of agents, or swarms of robots, need to make distributed decisions. In intelligent systems, individual agents usually coexist as part of a larger group and exploit their ability to communicate with one another in order to disseminate and aggregate knowledge from multiple sources. This is crucial in any decision-making process. Examples include sensor networks [57] with applications in environmental and industrial monitoring e.g. air pollution monitoring, earthquake detection, nuclear detonation detection etc. Autonomous vehicles are another example where groups of vehicles (agents) are able to communicate with one another in order to share information and to update their beliefs about the world, such that they are best able to select the most efficient route to their destination. The need to maintain accurate and up-to-date information is particularly necessary in order to achieve desired behaviour, given that acting upon inaccurate information (i.e. an inaccurate view of the current state of the world) may lead to undesired consequences e.g. vehicles taking heavily congested routes. In robotics, a slightly different set of constraints are placed on the individuals of the system. These could be a need to be fast and efficient in time-critical applications, robust to the presence of noise or erroneous behaviour, and often the process needs to be decentralised (i.e. distributed across the population, merging to form a consensus across the swarm) due to communication often being restricted or unreliable resulting from the limited hardware capabilities of robotic systems, or possibly environmental factors. One such example is collective motion, or 'flocking', in which agents must continuously select and distribute a shared direction of motion [30, 52, 71].

We intend to show that introducing a third truth value to consensus formation allows agents to avoid Boolean inconsistencies when combining their beliefs. By having agents adopt an intermediate truth value when there is a direct conflict of opinions regarding a proposition, they are able to reach a compromise where there would otherwise be no obvious and intuitive method of resolving a Boolean inconsistency without relying on the toss of a coin (a model we study in comparison to our proposed three-valued approach). Then, by combining agents' beliefs such that all agents involved in the consensus formation process (typically pairs of agents) adopt the same resulting belief, we can achieve population-wide consensus in a distributed manner where all agents ultimately share the same opinion. We will also show that when we combine consensus formation with evidential updating, agents are more effective at disseminating direct evidence compared with evidential updating alone, without any form of agent communication. Additionally, we propose that adopting a third truth value for the consensus process is more robust than its Boolean counter-part. In particular, we show that an intermediate truth state improves the robustness of the distributed decision-making process to the presence of malfunctioning agents. We go on to suggest that this approach may also be robust to other kinds of error, such as noisy sensors or malicious interference.

1.2 Background

In this section, we introduce a number of key concepts relevant to the research described in this thesis. In particular we discuss vagueness as it relates to our use of three-valued logic and the explicit representation of borderline cases. Then, we discuss the concept of limiting agent interactions based on the similarity of their opinions, a popular form of which is 'bounded confidence'.

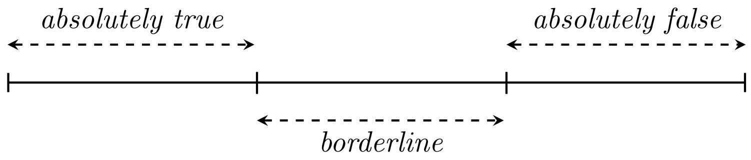


Figure 1.1: A depiction of truth-gaps allowing borderline cases in the definition of short.

Vagueness. A concept is said to be vague if it admits *borderline cases* i.e. where the proposition is neither absolutely true nor absolutely false [37], and where explicitly borderline cases are inherent to propositions involving vague concepts such as 'tall', 'young' and 'red'. This is depicted more clearly in Figure 1.1. Consider the following example: A search and rescue operation is taking place in a region following an earthquake. For this operation, a number of areas within the region have been identified as having been severely affected by the earthquake, which now require operatives to enter these areas in order to attempt to extract injured and trapped civilians. To do so, operatives must identify whether an area is considered to be 'accessible' (referring to their ability to enter the affected area safely). It is possible that a person, P_1, believes that 'the area is accessible' (i.e. that the proposition 'is accessible' is *true* with respect to the area being considered). The concept *accessible*, however, admits borderline cases such that the area may be neither *absolutely accessible* nor *absolutely not accessible*, but may instead be considered *borderline*. This is consistent with Parikh's [55] view that a sentence is absolutely true if it can be uncontroversially asserted, while a sentence being not absolutely false means that it is acceptable to assert.

It is important to emphasise here that vagueness is *not* meant to capture epistemic uncertainty. It is entirely possible for a person to believe that a proposition is borderline true with absolute certainty. Instead, it is helpful to think of vagueness and its relation to uncertainty in terms of conditioning: If we are informed that an area is 'borderline accessible', we learn that the safety level associated with the access of the area lies within the truth-gap between *absolutely not accessible* and *absolutely accessible* (i.e. that there is a fair amount of risk associated with entering the area). This is in contrast to being told that the area's accessibility is *unknown* from which we learn nothing. That is, being told that the area is 'borderline accessible' is not a precise description of the levels of safety concerning the area's access, but it does provide additional information than we would otherwise receive upon being told that the area's accessibility is unknown.

The use of Kleene logic as a representation of vagueness has been argued on the basis that its use as a logic of uncertainty tends to *overestimate* the level of uncertainty that one might possess. For example, suppose an operative possesses a belief in $\{0, \frac{1}{2}, 1\}^2$ about the accessibility of two areas $a_1, a_2 \in \{0, 1\}$, where the value $\frac{1}{2}$ denotes that the actual truth value of a proposition is *unknown*. Supposing that the operative believes $(\frac{1}{2}, 1)$, to be interpreted as 'a_1 is unknown' and 'a_2 is accessible'. Then it is clear that the operative is uncertain about whether a_1 is true or false, but that they are certain a_2 is true (i.e. that $(a_1, a_2) \in \{(0, 1), (1, 1)\}$). Similarly, any belief in $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ corresponds to an operative's complete certainty regarding the truth values of both a_1 and a_2. However, supposing instead that the operative believes $(\frac{1}{2}, \frac{1}{2})$, it becomes difficult to identify to which of the following uncertain sets an operative's belief corresponds:

$$(\frac{1}{2}, \frac{1}{2}) = \begin{cases} \{(0, 1), (1, 0)\}, \\ \{(0, 0), (1, 1)\}, \\ \{(0, 0), (0, 1), (1, 0)\}, \\ \{(0, 1), (1, 0), (1, 1)\}, \\ \{(0, 0), (0, 1), (1, 0), (1, 1)\}. \end{cases}$$

By using Kleene logic as a logic of uncertainty, we would interpret the belief $(\frac{1}{2}, \frac{1}{2})$ as representing $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$, the set of maximal uncertainty. Yet, the operative's belief could be representative of any of the above sets of possible beliefs, where some are more or less certain than others. Comparatively, if the third truth value represents a *borderline* state, as is the interpretation when using Kleene logic as a logic of vagueness, then the operative's belief does not represent a state of uncertainty regarding which truth value is assigned to each proposition, but that each proposition simply has a truth value of $\frac{1}{2}$, where it is considered to be borderline true/false. A more detailed analysis of the difference between these two possible interpretations of adopting a third truth state, representing *borderline* in this case, is given by Ciucci, Dubois and Lawry [10]. Lawry gives a more detailed discussion on the relationship between vagueness and uncertainty in [40].

Consensus. Throughout this thesis we will seek to form consensus between populations of agents. However, the term consensus is itself rather vague. It is therefore necessary to describe the properties of the kind of consensus that we are attempting to reach, and to compare those properties with other kinds of consensus that may be desirable in other approaches. Specifically, we define consensus by the following properties:

- *Precision*, where the resulting belief is completely precise (i.e. non vague, or Boolean);

- *Certainty*, such that the population converges to a single belief.

The first property ensures that the agents are completely precise in their beliefs such that they are able to come to a distinct decision about each proposition being either *true* or *false*. In the aforementioned search and rescue application, agents need to come to a firm decision whether an area is accessible or not, as to be vague is to risk indecision, and the operatives involved need to be unanimous in their agreement about whether an area is safe or unsafe.

The second property means that the agents have converged to a single belief; that is, every agent in the population has adopted the same belief with complete certainty. In combination with the property of precision, having converged on a single precise and certain belief, the population of agents is said to have reached a unanimous decision; to have achieved consensus.

Alternatively, some applications might require a different set of properties, such as:

- Preservation of vague beliefs;
- Convergence to a set of uncertain beliefs;
- A combination of the two, where beliefs can be both vague and uncertain.

These properties may well be useful for a variety of different scenarios, for example if you wish to deploy a swarm of robots for a similar search and rescue operation, you may want the robots to assign themselves to different areas depending on different features as mentioned earlier, including population density and time-sensitive accessibility, without the need for operatives to manage their resources manually. In such a scenario, it might be useful to allocate robots probabilistically based on their uncertain beliefs, such that they might choose to visit one area with probability 0.8 (an area believed to be highly accessible), while visiting a second area with probability 0.2 (an area believed to be fairly inaccessible). By allocating robots in this way, areas that are considered less safe are still allocated search and rescue resources, but the majority of the operation's resources are focussed on highly accessible areas. However, this extends beyond the initial problem of consensus and into task allocation, which is beyond the scope of this thesis. We assume instead that the process of task allocation, in this example, would follow from the population having already reached a consensus about the propositions that form the basis of further decision-making e.g. in task/resource allocation, planning etc.

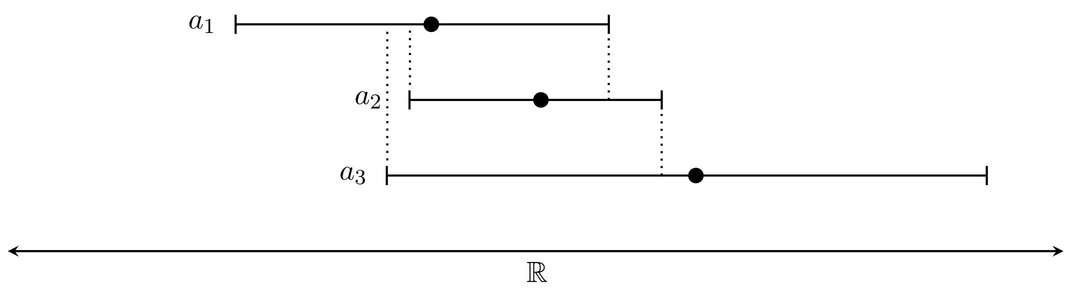


Figure 1.2: A depiction of bounded confidence where agents' opinions are some real number, and each agent defines their own *symmetric* confidence interval.

Agent interaction. In the Hegselmann-Krause model [31], the notion of 'bounded confidence' was introduced in order to limit agents to only interacting with other agents whose opinions differ by less than some distance, defined as a confidence interval, from their own opinion. Essentially, each agent a_i in the population has an opinion $o_i \in \mathbb{R}$, usually restricted to the interval $[0, 1]$ or $[-1, 1]$. An agent then defines a threshold value $\epsilon_i \geq 0$ such that, for any agent a_j with opinion o_j, agent a_j influences the opinion of agent a_i if $|o_i - o_j| \leq \epsilon_i$ (see [31] for further details). We can think of this bounded confidence as the uncertainty of an agent in their own opinion, such that a larger interval is indicative of the agent being uncertain of their opinion, and so is willing to allow differing opinions to influence them. Conversely, a very certain agent would have a small confidence interval such that they will only be influenced by those agents with very similar opinions to their own.

Figure 1.2 shows an example of three agents: a_1, a_2, and a_3. Each agent's opinon is represented by a black dot on the real number line, and also a symmetric confidence interval surrounding this opinion. In this example, a_1 is influenced by a_2, and a_2 by a_1, because each agent's opinion is contained within the confidence interval of the other agent. Neither a_1 nor a_2 are influenced by a_3, but a_3 is influenced by *both* a_1 and a_2, due to a_3 having a larger confidence interval than either of the other two agents. Notice that it is not always the case that if an agent a_i influences another agent a_j, that a_j in turn influences a_i.

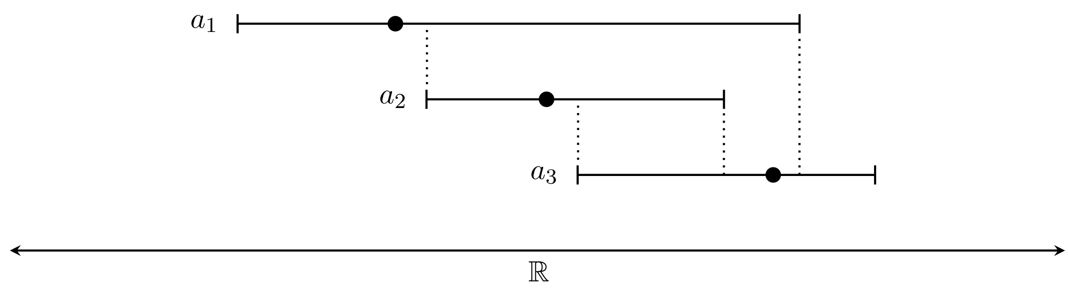


Figure 1.3: A depiction of bounded confidence where agents' opinions are some real number, and each agent defines their own *asymmetric* confidence interval.

Alternatively, in Figure 1.3 we see that the three agents possess asymmetric confidence intervals. In this example, a_1 is influenced by both a_2 and a_3, but both a_2 and a_3 have confidence intervals that are too small to be influenced by the opinions of other agents. In this example, an agent a_i would define both a lower and upper confidence interval, denoted $\underline{\epsilon}_i$ and $\overline{\epsilon}_i$, respectively. Then a_i is influenced by a_j if $o_j \in [o_i - \underline{\epsilon}_i, o_i + \overline{\epsilon}_i]$.

The notion of bounded confidence is important in the opinion dynamics literature because it allows the set of influencing agents to change over time, such that an agent might update their opinions based on a dynamic set of agents over the course of a simulation experiment. This was not the case in earlier models of opinion pooling, where an agent's set of influencers remained unchanged. This is discussed in more detail in Section 1.3.

Around the same time that Hegselmann and Krause introduced bounded confidence [31], related approaches proposed a global threshold adopted by the population [17, 80], rather than having each agent define their own levels of confidence in their opinions. In addition, the notion of relative agreement [18] was proposed as an extension of the bounded confidence model. In this extension, instead of an agent's confidence interval simply determining which opinions would influence them, the extent of the overlapping of agents' confidence intervals would also determine the extent to which those agents influenced one another. This is a natural generalisation of bounded confidence in that we no longer have a binary relation determining whether or not an agent is influenced by another. Instead, the level of influence is directly related to the degree to which agents overlap confidence intervals. One benefit of this kind of approach is that agents can assign individual weights to other agents, and that these weights can be based on the similarity of their opinions, as opposed to some a priori knowledge. This kind of model, however, is limited to opinions represented as real values and does not directly apply to the logical approach taken in this thesis. Nevertheless, the approach is an important step towards achieving intuitive combinations of opinions in a multi-agent system.

In this thesis, we adopt our own version of bounded confidence, based on an *inconsistency thresold* described in Chapter 2. This is similar to bounded confidence in that a pair of agents are either *consistent* according to some definition, in which case they go on to form consensus by combining their opinions, or they are not. Unlike in the model of relative agreement, our representation of opinions as truth assignments on some logical propositions means that the *extent* to which a pair of agents are consistent with one another does not influence the outcome of the consensus process between them. Agents combine their opinions in such a way that they both adopt the same, minimally altered opinion so as to become consistent with one another; their relative inconsistency only determines whether they attempt to reach consensus or not.

1.3 Related work

Opinion dynamics. A number of models for consensus have been proposed in the literature which have influenced the research described in this thesis. In some cases these are more directly comparable to our models for consensus, while others focus on an approach known as 'opinion pooling' where agents aggregate the opinions of other agents in the population. Opinion pooling dates back to the works of Stone [69], DeGroot [19] and Lehrer [45], with more extensive work on opinion pooling by Genest et al. [26, 27]. DeGroot introduced a model for reaching a consensus by iteratively aggregating a weighted (typically linear) combination of the opinions of the entire population until an agreement is reached. In such a model, agents assign a weight distribution to the population before forming a new opinion. By applying their assigned weights to the other agents' opinions, an agent can control the influence that others have on their own opinions.

Of course, when the weights are non-uniform it is assumed that agents possess a priori knowledge about the trustworthiness or expertise of the other agents in relation to the propositions concerned. This is difficult to justify outside of very specific scenarios, and in multi-agent systems it is often assumed that all agents begin with equally valid opinions until individuals receive further information (e.g. through forming consensus with other agents, or by directly sensing information) and adjust their opinions (both about the world and about one another) accordingly. Another assumption made in opinion pooling is that agents have the ability to communicate simultaneously with every other agent in the population. With the exception of physical systems where spatial constraints are placed on individuals, such as in robotics, it is unrealistic to model the dynamics of opinions in this way, given the volume of differing opinions that must be aggregated. As we have already discussed, this is typically achieved by means of weighted averaging, but more meaningful forms of agreement should be preferred, even at the cost of slower convergence due to limiting the number of opinions being aggregated at each iteration.

While opinion pooling typically refers to the aggregation of a large number of opinions (i.e. the entire population) at each iteration, the term *opinion dynamics* more generally refers to the study of an iterative process in which agents revise their opinions based on the opinions of other agents, and possibly other sources of information i.e. sensory information or external knowledge from experts). One of the most well known models of opinion dynamics is the Hegselmann-Krause model of bounded confidence [31], in which an agent updates their belief by averaging the opinions of only those other agents whose opinions do not differ from their own by more than a certain confidence level. This notion of bounded confidence has been adapted to other models of opinion dynamics, such as a model in which agents are Bayesian decision makers and bounded confidence is applied to agents' prior probabilities [78]. In [32], several opinion pooling functions are studied under bounded confidence, with axiomatic characterisations of the different operators given in [20, 21].

An alternative to opinion pooling was proposed by Deffuant et al. in [17], in which the authors sought to avoid the aggregation of opinions across the population. Instead, the proposed model selected pairs of agents at random from the population, with each agent then adjusting their opinion relative to the opinion of the other, provided that they were sufficiently similar according to some predefined threshold. Of course, this is close to the model of bounded confidence, except that agents update their opinions in a pairwise manner; an approach we adopt throughout this thesis. Then, in [18] the authors proposed an extension of bounded confidence, in which the extent to which an agent modifies their opinion so as to be more similar to the opinion of another agent depends on the amount of overlap between their confidence intervals.

It is worth noting here that the same assumption made in models of opinion pooling is also made in the previously mentioned models of opinion dynamics. These models assume that all agents have some pre-existing notion of confidence; either in other agents, for which a unique weight is assigned to each, or in themselves, such that an individual confidence interval is declared. This creates a neighbourhood around each agent in the opinion space, similar to that of a social network in which those of similar opinions are connected and may interact, dependent upon the level of confidence each individual has in their own opinion. Instead, we prefer to assume that all agents are initialised with equally valid opinions. Then, when we introduce the notion of an inconsistency threshold to limit agent interactions to those pairs of agents whose opinions are sufficiently similar, the same threshold is applied to all of the agents in the population.

Three-valued approaches. In the opinion dynamics literature, there is considerable diversity in regards to the representation of opinions. In opinion pooling, models typically represent opinions as probability distributions defined over some underlying parameter, but often this parameter is not explicitly defined beyond some abstract parameter space, as in [19]. More generally, representations in opinion dynamics are much more varied with many opting to represent opinions as bounded real values, usually either in the range $[0, 1]$ or $[-1, 1]$ (see [17] and [18], respectively). A number of more relevant studies exploit a third truth state to aid convergence of the system [1, 6, 16, 79]. A common motivation is that, when a population contains minority groups of highly opinionated individuals, resolving conflict between these groups requires an intermediate state for agents to transition more gradually between the more polarised states; a form of 'middle ground'. This is perhaps more relevant to discrete representations of opinions, in contrast to bounded real values, where an intermediate state between two polarised opinions does not already exist i.e. for binary opinions in $\{0, 1\}$, and similarly when opinions are truth assignments in $\{true, false\}$ on a (set of) propositional variable(s). A clear example of this is given by de la Lama et al. [16], in which agents are in either one of three states: A, B, or I, where A and B are states in which an agent is supporting either 'party A' or 'party B', respectively. State I then indicates that an agent is *undecided*. Agents in support of either party A or party B do not interact across parties, but instead interact with agents in the undecided state, where there exists a non-zero probability of each agent spontaneously switching from one party to the undecided state, and vice versa i.e. $A \rightleftarrows I$ or $B \rightleftarrows I$. Clearly the high-level intuition behind this intermediate state is directly related to our proposed adoption of a third truth value, despite the differences between the underlying models. In [79], Vazquez and Redner consider a similar model of 'leftists' and 'rightists' which do not interact, except through a the third group referred to as 'centrists'.

Perhaps less directly related is the work of Balenzuela et al. [1], who present a model in which agent opinions are represented as real values in $[-1, 1]$, similar to the relative agreement model [18]. However, in [1], the third truth state is defined by applying a partitioning threshold to the underlying real value. In this model, updating takes place iteratively between pairs of agents, where the magnitude and sign of the increments depends on the current truth states of the agents involved. Cho and Swami [6] adopt a model of beliefs based on Dempster-Shafer functions, essentially combining probability with a three-valued truth model. However, the consensus operator [36] used by Cho and Swami is quite different from the consensus operator described in Chapter 3 [41] and therefore results in rather different limiting behaviour, where the former consensus operator can only be applied in the presence of uncertainty. This model also employs iterative pairwise interactions in order for the population to converge towards consensus.

Modelling approach. We have highlighted several approaches to modelling consensus in the literature, some of which relate directly to our own approach. For example, while we mentioned how some models of opinion dynamics have opted for classical methods of aggregating opinions across the population, we described several models favouring iterative pairwise interactions between agents. In our approach we adopt a pairwise model of combining agents' opinions as in [1, 17, 18, 79]. We also propose our own method of limiting agent interactions related to the concepts of bounded confidence and relative agreement (see [31] and [18], respectively), and study the effect of this restriction on agent interactions. Our approach is most similar to that of Deffuant et al. [17] and Weisbuch et al. [80] in that a global threshold is set for the population. By varying an inconsistency threshold, we can study varying levels of connectivity in the population, ranging from those in which only very similar opinions can interact and form consensus, to a completely connected graph in which all agents can interact with any other agent in the population. In Chapter 2, we present results on how this threshold affects the overall convergence of a population of agents.

We adopt a propositional logic setting (specifically, using Kleene's strong three-valued logic), but unlike in the works of Cholvy [7, 8] and Grandi et al. [29] on propositional opinion diffusion which consider opinion aggregation, we assume pairwise agent interactions. In particular, Cholvy [7] models opinions as propositional formulae, such that if an agent has the opinion $(Can\_2026 \vee Norway\_2026) \wedge acroski$, then the agent believes that *either* Canada or Norway will host the 2026 Winter Olympics, and that there will be acroski trials. In Chapters 2 and 3, we assume a uniform distribution of truth values for each propositional variable, and that they are mutually independent. This necessarily differs in Chapters 4 and 5 due to the intended applications, where in Chapter 5 we model opinions as compound sentences, but these are still represented in terms of the underlying Kleene valuations on the propositional variables.

The consensus operator studied this thesis was first proposed by Lawry and Dubois [41] as an extension of the approach of Perron et al. [57], who examined the binary consensus problem on complete graphs in the single propositional variable case. The authors assumed unconstrained random interactions between individuals, and showed that extending both signalling and memory of individuals from two states to three dramatically improved the reliability and speed of convergence to a single shared Boolean opinion. In [13] we extended the operator of [57] to languages with multiple propositions. In [41], Lawry and Dubois studied the logical properties of this operator in greater depth, and also compared it to several other opinion combination operators highlighted in Chapter 2. The authors then extended the operators to take account of probabilistic uncertainty, which forms the basis of the model applied in Chapter 3. Indeed, the work of [41] has yet to be studied (to our knowledge) in a multi-agent setting until our recent work in [12], to be presented in greater detail in Chapter 3.

1.4 Contributions and outline

In this thesis we develop models of consensus for multi-agent systems and swarm robotics. In particular, we develop symmetric and asymmetric models of consensus for large populations of agents which are attempting to reach an agreement about a shared set of relevant propositions. We define *consensus* to mean where the population of agents has converged to a single shared viewpoint about the propositions, and where 'convergence' simply implies a significant reduction in the number of unique opinions in the population.

In this chapter we have introduced several areas of active research related to the general topic of consensus modelling, and we have given some background information and motivation for this thesis, supported by relevant work from the literature. We have discussed more thoroughly those works which have played an important role in sparking ideas and laying a foundation on which we have developed our own approaches, as well as those which attempt to answer the same questions but from rather different perspectives.

In **Chapter 2** we explore a symmetric model of consensus in a propositional logic setting. We examine relatively large populations of agents (up to 1 000) in which agents' beliefs are modelled as truth-valuations on propositional variables and where they combine their beliefs in pairs, such that each agent adopts the resulting combination and is therefore said to have formed a *pairwise consensus*. Subsequently we introduce a feedback mechanism whereby agents receive a 'payoff' reflecting the quality of their current beliefs, and use this to bias the agent-selection process in favour of agents with more accurate beliefs according to the payoff model. The intuition is that this payoff model reflects the true state of the world and agents are more likely to be selected to combine their beliefs with other agents if their opinions are closer to the truth. In order to generalise this approach to consensus, we investigate group models in which small subsets of the population are selected to combine their beliefs, with the intention being that increasing the number of agents involved in the consensus process should improve the speed at which the population converges on a single shared belief. As population sizes increase, so too does the time required for the population to reach a consensus, due to the nature of the iterative, pairwise process. By moving to groups of agents, we seek to circumvent this limitation. The first part of this chapter is based on the model presented in [13] with expansions to explain several aspects of the model in greater detail, as well as further discussions regarding the effects of increased population sizes and how the model for group-wide consensus performs as a proposed solution.

In **Chapter 3** we combine the three-valued propositional logic approach introduced in Chapter 2 with a probabilistic model of uncertainty. Here, beliefs exhibit both vagueness and uncertainty, and in this extended context we consider whether the addition of the third truth value continues to play a beneficial role in the process of consensus formation in multi-agent systems. The contents of this chapter are based entirely on the work presented in [12] which is an extension of [13].

In **Chapter 4** we investigate a different consensus scenario involving asymmetric belief-updating and inspired by biological systems of quorum sensing [48, 66]. We describe a model in which the population is divided into two sub-groups of agents: those in a disseminating state and those in an updating state. This model is applied to the 'best-of-n' problem [56, 75], a popular decision problem in swarm robotics in which the system must decide which is the best of n possible options, where each option is associated with a perceived quality. Initially we run experiments in a simulation environment for a swarm of simulated Kilobots, before implementing our approach on a Kilobot swarm consisting of 400 robots. The contents of this chapter are based on the three-valued 'voter model' presented in [15], a proposed extension of the 'weighted voter model' [73] with improved robustness characteristics.

In **Chapter 5** we explore how the three-valued model introduced in Chapter 2 can be extended to simulations in which the agents in the population are required to reach consensus about a set of compound sentences, rather than just about the propositional variables. Agents combine their truth assignments on a small set of sentences, and then adjust their underlying valuations on the propositional variables so as to remain consistent with their higher-level beliefs. We present two models for compound sentence consensus, and discuss some preliminary results obtained for both. This chapter is an extension of the work presented in [14].

Finally, in **Chapter 6** we give some conclusions identifying remaining challenges and discuss possible future avenues of research.

CHAPTER 2

A Three-Valued Model for Consensus

In this chapter we aim to exploit vagueness as a feature of natural language for consensus and apply this in a multi-agent setting. Typically, agents reason about propositions which are either *true* or *false*, and this reflects an entirely *precise* world in which truth-gaps do not exist and opinions are completely Boolean. Instead, we introduce a third truth value to represent the borderline cases inherent to vague concepts where propositions can instead be either *true*, *false* or *borderline*. Rather than simply being a more realist model of opinions, though, this third truth value also provides a means for agents to reach consensus by adopting a 'middle-ground' for propositions about which their opinions are inconsistent. Such an approach is natural in the presence of a third truth value but is not easy to achieve effectively when restricted to Boolean propositions. This is also an important part of reasoning intelligently and acting rationally amongst other agents in a system, where your opinions do not exist in isolation and your actions have consequences that affect others that exist in the world.

Initially, we introduce a model for representing opinions which can be vague (but aren't necessarily so) and a consensus operator which is applied between a pair of agents in order to merge their opinions. This operator allows even directly conflicting agents to form consensus by first resolving any inconsistencies via adoption of the third truth value. Here we are assuming that differences in opinions stem from differences in the ways in which each agent defines the underlying concepts and, more specifically, the boundary definitions for *absolutely true* and *absolutely false*; a separation between these two boundaries then indicates a truth-gap which we exploit to form consensus. We propose to model consensus as a pairwise interaction between two agents where the consensus operator is applied and both agents adopt the same resulting opinion, repeating this process between pairs of randomly selected agents for a fixed number of iterations or until consensus is reached. We opt to explore the use of a threshold parameter which limits agent interactions to pairs with sufficiently similar opinions and how this affects the convergence of the population. We then explore the idea of there being a 'true' state of the world which is unknown to the agents. By biasing the agent selection process via the weighting of an agent's probability of being selected, based on the accuracy or 'quality' of their opinions, we show that the three-valued consensus model outperforms a Boolean model in driving convergence of agents' opinions towards the opinion which reflects the true state of the world. Finally, we explore an alternative model in which, instead of pairs of agents, small groups are selected and consensus is applied iteratively between all consistent pairs in order to produce a probability distribution over the propositional variables. Each agent then adopts a new opinion generated from their resulting probability distribution which may differ from that of other agents in the group. While this is not *consensus* as we define it, this method should still allow us to increase the number of agents involved in the updating process with the intention being to achieve convergence to a smaller set of opinions in a shorter amount of time than is possible with pairwise interactions.

2.1 Related work

Early work on consensus formation appeared in the early 1960s and 1970s in the field of 'opinion pooling' [19, 69] where each agent aggregates beliefs across the entire population at each time step according to an agent's predefined distribution of weightings. In these models, beliefs are often represented by probability distributions on some underlying variable, which is rather different from the approach we take in this chapter, but nevertheless remains relevant to the general idea of consensus in large agent populations.

More recent approaches, however, look to model realistic interactions in which individuals do not assign static weights to others in the population [17, 18, 31, 32, 80]. Instead, these models of 'opinion dynamics' seek a more intuitive form of consensus in which individuals combine their beliefs only if they are sufficiently similar, and in some models they do so in a more distributed, pairwise manner [17, 18], repeating this process across the population until convergence occurs. Although these approaches still model agent opinions differently to our own, they form the basis of the model presented herein; more specifically, the way in which agents interact amongst the population is heavily inspired by [17]. We explore a method of limiting agent interactions in this chapter which is heavily inspired by bounded confidence [31] but does not influence the extent to which agents modify their beliefs as in [18]. Furthermore, these models do not assume some a priori knowledge about every agent in the population in order to assign weights as much of the opinion pooling literature does.

Most of the models mentioned here represent opinions as some continuous bounded real number, but even these models are not consistent in their representations. In this chapter, we model opinions in Kleene's strong three-valued logic where an opinion is some truth assignment (i.e. Kleene valuation) on the propositional variables of the language. Several existing models for consensus exploit a third truth state to aid convergence [1, 16, 79], though most interpret the third truth value as 'uncertain' or 'unknown', which we describe to be distinctly different from our approach to using a third truth value to model vagueness, as outlined in the introduction.

Finally, the original pairwise three-valued consensus operator studied in this paper was initially proposed by Perron et al. [57] for consensus across complete graphs, which is a special case of our model. The logical properties of this operator and its relationship to other similar aggregation functions are investigated by Lawry and Dubois in [41], where the authors also introduce several other pairwise operators for combining opinions. The concept of orthopairs used in this thesis to simplify our notation was introduced in [42].

¬	1	0
	$\frac{1}{2}$	$\frac{1}{2}$
	0	1

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

Table 2.1: Kleene truth tables.

2.2 Model

Overview. We introduce Kleene's three-valued logic as a model of explicitly borderline cases resulting from the vagueness of natural language propositions. We then exploit these truth-gaps to allow agents to adopt a more vague interpretation of the underlying concepts about which they are attempting to reach a consensus. Consensus is a pairwise process between agents, with each agent either chosen at random or via a weighted selection process. We deal explicitly with a symmetric updating process for consensus in which both agents adopt the newly formed opinion as a combination of their previously held opinions. This differs to asymmetric models where agents may adopt different opinions from others involved in the same updating process.

2.2.1 A three-valued model for pairwise consensus

We adopt a propositional logic setting based on Kleene's strong three-valued logic [39] as follows: Let $\mathcal{L}$ be a finite language of propositional logic with connectives $\wedge$, $\vee$ and $\neg$, and propositional variables $\mathcal{P} = \{p_1, ...p_n\}$. Also, let $S\mathcal{L}$ denote the sentences of $\mathcal{L}$ generated by recursive application of the connectives to the propositional variables in the usual manner. A Kleene valuation then allocates the truth values 0, $\frac{1}{2}$, and 1 to the sentences of $\mathcal{L}$ denoting false, borderline and true, respectively as follows:

Definition 2.1. Kleene valuations

A Kleene valuation $\mathbf{v}$ on $\mathcal{L}$ is a function $\mathbf{v} : S\mathcal{L} \to \{0, \frac{1}{2}, 1\}$ such that $\forall \theta, \varphi \in S\mathcal{L}$ the following hold:

- $\mathbf{v}(\neg\theta) = 1 - \mathbf{v}(\theta)$
- $\mathbf{v}(\theta \wedge \varphi) = \min(\mathbf{v}(\theta), \mathbf{v}(\varphi))$
- $\mathbf{v}(\theta \vee \varphi) = \max(\mathbf{v}(\theta), \mathbf{v}(\varphi))$.

The truth table for Kleene valuations are shown in Table 2.1. Note that given this definition, a Kleene valuation $\mathbf{v}$ on $S\mathcal{L}$ is completely characterised by its values on $\mathcal{P}$. We also denote the set of all possible Kleene valuations by $\mathbb{V}$.

Orthopairs

It can also be convenient to represent a Kleene valuation $\mathbf{v}$ by its associated *orthopair* [42], (P, N), where $P = \{p_i \in \mathcal{P} : \mathbf{v}(p_i) = 1\}$ and $N = \{p_i \in \mathcal{P} : \mathbf{v}(p_i) = 0\}$. Then $\mathbf{v}_{(P,N)}$ denotes the valuation characterised by the orthopair (P, N). Notice that $P \cap N = \emptyset$ and that $(P \cup N)^c$ corresponds to the set of borderline propositional variables. Orthopairs are particularly helpful when discussing valuations as applied to the underlying propositional variables. Consider the following example: Given two propositional variables p_i and p_j for $j \neq i$, an orthopair $(\{p_i\}, \emptyset)$ represents the valuation $\mathbf{v}$ such that $\mathbf{v}(p_i) = 1$ and $\mathbf{v}(p_j) = \frac{1}{2}$, since $p_i \in \mathcal{P}$ and $p_j \in (P \cup N)^c$.

Why Kleene valuations?

Kleene valuations have been proposed as a suitable formalism in which to capture explicitly borderline cases, as resulting from the inherent flexibility in the definition of vague concepts in natural language [42, 44]. The concept 'safe', for example, is vague such that there exist cases that would be considered neither *absolutely safe* nor *absolutely not safe*, but would instead be considered *borderline safe/not safe*. Using Kleene's three-valued logic to represent vagueness, we can enable agents with conflicting beliefs to resolve their inconsistencies by having them adjust their underlying definitions and adopt a more vague interpretation of the concept 'safe'. For a more detailed study of vague concepts see [70] which motivates the use of Kleene's three-valued logic in terms of lower and upper thresholds of distances from a prototype.

2.2.2 An overview of operators for combining valuations

A number of combination operators for Kleene valuations are proposed in [41] which aim to combine a pair of valuations according to some underlying principle. However, in order to adequately define these operators we must first introduce the notion of *consistency*.

Definition 2.2. Consistency [41, def. 6]

Kleene valuations $\mathbf{v}_1$ and $\mathbf{v}_2$ are consistent if and only if $\forall \theta \in S\mathcal{L}$,

$$\min(\max(\mathbf{v}_1(\neg\theta), \mathbf{v}_2(\theta)), \max(\mathbf{v}_2(\neg\theta), \mathbf{v}_1(\theta))) \neq 0.$$

That is, given valuations $\mathbf{v}_1$ and $\mathbf{v}_2$ and associated orthopairs (P_1, N_1) and (P_2, N_2), then two valuations are consistent if and only if $P_1 \cap N_2 = P_2 \cap N_1 = \emptyset$. In other words, two valuations are consistent provided that, if a proposition is *absolutely true* according to one valuation then it is *not absolutely false* according to the other [41]. Therefore borderline valuations do not contribute a source of inconsistency. For the following operator, consistency is a necessary condition by the definition of orthopairs as we have that $P \cap N = \emptyset$ and so the union of two valuations characterised by their respective orthopairs requires that $(P_1 \cup P_2) \cap (N_1 \cup N_2) = \emptyset$. We can then go on to define the optimistic operator below.

The optimistic operator seeks to combine opinions by simply taking the union of two consistent valuations as follows.

Definition 2.3. Optimistic operator [41, def. 14]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be consistent Kleene valuations on $\mathcal{L}$. Then the optimistic combination $\mathbf{v}_1 \oplus \mathbf{v}_2$ is defined as follows:

$$\mathbf{v}_1 \oplus \mathbf{v}_2 = \mathbf{v}_{(P_1 \cup P_2, N_1 \cup N_2)}.$$

$\oplus$	1	$\frac{1}{2}$	0
1	1	1	$-$
$\frac{1}{2}$	1	$\frac{1}{2}$	0
0	$-$	0	0

Table 2.2: Truth table for the optimistic operator.

$\ominus$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	1	$\frac{1}{2}$	0
0	$\frac{1}{2}$	$\frac{1}{2}$	0

Table 2.3: Truth table for the difference operator.

We can see here that given a consistent pair of valuations, we are preserving all agreed upon truth values for all propositions, while adopting more precise (i.e. non-borderline) truth values where one agent believes the proposition to be true/false and the other believes it to be borderline. For inconsistent valuations $\mathbf{v}_1$, $\mathbf{v}_2$, $\oplus$ is undefined.

The second operator to be introduced is an asymmetric operator by which, for a pair of agents, one agent can be made consistent with the other by minimally adapting their opinions to align with those of the other agent.

Definition 2.4. Difference operator [41, def. 17]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be Kleene valuations on $\mathcal{L}$. Then the difference combination $\mathbf{v}_1 \ominus \mathbf{v}_2$ is defined as follows:

$$\mathbf{v}_1 \ominus \mathbf{v}_2 = \mathbf{v}_{(P_1 \setminus N_2, N_1 \setminus P_2)}.$$

Hence, this operator softens the precise values of $\mathbf{v}_1$ by adopting more vague interpretations of the conflicting propositions, so as to become consistent with $\mathbf{v}_2$.

An operator for reaching consensus

We now introduce the consensus operator which seeks to merge two potentially conflicting opinions into a single, consistent opinion.

The following operator is designed in such a way as to ensure that both agents, when applying the operator to their opinions, receive the same resulting opinion which they then both adopt. This is similar to the optimistic combination operator, only without requiring that both valuations be consistent, and is different to the difference operator in that the consensus operator is commutative and the resulting valuation is a minimally consistent combination of both valuations.

Definition 2.5. Consensus operator [41, def. 22]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be Kleene valuations on $\mathcal{L}$. Then the consensus combination $\mathbf{v}_1 \odot \mathbf{v}_2$ is the Kleene valuation:

$$\mathbf{v}_1 \odot \mathbf{v}_2 = \mathbf{v}_{((P_1 \cup P_2) \setminus (N_1 \cup N_2), (N_1 \cup N_2) \setminus (P_1 \cup P_2))}.$$

The corresponding truth table for this operator is shown in Table 2.4. The intuition behind the operator is as follows: In the case that the two agents disagree then if one has allocated a non-borderline truth value to p_i, while the other has given p_i a borderline truth value then the

$\odot$	1	$\frac{1}{2}$	0
1	1	1	$\frac{1}{2}$
$\frac{1}{2}$	1	$\frac{1}{2}$	0
0	$\frac{1}{2}$	0	0

Table 2.4: Truth table for the consensus operator.

I	1	$\frac{1}{2}$	0
1	0	0	1
$\frac{1}{2}$	0	0	0
0	1	0	0

Table 2.5: Truth table for the inconsistency measure.

non-borderline truth value is adopted in the agreed compromise. In other words, if one agent has a strong view about p_i while the other is ambivalent then they will both agree to adopt the strong viewpoint. In contrast if both agents have strong but opposing views i.e. with one valuation giving p_i truth value 0 and the other 1, then they will agree on a compromise truth value of $\frac{1}{2}$. Alternatively, from Definition 2.5 we can think of $\odot$ as an operator which merges both the optimistic and difference operators into a two-step process: initially, the operator weakens both opinions so as to remove direct inconsistencies, before then combining them to form a single, consistent opinion. An alternative formulation representing this two-step process is given by:

$$\mathbf{v}_1 \odot \mathbf{v}_2 = (\mathbf{v}_1 \ominus \mathbf{v}_2) \oplus (\mathbf{v}_2 \ominus \mathbf{v}_1).$$

Measuring consensus

In order to analyse the resulting consensus we introduce two measures that will be used throughout the subsequent simulation experiments. We begin with a measure of vagueness in order to help us understand the effect that vagueness has on the resulting consensus, and also to verify that the resulting consensus is meaningful in the sense that we do not converge to a completely borderline valuation on the propositions. A vagueness measure of 1 indicates a *completely vague* valuation such that every proposition is considered borderline. Conversely, a vagueness measure of 0 is therefore reflective of a completely *crisp* valuation in which all propositional variables have truth values either 0 or 1.

Definition 2.6. A vagueness measure

Let $\mathbf{v}$ be a Kleene valuation on $\mathcal{L}$ with n propositional variables. Then we measure the vagueness of $\mathbf{v}$ by the proportion of propositional variables which it classifies as being borderline. That is, for $p_i \in \mathcal{P}$:

$$V(\mathbf{v}) = \frac{\left|\{p_i \colon \mathbf{v}(p_i) = \frac{1}{2}\}\right|}{n} = \frac{|(P \cup N)^c|}{n}.$$

Definition 2.7. An inconsistency measure

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be Kleene valuations on $\mathcal{L}$. Then we define the inconsistency measure of $\mathbf{v}_1$ and $\mathbf{v}_2$ to be the proportion of propositional variables which are in direct conflict between the two valuations i.e. $\mathbf{v}_1(p_i) \neq \frac{1}{2}$, $\mathbf{v}_2(p_i) \neq \frac{1}{2}$ and $\mathbf{v}_1(p_i) = 1 - \mathbf{v}_2(p_i)$. That is, for $p_i \in \mathcal{P}$:

$$I(\mathbf{v}_1, \mathbf{v}_2) = \frac{|\{p_i \colon |\mathbf{v}_1(p_i) - \mathbf{v}_2(p_i)| = 1\}|}{n}.$$

Notice that the inconsistency measure here is related to consistency (Definition 2.2) where the measure of consistency of two Kleene valuations $\mathbf{v}_1$ and $\mathbf{v}_2$ is simply $1 - I(\mathbf{v}_1, \mathbf{v}_2)$.

Table 2.5 shows the inconsistency truth table of two valuations for a propositional variable, highlighting the cases where two valuations are *inconsistent*, and consistent otherwise. Assuming that two valuations are picked at random from $\mathbb{V}$, we can see that there is a probability of $\frac{2}{9}$ that two valuations will be inconsistent for each propositional variable in the language. In the following section we will propose a threshold $\gamma \in [0, 1]$ on inconsistency so that valuations $\mathbf{v}_1$ and $\mathbf{v}_2$ can be combined only if $I(\mathbf{v}_1, \mathbf{v}_2) \leq \gamma$.

2.3 Random selection experiments

In this section we introduce simulation experiments in order to investigate the convergence properties of the three-valued consensus operator (Definition 2.5) when implemented across a multi-agent system. The experimental set up is loosely based on those proposed in [18] and [50], although our representation of opinions is quite different with opinions taking the form of Kleene valuations on $\mathcal{L}$, rather than vectors of bounded real numbers.

We will consider two distinct initialisations of the opinions of a population of agents, where an opinion is simply a particular allocation of truth values to all of the propositions. The *random three-valued initialisation* allocates the truth values 0, $\frac{1}{2}$ and 1 to each agent and each propositional variable at random i.e. with probability $\frac{1}{3}$ for each truth value. In contrast, the *random Boolean initialisation* only allocates the binary truth values 0 and 1, each with a probability of $\frac{1}{2}$. This latter initialisation will be required in order to directly compare the proposed three-valued combination operator with a similar two valued operator. In this section we will use the random three-valued initialisation in order to investigate the extent to which the three-valued operator results in convergence to a shared set of opinions across the population of agents.

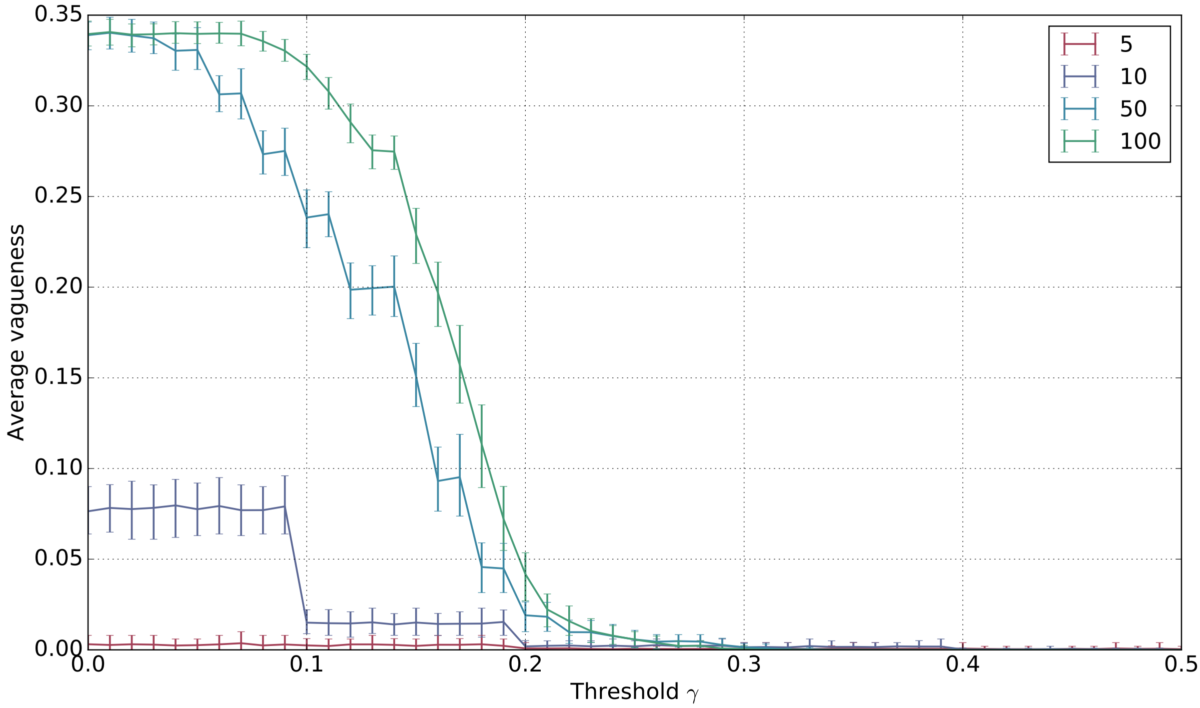
We set a fixed maximum number of 50 000 iterations[1]. At each time step a pair of agents are selected at random from the population. An inconsistency threshold value $\gamma \in [0, 1]$ is set, so that for any pair of agents with respective valuations $\mathbf{v}_1$ and $\mathbf{v}_2$, if $I(\mathbf{v}_1, \mathbf{v}_2) \leq \gamma$ then both agents replace their opinions with the consensus valuation $\mathbf{v}_1 \odot \mathbf{v}_2$, while if $I(\mathbf{v}_1, \mathbf{v}_2) > \gamma$ then no combination is performed and both agents retain their original opinions. This is a form of bounded confidence [17, 31] as detailed in Chapter 1 whereby we limit agent interactions based on the consistency (and therefore similarity) of their opinions. For $\gamma = 1$ we obtain what is equivalent to the totally connected graph model described in [57], in which any pair of agents can combine their opinions, whilst taking $\gamma = 0$ corresponds to the most conservative scenario in which only absolutely consistent opinions can be combined. It should be noted that forming consensus between consistent opinions still allows for the merging of differing opinions and a new consensus opinion to emerge, as borderline valuations maintain consistency and yet will be replaced by a more precise value when merged. The parameters for the simulation experiments are then as follows: we study a population of 100 agents, with language sizes (i.e. $|\mathcal{P}| = n$) of $5, 10, 50$, and 100. opinions are initialised at random according to the three-valued

[1]In preliminary experiments we found that 50 000 was an upper bound on the number of iterations required for the system to reach steady state across a range of parameter settings.

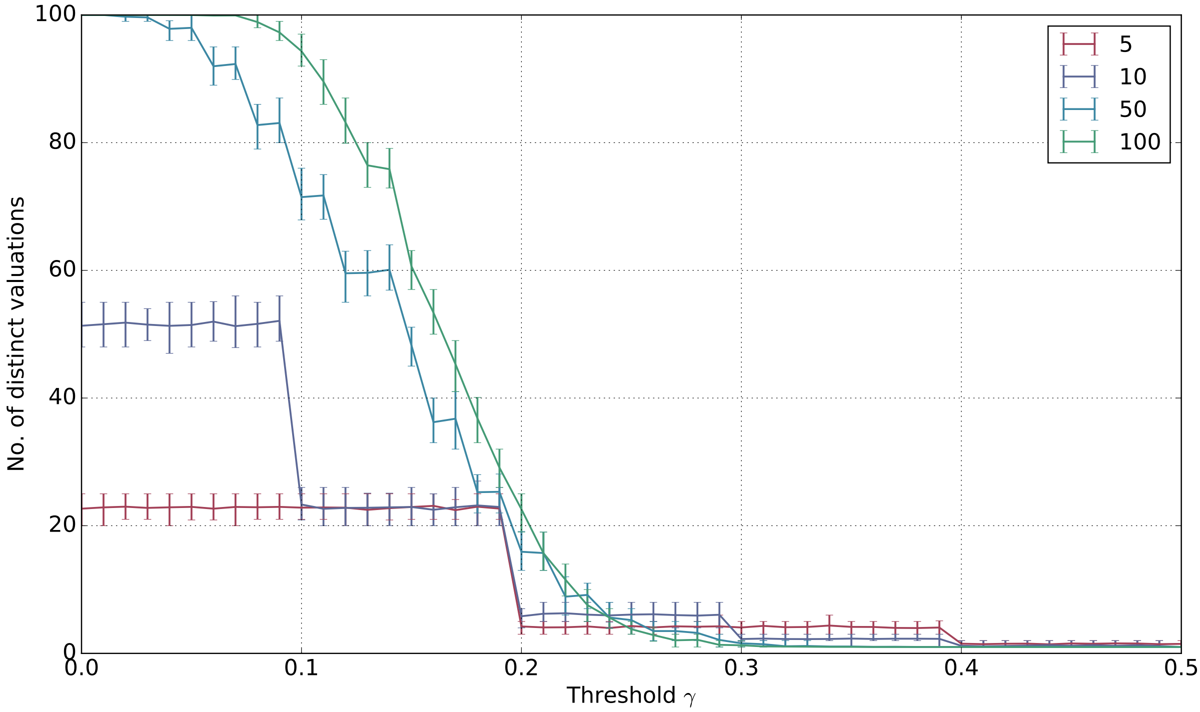
initialisation as described earlier. We also set an inconsistent threshold $\gamma \in [0, 1]$ in order to study how a population of agents is affected by forming consensus with opinions of varying levels of inconsistency.

2.3.1 Results with random three-valued initialisation of opinions

Figure 2.1 shows the results for the experiments after 50 000 iterations. In each case the plots show mean values with error bars indicating the 10^{th} and 90^{th} percentiles across 100 independent runs of the simulation. Figure 2.1a shows the average vagueness determined by taking the mean value of $V(\mathbf{v})$ (Definition 2.6) across the population. Note that for a random three-valued initialisation of opinions we expect a mean vagueness value of $\frac{1}{3}$ at the start of the simulation. As the threshold γ increases then the average vagueness decreases to zero, so that for $\gamma \geq 0.3$ we are left with almost entirely crisp (i.e. Boolean) opinions. In general the more conservative the combination rules (i.e. requiring higher levels of consistency) then the more it is that vague opinions are maintained in the population. Figure 2.1b shows the number of distinct valuations (i.e. different opinions) remaining in the population after 50 000 iterations. Again this decreases with γ and for $\gamma > 0.4$ agents have on average converged to a single shared opinion. This is consistent with the analytical results presented in [57] for the single propositional, $\gamma = 1$ case.



(a) Average vagueness.

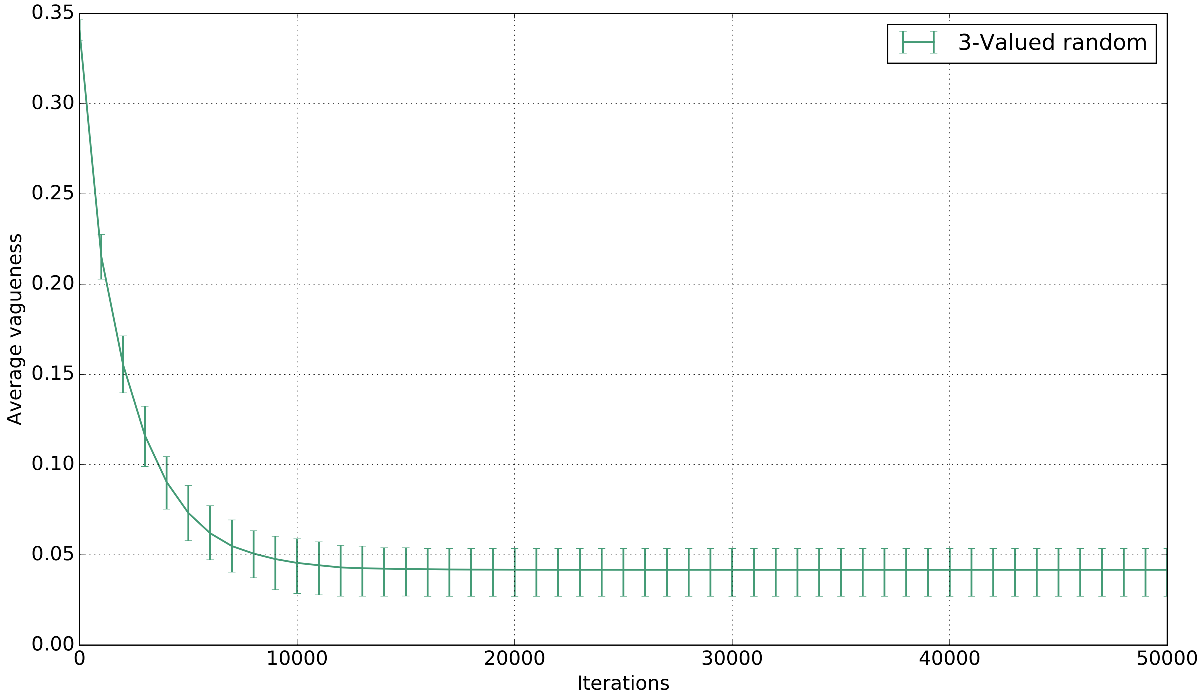


(b) Number of distinct valuations.

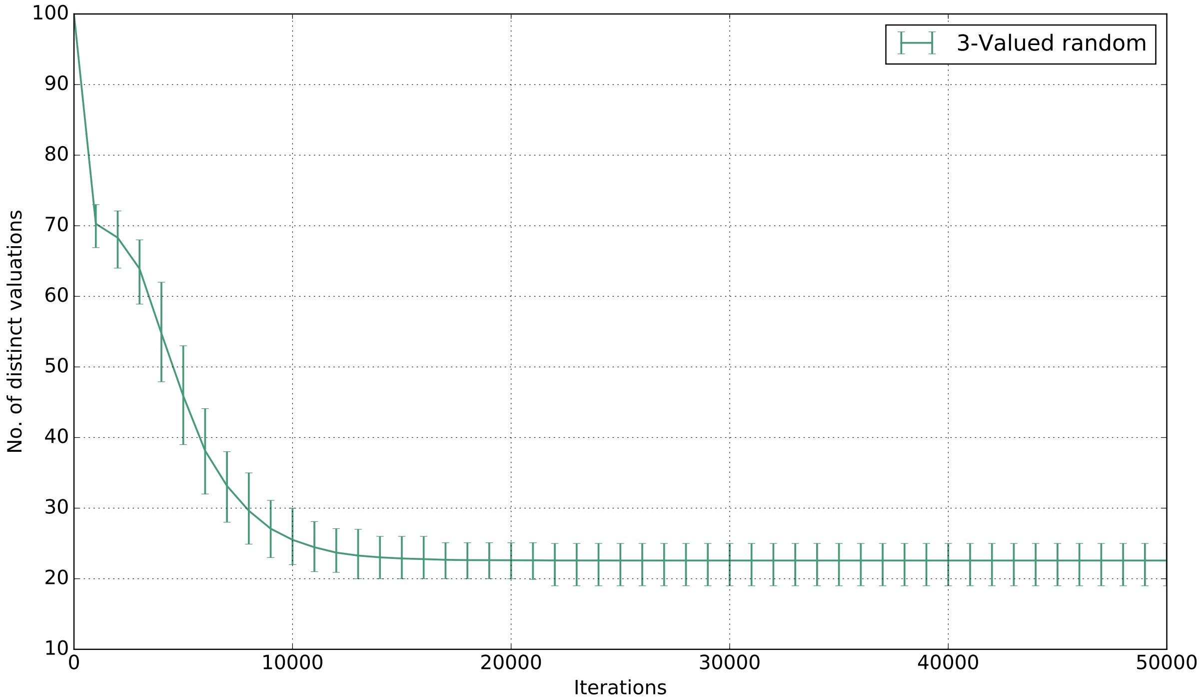
Figure 2.1: Three-valued consensus model with random selection and random three-valued initialisation at steady state for different inconsistency thresholds γ and different language sizes $|\mathcal{P}|$.

To provide a more complete picture, we now examine trajectory results of the model for the $|\mathcal{P}| = 100$ case. In Figure 2.2 we show trajectory results of the three-valued consensus model for the same experiments for an inconsistency threshold $\gamma = 0.2$. We choose $\gamma = 0.2$ specifically so that we can see more clearly the convergence properties of the model for an inconsistency value that does not lead to complete consensus to a single valuation. We see that even for a rather strict inconsistency threshold, where only relatively consistent agents are able to interact and to form consensus, the population converges on a much smaller set of distinct valuations which are more precise than the initial distribution of opinions. In just over 13 000 iterations the average vagueness of the population of 100 agents has been reduced from 0.33 at initialisation to just above 0.04 in Figure 2.2a. In Figure 2.2b we see that, at approximately 17 000 iterations the population converges to a steady state in Figure 2.2b, averaging below 25 distinct valuations across the 100 independent runs.

In Figure 2.3 we see that a small increase in γ from 0.2 to 0.3 leads to a large difference in the convergence of the system. The population now reaches consensus, converging on a single valuation in approximately 2 000 iterations (Figure 2.3b), this being considerably fewer than the 17 000 iterations it took for $\gamma = 0.2$. Similarly, we see in Figure 2.3a that for a sufficiently high inconsistency threshold γ vagueness is eliminated at steady state but with almost immediate convergence to crisp opinions in under 2 000 iterations. It seems to be the case that applying a more restrictive inconsistency threshold slows convergence and maintains a certain level of vagueness in the opinions of the population, albeit a reduced amount. It is only for inconsistency thresholds $\gamma \geq 0.4$ that we see complete convergence occurring incredibly quickly.

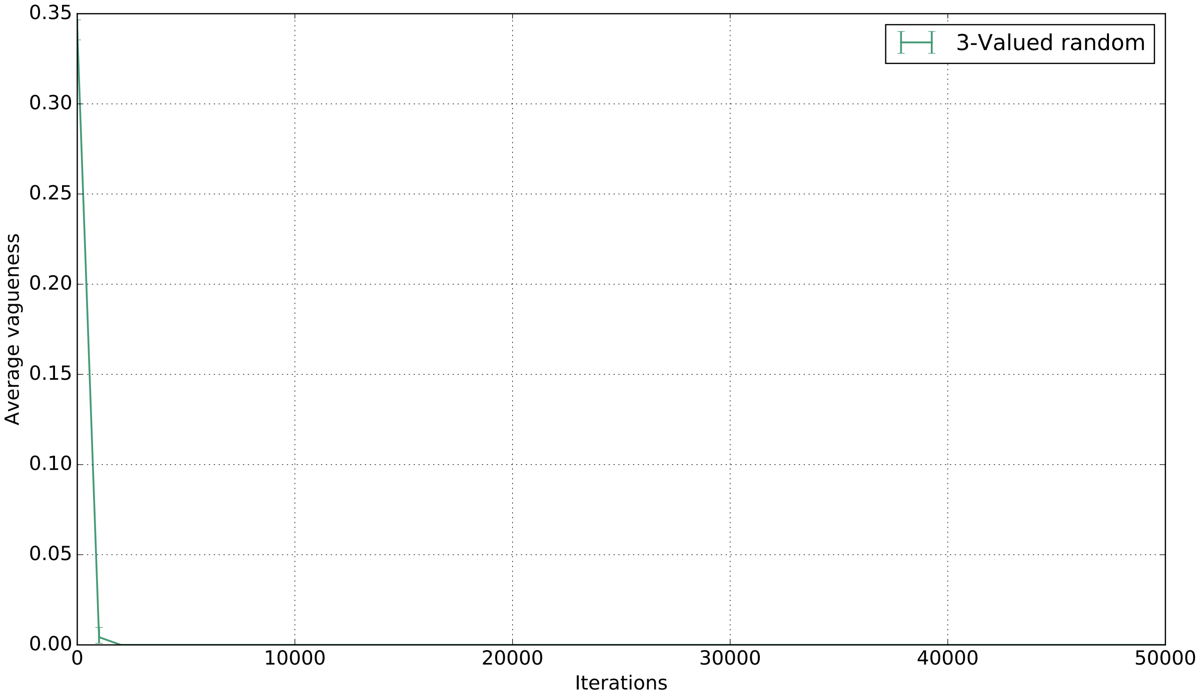


(a) Average vagueness.

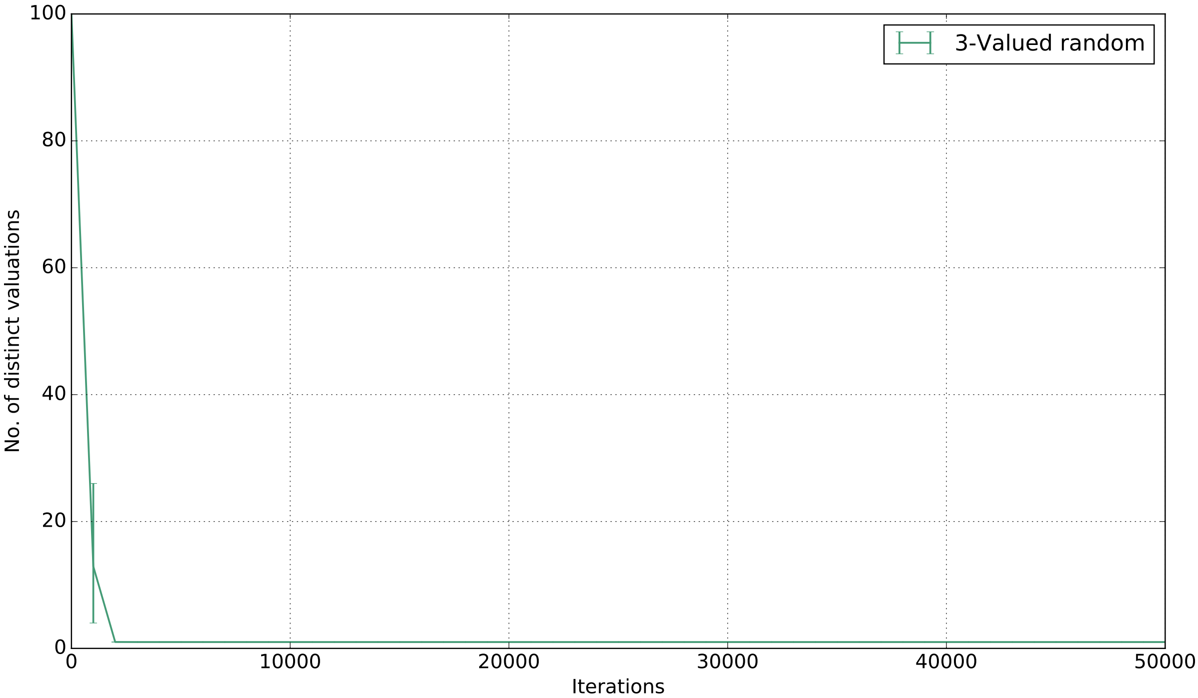


(b) Number of distinct valuations.

Figure 2.2: Three-valued consensus model with random selection and random three-valued initialisation shown as trajectories for an inconsistency threshold $\gamma = 0.2$ and $|\mathcal{P}| = 100$.



(a) Average vagueness.



(b) Number of distinct valuations.

Figure 2.3: Three-valued consensus model with random selection and random three-valued initialisation shown as trajectories for an inconsistency threshold $\gamma = 0.5$ and $|\mathcal{P}| = 100$.

2.3.2 Summary

From experiments in the previous section, we can see that effective convergence occurs in the three-valued model and that a population of agents converges to a single valuation for a sufficiently high inconsistency threshold γ. Furthermore, we see that this valuation is completely precise i.e. has non-borderline truth values for all propositional variables. We say this population has reached consensus because it has converged to a single, shared position regarding the state of the world by agreeing on a single truth assignment on all of the propositions in $\mathcal{L}$, and that this truth assignment is not vague. The consensus operator is therefore best suited to decision problems where the true state of the world may be assumed to be some precise truth assignment over the propositional variables, or where the agents reaching a precise consensus allows them to act with certainty as to which choices the rest of the population will make. There may be scenarios in which borderline propositions are made more precise via some stochastic reduction to Boolean truth values, and therefore where some agents may assume the proposition is true while others assume it to be false, causing a split in the population where agents are no longer acting in unison on the same underlying opinions.

These experiments also show the effect that restricting agent interactions has on the convergence of the population overall. In general, the effect of the inconsistency threshold is dependent on a combination of the language size, population size and the inconsistency measure in Definition 2.7 (Table 2.5). Given a small language size (e.g. $|\mathcal{P}| = 5$) and a relatively small population of 100 agents, then the likelihood of two *consistent* agents being selected, even for an inconsistency threshold $\gamma = 0$, is sufficiently high to allow agents whose opinions only differ by borderline truth values to form consensus. Indeed we see this in Figure 2.1 where for a language size of 5 the population of agents have reached an average vagueness value of approximately 0 for an inconsistency threshold $\gamma = 0$. Meanwhile, there are more than 20 distinct opinions that persist at steady state. For a much larger language size of 100 propositional variables, however, this is no longer the case. Instead, the average vagueness at steady state is essentially unchanged from the initial distribution of opinions and, similarly, there are 100 distinct valuations despite the ability for agents to form consensus between opinions where truth values differ only by borderline valuations. Given that there are 100 propositional variables, the agents are extremely unlikely to be paired with other agents whose valuations differ only by vague propositions during consensus formation. This is due to the limited pool of valuations present at initialisation: For 100 agents there can be, at most, 100 distinct valuations, yet there are a possible 3^{100} valuations for initialisation. Given the disparity between the population size and the number of possible valuations, it becomes increasingly unlikely that any two valuations will be consistent for $\gamma = 0$ in the case of 100 agents and 100 propositional variables. In contrast, for $|\mathcal{P}| = 5$, there are only $3^5 = 243$ distinct opinions possible for initialisation. As such, the likelihood of one of the 100 agents encountering another consistent agent in the population is far greater. If we briefly examine the single propositional variable case, there are just $3^1 = 3$ possible valuations, and from Table 2.5 we see that the likelihood of two randomly selected truth values being inconsistent is $\frac{2}{9}$ or 0.22 so it is closer to this value for the inconsistency threshold that we see a dramatic increase in convergence in the system. We believe it is also likely that as $|\mathcal{P}|$ increases, the steady state results of Figure 2.1 will come to greatly resemble the step function.

2.4 Incorporating feedback

In the following section we extend the three-valued model to incorporate a form of feedback based on the quality of the opinions of agents, such that higher quality opinions increase the likelihood of the agents holding those opinions being selected during the consensus formation process[2]. In this section, 'payoff' is introduced as a proxy for performance, and is motivated by the intuition that different opinions result in different actions which then, over time, lead to different levels of performance and different accuracies of opinions about the true state of the world. Here we adopt an abstract simplification of this process in which each Kleene valuation is allocated a real valued payoff. Then, instead of being selected at random for combination, an agent is picked from the population according to a probability which is proportionate to the payoff value of their opinions. The idea, then, is that agents with 'better' or more useful/well-informed opinions will be more successful and furthermore, it will be these successful agents who will be most likely to need to reach a consensus between them. Here the underlying intuition is that, in real systems it is the most successful agents, with the highest payoff values, who are most likely to find themselves in conflict with one another, and who will most benefit from reaching an agreement.

To better motivate the introduction of a payoff model, consider the example from Section 1.2 in which a group of operatives, who are about to conduct a search and rescue operation, must reach a consensus about which areas in a region are accessible. We might then consider a scenario in which certain areas are prioritised, perhaps due to the population density of a given area, or due to the degradation of an area's accessibility over time. In this scenario, a payoff model is a method of biasing the selection process in an attempt to rescue a greater number of people, or to prioritise areas that will only remain accessible for a limited period of time. Similarly, the payoff model acts to suppress low-priority areas so that resources can be prioritised in favour of areas with greater need. Consider the following scenario in this context: An operative believes that an area is currently inaccessible due to the low safety levels associated with that area. However, the area is known to have been densely populated and so to conduct a search and rescue operation in the area would result in a large number of potential lives being saved. Consider, now, that there are other operatives that believe the area is in fact safe enough to be accessible. Then it would be preferable to disseminate those operative's opinions more often than others so that there is a greater potential to save additional lives. Of course, if the population believes overwhelmingly that the area is unsafe, then it is likely that this opinion will remain dominant, despite the negative feedback associated with having that opinion.

2.4.1 Extending the three-valued model

We assume that the true state of the world is a Boolean valuation $\mathbf{v}^*$ on $\mathcal{L}$ so that $\mathbf{v}^*(p_i) \in \{0, 1\}$ for $p_i \in \mathcal{P}$. Given the interpretation of the third truth value as meaning 'borderline',

[2] This is typically referred to as 'roulette wheel selection' for fitness proportionate selection in genetic algorithms [28].

this is clearly a simplification from that perspective. For example, consider the proposition 'Ethel is short', then an experiment could consist of measuring Ethel's height according to some mechanism, and then comparing it to the experimenter's definition of the term 'short' in order to determine the truth value of the proposition. If that definition is three-valued then the outcome of the experiment could well be to identify a borderline truth value for the proposition. However, the convention in science is to establish an agreed crisp definition of all the terms used to express a hypothesis so that the resulting proposition is falsifiable. This would then be consistent with our identifying the true state of the world with a Boolean valuation. In the following definition we adopt a simple summative payoff model which we use to determine how close a valuation is to the truth. We use this measure in order to bias the selection of agents during the consensus process (as detailed later in this section), and also to analyse the resulting convergence in relation to the true state of the world.

Definition 2.8. A quality measure
Let $f : \mathcal{L} \to \{-1, 1\}$ be such that $f(p_i) = 2\mathbf{v}^*(p_i) - 1$ is the payoff for believing that p_i has truth value 1 and $-f(p_i)$ is the payoff for believing that the truth value of p_i is 0. Furthermore, it is always assumed that believing that p_i has truth value $\frac{1}{2}$ has payoff 0. Then we define the quality or payoff for the valuation $\mathbf{v}$ by:

$$f(\mathbf{v}) = \sum_{p_i \in \mathcal{P}:\mathbf{v}(p_i)=1} f(p_i) \quad - \sum_{p_i \in \mathcal{P}:\mathbf{v}(p_i)=0} f(p_i).$$

Another perspective on this type of payoff function is as follows: For each propositional variable p_i, a truth value of 1 results in a payoff $f(p_i)$ (which can be either positive or negative), a truth value of 0 results in the opposite signed payoff $-f(p_i)$, and a borderline truth value $\frac{1}{2}$ results in a payoff of 0. The payoff value for a Kleene valuation $\mathbf{v}$ is then simply taken to be the sum of the payoffs for each propositional variable under the truth values allocated by $\mathbf{v}$. This payoff function is symmetric as only a single payoff value is assigned to a proposition, and a positive or negative payoff for a given valuation differs by sign only. Additionally, a borderline valuation resulting in a neutral payoff of 0 maintains this symmetry. Alternatively, an asymmetric payoff model would give payoff of different magnitudes to the truth values 0 and 1 for given propositions, and/or non-zero payoff for the truth value $\frac{1}{2}$.

2.4.2 Adapting the selection process

We now detail a payoff-based weighted selection process as a replacement for the previous random selection process described in Section 2.3. Based on payoff values we define a probability distribution over the agents in the population according to which the probability that an agent with valuation $\mathbf{v}$ is selected for possible consensus combination is proportional to $f(\mathbf{v}) + n + 1$, where n is the language size. At each iteration a pair of agents are selected at random according to this distribution. For each such pair the inconsistency measure (Definition 2.7) is evaluated and either both the valuations are replaced with the consensus valuation, or both are left unchanged, depending on the threshold γ as in Section 2.3. The parameters for the simulation experiments are as follows: We show results for populations of 100 agents with

binary operator	0	1
0	0	$0 : \frac{1}{2}, 1 : \frac{1}{2}$
1	$0 : \frac{1}{2}, 1 : \frac{1}{2}$	1

Table 2.6: Truth table for the stochastic Boolean consensus operator.

language sizes in $\{5, 10, 50, 100\}$ are initially studied for comparisons with the random selection model, before settling on $|\mathcal{P}| = 5$ for a more detailed analysis of results. This time opinions are initialised as random Boolean opinions, whereby we do not allocate borderline valuations as part of agents' opinions; instead we allocate Boolean true/false truth values for all propositions $p_i \in \mathcal{P}$ for both the Boolean consensus model and the three-valued consensus model. As before, we study various values for an inconsistency threshold $\gamma \in [0, 1]$.

2.4.3 Random Boolean opinion initialisation

Notice that here we are initialising the opinions as random Boolean valuations (see Section 2.3)[3]. This allows us to make a direct comparison between the performance of the three valued combination operator and a similar two valued operator. For the latter we assume that only binary truth values are available to represent an agent's opinions. In this context, in order for two agents with conflicting truth values for p_i (i.e. one 0 and the other 1) to reach consensus, we propose that they simply agree to pick one of the truth values at random e.g. by tossing a fair coin. Table 2.6 gives the truth table for the operator in which directly conflicting truth values leads to a stochastic outcome.

2.5 Payoff-based selection experiments

We now present results for the three-valued model with payoff-based selection, briefly comparing this payoff model to that of the random selection model of Section 2.3 under the same random three-valued opinion initialisation. We study the cases of $|\mathcal{P}| \in \{5, 10, 50, 100\}$ for which we compare the three-valued model to the newly introduced Boolean model, with the opinions of both models being initialised as random Boolean valuations.

2.5.1 Comparing the three-valued payoff model with the random selection model

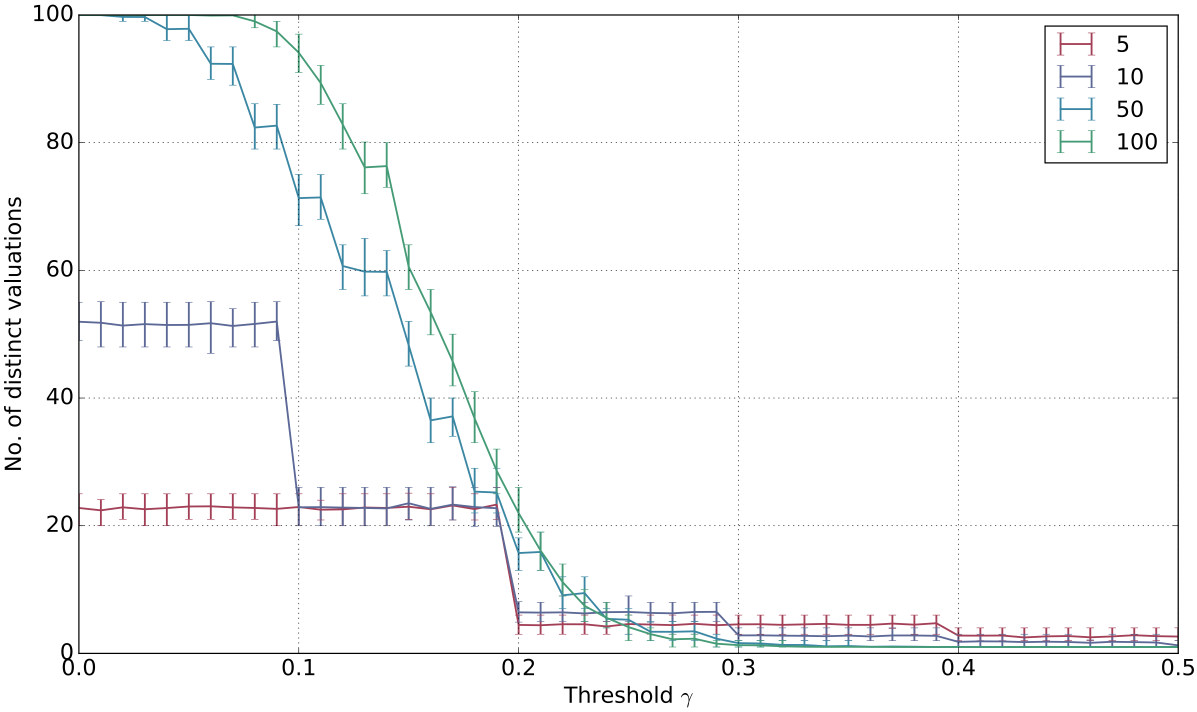
In Figure 2.4 we show results for the population at steady state (after 50 000 iterations) for various language sizes as we did in Figure 2.1. The results shown are mean values with error bars taken over 100 independent runs of the simulation, indicating the 10^{th} and 90^{th} percentiles. The model changes little in terms of convergence, as is clear when comparing Figure 2.4a with Figure 2.1b. Here we see very similar reductions in the number of distinct valuations across all language sizes as the inconsistency threshold increases, with each language size following a

[3] As a result of this Boolean initialisation, a language size of 5 now produces a total of 2^5 (32) possible valuations, as opposed to 3^5 (243) possible valuations.

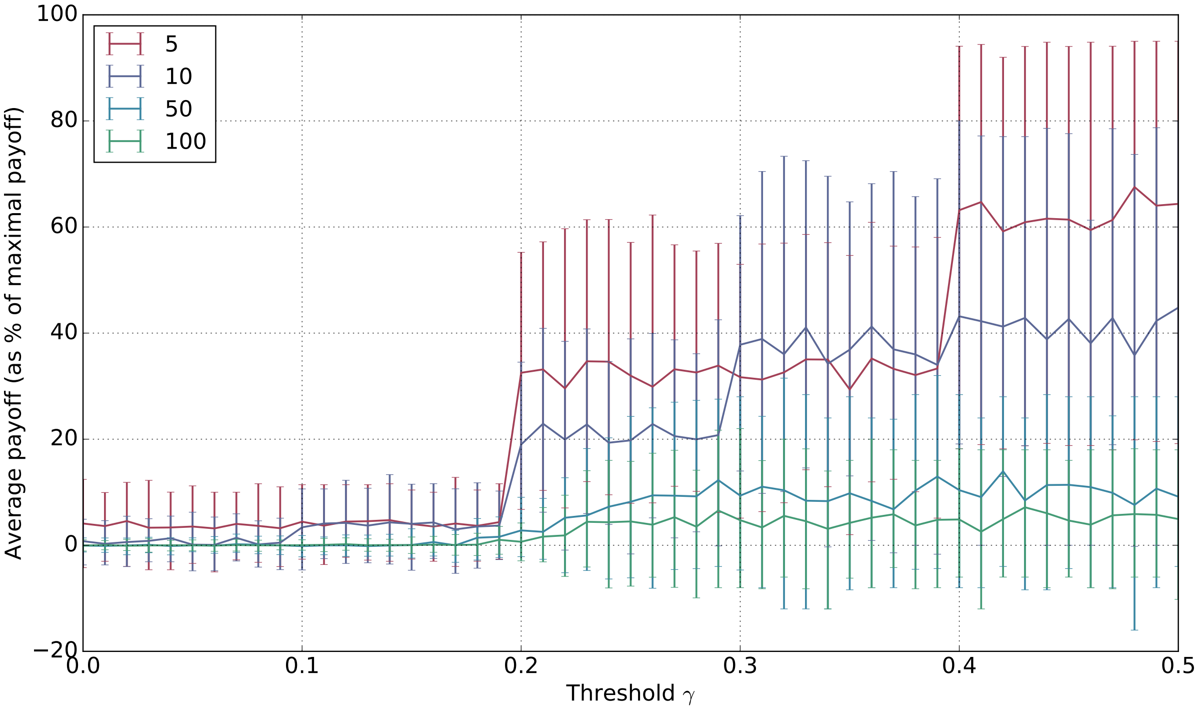
similar curve as for random agent selection. However, for $\gamma \geq 0.4$, convergence is not quite as strong as in the random selection case where most models had converged to just a single valuation.

Figure 2.4b shows the average payoff of the population for different language sizes as a percentage of the maximal possible payoff value i.e. the payoff for the valuation $\mathbf{v}^*$ where $f(\mathbf{v}^*) = |\mathcal{P}|$. It is here that we see the effect of the payoff-based selection process, as well as the impact that the language size has on performance. We can see that payoff does indeed increase as the population begins to converge towards a single valuation, and that for an inconsistency threshold $\gamma = 0.5$ where the system has almost completely converged, for $|\mathcal{P}| = 5$ the average payoff of the population of agents is averaging above 60%, and with error bars indicating that for some runs of the experiment, this can even exceed 90% of the maximal payoff. It becomes immediately clear, however, that for different language sizes this effect diminishes greatly. For 10 propositional variables we see an average payoff of below 50%. For 100 propositional variables we see that the average payoff of the population is reduced even further to around 5% of the maximal possible payoff value, with a language size of 50 propositional variables performing a little better with an average of around 10% for $\gamma = 0.5$. As discussed in the previous section, this is most likely due to the sparsity of the initial opinions for a small population size of 100 agents relative to the number of possible three-valued valuations. We believe that, for a fixed population size, as the language size increases, the effect that the payoff-based selection process has on the average payoff of the system is reduced dramatically to the extent that it will eventually achieve no additional performance gain over the random selection model.

Finally, it is worth noting that the weighted stochasticity of the agent selection process results in a huge variation in average payoff across the different independent runs of the simulation, as is clear from the large error bars in Figure2.4b. This suggests that the performance of the payoff model is hugely dependent on the initial distribution of opinions being sufficiently diverse, so as to possess enough high quality opinions (i.e. valuations associated with large payoffs) to reinforce the payoff-based selection process. An increased number of high quality opinions allows the payoff-based selection process to more easily disseminate such opinions throughout the population. If the initial population of opinions is sparse in relation to the total number of possible opinions for that language size, then it is likely that for some runs of the simulation, the valuation with maximal payoff is not present at initialisation and so the system faces increased difficulty in converging to a high-payoff valuation.



(a) Number of distinct valuations.



(b) Average payoff shown as a percentage of the maximal possible payoff.

Figure 2.4: Three-valued consensus model with payoff-based selection and random three-valued initialisation at steady state, for different inconsistency thresholds γ and different language sizes $|\mathcal{P}|$.

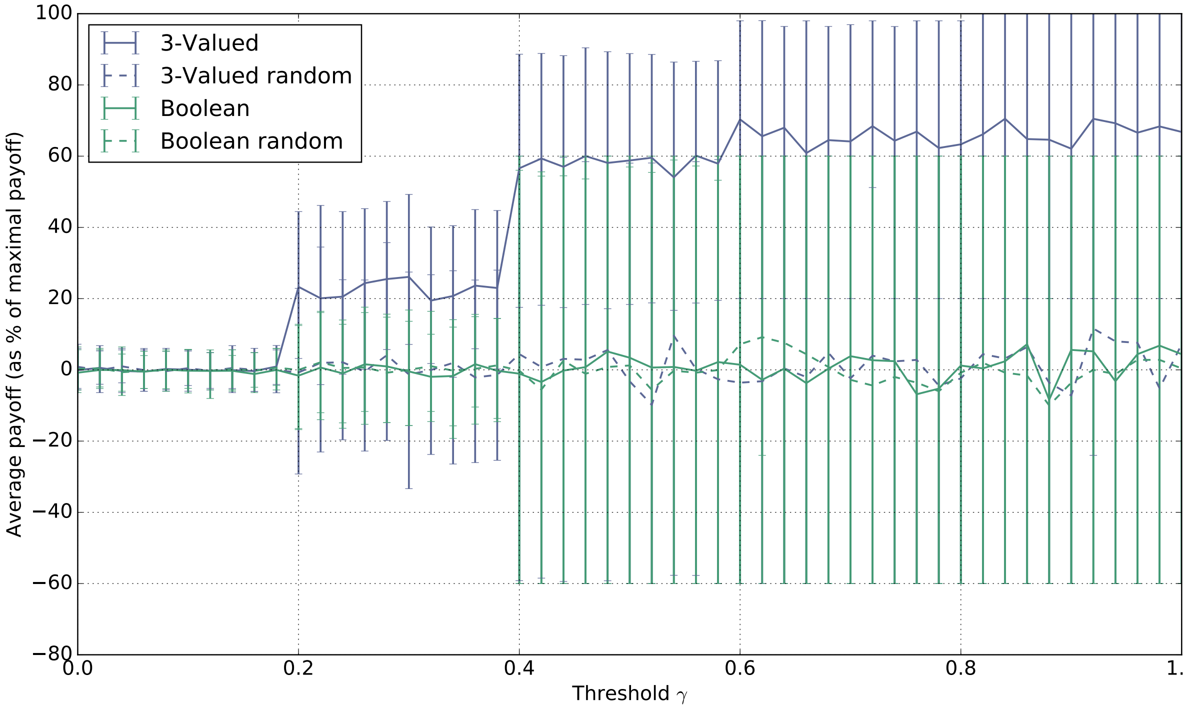


Figure 2.5: Average payoff for the three-valued and Boolean consensus models with random and weighted selection at steady state for a $|\mathcal{P}| = 5$ and different inconsistency thresholds γ.

2.5.2 Results with random Boolean initialisation of opinions

We now look exclusively at simulation experiments for 5 propositional variables and random *Boolean* opinion initialisation, where each proposition $p_i \in \mathcal{P}$ can have a value of either 0 or 1 at initialisation (at the 0^{th} iteration). For the Boolean model, valuations will only be able to form opinions in $\{0, 1\}^n$ according to the Boolean consensus operator defined in Table 2.6, whereas for the three-valued model, from iteration 1 onwards, agents are free to adopt opinions in $\{0, \frac{1}{2}, 1\}^n$ which naturally occur through application of the three-valued consensus operator of Definition 2.5 (Table 2.4), as normal.

Simulations with a population of 100 agents

In Figures 2.5 and 2.6 we present results comparing the three-valued payoff model with the Boolean payoff model, and for each of these we also include comparisons to their random selection variants. As such, those labelled '3-valued' and 'Boolean' refer to the models with payoff-based selection, while those with 'random' appended refer to the random selection models. Immediately, from Figure 2.5, we see that one model in particular - the three-valued payoff model - far exceeds the others in average payoff for an inconsistency threshold $\gamma \geq 0.2$. All the other models are averaging close to 0 for all values of γ, though with large variation as shown by their error bars. The three-valued model, however, is able to average above 60% for $\gamma \geq 0.6$; an increase in inconsistency threshold from $\gamma = 0.5$ as required by the three-valued model for similar performance under random three-valued opinion initialisation in Figure 2.4a. It is natural to expect that, averaged over 100 independent runs, both the three-valued and Boolean models

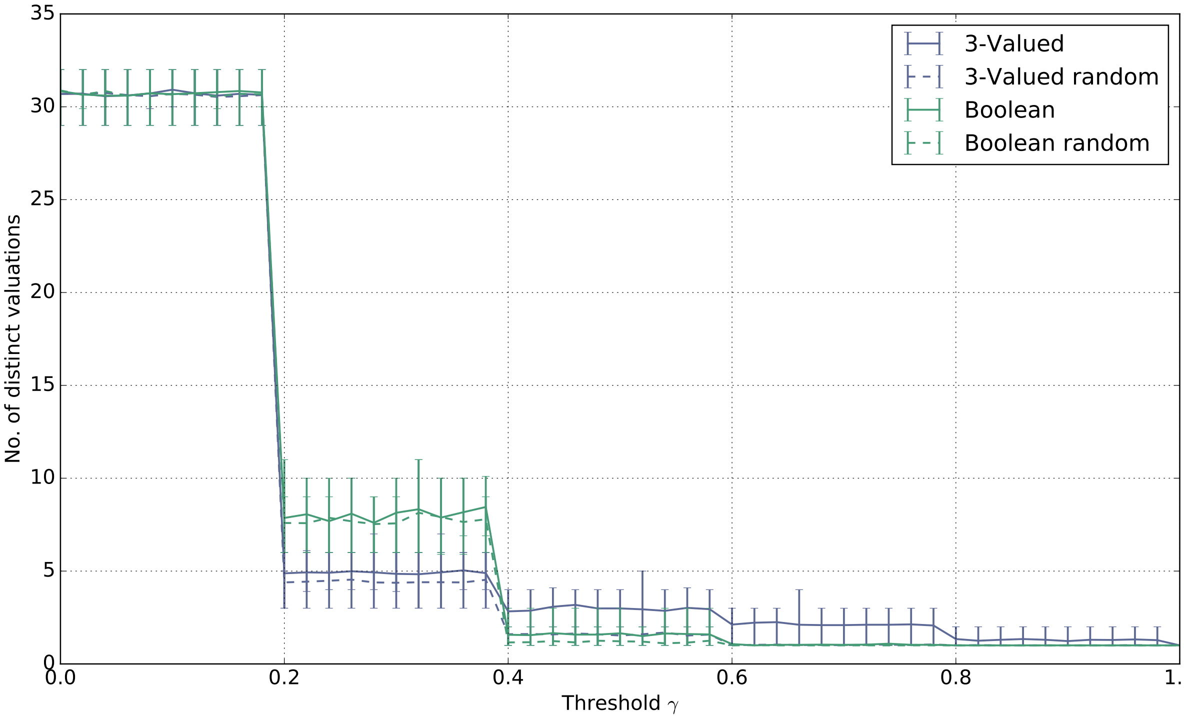


Figure 2.6: Number of distinct valuations for the three-valued and Boolean consensus models with random and weighted selection at steady state for different inconsistency thresholds γ.

with random selection would show an average payoff around 0 as there is no bias present in the population to favour opinions of greater payoff. However, the Boolean payoff model also averages around 0 for all values of γ. This is probably due to the stochastic nature of the Boolean consensus operator, which compensates for agents being unable to adopt an intermediate truth value as a means of compromise, making them more likely to adopt completely opposite opinions during their next pairing. Instead, one agent forces the other to adopt their truth value. This dominance is not maintained across all propositions, however, and so the resulting opinion is likely to be a mixture of truth values between the two agents, possibly negating any positive gain that could have occurred had one agent simply adopted the valuation of the other, and potentially resulting in a valuation of lower payoff than either of the opinions prior to forming consensus. The stochastic resolution to conflicting opinions is the most obvious approach to take for this model, and yet it is clear that even by weighting agent selection based on payoff, this stochasticity prevents any measurable convergence to higher-payoff opinions from occurring.

To obtain a more complete picture, we can also consider other properties of the system. Figure 2.6 shows the number of distinct valuations at steady state for the four models, and we can see immediately that for $\gamma = 0.0$, there is some noise around the 32 possible opinions resulting from the random Boolean initialisation, with the models averaging around 31 distinct valuations at steady state as a result of some duplication (all models) or a minute amount of convergence (three-valued models only) occurring in the populations. As γ reaches 0.2, or $\frac{1}{5}$, we start to see a drastic reduction in the number of distinct valuations present at steady state. This is due to the language consisting of 5 propositional variables where, as soon as the inconsistency threshold reaches or exceeds $\frac{1}{n}$, then Boolean opinions become consistent when

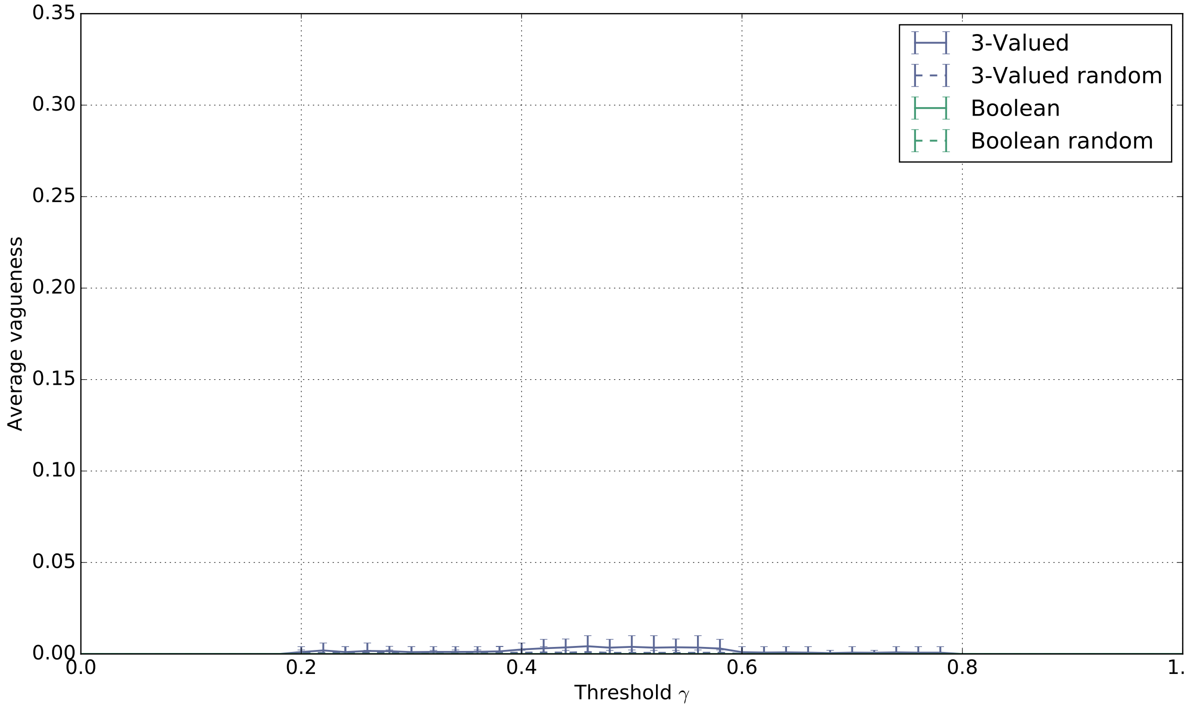
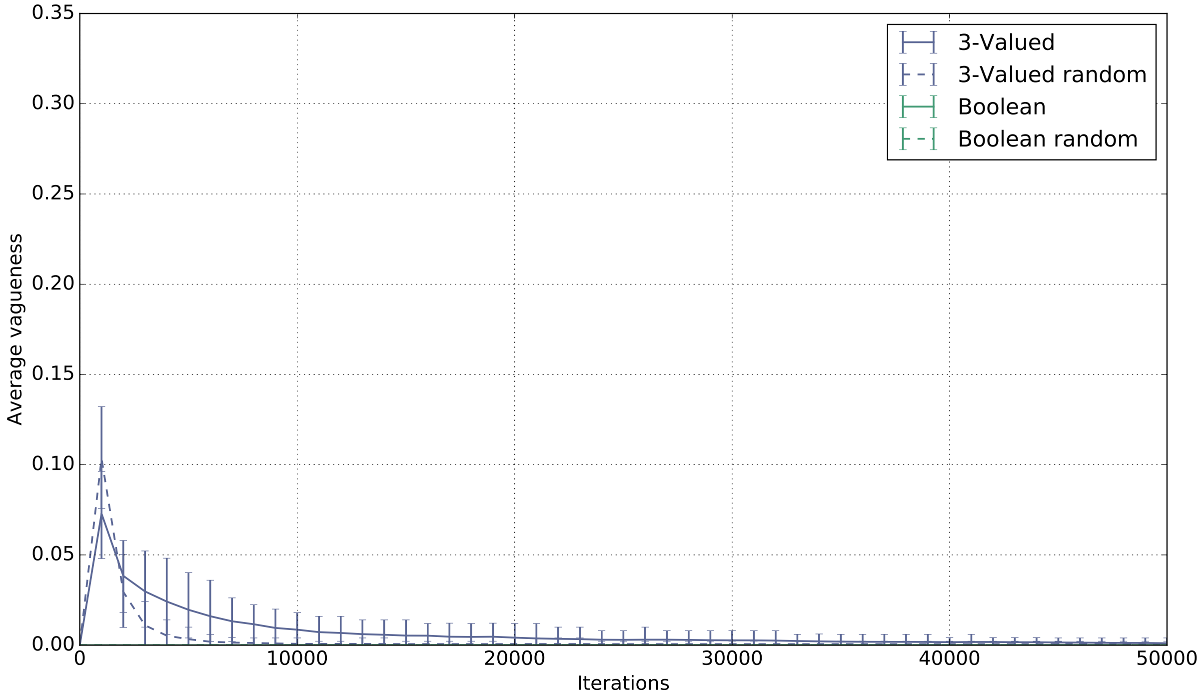


Figure 2.7: Average vagueness for the three-valued and Boolean consensus models, with random and weighted selection at steady state for different inconsistency thresholds γ.

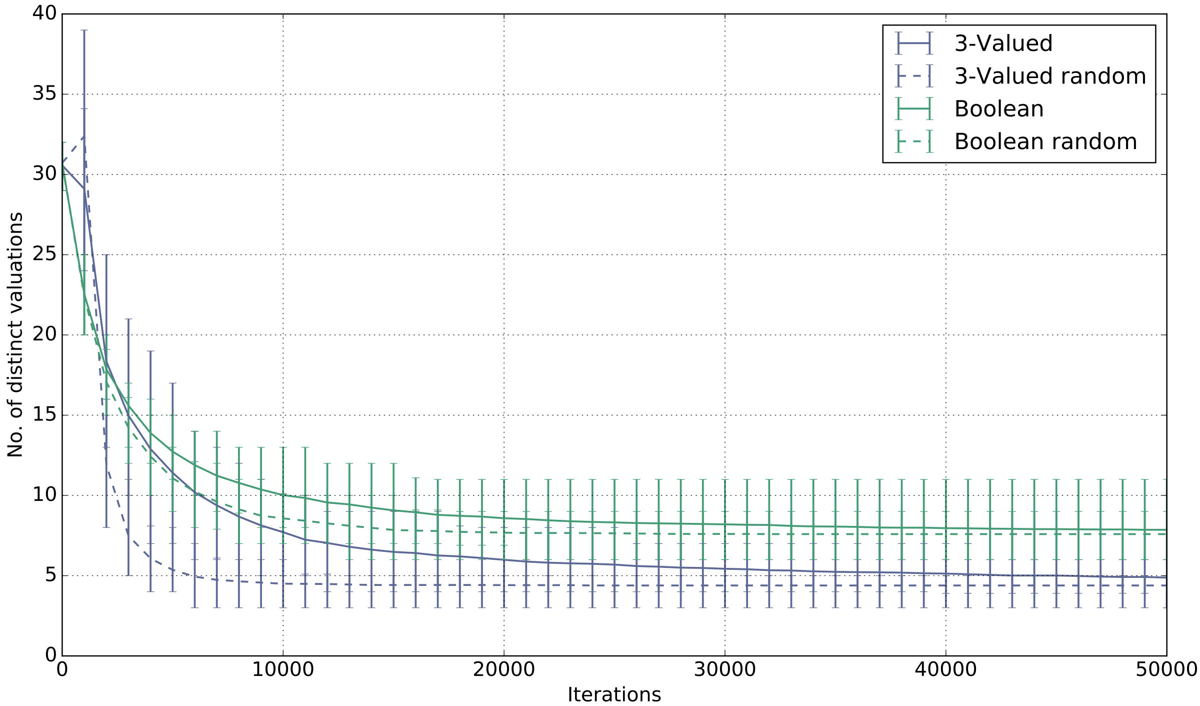
just one proposition differs in truth value between a pair of valuations. This is why we see convergence for $\gamma = 0.2$.

While all four models are initialised with random Boolean opinions, it is worth considering the level of vagueness that might be present at steady state for the three-valued models as a result of applying the three-valued consensus operator. In Figure 2.7 we see that the vagueness levels across both three-valued models are negligible at steady state. The three-valued payoff model does exhibit some vagueness of opinions at steady state for $0.2 \leq \gamma \leq 0.8$, but the level of vagueness is small, and the population has effectively converged to Boolean opinions. We can also consider trajectory results in Figure 2.8. Here we fix the inconsistency threshold at $\gamma = 0.2$ and we note that a large and sudden increase in vagueness occurs within the first 1 000 iterations from the beginning of the simulation experiment, as shown in Figure 2.8a. In this period agents go from having completely Boolean valuations to valuations with average levels of vagueness between 0.07 and 0.10 for the three-valued payoff model and three-valued random model, respectively. This sharp increase is then followed by a slow decline in vagueness until the system eventually converges to close to 0 vagueness on average after 50 000 iterations.

From Figure 2.8b, we can see that this sudden increase in vagueness is aligned with a similar increase in the variation in the number of distinct valuations in both of the three-valued models. Unlike in the Boolean models, where the number of distinct opinions consistently decline from the beginning of the experiments, the three-valued models vary between runs, with the three-valued random model seeing increases in the number of opinions on average after 1 000 iterations, and the three-valued payoff model seeing similar variation in the number of distinct opinions, though still reducing to below 30 on average. Interestingly Figure 2.8b suggests that convergence of the Boolean models are initially much quicker, and slow as the size of the opinion pool for the population reduces, levelling off to below 10 distinct valuations. For the three-valued model, we see that there is slower convergence early on in the experiments where vagueness and the number of distinct valuations initially increases, before both models begin converging more quickly. The three-valued random model converges extremely quickly compared with the other models, while the three-valued payoff model converges more slowly than its random counterpart, but still converges more quickly to a smaller set of distinct opinions than both the Boolean payoff model and the Boolean random model. It may therefore be the case that by introducing a small level of vagueness into the population of opinions allows for the agents to form consensus more easily and for the system to converge more quickly overall when compared with a strictly Boolean population of opinions.



(a) Average vagueness.



(b) Number of distinct valuations.

Figure 2.8: Comparison of the three-valued and Boolean consensus models, with random and weighted selection as trajectories for an inconsistency threshold $\gamma = 0.2$.

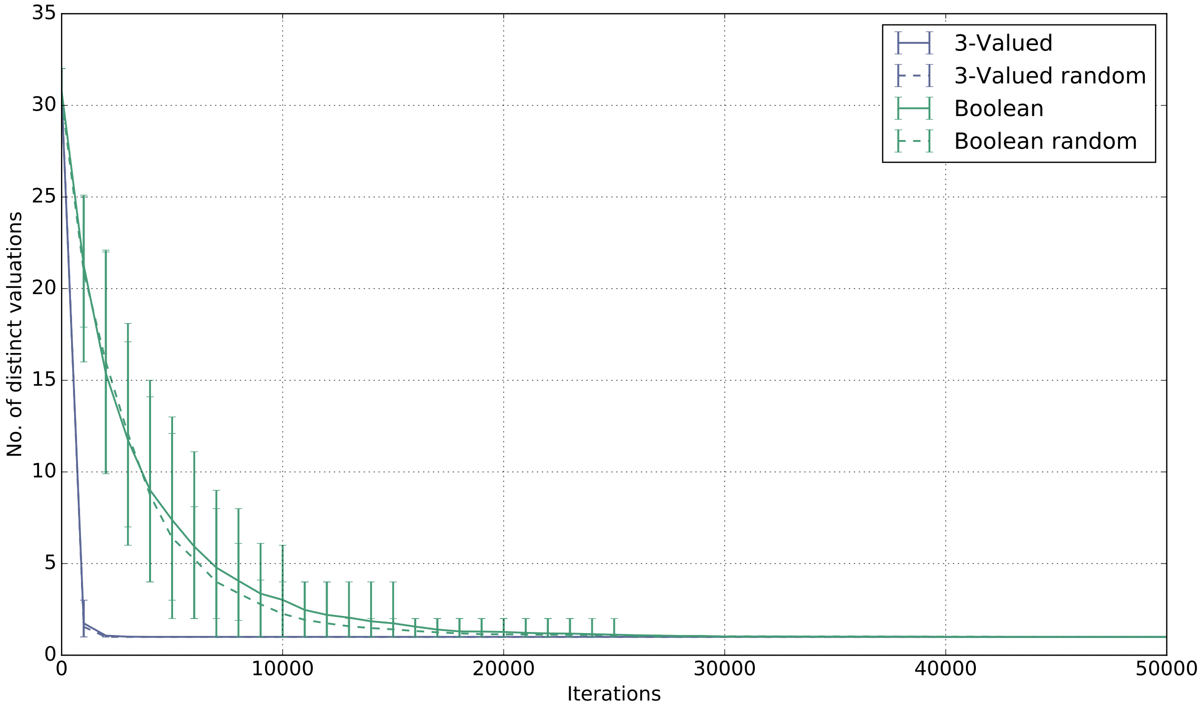


Figure 2.9: Comparison of the three-valued and Boolean consensus models, showing the number of distinct valuations with random and weighted selection as trajectories for an inconsistency threshold $\gamma = 1.0$.

Finally, we briefly analyse the convergence properties of all four models when all selected pairs of agents combine their opinions, regardless of how inconsistent they are. Figure 2.9 shows the number of distinct valuations for both the three-valued model and the Boolean model, and for both agent selection strategies with an inconsistency threshold $\gamma = 1.0$. We can see that both of the three-valued models converge very quickly, reaching 2 distinct valuations on average after just 1 000 iterations and fully converging by 2 000 iterations. This is in contrast to the much slower convergence of both of the Boolean models which do not converge until after 30 000 iterations. It is to be expected that, as the population sizes increase, so too does the time required by both models to reach consensus, as each iteration corresponds to only a single pair of agents combining their opinions.

2.5.3 Summary

With the introduction of payoff we have demonstrated that the three-valued payoff model (where agent selection is based on their associated payoff values) enables populations of agents to reach a more meaningful consensus. Opinions of agents in this setting can be thought of as representing some perceived states of the world which the agents believe to be true. Some opinions will inherently be more accurate than others, and this should be reflected in the consensus formation process. To this end, payoff serves to bias the agent selection process to favour opinions which more accurately represent the true state of the world, as specified by the underlying payoff model. This way, agents which possess high-payoff opinions are favoured in the consensus formation process proportional to the accuracy of their opinions, resulting in being chosen more often to combine their opinions with others in the population. We have shown that the effect of payoff-based selection of the agents is reduced as the language size increases, and is due to the diversity of opinions in the population. For larger language sizes and a proportionately small population size, it becomes increasingly likely that more accurate opinions are not present in the population at initialisation and so are unable to drive initial convergence towards the more accurate valuations.

We also introduced a Boolean model for consensus which is stochastic in its resolution of inconsistent truth values during opinion combination. Simulation experiments highlighted that the three-valued payoff model was the most effective at increasing the average level of payoff in the population and that for large inconsistency threshold γ the three-valued model achieved over 65% of the maximal possible payoff, on average, for a language size of 5. Meanwhile, both the three-valued random model and both of the Boolean models were ineffective at achieving any substantial improvements in performance, remaining close to 0% payoff averaged across the population. For the same population size, we saw how payoff-based selection did not come at a cost to convergence time for either of the three-valued or Boolean consensus models, as most of the convergence properties remained unchanged.

While convergence does appear to be quick for sufficiently large inconsistency values, particularly for the three-valued models, we would expect for there to be a proportional increase in the time required to reach consensus for both the three-valued and Boolean models as the population size increases. An iteration corresponds to a single pair of agents combining their opinions and this is restricted further by the inconsistency threshold. As such, a dramatic increase in population size would require much longer running times to reach similar convergence across the populations. Of course, the length of time to convergence is a significant issue with the models presented here. Given that the consensus process taking place is *symmetric*, meaning that both agents adopt the resulting opinion combination as their new opinions, attempting to apply the consensus operator to additional pairs of agents within the same iteration will inevitably lead to issues stemming from the fact that the consensus operator is not associative, and so the order of interactions between agents will likely change the outcome of the consensus process, where different orderings may lead to different opinions as a result. Should consensus be a serial process, then one could expect the agents to adopt their new opinions before the operator is applied to the next pair of agents, which may contain reoccurring agents. However, this would negate the payoff-based selection process, as agents selected multiple times will not necessarily possess the same payoff value for each interaction processed during the iteration. The weighted selection process is based on an agent's current payoff value, and so is only effective at selecting agents when it has accurate information. Altering an agent's opinion, and therefore payoff value, after selection would adversely affect this process. Similarly, it would not be possible to process multiple opinion combinations in parallel because reoccurring agents would receive multiple opinions and there would be no intuitive method of determining which opinion should be adopted, and which ones should be discarded. Given that the purpose of the consensus operator is to combine two potentially different opinions into a single, shared opinion adopted by both agents, this would defeat the purpose. As such, we now look to improve the speed of convergence for larger populations through means of group consensus formation.

2.6 A model for group-wide consensus

Pairwise interactions form a natural approach to multi-agent consensus where pairs of agents combine their opinions through a combination operator. The consensus operator introduced in Definition 2.5 is an extreme form of operator which merges two different opinions into a single, shared opinion which is then adopted by both agents. This merging of opinions inevitably leads to a reduction in the number of unique opinions that exist in the population as two differing opinions are replaced by a single, new opinion which may or may not already be present in the population. At the end of Section 2.5 we discussed some problems that arise when the population size increases with regards to the efficiency of this kind of symmetric model of opinion updating. The pairwise property of the three-valued consensus model is highly desirable for modelling human behaviour when it comes to opinion formation and dissemination in populations of individuals, but is somewhat of a hindrance when the system ought to converge quickly. For decision-making applications, agents typically need to make a collective decision by forming consensus about the state of the world, on which they base their future actions. The proposed three-valued model is restricted by this pairwise property to smaller populations if the system is time-sensitive, and so we now propose a new model for group consensus in the hopes of addressing this limitation.

2.6.1 Model

Overview. Pairwise conversations are just one way in which people discuss their opinions and try to change the opinions of others. Perhaps a more efficient way of achieving a similar result is through group-wide discussions; either where a speaker is disseminating their opinion to a group of people (a sort of one-to-many relation, such as a lecture), or where each member of the group converses with one another (many-to-many). In this model, we opt to implement the latter as a means of forming 'consensus' between groups of agents in the population, rather than restricting interactions to a single pair of agents at each iteration. This should be effective at speeding up the convergence of opinions in a population by involving more agents in the consensus process at each iteration, as is the motivation behind this proposed model. However, it is no longer necessarily the case that agents in the group adopt the same opinions. Instead, each agent in the group adopts an individualised opinion of their own: this opinion is based on a probability distribution generated from a set of consensus valuations formed by systematically applying the consensus operator to all of the agent pairs in the group, where each agent stores the resulting valuations for pairs in which they participate. For this model, consensus is not enforced at the group level, or even between pairs of agents. Instead, agents attempt to align their opinions closer to one another as a collective.

Group-wide consensus as collective pairwise interactions

As in Sections 2.2 and 2.4 the underlying model remains the same, with the predominant change affecting the interaction of agents and the way in which the consensus operator is applied. At every iteration in the simulation, a set of k agents is selected according to the chosen selection strategy (e.g. random or payoff-based). For each agent $r \in \{1, ..., k\}$ and associated valuations $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$, we then systematically identify all agents $s \in \{1, ..., k \colon s \neq r\}$ which are consistent with agent r according to the inconsistency threshold $\gamma \in [0, 1]$. For each consistent pair of agents r, s, such that $I(\mathbf{v}_r, \mathbf{v}_s) \leq \gamma$, the chosen consensus operator is applied to the agents' valuations $\mathbf{v}_r \odot \mathbf{v}_s$ (Table 2.4 for the three-valued model and Table 2.6 for the Boolean model), forming a set of consensus valuations for each agent in the group. Then, for every agent $r \in \{1, ..., k\}$ that is consistent with at least one other agent, we generate a probability distribution P_r on $\mathbb{V}$ based on agent r's set of consensus valuations such that:

$$P_r(\mathbf{v}) = \frac{|\{s \in \{1, ..., k\} \colon s \neq r, I(\mathbf{v}_s, \mathbf{v}_r) \leq \gamma, \mathbf{v}_s \odot \mathbf{v}_r = \mathbf{v}\}|}{|\{s \in \{1, ..., k\} \colon s \neq r, I(\mathbf{v}_s, \mathbf{v}_r) \leq \gamma\}|}.$$

Agent r then adopts a new valuation $\mathbf{v}'_r$ as follows: For each propositional variable $p_i \in \mathcal{P}$ we select a valuation at random according to P_r, and set $\mathbf{v}'_r(p_i) = \mathbf{v}(p_i)$. Each agent therefore generates a new valuation according to their own probability distribution, which may be unique amongst the group, or may be similar to other agents should they have interacted with one another.

Notice that for the group model, a group size of $k = 2$ is equivalent to the pairwise model of Section 2.2. That is, for a pair of agents, either they are consistent with one another, in which case they form consensus and the probability that they each select the resulting consensus valuation is equal to 1, or they are inconsistent and the agents do not form consensus. Additionally, in the case where every agent in the group is only consistent with one other agent, then the model behaves as if we had selected multiple pairs of agents per iteration, rather than a single pair, where each pair forms consensus once only, and do not interact with agents of other pairs within the group.

2.7 Simulation experiments for iterative group-wide consensus

In this Section we focus on comparing the three-valued group model of Section 2.6 with its pairwise version of Section 2.2. There is likely to be a trade-off in terms of the speed of convergence and performance with regard to payoff between the two models, due to there being a difference between symmetric and asymmetric opinion updating, where pairwise interactions are of the former and group-wide the latter. For larger group sizes, we might expect to see faster convergence than for smaller groups, simply because of the increase in the number of agents realigning their opinions to be closer to those of other agents in the population. Of course, this is not necessarily the case, given that for larger groups, there will be a greater variety of opinions presented and this may even detract from the convergence of the group depending on their similarity, leading to less effective convergence as a result. Larger groups will also suffer under more restrictive inconsistency thresholds: if a group of agents contains a number of dissimilar opinions, and an inconsistency threshold prevents most agents from interacting with more than just one or two other agents, then it can be expected that there will be smaller subgroups of the selected agents that converge closer to one another, but may in fact be diverging from the more 'central' opinion for all opinions in the group. Following these initial comparisons between the

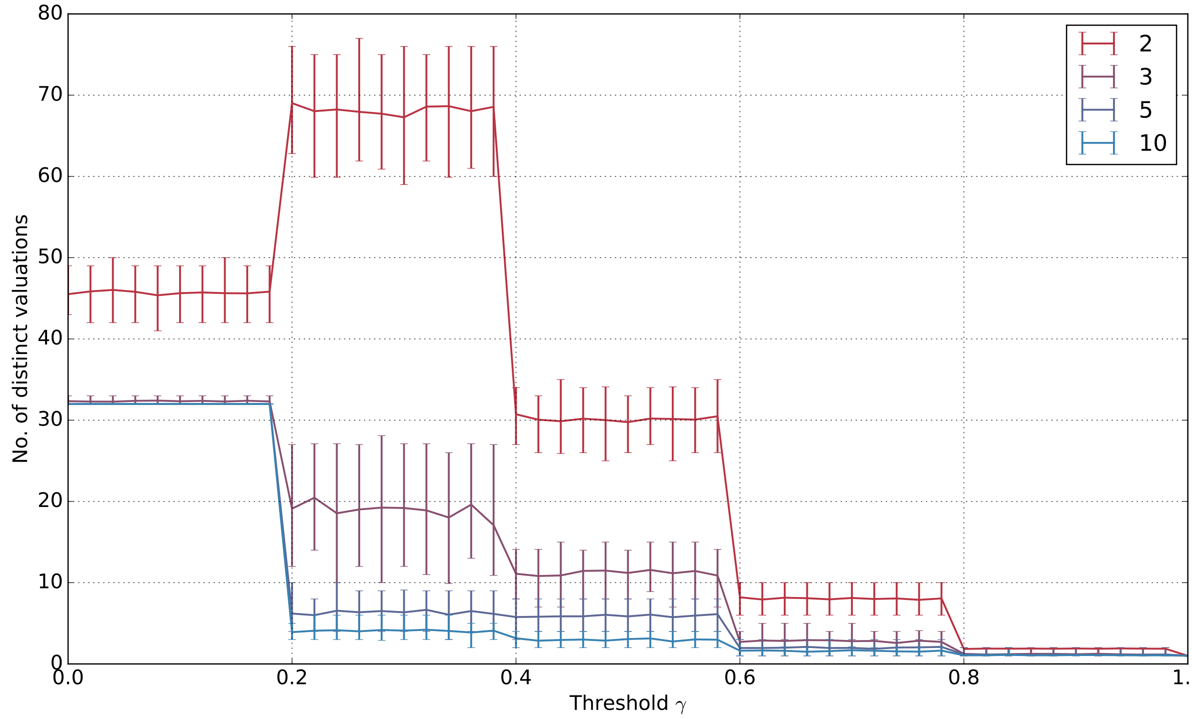


Figure 2.10: Number of distinct valuations at steady state for the three-valued group model, with payoff-based selection and random three-valued opinion initialisation for group sizes $k \in \{2, 3, 5, 10\}$ and different inconsistency thresholds γ.

pairwise and group model, we compare the three-valued group model with a Boolean variant, just as we did in Section 2.4 where the operator applied between each pair of agents is the same as in Table 2.6.

Simulation experiments for the group model are conducted for a population of 1 000 agents, ten times the population size of previous experiments as we are interested in seeing if there is a scalable alternative to the pairwise three-valued model. We therefore also restrict studies to a language size of 5 propositional variables just as before, alongside the same range for the inconsistency threshold $\gamma \in [0, 1]$. We are interested in the payoff-based selection strategy as a measure of performance, as we expect the group model to lead to faster convergence for a large population, but do not expect that this will necessarily correlate with improved average payoff if the population converges quickly to an opinion of low payoff, compared with a slower convergence to a more 'correct' opinion of higher payoff. We examine group sizes of $k \in \{2, 3, 5, 10\}$ agents, with $k = 2$ a special case of the general group model in which consensus is formed at every iteration (provided both agents are sufficiently consistent according to γ) just as in the pairwise three-valued model, allowing us to make a direct comparison between the two models. Experiments are run for 50 000 iterations, and with random three-valued opinion initialisation unless otherwise stated (i.e. for comparisons with the Boolean group model).

2.7.1 Results

Under the group model, we wish to examine whether increasing the number of agents involved in the agreement process improves the speed of convergence for larger populations, which necessarily require longer running times under pairwise consensus. If the model fails to achieve significant convergence, then it is unfit for use in multi-agent settings where *no significant convergence* provides no additional information over *partial convergence*, which might even be preferable depending on the scenario concerned (e.g. for workload division between agents). In Figure 2.10 we show the number of distinct valuations for three-valued group model at steady state with payoff-based agent selection plotted against inconsistency thresholds γ. For a group size of 2, we see the pairwise three-valued payoff model for a larger population than was previously studied. For 1 000 agents, it is clear that convergence across the range of γ differs significantly. Rather than being monotonically decreasing in the number of distinct valuations (as seen in Figure 2.1b for a population size of 100 agents) we instead see an increase for $0.2 \leq \gamma < 0.4$. This is unusual when compared with the number of valuations for $\gamma = 0.0$ where the consensus operator is applied only to the most consistent opinions, as we see greater convergence for a more strict inconsistency threshold than we do for one that allows for the combination of opinions which differ by more than just borderline truth values. Despite this difference, we do still see quite strong convergence in terms of reducing the number of distinct valuations present in the population. For 5 propositional variables and 1 000 agents we expect the 243 possible valuations to be present at least once for the majority of the simulation runs, with roughly 4 copies per population on average at initialisation. Yet, there is a reduction to under 50 distinct opinions at steady state for $\gamma = 0.0$, and still below 70 on average even with the increase for $0.2 \leq \gamma < 0.4$. The pairwise model, as expected, struggles to achieve the same performance as it did for a population of 100 agents without extending the simulation runs beyond 50 000 iterations for $\gamma < 1.0$. When the inconsistency threshold does allow for all agents to form consensus ($\gamma = 1.0$), however, we do finally see the model achieve consensus.

For a group size of 3 and above, the model achieves a decrease in the number of distinct valuations as the inconsistency threshold increases. This is because a larger inconsistency threshold increases the likelihood that agents selected as part of the group are able to combine their opinions with the other agents. The more agents in the group that are sufficiently consistent, the closer their resulting opinions will be to one another and the stronger the convergence at that iteration. Even an increase from 2 agents to 3 provides considerable performance improvements in terms of convergence. Despite the lack of consensus for $\gamma < 0.8$, after 50 000 iterations and for group sizes 3, 5 and 10, the model shows effective partial convergence, achieving complete consensus for $\gamma \geq 0.8$ and even outperforming the pairwise model across all inconsistency thresholds $\gamma \in [0, 1]$. It should be noted here that for all group sizes above 2, the population converges to 32 distinct valuations, on average, for $\gamma = 0.0$. Given that this inconsistency threshold limits combinations to completely consistent valuations which differ by borderline truth values only, it is expected that the population has essentially converged from all possible three-valued valuations at initialisation, to the set of Boolean valuations by simulation's end. Indeed we can verify this in Figure 2.11 where, under the three-valued initialisation of opinions, we see the average vagueness in the population. For $\gamma < 0.2$, the average vagueness is reduced from 0.33 to 0 for the group model, leaving 32 completely precise valuations in the population: the Boolean valuations. For $0.2 \leq \gamma < 0.4$ we see that for both $k = 2$ and $k = 3$ the average vagueness in the population by the end of the simulations remains above 0. While still quite precise, given the

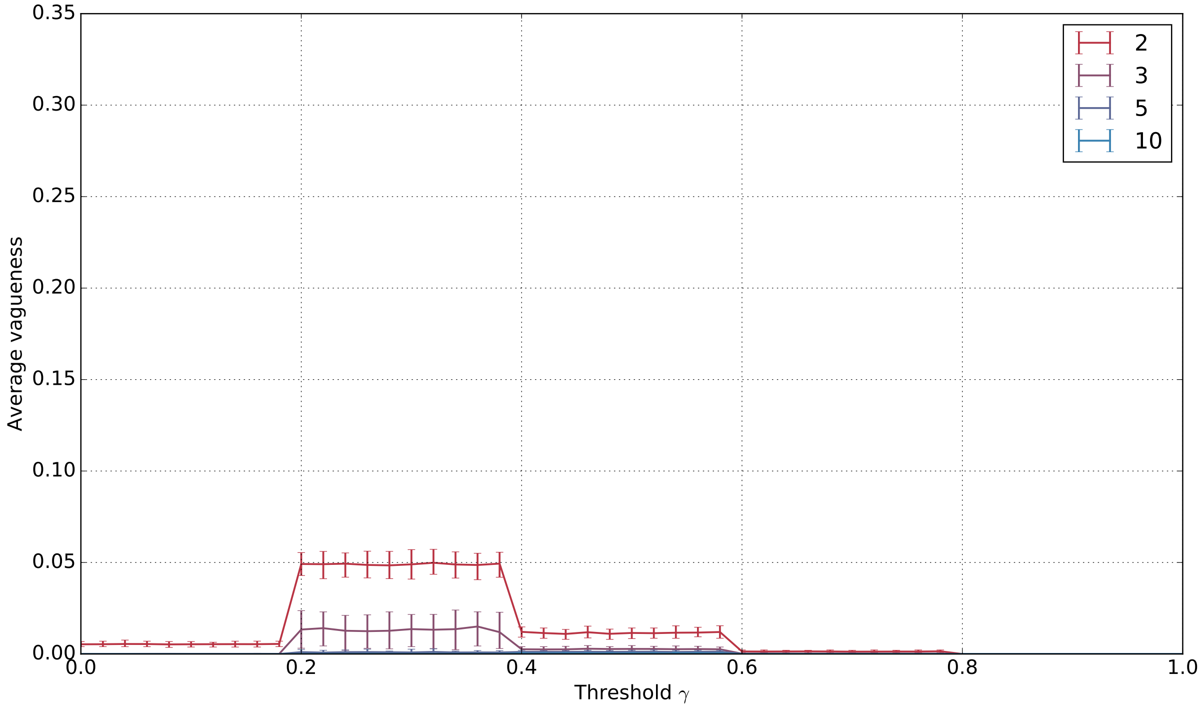
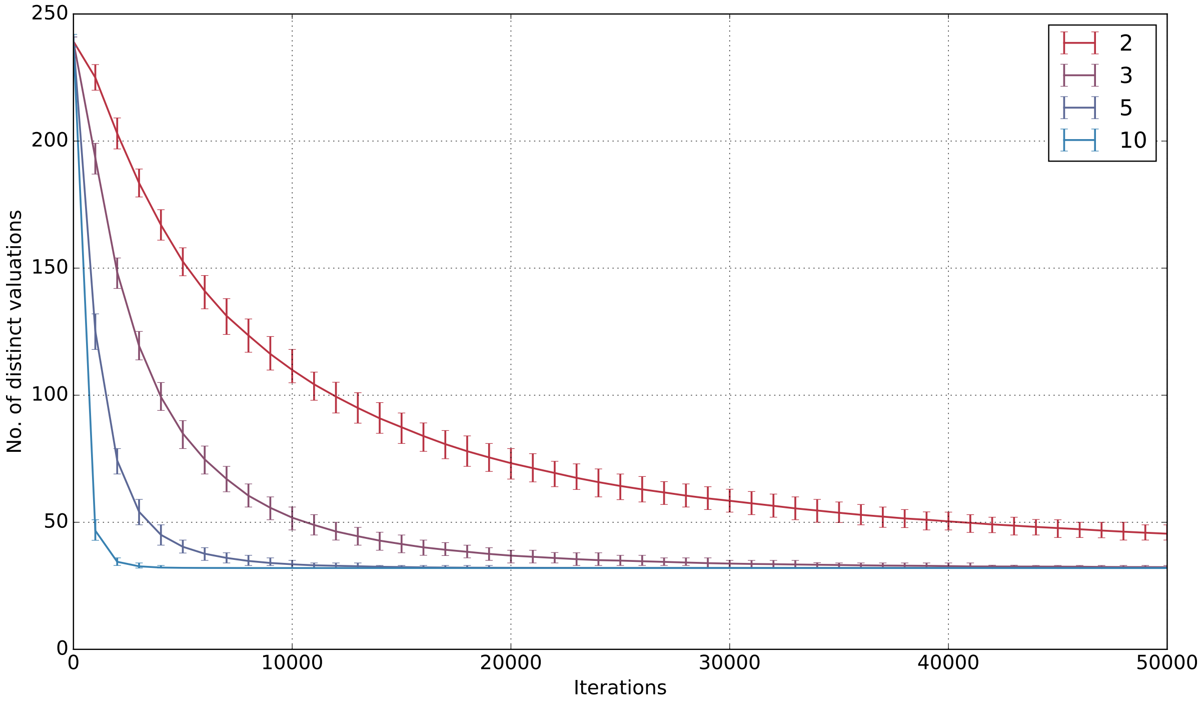


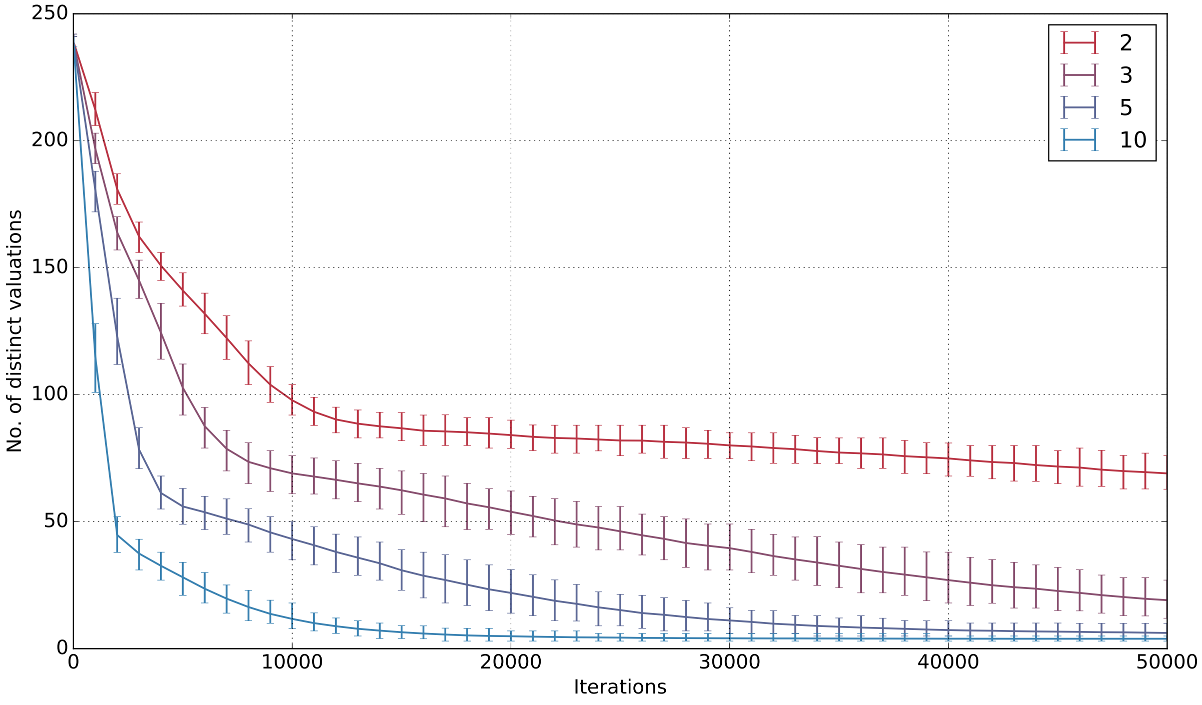
Figure 2.11: Average vagueness at steady state for the three-valued group model, with payoff-based selection and random three-valued opinion initialisation for group sizes $k \in \{2, 3, 5, 10\}$ and different inconsistency thresholds γ.

consensus operator used this is indicative of a population having not reached a steady state.

In Figure 2.12 we show the number of distinct valuations as trajectories for two different inconsistency thresholds: $\gamma \in \{0.0, 0.2\}$. It becomes clear from Figure 2.12a that, for $\gamma = 0.0$, the model converges increasingly quickly for larger group sizes, whereas the pairwise model fails to converge before 50 000 iterations, even though it is likely to eventually converge to the set of Boolean valuations at steady state, given sufficient time. We see that with a group size of 3, the model converges fully in just under 50 000 iterations, while for a group size of 10 convergence occurs in less than 5 000 iterations. We have therefore confirmed our predication that by increasing the number of agents involved in the consensus formation process at each iteration, we can reduce the time required to achieve significant convergence and can even reach consensus for a population of 1 000 agents. For an inconsistency threshold $\gamma = 0.2$ (Figure 2.12b), the models behave very differently with less consistent opinions being allowed to combine. We see fast convergence initially, particularly for the larger group sizes, but then a significant slow down in the reduction of distinct valuations at various stages for the different group sizes. While we do see convergence to a smaller set of valuations for $\gamma = 0.2$ in the group model, it appears that for pairwise and $k = 3$ group models, neither model reaches a steady state before the simulations end.



(a) Number of distinct valuations for $\gamma = 0.0$.



(b) Number of distinct valuations for $\gamma = 0.2$.

Figure 2.12: Three-valued group model with payoff-based selection and random three-valued opinion initialisation shown as a trajectory for group sizes $k \in \{2, 3, 5, 10\}$.

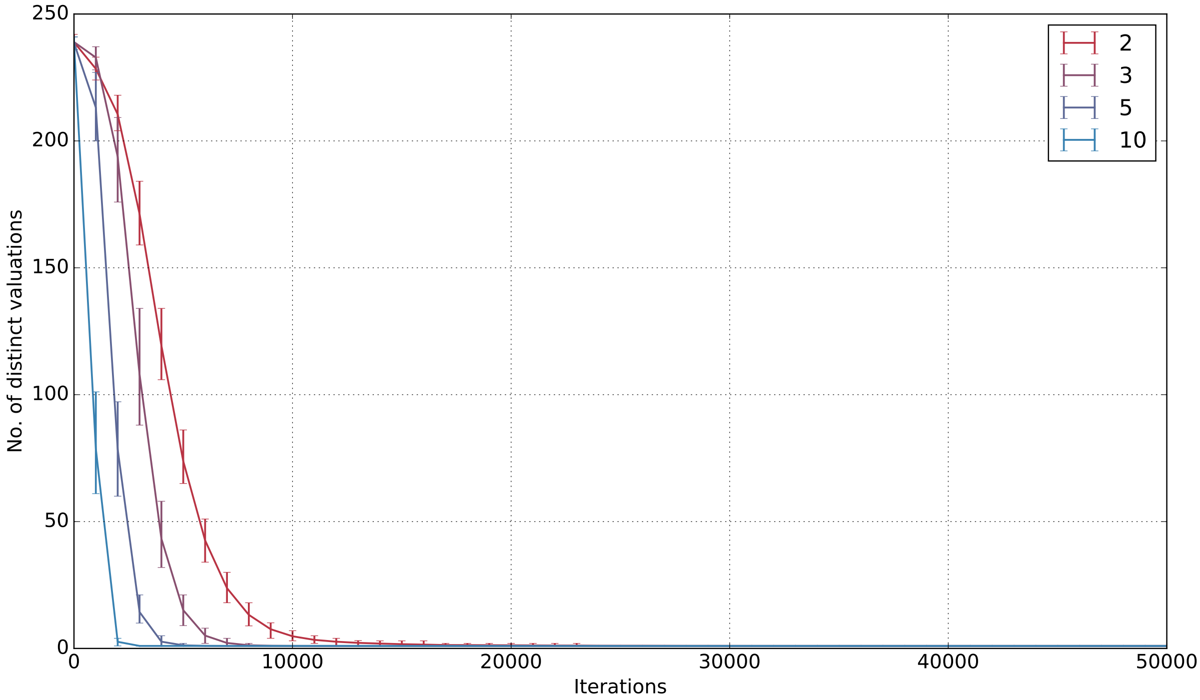


Figure 2.13: Number of distinct valuations for the three-valued group model with payoff-based selection and random three-valued opinion initialisation, shown as a trajectory for group sizes $k \in \{2, 3, 5, 10\}$ and $\gamma = 1.0$.

Finally, in Figure 2.13 we show the same graph but for an inconsistency threshold $\gamma = 1.0$ where all agent pairs in the group combine their opinions via the consensus operator, where the resulting opinions are then used to generate the probability distribution from which their new opinions are formed. Of course, these opinions are not necessarily the same, but instead the agents simply generate their new opinions from the same probability distribution in the case of $\gamma = 1.0$. As we would expect, all values of k reach a consensus under this extreme inconsistency threshold, with the pairwise model taking under 25 000 iterations, while a group size of 10 agents achieves consensus in under 3 000 iterations. We can see clearly from this graph that a larger value of k increases the speed of convergence when presented with a large population size, while pairwise interactions are no longer sufficient to reach a steady state within 50 000 iterations and would require much much longer simulation times in order to converge fully to either a steady state or to reach a consensus amongst the population. We now look to examine the performance of the group model in terms of payoff, and consider whether an increased convergence speed affects the overall quality of the opinions persistent in the population.

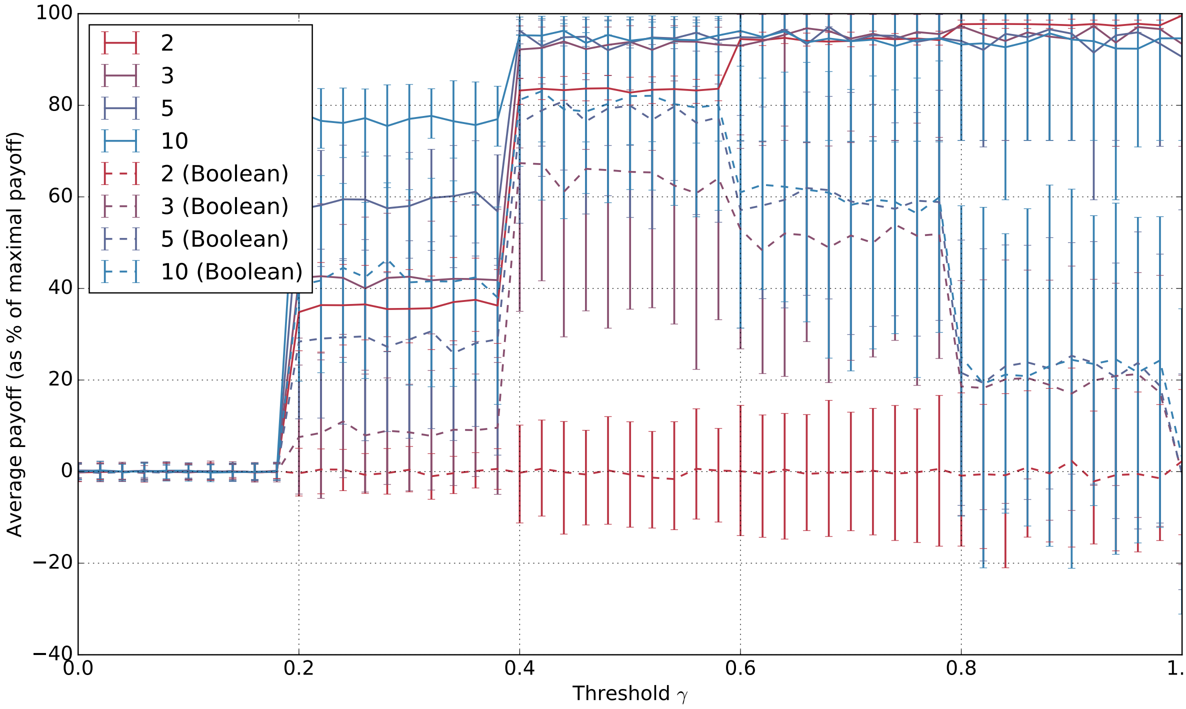


Figure 2.14: Average payoff for the three-valued and Boolean group models with random selection and random Boolean opinion initialisation at steady state for different inconsistency thresholds γ and different group sizes.

Comparing the Three-valued and Boolean group models

We analyse the performance of the three-valued group model alongside a Boolean group model under random Boolean opinion initialisation, and use the same operator as defined in Table 2.6 to apply to each pair of agents in the group, just as we do the three-valued consensus operator (Table 2.4). Figures 2.14 and 2.15 show results for both the three-valued and Boolean models after 50 000 iterations and for different group sizes $k \in \{2, 3, 5, 10\}$. Specifically, Figure 2.14 shows the average payoff as a percentage of the maximal possible payoff for both models. Examining the three-valued model first, we see that the average payoff is increasing with the inconsistency threshold γ. This increases in steps of $\frac{1}{5} = 0.2$ as we have come to expect for a language size $|\mathcal{P}| = 5$, as this increase allows for a propositional variable to be inconsistent between valuations and still be combined. Even for a strict inconsistency threshold $\gamma = 0.2$ we see an immediate increase in average payoff, greater than that seen in the same experiments for a population of 100 agents in Section 2.5. The increased density of opinions means there are a greater number of high quality opinions, and this seems to have a strong effect on the convergence of the system towards valuations which more accurately reflect the payoff model. As γ increases, we see that this increase continues with more severity, such that for $\gamma = 0.4$, the smallest value of average payoff across all values of k is above 80% for groups of 2 agents, with all other group sizes averaging above 90% of the maximal payoff. However, for group sizes 3, 5 and 10, payoff appears to remain unchanging for more lenient values of $\gamma > 0.4$. For $k = 2$, however, we see that payoff continues to increase in steps, eventually reaching a consensus of 100% payoff. This is an impressive result for mere pairwise interactions: given its slow convergence, the three-valued model manages to

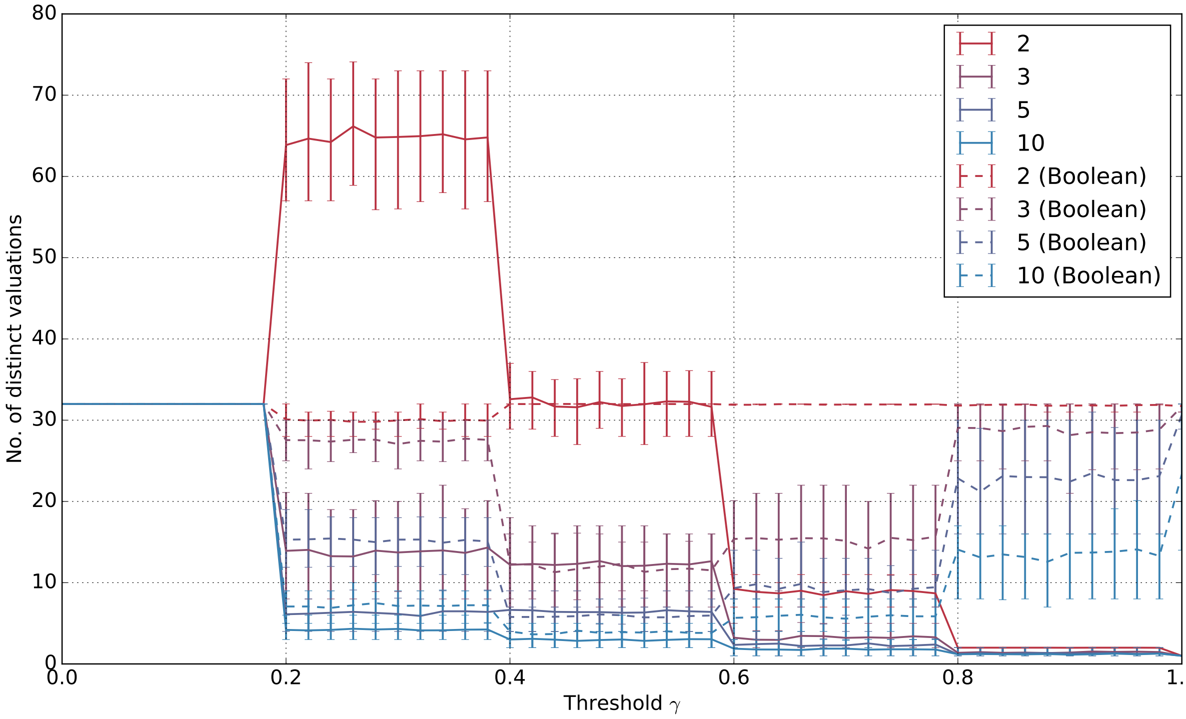


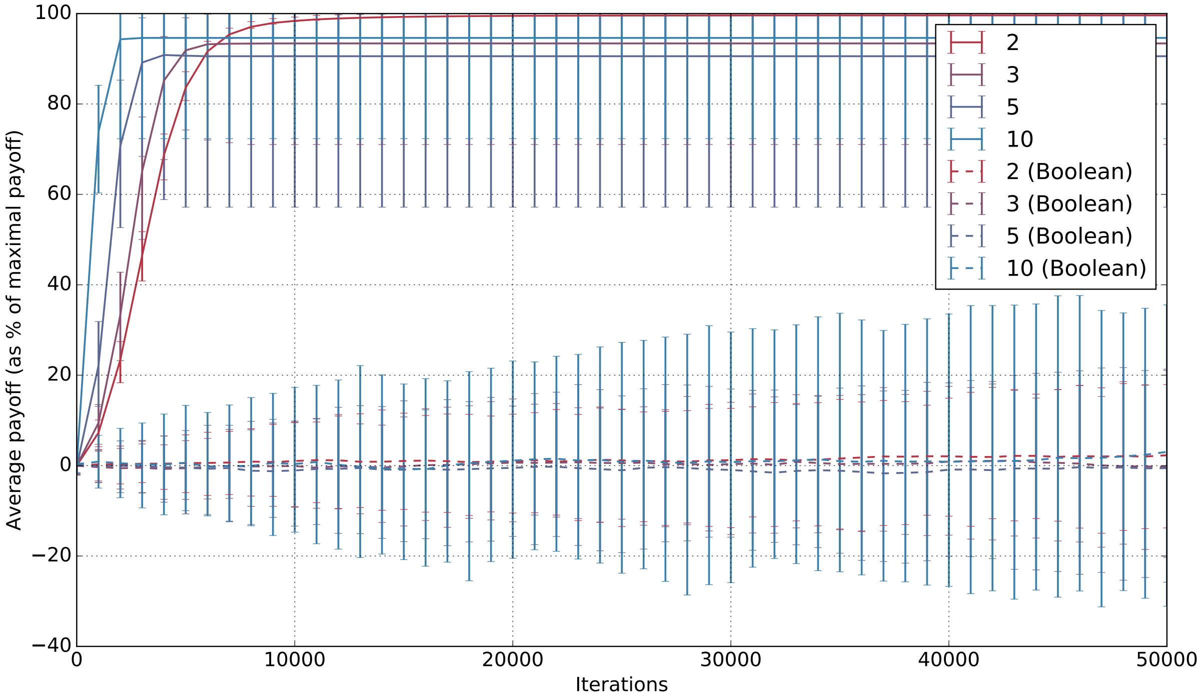
Figure 2.15: Number of distinct valuations for the three-valued and Boolean group models with random selection and random Boolean opinion initialisation at steady state, for different inconsistency thresholds γ and different group sizes.

converge to a single, shared valuation which is of maximal payoff and it does so in under 50 000 iterations. Even though larger group sizes converge faster for all values of γ compared with the pairwise model, the system performs more poorly when a meaningful consensus is sought.

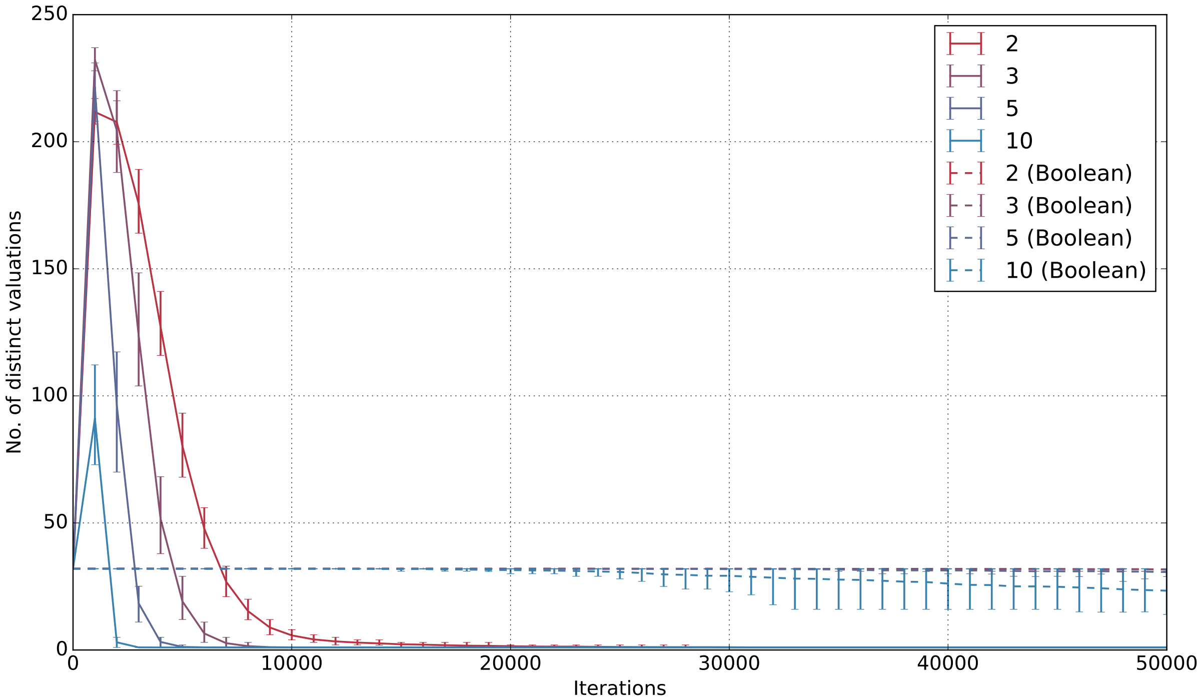
For the same experiments and parameters, the Boolean model performs rather differently. In Figures 2.14 and 2.15 the Boolean model is indicated by dashed lines of the same colour as three-valued model. We see that the pairwise Boolean model (dashed red) behaves the same as it did for a population size of 100 agents as it now does for 1 000 in Figure 2.14, but the rest of the results differ greatly. For the group model, we see the Boolean model increase in average payoff for $0.2 \leq \gamma < 0.6$, although the model averages less payoff for each group size than the three-valued model does. For $\gamma \geq 0.6$, the Boolean model behaves very differently, with average payoff decreasing for larger values of γ until the model eventually averages back around 0% for $\gamma = 1.0$. This surprising result can be better explained by Figure 2.15 where we present the number of distinct valuations for both the three-valued and Boolean variants of the group model. As we again see the three-valued model converging towards a smaller subset of distinct valuations with each increase in γ of 0.2 (except for in the pairwise case), the Boolean model behaves rather differently, just as it does in relation to average payoff. The pairwise Boolean model retains all 32 Boolean valuations across most of the range of inconsistency thresholds γ, never achieving consensus as in the three-valued model for $\gamma = 1.0$, and only deviating from the 32 valuations for $0.2 \leq \gamma < 0.4$ where the inconsistency allows for the combination of opinions for which one propositional variable is conflicting. What is more interesting perhaps is the performance of the group model for $k > 2$ where we see the model begin to converge for $\gamma \geq 0.2$, though never quite reaching consensus. Then for $\gamma = 0.6$ we see the opposite effect as the Boolean group model fails to reach any further convergence, and instead convergence is weakened across the population, until we reach $\gamma = 1.0$ where once again all 32 Boolean valuations persist amongst the population, just as at initialisation.

In Figure 2.16 we study the case of $\gamma = 1.0$ more closely as trajectories for both the average payoff and the number of distinct valuations so that we might understand why the Boolean model performs so poorly after having performed much better for lower inconsistency thresholds. For Figure 2.16a we see that the three-valued model converges to high average payoff in around 5 000 iterations for $k > 2$, and the pairwise model looks to have converged in under 18 000 iterations. For the Boolean model, however, we notice very little change regarding performance. Across the full running times of the experiments, all 50 000 iterations, we do see variation increasing as the number of iterations increases, but we see that this leads to very little performance gain which is likely as much noise as it is improved performance, with small deviations from 0% matched with increasing variation both in positive and negative payoff. In Figure 2.16b, for the number of distinct valuations, we see the three-valued model increase this number dramatically within the first 1 000 iterations where vagueness is introduced in the population, followed by a rather swift decline. All values of $k > 2$ for the three-valued group model converge by 10 000 iterations, with the pairwise model converging later at just under 30 000 iterations. The Boolean model presents a stark contrast with the three-valued model. The inability of the Boolean model to introduce vagueness into the opinions of agents causes the population to change little for well over 20 000 iterations. By 30 000 iterations the group size of 10 sees some convergence to a smaller subset of distinct valuations but by 50 000 iterations the model remains far from consensus, with the other group sizes achieving little to no reduction in the number of distinct valuations.

Clearly, then, the Boolean model struggles to achieve any meaningful convergence for more extreme values of γ within a reasonable time, when compared with the three-value model. it is likely that the model would eventually converge provided enough time, but we have seen that the three-valued *pairwise* model achieves 100% of the maximal possible payoff at steady state for a language size of 5 and an inconsistency threshold $\gamma = 1.0$. Furthermore, it does so in under 20 000 iterations, where the Boolean model with a group size of 10 agents would likely require beyond 150 000 iterations to reach a consensus and smaller group sizes further still. For larger group sizes, the three-valued model converges more quickly, and it does so with higher average payoff for lower values of γ, but is eventually succeeded by the pairwise model for $\gamma = 1.0$. Therefore, there is a trade-off to be made between speed of convergence to a valuation of high quality, and slower convergence to maximal payoff; the most accurate of valuations.



(a) Average payoff.



(b) Number of distinct valuations.

Figure 2.16: Three-valued and Boolean group models with random selection and random Boolean opinion initialisation at steady state for an inconsistency thresholds $\gamma = 1.0$ and different group sizes.

2.8 Conclusions

At this point it is worth discussing the contributions of the work so far, as well as some motivations for the following chapters of this thesis. We have shown, through simulation studies, how a three-valued model for consensus exhibits strong convergence in a multi-agent setting, enabling the system to form consensus (that is, to converge unanimously to a single, shared valuation) for a range of different language sizes and sufficiently large inconsistency threshold values. We discussed some properties of the inconsistency threshold applied throughout our experiments as a means of preventing inconsistent combinations between pairs of strongly conflicting agents, and how, for a fairly conservative inconsistency threshold $\gamma = 0.2$, the population manages to achieve strong convergence across the range of language sizes studied. We also mentioned earlier that the inconsistency threshold is inspired by similar means of limiting agent interactions, such as bounded confidence [31] and relative agreement [18]. Perhaps more interestingly, we have been able to show that the introduction of vagueness into the opinions of agents does not negatively affect the decision making process, either by causing the population to adopt completely vague opinions (i.e. where all propositional variables are assigned the borderline truth value $\frac{1}{2}$) as a means of avoiding further conflict, or by allowing highly conflicting agents to diverge and form a population of bipolar opinions. Indeed at lower values of γ it may be surprising to still see reasonable convergence occurring, but due to the way the inconsistency measure is defined, borderline valuations remain consistent with more strong valuations. As such, the three-valued model is capable of reducing the initial population of uniformly distributed three-valued opinions from 100 (for a population of 100 agents) to under 25 unique opinions on average, at steady state, for a language size of 5. Similarly, for a language size of 10 we see the population converges on a subset of the initial opinions, averaging just over 50 unique opinions at steady state. While this had been shown to be the case for the single propositional variable case in [57], it has not, until now, been extended for the multi-propositional variable case.

We have also introduced a feedback mechanism in the form of payoff which allows us to bias agent selection in order to drive convergence to opinions that more accurately reflect the 'true' state of the world, according to the chosen payoff model. We also introduced a stochastic Boolean version of the consensus operator with which we compared the three-valued consensus model with a Boolean version to further study how the ability to adopt a more vague interpretation of propositions improves the convergence of agents with highly conflicting opinions. As a result, we have shown through direct comparisons that the three-valued model outperforms the Boolean model both in terms of convergence speed, as well as average payoff, with the Boolean model often achieving no significant gain in average payoff and the three-valued model achieving an average payoff above 60% for a population of 100 agents and a language size of 5. This supports the motivating hypothesis of this work which is that a three-valued model of consensus improves convergence across both language sizes and population sizes. We see that the third truth value allows for the introduction of vagueness as a means of smoothing the combination of agent opinions, allowing both agents to adopt a consistent, intermediary valuation. Otherwise, agents would either be unable to reach an agreement, or one agent would be forced to adopt the opposite valuation for each inconsistent propositional variable, which seems counter-intuitive given that we are attempting to make the minimal changes to achieve consistency, without agents having to adopt a new opinion which is in direct conflict with their current opinion.

Finally we introduced a group-based consensus model in which small groups of agents are selected from the population at each iteration, rather than a single pair of agents. Within the group, agents attempt to update their opinions to reflect the opinions of those other agents with which their opinions are consistent. The intention of this model is to improve the speed of convergence in large populations of agents where the pairwise model slows with the increasing population size. By involving a larger number of agents in the consensus process, we achieved a speed up in convergence over the pairwise model, albeit at the cost of reduced average payoff after the 50 000 iterations. We saw that for the three-valued model, pairwise combinations of opinions actually achieves 100% of the maximal payoff in the population whereas for the group model, payoff suffers as the group size increases. However, convergence occurs more quickly for larger group sizes. Therefore we see that both models have their advantages and the context of the consensus problem should be considered when deciding which approach to take for large agent populations.

CHAPTER 3

From Vague Opinions to Uncertain Beliefs

In the preceding chapter, we introduced a formalism for representing vague opinions using Kleene's three-valued logic where the third truth value represented an intermediate state between *absolutely true* and *absolutely false* meaning *borderline*. We then exploited this third truth value through the use of a consensus operator. This allowed conflicting agents to reach an agreement about a set of propositions for which their opinions were inconsistent i.e. where one believes a proposition to be true and the other believes it to be false; an issue that has no obvious resolution when using Boolean logic. In this chapter we transition from Kleene valuations as a representation of vague opinions to Kleene belief pairs as a representation of vague and uncertain *beliefs* [41]. This is an important distinction as we are allowing agents to express uncertainty as to *which* Kleene valuation they believe to be correct as opposed to a single, certain opinion being held by each agent, as was the case throughout the previous chapter.

We begin this chapter by redefining many aspects of the model presented in Chapter 2 in order to take account of the presence of probabilistic uncertainty. We then present simulation results in order to examine how well our new model is able to combine vagueness with uncertainty across a population of 1 000 agents in order to assess the ability of our model to scale to such large numbers of agents and to still form consensus. Following this, we examine how our model performs when we select agents based on the accuracy of their beliefs according to the 'true' state of the world, just as we did in Chapter 2. Given that agents may now express uncertainty, we must re-examine the model's ability to converge towards a perceived 'true' belief as opposed to random consensus. Finally we highlight how consensus in multi-agent systems may provide additional utility beyond forming an agreement for the purpose of decision-making. We do so by combining our model of consensus for vague and uncertain beliefs with evidential belief updating. We then study the resulting model's ability to disseminate evidence when compared with an evidence-only model in which agents do *not* form consensus, but instead receive direct evidence with which they update their beliefs. We believe that combining evidential updating with consensus greatly improves evidence dissemination across a large population of agents, even for very low rates of evidence delivery.

3.1 Related work

This chapter is an extension of the model presented in Chapter 2 and hence all of the work highlighted in Section 2.1 remains relevant. For example, we continue to employ an inconsistency measure as in Definition 2.7 which (although adapted) still follows from the similar idea of bounded confidence [18, 31] as a means of limiting agent interaction to those pairs of agents whose inconsistency measure is below a threshold denoted by γ. Several models for consensus also exploit a third truth state to aid convergence as also mentioned in the previous chapter [1, 16, 57, 79].

The aggregation of uncertain beliefs in the form of a probability distribution over some underlying parameter has been widely studied with work on opinion pooling dating back to DeGroot [19] and Stone [69]. Usually this aggregate of a set of opinions takes the form of a weighted linear combination of the associated probability distributions. Aside from the aforementioned assumption that agents are able to assign weights a priori to all other agents in the population, such a method relies on the ability of agents to communicate with all other agents in the population, as well as the ability to merge large numbers of probability distributions in a more meaningful manner. The convergence of alternative opinion pooling functions therefore has been studied by Hegselmann and Krause [32] and axiomatic characterizations of different operators are given in [20]. However, all of these approaches assume Boolean truth states; indeed, there are very few studies in this context that combine probability with a three-valued truth model. One such is [6] in which the authors adopt a model of beliefs in the form of Dempster-Shafer functions. The combination operator proposed in [6], however, is quite different from those described in this thesis and results in quite different limiting behaviour. The consensus operator investigated in this chapter was first proposed by Lawry and Dubois in [41], as the previous version was in Chapter 2, as an extension of the approach of [57] to take account of probabilistic uncertainty.

Later in this chapter we consider how consensus can be used to propagate a form of direct evidence through a large population of agents and compare this with only evidential updating taking place. A recent review by Douven [22] covers several computation models in the field of social epistemology that combine a form of agent compromise and evidential updating, including [31, 32] and [61], before concluding that models which combine both aspects of belief updating lead to agents having more accurate belief states than systems which rely solely upon evidential updating.

3.2 Model

Overview. Consensus formation is investigated for multi-agent systems in which agents' beliefs are both vague and uncertain. Vagueness is represented by a third truth state meaning *borderline*, as discussed in Chapter 2. We combine this model with a probabilistic model of uncertainty, and then propose a modified belief combination operator which exploits borderline truth values to enable agents with conflicting beliefs to reach a compromise. It should be noted that there is an important distinction to be made between the role of three-valued logic in conveying vagueness and the role of probability representing uncertainty. In this model, the third truth state models inherently borderline cases resulting from the existence of truth-gaps for vague propositions, while probability quantifies uncertainty about the state of the world i.e. it is possible for an agent to be completely certain that a proposition is borderline.

3.2.1 A restatement of the three-valued model for consensus of vague beliefs

As in Chapter 2 we adopt a propositional logic setting based on Kleene's strong three-valued logic [39] and combine this with a probabilistic model of uncertainty. We consider a finite language $\mathcal{L}$ consisting of the connectives $\wedge$, $\vee$ and $\neg$, and n propositional variables $\mathcal{P} = \{p_1, \ldots, p_n\}$, where each propositional variable can have one of the three truth values 0, denoting false, $\frac{1}{2}$, denoting borderline and 1, denoting true. A valuation on $\mathcal{L}$ corresponds to an allocation of a truth value to each of the propositional variables, following from Definition 2.1. Consequently, a valuation may be naturally represented as an n dimensional vector $\mathbf{v} \in \{0, \frac{1}{2}, 1\}^n$; a simplification we adopt for the remainder of this chapter. We let $\mathbf{v}(p_i)$ denote the i'th dimension of $\mathbf{v}$ as corresponding to the truth value of the propositional variable p_i in the valuation $\mathbf{v}$. In the absence of any uncertainty we assume than an agent's opinion is represented by a single valuation. For two agents with distinct and possibly conflicting opinions $\mathbf{v}_1, \mathbf{v}_2 \in \{0, \frac{1}{2}, 1\}^n$ to reach a compromise position or consensus we apply the operator in Definition 2.5 and based on the truth table given in Table 2.4 which is applied to each propositional variable independently so that:

$$\mathbf{v}_1 \odot \mathbf{v}_2 = (\mathbf{v}_1(p_1) \odot \mathbf{v}_2(p_1), \ldots, \mathbf{v}_1(p_n) \odot \mathbf{v}_2(p_n)).$$

See Section 2.2 for additional information regarding the intuition behind this operator.

3.2.2 A representation of vague and uncertain beliefs

Here we extend the model described in Chapter 2 so as to allow agents to hold opinions which are *uncertain* as well as vague. More specifically, an integrated approach to uncertainty and vagueness is adopted in which an agent's belief is characterised by a probability distribution w over $\{0, \frac{1}{2}, 1\}^n$ so that $w(\mathbf{v})$ quantifies the agent's belief that $\mathbf{v}$ is the correct valuation of $\mathcal{L}$. This naturally generates lower and upper belief measures on $\mathcal{L}$ quantifying the agent's belief that a given proposition is true and that it is not false respectively [44]. That is; for $p_i \in \mathcal{P}$, [1]

$$\underline{\mu}(p_i) = w(\{\mathbf{v} : \mathbf{v}(p_i) = 1\}) \text{ and}$$
$$\overline{\mu}(p_i) = w(\{\mathbf{v} : \mathbf{v}(p_i) \neq 0\}).$$

The probability of each of the possible truth values for a propositional variable p_i can be recaptured from the lower and upper belief measures such that the probabilities that p_i is true, borderline and false are given by $\underline{\mu}(p_i)$, $\overline{\mu}(p_i) - \underline{\mu}(p_i)$ and $1 - \overline{\mu}(p_i)$, respectively. Hence, if we assume that the truth value of each propositional variable is independent, we can represent an

[1]In the following we slightly abuse notation and also use w to denote the probability measure generated by the probability distribution w.

agent's belief by a vector of pairs of lower and upper belief values for each variable as follows:

$$\boldsymbol{\mu} = ((\underline{\mu}(p_1), \overline{\mu}(p_1)), \ldots, (\underline{\mu}(p_n), \overline{\mu}(p_n))).$$

Here we let $\boldsymbol{\mu}(p_i)$ denote $(\underline{\mu}(p_i), \overline{\mu}(p_i))$, the pair of lower and upper belief values for p_i. In the case that a belief $\boldsymbol{\mu}$ gives probability zero to the borderline truth value for every propositional variable in $\mathcal{P}$ so that $\underline{\mu}(p_i) = \overline{\mu}(p_i) = \mu(p_i)$ for $i = 1, \ldots, n$, then we call $\boldsymbol{\mu}$ a *crisp belief.*

The following definition expands the consensus operation $\odot$ from three-valued valuations to this more general representation framework.

Definition 3.1. Consensus operator for beliefs

For beliefs μ_1 and μ_2 we define the consensus operator as follows:

$$\boldsymbol{\mu}_1 \odot \boldsymbol{\mu}_2 = \\ ((\underline{\mu}_1 \odot \underline{\mu}_2(p_1), \overline{\mu}_1 \odot \overline{\mu}_2(p_1)), \ldots, (\underline{\mu}_1 \odot \underline{\mu}_2(p_n), \overline{\mu}_1 \odot \overline{\mu}_2(p_n)))$$

where

$$\underline{\mu}_1 \odot \underline{\mu}_2(p_i) = \underline{\mu}_1(p_i) \times \overline{\mu}_2(p_i) + \overline{\mu}_1(p_i) \times \underline{\mu}_2(p_i) - \underline{\mu}_1(p_i) \times \underline{\mu}_2(p_i)$$

and

$$\overline{\mu}_1 \odot \overline{\mu}_2(p_i) = \underline{\mu}_1(p_i) + \underline{\mu}_2(p_i) + \overline{\mu}_1(p_i) \times \overline{\mu}_2(p_i) - \overline{\mu}_1(p_i) \times \underline{\mu}_2(p_i) \\ -\underline{\mu}_1(p_i) \times \overline{\mu}_2(p_i).$$

If $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are generated by the probability distributions w_1 and w_2 on $\{0, \frac{1}{2}, 1\}^n$ respectively, then $\boldsymbol{\mu}_1 \odot \boldsymbol{\mu}_2$ corresponds to the lower and upper measures generated by the following combined probability distribution on $\{0, \frac{1}{2}, 1\}^n$ [41]:

$$w_1 \odot w_2(\mathbf{v}) = \sum_{\mathbf{v}_1, \mathbf{v}_2 : \mathbf{v}_1 \odot \mathbf{v}_2 = \mathbf{v}} w_1(\mathbf{v}_1) \times w_2(\mathbf{v}_2).$$

In other words, assuming that the two agents are independent, all pairs of valuations supported by the two agents are combined using the consensus operator for valuations and then aggregated. Interestingly, this operator can be reformulated as a special case of the union combination operator in Dempster-Shafter theory (see [67]) proposed by [24]. To see this notice that given a probability distribution w on $\{0, \frac{1}{2}, 1\}$ we can generate a Dempster-Shafer mass function m on the power set of $\{0, 1\}$ for each propositional variable p_i such that:

$$\begin{aligned} m(\{1\}) &= w(\{\mathbf{v} : \mathbf{v}(p_i) = 1\}) &= \underline{\mu}(p_i) \\ m(\{0\}) &= w(\{\mathbf{v} : \mathbf{v}(p_i) = 0\}) &= 1 - \overline{\mu}(p_i) \\ m(\{0,1\}) &= w(\{\mathbf{v} : \mathbf{v}(p_i) = \frac{1}{2}\}) &= \overline{\mu}(p_i) - \underline{\mu}(p_i). \end{aligned}$$

$\odot$	$1 : 0.6$	$\frac{1}{2} : 0.2$	$0 : 0.2$
$1 : 0.4$	$1 : 0.24$	$1 : 0.08$	$\frac{1}{2} : 0.08$
$\frac{1}{2} : 0.3$	$1 : 0.18$	$\frac{1}{2} : 0.06$	$0 : 0.06$
$0 : 0.3$	$\frac{1}{2} : 0.18$	$0 : 0.06$	$0 : 0.06$

Table 3.1: Probability table for the consensus operator.

In this reformulation then the lower and upper measures $\underline{\mu}(p_i)$ and $\overline{\mu}(p_i)$ correspond to the Dempster-Shafer belief and plausibility of the focal set $\{1\}$, as generated by m, respectively. Now in this context the union combination operator is defined as follows: Let m_1 and m_2 be two mass functions generated as above by probability distributions w_1 and w_2. Also let c be a set combination function defined as:

$$c(A, B) = \begin{cases} A \cap B : A \cap B \neq \emptyset \\ A \cup B : \text{otherwise.} \end{cases}$$

Then the combination of m_1 and m_2 is defined by:

$$m_1 \odot m_2(D) = \sum_{A,B \subseteq \{0,1\}: c(A,B)=D} m_1(A) \times m_2(B).$$

The belief and plausibility of the focal set $\{1\}$ generated by $m_1 \odot m_2$ then respectively correspond to $\underline{\mu}_1 \odot \underline{\mu}_2(p_i)$ and $\overline{\mu}_1 \odot \overline{\mu}_2(p_i)$ as given in Definition 3.1.

Example 3.1. Suppose two agents have the following beliefs about propositional variable p_i: $\boldsymbol{\mu}_1(p_i) = (0.6, 0.8)$ and $\boldsymbol{\mu}_2(p_i) = (0.4, 0.7)$. The associated probability distributions on valuations, w_1 and w_2, are then such that:

$$\begin{aligned} &w_1(\{\mathbf{v} : \mathbf{v}(p_i) = 1\}) = 0.6, \\ &w_1(\{\mathbf{v} : \mathbf{v}(p_i) = \frac{1}{2}\}) = 0.8 - 0.6 = 0.2, \\ &w_1(\{\mathbf{v} : \mathbf{v}(p_i) = 0\}) = 1 - 0.8 \quad = 0.2, \quad \text{and} \\ &w_2(\{\mathbf{v} : \mathbf{v}(p_i) = 1\}) = 0.4, \\ &w_2(\{\mathbf{v} : \mathbf{v}(p_i) = \frac{1}{2}\}) = 0.7 - 0.4 = 0.3, \\ &w_2(\{\mathbf{v} : \mathbf{v}(p_i) = 0\}) = 1 - 0.7 \quad = 0.3. \end{aligned}$$

From this we can generate the probability table shown in Table 3.1. Here the corresponding truth values are generated as in Table 2.4 and the probability values in each cell are the product of the associated row and column probability values. From this table we can then determine the consensus belief in p_i by taking the sum of the probabilities of the cells with truth value 1 to give the lower measure and the sum of the probabilities of the cells with truth values of either

1 or $\frac{1}{2}$ to give the upper measure. That is:

$$\underline{\mu}_1 \odot \underline{\mu}_2(p_i) = 0.24 + 0.08 + 0.18 = 0.5$$
$$\overline{\mu}_2 \odot \overline{\mu}_2(p_i) = 0.24 + 0.08 + 0.18 + 0.18 + 0.06 + 0.08 = 0.82.$$

3.2.3 Measures for uncertain beliefs

We now introduce three measures which will subsequently be used to analyse the behaviour of multi-agent systems applying the operator given in Definition 3.1.

Definition 3.2. A measure of vagueness
The degree of vagueness of the belief $\boldsymbol{\mu}$ is given by:

$$\frac{1}{n}\sum_{i=1}^{n}(\overline{\mu}(p_i) - \underline{\mu}(p_i)).$$

Definition 3.2 is simply the probability of the truth value $\frac{1}{2}$ averaged across the n propositions in $\mathcal{P}$. Since in this model vagueness is associated with borderline truth values then this provides an intuitive measure of the degree of vagueness of an opinion. Accordingly the most vague belief has $(\underline{\mu}(p_i), \overline{\mu}(p_i)) = (0, 1)$ for $i = 1, \ldots, n$.

Definition 3.3. A measure of uncertainty
The entropy of the belief $\boldsymbol{\mu}$ is given by:

$$\frac{1}{n}\sum_{i=1}^{n} H(p_i)$$

where

$$H(p_i) = -\underline{\mu}(p_i)\log_2(\underline{\mu}(p_i)) - (\overline{\mu}(p_i) - \underline{\mu}(p_i))\log_2(\overline{\mu}(p_i) - \underline{\mu}(p_i))$$
$$-(1 - \overline{\mu}(p_i))\log_2(1 - \overline{\mu}(p_i)).$$

Definition 3.3 corresponds to the entropy of the marginal distributions on $\{0, \frac{1}{2}, 1\}$ averaged across the n propositions. Hence, according to this measure the most uncertain belief allocates probability $\frac{1}{3}$ to each of the truth values for each proposition so that:

$$\boldsymbol{\mu} = \left(\left(\frac{1}{3}, \frac{2}{3}\right), \ldots, \left(\frac{1}{3}, \frac{2}{3}\right)\right).$$

The most certain beliefs then correspond to those for which for every proposition $(\underline{\mu}(p_i), \overline{\mu}(p_i)) = (0, 0)$, $(0, 1)$ or $(1, 1)$.

Definition 3.4. A measure of inconsistency
The degree of inconsistency of two beliefs $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ is given by:

$$\frac{1}{n}\sum_{i=1}^{n}\left(\underline{\mu}_1(p_i) \times (1 - \overline{\mu}_2(p_i)) + (1 - \overline{\mu}_1(p_i)) \times \underline{\mu}_2(p_i)\right).$$

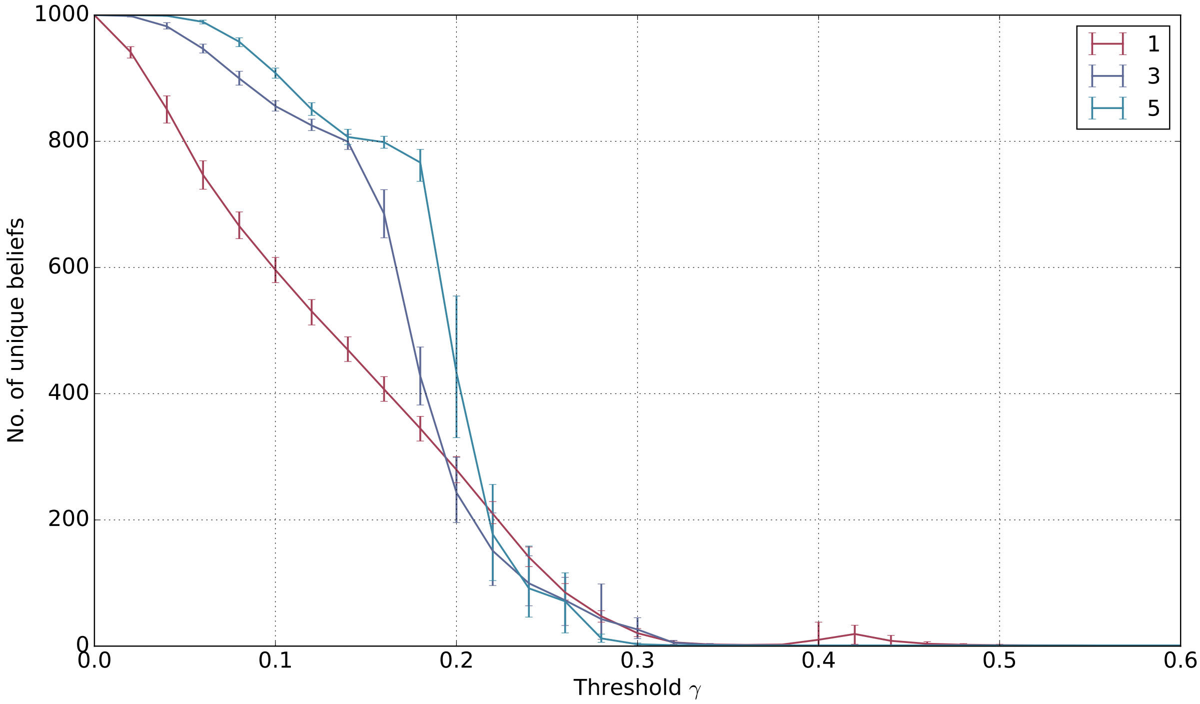


Figure 3.1: Number of unique beliefs after 50 000 iterations for varying inconsistency thresholds γ and various language sizes $|\mathcal{P}|$.

Definition 3.4 is the probability of a direct conflict between the two agents' beliefs, i.e. with agent 1 allocating the truth value 1 and agent 2 the truth value 0 or vice versa, this being then averaged across all n propositions.

3.3 Experiments with random agent selection

We now describe simulation experiments in which pairs of agents are selected to interact at random. For each selected pair of agents the consensus operation (Definition 3.1) is applied if and only if the measure of inconsistency between their beliefs, as given in Definition 3.4, does not exceed a threshold parameter $\gamma \in [0, 1]$. Notice that with $\gamma = 0$ we have a very conservative model in which only entirely consistent beliefs can be combined, while for the case that $\gamma = 1$ we have a model which is equivalent to a totally connected interaction graph, whereby any pair of randomly selected agents may combine their beliefs. In the following, results are presented for a population of 1 000 agents and for the language sizes $|\mathcal{P}| \in \{1, 3, 5\}$. The agents' beliefs are initialized by sampling at random from the space of all possible beliefs $\{(x, y) \in [0, 1]^2 : x \leq y\}^n$. Each run of the simulation is terminated after 50 000 iterations[2] and the results are averaged over 100 independent runs, with error bars indicating the 10^{th} and 90^{th} percentiles.

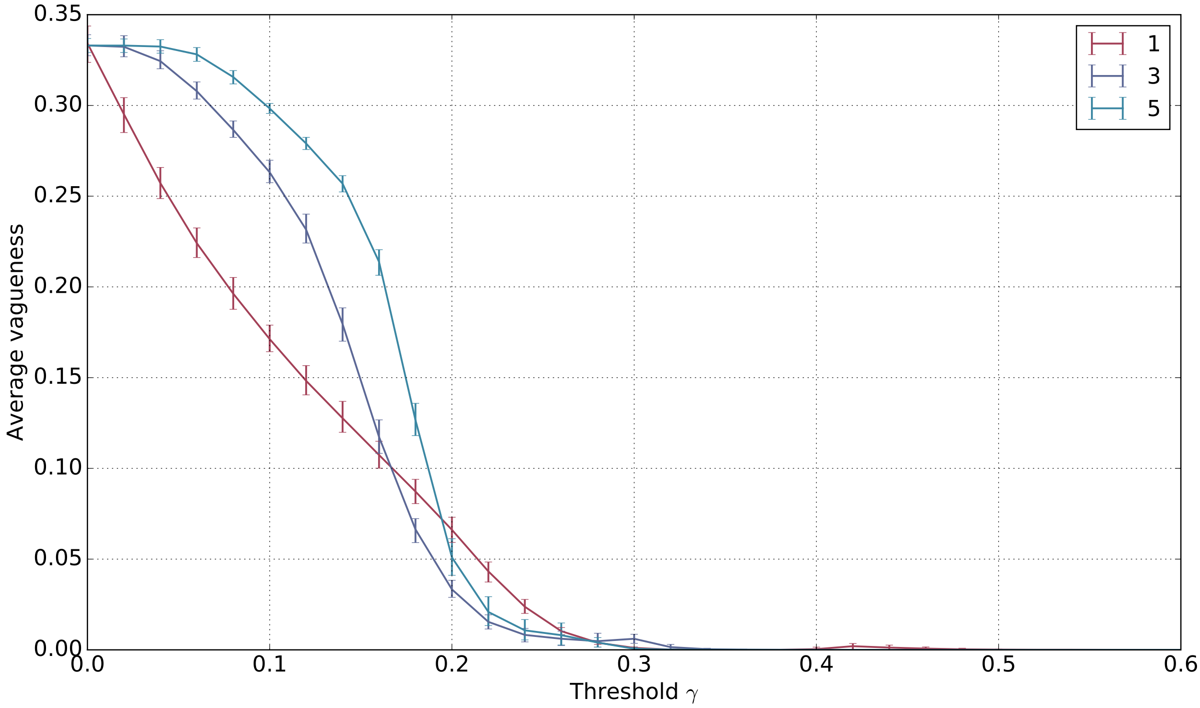


Figure 3.2: Average vagueness after 50 000 iterations for varying inconsistency thresholds γ and various language sizes $|\mathcal{P}|$.

3.3.1 Results

Figure 3.1 shows that the mean number of unique beliefs after 50 000 iterations decreases with γ and for $\gamma \geq 0.5$ there is on average a single belief shared across the population for all language sizes $|\mathcal{P}|$. Furthermore, Figure 3.2 shows that the vagueness of beliefs, as given in Definition 3.2, averaged both across the different agents and across the independent simulation runs, also decreases with γ so that for $\gamma \geq 0.5$ the population has converged to crisp beliefs, i.e. those with a vagueness measure value of 0. Similarly, from Figure 3.3 we can see that the entropy of beliefs, as given by Definition 3.3, decreases with γ and for $\gamma \geq 0.5$ at the end of the simulation the population hold beliefs with mean entropy 0. Hence, summarising Figures 3.1 to 3.3, we have that provided the consistency restrictions are sufficiently relaxed, i.e. for $\gamma \geq 0.5$, then a population with beliefs initially allocated at random and with random interactions will converge to a single belief which is both crisp and certain. Unsurprisingly, given the random nature of the agent interactions, the 2^n beliefs of this form occur with a uniform distribution across the 100 independent runs of the simulation.

In addition to the overall consensus reached between agents when $\gamma \geq 0.5$, intermediate values of γ between 0.15 and 0.35 tend to result in a population with highly polarised opinions. To see this consider Figure 3.4 showing the average pairwise inconsistency measure value between agents at the end of this simulation and plotted against γ. For example, consider the case when $|\mathcal{P}| = 1$ shown as the red line in Figures 3.1 to 3.4. In this case we see that the mean inconsistency value obtains a maximum of 0.5 at around $\gamma = 0.3$. Furthermore, from Figures 3.1 to 3.3 we

[2] We found that 50 000 iterations was sufficient to allow simulations to converge across a range of parameter settings.

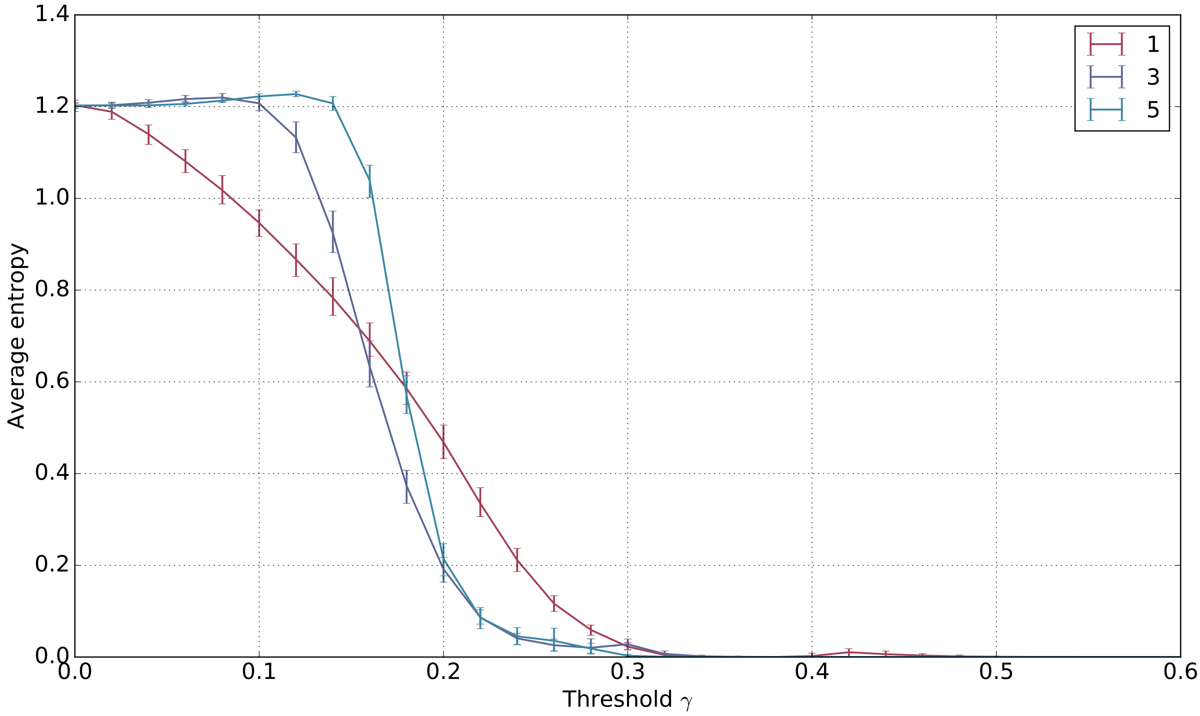


Figure 3.3: Average entropy after 50 000 iterations for varying inconsistency thresholds γ and various language sizes $|\mathcal{P}|$.

see that for this value of γ the average number of unique beliefs, the vagueness, and entropy are all relatively low. Consequently, we are seeing a polarisation of opinions where individuals are holding a small number of highly inconsistent beliefs which are also relatively crisp and certain. Such behaviour, while still present, is less pronounced for language sizes $|\mathcal{P}| = 3$ and 5. This may be due to the fact that, since Definition 3.4 is an average of inconsistency values across the propositions in $\mathcal{P}$, increasing the language size reduces the variance of the inconsistency values in the initial population. Furthermore, as $|\mathcal{P}|$ increases, the distribution of inconsistency values is approximately normal with mean $\frac{2}{9}$. Hence, for $\gamma \geq \frac{2}{9}$ the probability that a randomly selected pair of agents will have an inconsistency value exceeding γ decreases as $|\mathcal{P}|$ increases. This in turn will increase the probability of agreement in any interaction, reducing the likelihood of opinion polarisation for $\gamma \geq \frac{2}{9}$.

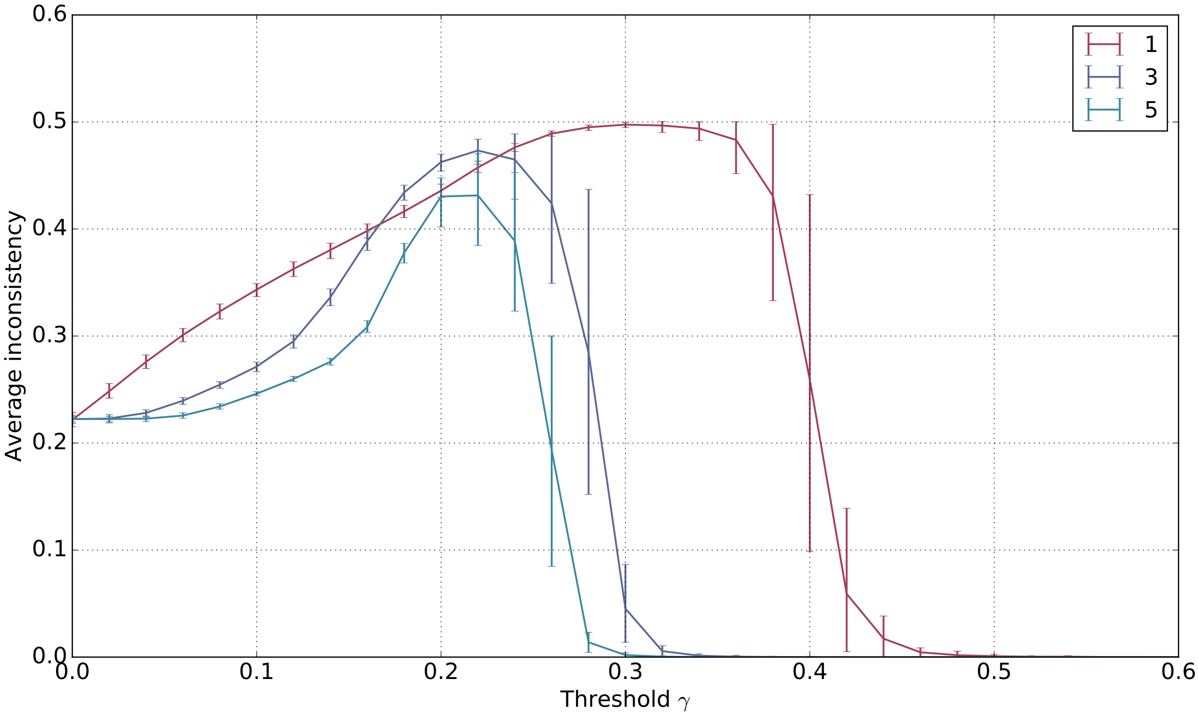


Figure 3.4: Average pairwise inconsistency after 50 000 iterations for varying inconsistency thresholds γ and various language sizes $|\mathcal{P}|$.

3.4 Experiments with agent selection influenced by belief quality.

In the following definition we propose a measure of belief quality which quantifies the similarity of an agent's beliefs to the true state of the world. This will subsequently be used to assess the extent to which the population has converged to the truth. Furthermore, it will also be employed in the next section as a mechanism for providing indirect information about the state of the world. As in Section 2.4.1 we assume that the true state of the world is chosen at random from $\{0,1\}^n$. This represents complete certainty in the true state being reflected by a Boolean valuation $\mathbf{v}^*(p_i) \in \{0,1\}$ for $p_i \in \mathcal{P}$. Then we can define a measure of quality for vague and uncertain beliefs.

Definition 3.5. A measure of quality
Let $f : \mathcal{L} \to \{-1, 1\}$ be such that $f(p_i) = 2\mathbf{v}^*(p_i) - 1$ is the payoff for believing that p_i has truth value 1 and $-f(p_i)$ is the payoff for believing that the truth value of p_i is 0. Furthermore, it is always assumed that believing that p_i has truth value $\frac{1}{2}$ has payoff 0. Then we define the quality or payoff for the belief $\boldsymbol{\mu}$ by:

$$\sum_{i=1}^{n} \left(f(p_i)(\underline{\mu}(p_i) + \overline{\mu}(p_i) - 1) \right).$$

Notice that $f(p_i)(\underline{\mu}(p_i) + \overline{\mu}(p_i) - 1) = f(p_i)\underline{\mu}(p_i) + (-f(p_i))(1 - \overline{\mu}(p_i))$ corresponding to the agent's expected payoff from their beliefs about proposition p_i. Definition 3.5 then takes the sum of this expected payoff across the propositions in $\mathcal{P}$ as a reformulation of Definition 2.8 incorporating probabilistic uncertainty.

In this section we consider a scenario in which agents receive indirect feedback about the accuracy of their beliefs in the form of payoff or reward obtained as a result of actions that they have taken on the basis of these beliefs. Furthermore, we assume that the closer that an agent's beliefs are to the actual state of the world then the higher their rewards will be on average. Hence, we use the payoff measure given in Definition 3.5 as a proxy for this process so that agent selection in the consensus formation process is guided by the payoff or quality measure of their beliefs. More specifically, we now investigate an agent-based system in which pairs of agents are selected for interaction with a probability that is proportional to the product of the quality of their respective beliefs. For modelling societal opinion dynamics this captures an assumption that better performing agents, i.e. those with higher payoff, are more likely to interact in a context in which both parties will benefit from reaching an agreement. In biological systems there are examples of a similar quality effect on distributed decision making. For instance, honeybee swarms collectively choose between alternative nesting sites by means of a dance in which individual bees indicate the direction of the site that they have just visited [48]. The duration of the dance is dependant on the quality of the site and this in turn affects the likelihood that the dancer will influence other bees. Artificial systems can of course be designed so that interactions are guided by quality provided that a suitable measure of the latter can de defined, as is typically the case in evolutionary computing.

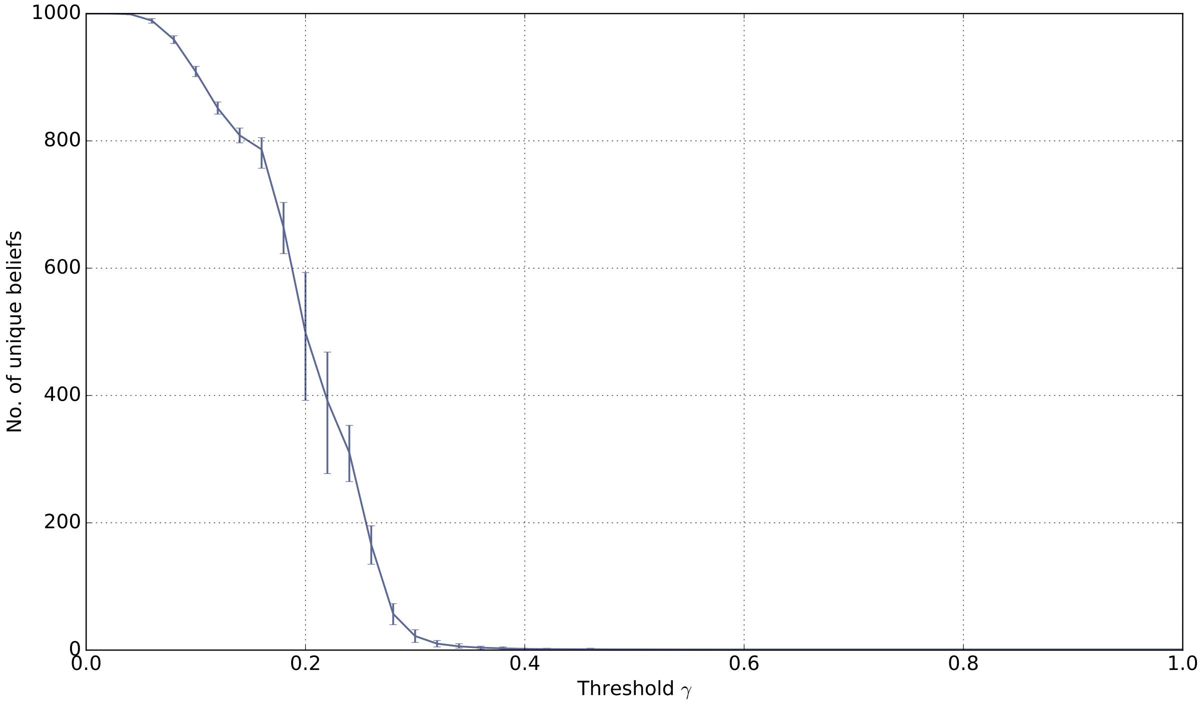


Figure 3.5: Number of unique beliefs after 50 000 iterations for varying inconsistency thresholds γ and $|\mathcal{P}| = 5$.

3.4.1 Results

We now describe the results from running agent-based simulations mainly following the same template as described in Section 3.3 but with an important difference. Instead of being selected at random, agents were instead selected for interaction with probability proportional to the quality value of their beliefs as given in Definition 3.5. These experiments therefore follow the same template as those in Section 2.3 but where beliefs are now both vague and uncertain. The true state of the world $\mathbf{v}^*$ was chosen at random from $\{0, 1\}^n$ prior to running the simulation and the payoff function f was then determined as in Definition 3.5. As in the previous sections the population consisted of 1 000 agents with initial beliefs selected at random from $\{(x, y) \in [0, 1] : x \leq y\}^n$. All results in this section relate to the language size $|\mathcal{P}| = 5$.

Figure 3.5 shows the mean number of unique beliefs for the consensus operator after 50 000 iterations plotted against the inconsistency threshold γ. For $\gamma \geq 0.5$ applying the consensus operator results in the population of agents converging on a single shared belief. Figures 3.6 and 3.7 show the average vagueness and entropy of the beliefs held across the population of agents at the end of the simulation. In Figure 3.6 we see that for $\gamma \geq 0.5$ the beliefs resulting from applying the consensus operator are crisp. Figure 3.7 shows that the mean entropy values decreases as γ increases resulting in an average entropy of 0 for $\gamma \geq 0.5$. Overall then, as in Section 3.3, for $\gamma \geq 0.5$ the population of agents converge on a single shared belief which is both crisp and certain.

Figure 3.9 shows the average quality of beliefs (Definition 3.5) at the end of the simulation, plotted against γ and given as the percentage of the maximum possible quality value. For $\gamma \geq 0.5$ the consensus operator converges on a single shared crisp and certain belief with a

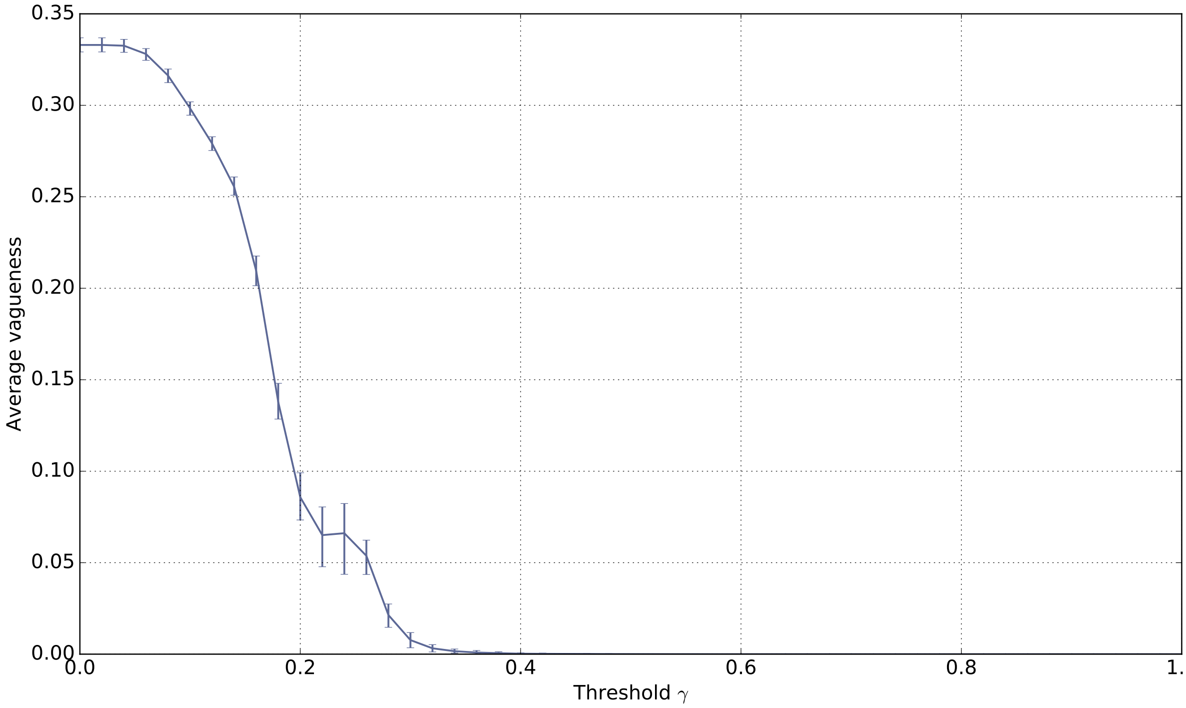


Figure 3.6: Average vagueness after 50 000 iterations for varying inconsistency thresholds γ and $|\mathcal{P}| = 5$.

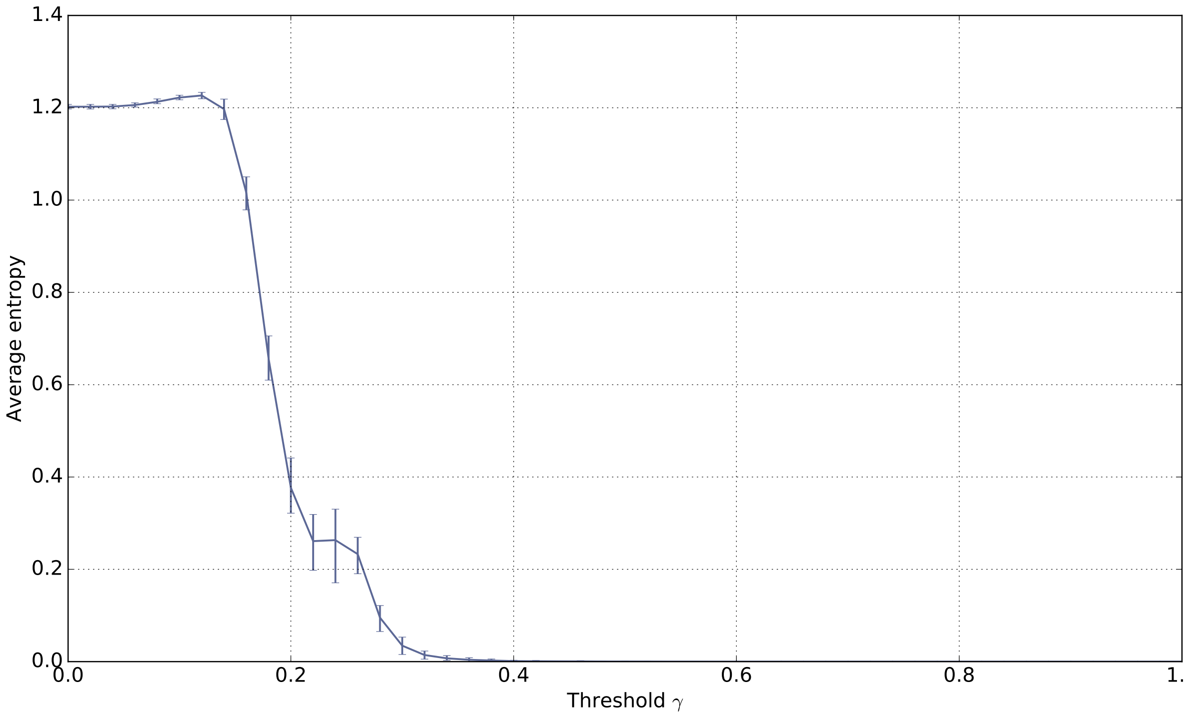


Figure 3.7: Average entropy after 50 000 iterations for varying inconsistency thresholds γ and $|\mathcal{P}| = 5$.

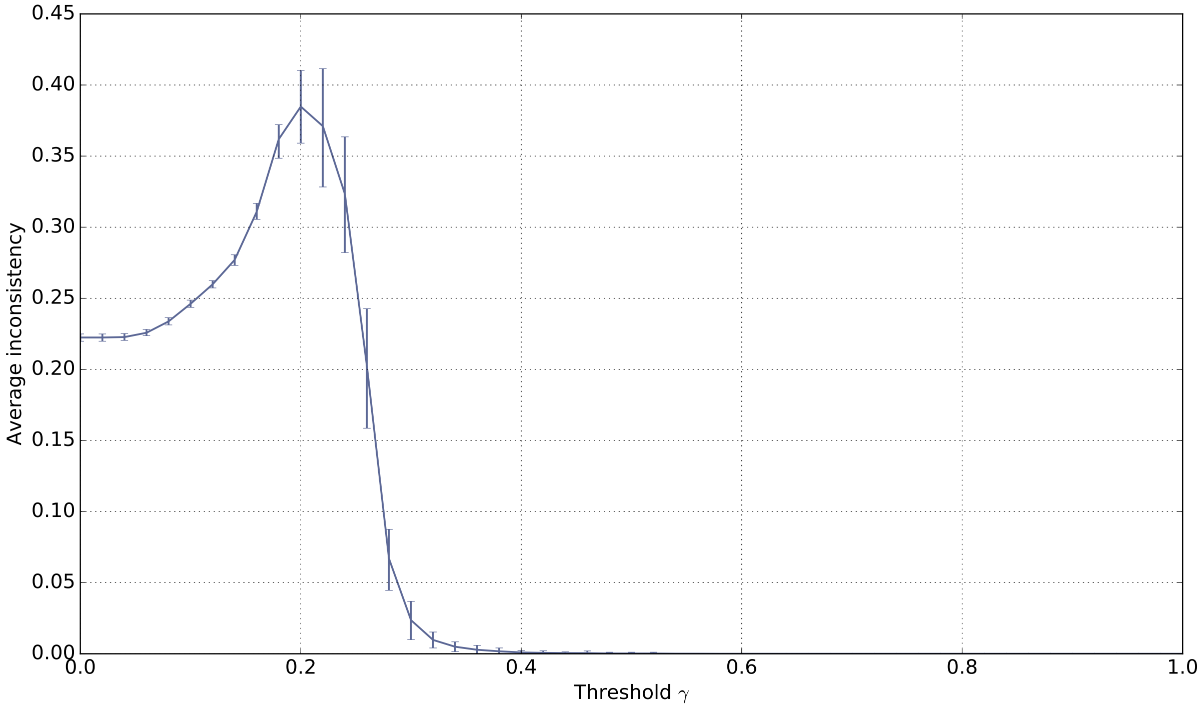


Figure 3.8: Average pairwise inconsistency after 50 000 iterations for varying inconsistency thresholds γ and $|\mathcal{P}| = 5$.

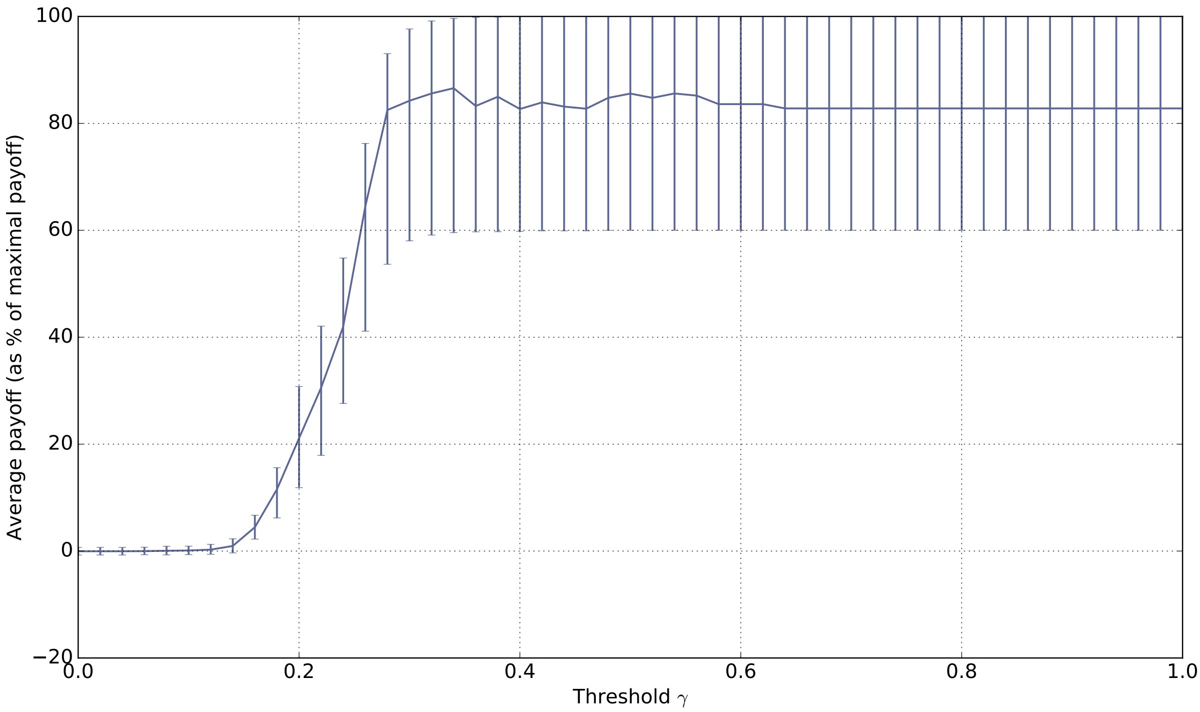


Figure 3.9: Average payoff after 50 000 iterations for varying inconsistency thresholds γ and $|\mathcal{P}| = 5$.

quality value which is on average over 80% of the maximum. Hence, unlike in Section 3.3 in which convergence can be to any of the 2^n crisp and certain beliefs at random, agent interactions guided by relative quality converge with higher probability to those beliefs amongst the 2^n that are the closest to the actual state of the world.

3.5 Combining consensus formation and evidential belief updating

Hegselmann and Krause [32] investigated an opinion model in which agents receive direct evidence about the state of the world, perhaps from an ongoing measurement process, as well as pooling the opinion of others with similar beliefs. The original Hegselmann-Krause model [31] involves real valued beliefs but this has been adapted to the case in which beliefs and evidence are theories in a propositional logic language [61]. The fundamental question under consideration is whether or to what extent dialogue between individuals, for example scientists, helps them to find the truth or instead whether they are better off simply to wait until they receive direct evidence? In this section we investigate this question in the context of vague and uncertain beliefs and where consensus formation is modelled using the combination operator in Definition 3.1. Direct evidence is then provided to the population at random instances when an individual is told the truth value of a proposition. That agent then updates their beliefs by adopting a compromise position between their current opinions and the evidence provided.

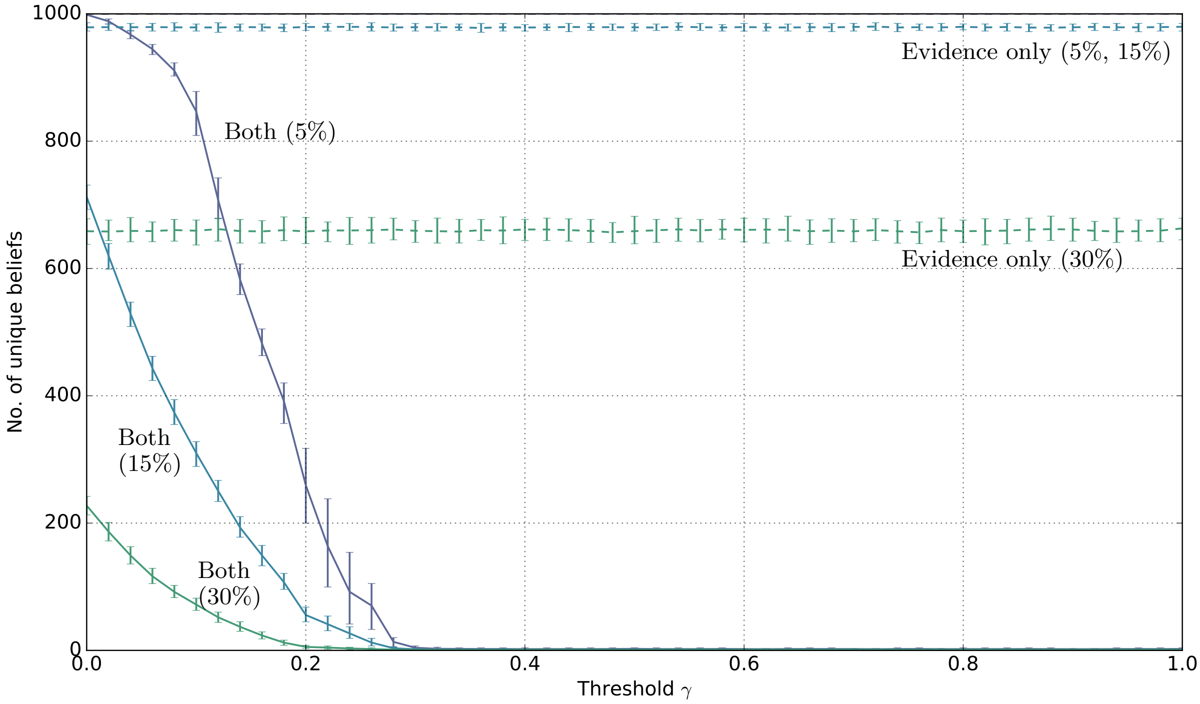
The simulations consist of 1 000 agents with beliefs initially picked at random from $\{(x, y) \in [0,1]^2 : x \leq y\}^n$ as before. Furthermore, the true state of the world $\mathbf{v}^*$ is picked at random from $\{0,1\}^n$ prior to the simulation and the payoff f calculated as in Definition 3.5. Each run of the simulation is terminated after 50 000 iterations and the results are averaged over 100 runs. At each iteration two agents are selected at random and apply the consensus operator (Definition 3.1) provided that the inconsistency level of their current beliefs does not exceed γ. Furthermore, at each iteration there is a fixed α% chance of direct evidence being presented to the population. In the case that it is, an agent is selected at random and told the value of $\mathbf{v}^*(p_i)$ for some proposition also selected at random from those in $\mathcal{P}$. The agent then updates their current beliefs $\boldsymbol{\mu}$ to $\boldsymbol{\mu}'$ where

$$\boldsymbol{\mu}' = \boldsymbol{\mu} \odot \boldsymbol{\mu}^* = \boldsymbol{\mu} \odot ((0,1), \ldots, (\mathbf{v}^*(p_i), \mathbf{v}^*(p_i)), \ldots, (0,1)) .$$

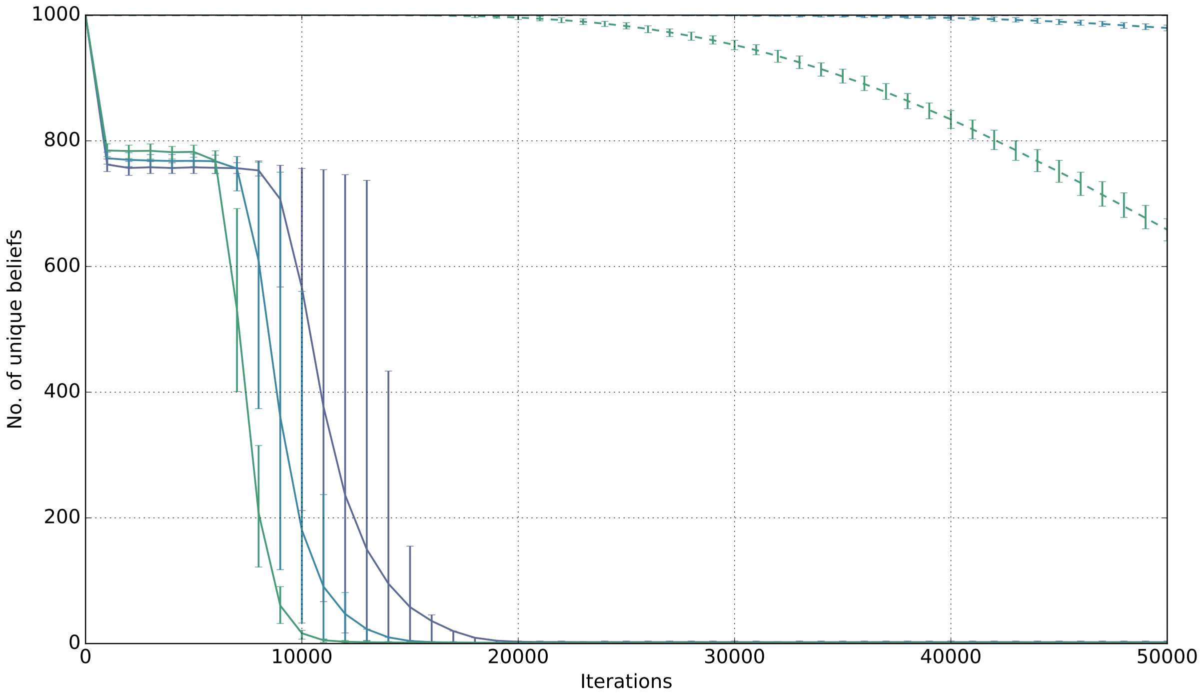
In other words, the agent adopts a new set of beliefs formed as a compromise between their current beliefs and the evidence, the latter being interpreted as a set of beliefs where $\boldsymbol{\mu}^*(p_i) = (\mathbf{v}^*(p_i), \mathbf{v}^*(p_i))$ and $\boldsymbol{\mu}^*(p_j) = (0,1)$ for $j \neq i$. That is they form consensus with an alternative opinion which is certain about the truth value of p_i and is neutral about the other propositions. Notice that in this case it follows from Definition 3.1 that $\boldsymbol{\mu}'(p_i) = (\overline{\mu}(p_1), 1)$ if $\mathbf{v}^*(p_i) = 1$, $\boldsymbol{\mu}'(p_i) = (0, \underline{\mu}(p_i))$ if $\mathbf{v}^*(p_i) = 0$ and $\boldsymbol{\mu}'(p_j) = \boldsymbol{\mu}(p_j)$ for $j \neq i$. The combined consensus and evidential belief updating approach can then be compared with simulations in which only the above belief updating model is applied and in which there is no consensus formation.

3.5.1 Results

In this section we focus on evidence rates of $\alpha = 5,\ 15$ and 30% and we assume that the language size is $|\mathcal{P}| = 5$. For these parameter settings Figure 3.10a shows that for $\gamma \geq 0.4$, all three cases in which evidential updating is combined with consensus formation converge on shared belief across the population. Furthermore, the higher the evidence rate α, the greater the convergence for any given threshold value γ. It is also clear from Figure 3.10a that combining consensus formation with evidential updating leads to much better convergence than evidence based updating alone. For instance, we see that for evidential updating alone it is only with an evidence rate of 30% that there is a large reduction in the number of distinct beliefs in the population after 50 000 iterations, with the population still containing over 950 different opinions for both the 5% and 15% rates. Furthermore, Figure 3.10b shows a typical trajectory for the average number of unique beliefs against iteration when $\gamma = 0.8$. Notice that after $25,000$ iterations all three of the combined models have converged to a single shared belief. In contrast the evidence-only approaches have still not converged after 50 000 iterations, where even with a 30% evidence rate there are still over 600 distinct opinions remaining in the population.



(a) Number of unique beliefs after 50 000 iterations for varying inconsistency thresholds γ.



(b) Number of unique beliefs over 50 000 iterations for $\gamma = 0.8$.

Figure 3.10: Unique belief results for $|\mathcal{P}| = 5$ and evidence rates $\alpha = 5, 15$ and 30%. The solid lines refer to evidential updating combined with consensus formation while the dotted lines refer to evidential updating only.

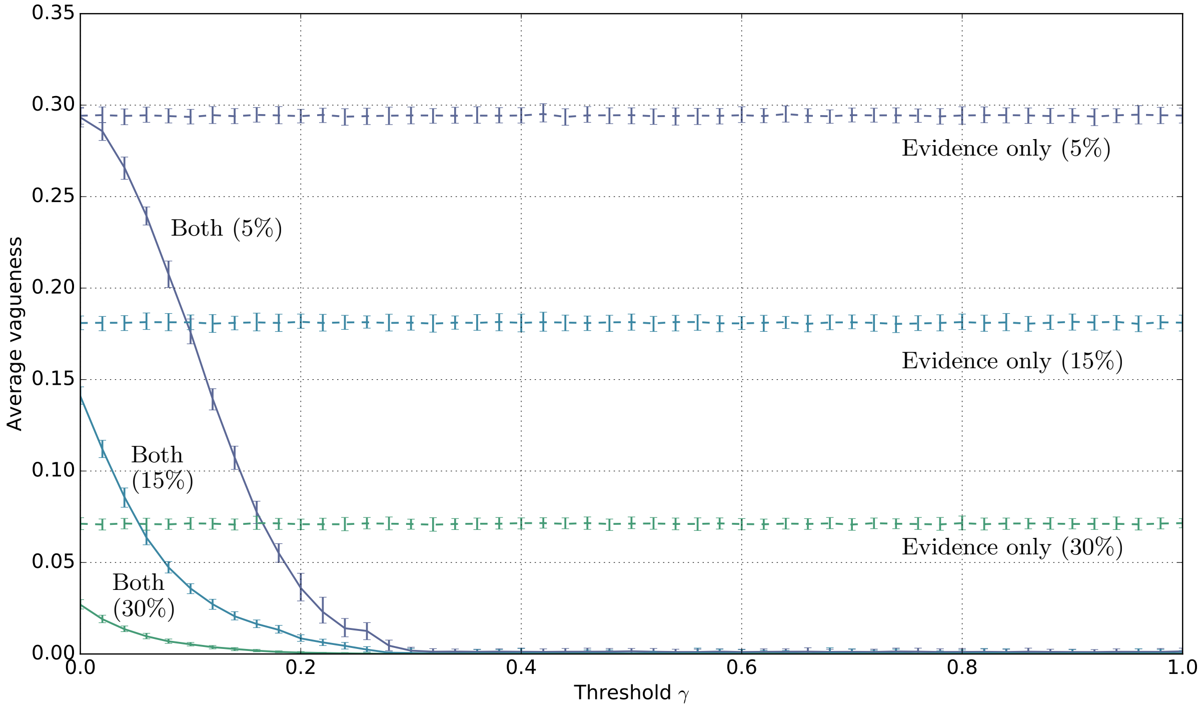


Figure 3.11: Average vagueness after 50 000 iterations for varying inconsistency thresholds γ, $|\mathcal{P}| = 5$ and evidence rates $\alpha = 5, 15$ and 30%. The solid lines refer to evidential updating combined with consensus formation while the dotted lines refer to evidential updating only.

Taken together with Figure 3.10a, Figures 3.11 and 3.12 show that, assuming a sufficiently high threshold value $\gamma \geq 0.4$, the combined consensus formation and updating approach results in convergence to a shared belief which is both crisp and certain. Again, increasing the evidence rate leads to a reduction in both the average vagueness and average entropy for any given threshold value and evidence based updating alone results in much higher values for the same evidence rate. The overall convergence of the population is also shown by the average pairwise inconsistency values in Figure 3.13. The convergence of the combined approach to a shared opinion for all evidence rates and thresholds $\gamma \geq 0.4$ is reflected in a zero average inconsistency level for this range of parameters. Notice, however, that for all evidence rates the average inconsistency for the combined approach has a peak value in the range $0 < \gamma < 0.4$, suggesting that there is some polarisation of opinion for thresholds in this range. For evidence updating only the level of inconsistency is relatively higher than for the combined approach for all evidence rates suggesting that there is a much higher level of disagreement remaining between agents after 50 000 iterations.

Finally, Figure 3.14 shows the average payoff values calculated as in Definition 3.5 and given as a percentage of the maximum possible value i.e. in this case 5. These values reflect the extent to which the population has converged to a set of beliefs close to the true state of the world. For each of the three evidence rates, given a sufficiently high threshold value, the combined approach results in an average payoff which is significantly higher than for evidential updating alone. Indeed for a 30% evidence rate and $\gamma \geq 0.3$ the combination of consensus formation and belief updating results in close to the maximum payoff value on average i.e. the population

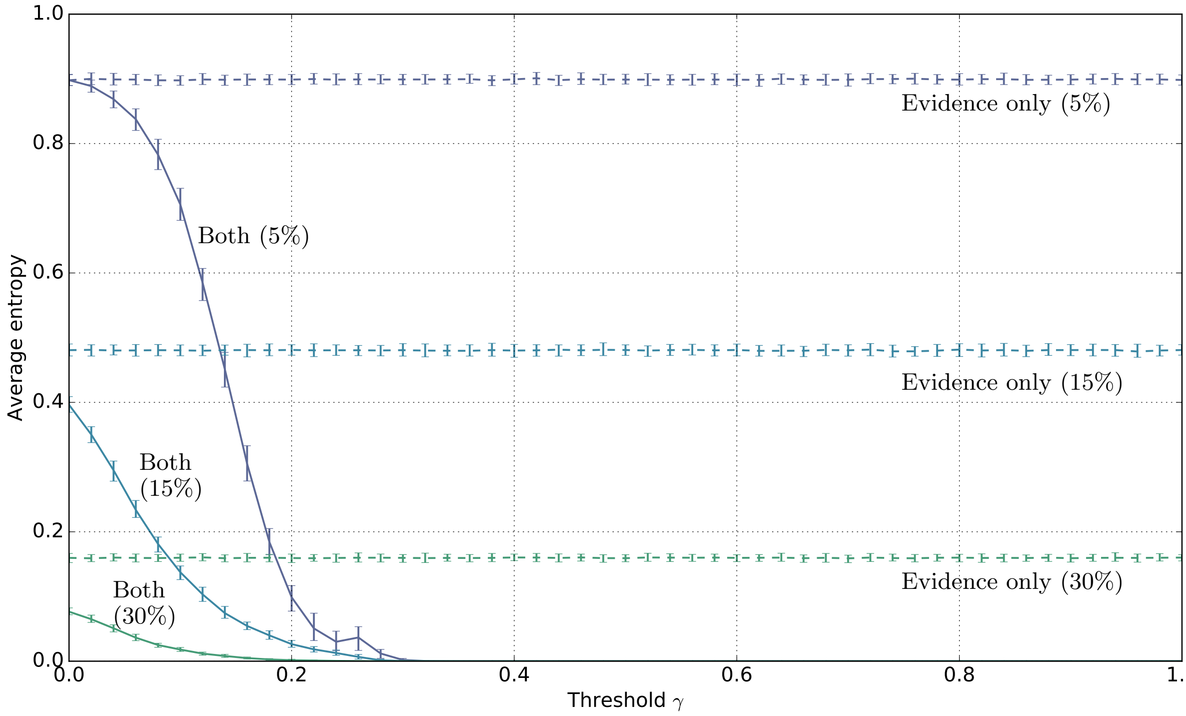


Figure 3.12: Average entropy after 50 000 iterations for varying inconsistency thresholds γ, $|\mathcal{P}| = 5$ and evidence rates $\alpha = 5, 15$ and 30%. The solid lines refer to evidential updating combined with consensus formation while the dotted lines refer to evidential updating only.

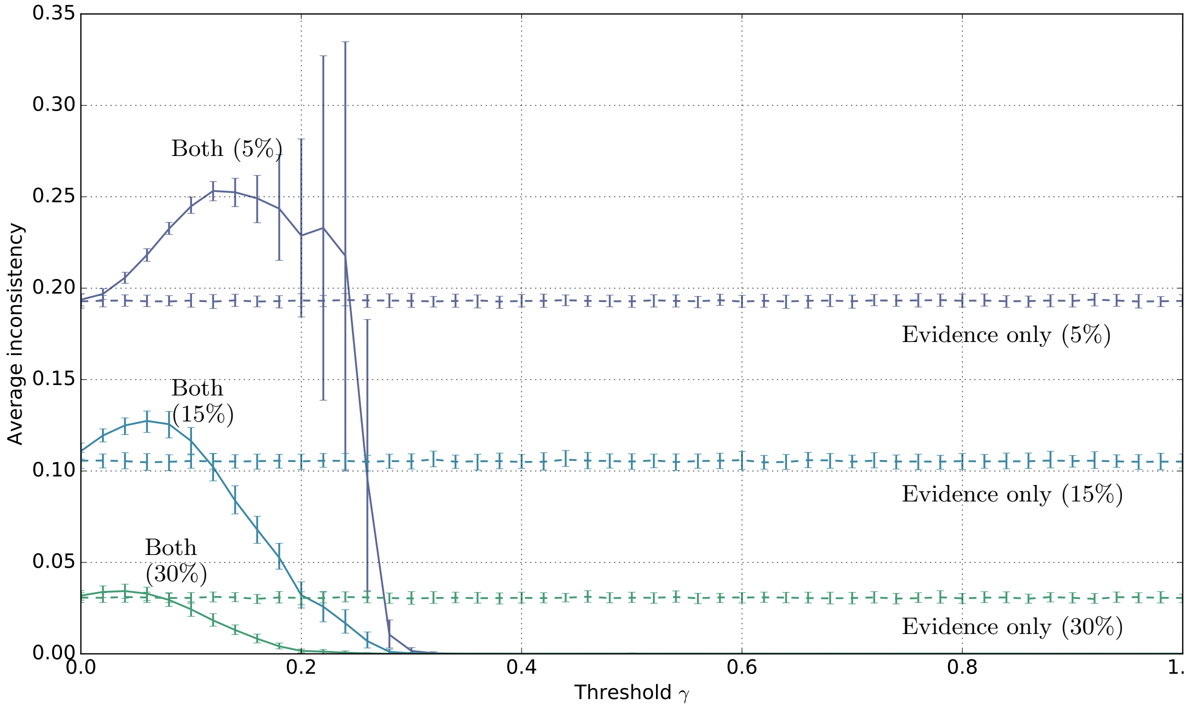


Figure 3.13: Average pairwise inconsistency after 50 000 iterations for varying inconsistency thresholds γ, $|\mathcal{P}| = 5$ and evidence rates $\alpha = 5, 15$ and 30%. The solid lines refer to evidential updating combined with consensus formation while the dotted lines refer to evidential updating only.

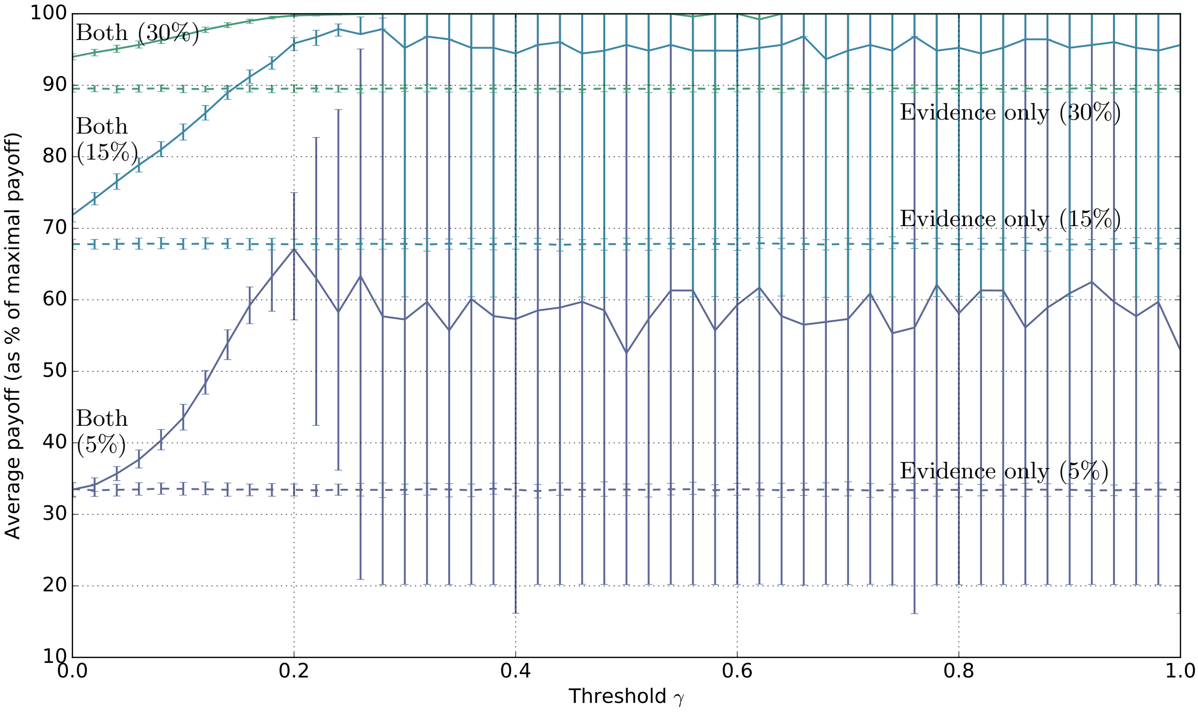


Figure 3.14: Average payoff after 50 000 iterations for varying inconsistency thresholds γ, $|\mathcal{P}| = 5$ and evidence rates $\alpha = 5, 15$ and 30%. The solid lines refer to evidential updating combined with consensus formation while the dotted lines refer to evidential updating only.

has learnt the state of the world with an average accuracy of close to 100%. Perhaps a more interesting result is that combining an evidence rate of 15% with consensus outperforms double the evidence rate of 30% for $\gamma \geq 0.6$. That is, with a 15% rate of evidence dissemination, the process of consensus propagates the evidence more effectively than if we double the evidence rate without consensus taking place. In comparison to the payoff-based experiments described in Section 3.4 we see that the payoff shown in Figure 3.5 when combining a 15% direct evidence rate with consensus formation and based on random interactions, is similar to that obtained when biasing agent selection based on belief quality (see Figure 3.9).

3.6 Conclusions

In this chapter we have investigated consensus formation for a multi-agent system in which agents' beliefs are both vague and uncertain. For this we have adopted a formalism which combines three truth states with probability, resulting in opinions which are quantified by lower and upper belief measures. The combination operator from Chapter 2 is extended in order to incorporate this additional uncertainty, as have the several measures that are used to analyse the convergence results of the model. In simulation experiments we have applied this operator to random agent interactions constrained by the requirement that agreement can only be reached between agents holding beliefs which are sufficiently consistent with each other. Provided that this consistency requirement is not too restrictive then the population of agents is shown to converge on a single shared belief which is both crisp and certain. While we had shown that agent populations converge to crisp beliefs in Chapter 2 under similar constraints, we have now shown additional convergence results for an enlarged population of 1 000 agents with uncertain beliefs. We have also studied smaller values of $|\mathcal{P}|$, including the single propositional variable case for completeness.

Furthermore, we have investigated the use of consensus as a means of propagating information through a distributed system of agents, where at each iteration and for a given evidence rate α, there is an $\alpha\%$ chance that a randomly selected agent shall receive evidence about the state of the world and then incorporate it into their current belief. We show that, if combined with evidence about the state of the world, either in a direct or indirect way, then consensus formation of the kind presented in this chapter results in better convergence to the truth than just evidential belief updating. This reaffirms the claims of Douven [22] and Kelp [23] in which the authors argue that evidence propagation is greatly improved when combining evidential updating with some form of belief compromise between agents, after highlighting several other models which also support this argument.

An avenue for future research in this context is to consider noisy evidence. Evidential updating is rarely perfect and, for example, experiments can be prone to measurement errors. An interesting question is therefore, how does the combined consensus formation and updating approach described in Section 3.5 cope with such noise? Later in Chapter 4 we apply our approach to distributed decision making scenarios such as, for example, in swarm robotics, where we consider another kind of robustness in our approach, namely robustness to malfunction.

Overall, these results provide some evidence for the beneficial effects of allowing agents to hold beliefs which are both vague and uncertain, in the context of consensus formation. However, in this chapter we have only studied pairwise interactions between agents, while in the literature it is normally intended that pooling operators should be used to aggregate uncertain beliefs across a group of agents [19, 20]. In Chapter 2 we investigated a form of group-wide belief updating and showed that this detracted from the consensus formation process in relation to increasing the average payoff of the population, although it did speed up the convergence process for larger groups. We have not repeated similar experiments in this chapter, however, due to the similarity of the results between both models as we felt that this similarity would be reproduced in group-wide consensus experiments. It is also not entirely obvious how we would develop a group model which takes into account the uncertainty of agents' beliefs, and so further work is needed in this area. Probabilistic pooling operators also take account of different weights associated with the beliefs of different agents and it would be interesting to investigate if this can be incorporated in our approach. It is our belief, however, that to assume agents possess preassigned weights about one another is unintuitive, though certain scenarios may invoke such a requirement of the system.

CHAPTER 4

Distributed Decision-Making

In preceding chapters we have proposed a model for consensus in multi-agent systems, extended the model by combining vagueness with epistemic uncertainty, and studied some of the system-level convergence properties. We have focused primarily on a pairwise model for combining beliefs, but have also proposed a model for combining beliefs in small groups of agents and have identified several limitations associated with this model, which was developed in order to achieve faster convergence by combining beliefs in larger numbers per iteration. In these simulation experiments, the only restriction on agent interactions is that the beliefs of the agents involved are relatively consistent with each other, where the latter is modelled using a threshold on the inconsistency measure (see Chapter 2, Definition 2.7). However, in robotic systems there are often additional constraints on interactions, the effects of which are disregarded in most theoretical models. For example, there are typically communication constraints according to which individuals can only communicate directly with others within their range of communication. This can be either due to hardware limitations, or a feature of the environment in which these robotic systems are deployed. Therefore, in this chapter we explore a model of consensus applied to robotic systems which includes these spatial constraints.

Decentralised methods are also considered to be more robust than their centralised counterparts because there is no single point of failure, but this is seldom tested [51]. In many applications it is crucial for robot swarms to be robust to a variety of different types of noise as well as hardware and software failure. In particular, the lack of calibration and the use of low-cost hardware can sometimes cause catastrophic failure [63]. In [81] five distinct ways are identified in which a swarm can be robust, including being tolerant to noise and uncertainties in the environment, or because it has no common-mode point of failure. The notion of robustness that concerns us here relates to 'individual robots who fail in such a way as to thwart the overall desired swarm behaviour'. For the best-of-n problem [56] (discussed in Section 4.1), the desired swarm behaviour is that of convergence to the best decision, and we will investigate the effect of malfunctioning robots with the potential to disrupt this desired behaviour by making decisions on the basis of random beliefs.

One way of building fault tolerance into robot swarms is to enable individual robots to detect faults in their neighbours so that they can compensate for them. This approach is referred to as *exogenous* fault detection [51]. For example, [9] propose an approach inspired by the synchronised flashing behaviour of fireflies in which each robot flashes by lighting up its on-board LEDs. Neighbouring robots then flash in synchrony unless they are malfunctioning. The non-periodic flashing of faulty individuals can then be detected by other members of the swarm. However, in the following we do not allow for exogenous fault detection and we assume that individual robots have no way of distinguishing between neighbours which are functioning correctly and those which are malfunctioning. Instead, robustness to the presence of faulty individuals should be inherent to the distributed decision-making algorithm employed. In this context we aim to study our model for distributed decision-making in large robot swarms and examine its robustness to the presence of malfunctioning robots when compared with a modified version of an existing two-valued approach: the weighted voter model [73].

In this chapter we deviate from a *symmetric* model for consensus, in which all agents involved in the consensus process adopt a newly formed belief, to an *asymmetric* model, in which agents are either in an updating state (and therefore update their beliefs), or in a signalling state, such that they do not update their beliefs, but instead signal their beliefs to other agents. This differs from the kind of subgroup consensus considered in earlier chapters in that while combinations remain pairwise, unlike the serial nature of symmetric pairwise combinations, a large number of combinations can occur in parallel due to the separation of the population into two discrete subgroups of agents, where the agents are not synchronised in their states, such that a portion of the population will be in the updating state, while the remainder will be in the signalling state. This can potentially increase speed of convergence without having to be concerned with the order in which agents combine beliefs.

Initially, we explore distributed decision-making in a multi-agent setting before applying this to robotic systems; specifically swarm robotics. Inspired by biological systems, a large number of robots are programmed with a set of simple updating and signalling rules, from which complex behaviour emerges at the population level. Specifically, we address the best-of-n problem in which a set of agents must choose between n alternatives where each choice differs in quality and there exists a preferred, or *correct*, choice. We develop a three-valued model for distributed decision-making based on the same consensus operator introduced in Chapter 2. Initially, we study the $n = 2$ case of the best-of-n decision-problem for a range of parameter values and compare our model directly to a Boolean model. Finally, we explore the robustness of both models in this setting to a particular type of malfunction, before extending these models to the $n > 2$ case.

4.1 Related work

There has been considerable interest in the 'best-of-n' decision problem in the literature [56, 65, 75] with a number of approaches heavily inspired by social-insects searching for potential nest sites e.g. honey bees and Temnothorax ants [4, 38, 48, 49, 54, 58, 66]. In this decision problem a population of individuals is concerned with identifying which is the best choice (e.g. nest site) and the individuals receive feedback on their choices based on the quality of the n alternatives. System convergence to the optimal choice is then driven by this feedback mechanism. For example, in [48] honey bees advertise their nest site of choice by dancing for a length of time proportional to the quality of that nest site. As such, honey bees that choose the best site are dancing for longer periods of time and in turn are recruiting more members of the swarm to favour their chosen site. Those dancing for sites of lower quality are less effective at recruitment and so support for these sites eventually declines.

There have been numerous studies applying the bio-inspired approaches to swarm robotics [2, 59, 71, 72] where small, affordable robots can be deployed in large numbers for carrying out distributed tasks. Distributed decision-making on this scale tends to greatly improve redundancy and robustness in such systems as faulty individuals are limited in their affects on the rest of the system so long as the majority remain functional. The work presented in this chapter is most closely related to the weighted voter model [73], itself an extension of the classic voter model [11, 33]. Network science methods are commonly applied to voter models, to understand the coupling between states of the agents and the dynamics of the network which connects them [5, 34]. In contrast to this previous work our proposed model will be three-valued and based on the consensus operator given in Chapter 2, Definition 2.5. For both the weighted voter model and the proposed three-valued model, a population of agents attempts to reach consensus about which is the best of n discrete choices by individuals sampling a portion of the population that are signalling their preference for one of the n choices, before updating their beliefs accordingly. In this setting, the voting is weighted by the number of individuals in the sample signalling for each of the n different sites, while the feedback mechanism is directly inspired by the same mechanism found in honey bee swarms [48, 65] where signalling times are directly proportional to the quality associated with each choice. For the weighted voter model, an agent randomly selects an agent that is signalling within its communication radius and adopts that agent's choice, before transitioning into the signalling state where it begins to signal for the newly adopted choice. In the proposed three-valued model, however, agents instead combine their beliefs for which choice they believe to be *correct* and, when an agent attempts to combine with an agent which is signalling for a conflicting choice, a compromise is adopted according to Definition 2.5. The intuition here is that an intermediate state prevents those signalling for lower quality choices from affecting the population signalling in favour of high quality choices. We see in [73] that the weighted voter model indeed has strong convergence properties, as we confirm later in this chapter, and we directly compare this model with the proposed three-valued model both in convergence speed and in accuracy of convergence, as well as the extent to which each model is robust in the presence of malfunctioning agents in the population.

In recent robotics literature, a general overview of swarm robotics can be found in [3]. More relevant to our focus, Valentini et al. [74] investigate the trade-off between convergence speed and accuracy in the context of the majority rule as applied to a swarm of 100 Kilobots, while in [60] the effects of spatiality are considered for a population of 150 Kilobots. Both studies are conducted in the context of the best-of-n decision problem. Other related work on robot swarms includes a honey bee nest-site inspired decision model implemented on Jasmine micro-robots [38]. An extensive overview of research on the best-of-n problem can be found in [75]. Finally, [59] presents a general model of decentralised decision-making for the best-of-n problem. Interestingly this also employs a third truth state representing 'uncommitted', and is perhaps inspired by the earlier work of Seeley et al [66] in which honey bees are observed to employ use of a 'stop signal' in which they revert from a signalling state to an uncommitted state. The updating model proposed, however, is inherently probabilistic with probabilities dependent on quality values. This is in contrast to our approach in which updating is a purely logical operation (as given in Table 4.1). Another significant difference concerns the case in which $n > 2$, for which according to [59] an agent must be in one of $n + 1$ states; one for each choice and an overall uncommitted state. However, in our approach there are $2^n - 1$ states which include the cases in which certain choices are ruled out by the agent but where it is uncertain about the remaining options.

4.2 Model

Overview. In this section we assume a best-of-n decision problem where $n = 2$ and associate each choice with a discrete quality value. The aim is for a population of agents to be able to reach a consensus (that is, to make a definitive decision) about which site is the best choice compared to all other alternatives, and that this process of consensus formation be efficient both in terms of time and accuracy. Where in Section 2.4 of Chapter 2 and Section 3.4 of Chapter 3 agents received feedback on their beliefs in the form of biased selection taking account of payoff, feedback now forms a more integral role in the decision-making process.

Unlike previous approaches to consensus discussed in this thesis, we divide the population into two states: a signalling state; and an updating state. Consensus is therefore asymmetric with only the agents in the updating state adopting a new belief. We also deviate from the previous approaches in that the problem has only a single, correct choice, and so at a propositional level it cannot be the case that both propositions 'site A is the best site' and 'site B is the best site' are true simultaneously. This is straightforward in our model for the $n = 2$ case, as it is similar to the single propositional variable case of the model introduced in Chapter 2, however we later consider cases for $n > 2$ in this chapter and highlight the difficulties that arise when extending beyond the case where $n = 2$.

4.2.1 Best-of-n decision problem

In this section we describe the decision-making procedure at the individual level, where each agent has the same state-transitioning behaviour, just as would individuals of an insect colony. This is the fundamental property that makes systems such as these scalable to large populations i.e. 1 000 or more individuals. Here we directly compare the three-valued model with the weighted voter model for the $n = 2$ case. It may be helpful to think of the weighted voter model as a Boolean version of the proposed three-valued model, where instead of applying the three-valued updating operator (Table 4.1), an agent in the updating state will simply adopt a signalling agent's choice as their own, having selected a signalling agent at random from within their communication radius.

	signalling agent		
updating agent	1	$\frac{1}{2}$	0
1	1	1	$\frac{1}{2}$
$\frac{1}{2}$	1	$\frac{1}{2}$	0
0	$\frac{1}{2}$	0	0

Table 4.1: Truth table for the three-valued updating operator.

Three-valued voter model

For the $n = 2$ case, a population of agents must decide between two choices labelled A and B. Each agent is then in one of three possible belief states: 0, meaning 'B is the best choice'; 1, meaning 'A is the best choice'; or $\frac{1}{2}$ meaning 'I have no preference between A and B', suggesting that the difference between the two sites is indeterminate. Each choice is associated with a fixed quality value in the form of a positive integer that agents receive as feedback once a choice has been made. Agents receive the feedback ρ_A for choosing A and ρ_B for choosing B. Once a decision is made, the agent begins to broadcast their current belief to all other agents within communication range for a number of iterations based on the feedback they received. After signalling their belief, the agent then transitions to an updating state in which they observe the broadcasts of other agents before updating its belief by randomly selecting a signalling agent within its radius of communication and applying the consensus operator in Definition 2.5. The updating agent's new belief is thus a function both of its current belief and that of the selected signalling agent. The difference between the models in Chapters 2 and 3 and the model we present here is that in this model only the updating agent adopts the new consensus belief. It should be noted that, in order to maintain parity between the simulation and robotic experiments, agents are asynchronous across the population with respect to their transitioning between states.

As before, the underlying intuition behind the consensus operator (the truth-table for which is shown in Table 4.1) is that the stronger belief dominates, as is the case where a belief of 1 (0) combines with a belief of $\frac{1}{2}$. Here, the stronger belief dominates the weaker belief with the updating agent adopting 1 (0) as their belief. The exception to this is when there exists a direct conflict between two beliefs, in which case the intermediate state of $\frac{1}{2}$ is adopted to form a compromise. It is through this method of compromise that agents are able to reach an agreement with the intention that a consensus can begin to form in a population with initially diverse and conflicting beliefs.

State transitions

In this model, there are 4 distinct states following the state transition diagram in Figure 4.1: a signalling state, S, in which agents broadcast their current belief for a length of time directly proportional to the feedback they received when they made their most recent choice; state U is the updating state in which agents update their beliefs based on a combination of their currently

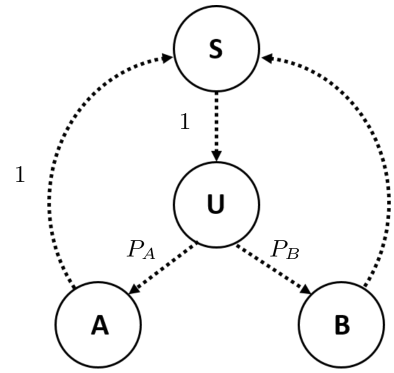


Figure 4.1: State transition diagram for the three-valued voter model, with states S (signalling), U (updating), A and B (choose A and B respectively).

held belief and the belief of a randomly selected agent in the signalling state; and states A and B corresponding to the agents choosing A and B respectively. Those agents in state A receive the feedback ρ_A while those in state B receive ρ_B. Choices are made immediately after agents update their beliefs and feedback is received thereafter, before agents transition back into the signalling state.

Given the three-valued updating operator in Table 4.1, we can identify the probabilities for choosing A and B, denoted by P_A and P_B respectively, based on the belief states of signalling agents within communication range: Letting Δ_i denote the proportion of other signalling agents in an updating agent's communication radius with belief state i, for $i = 0, \frac{1}{2}$ or 1, the probabilities that the agent will choose A or B after updating are given by

$$P_A = \begin{cases} \Delta_1 + \Delta_{\frac{1}{2}} + \frac{1}{2}\Delta_0 : \text{current belief} = 1, \\ \Delta_1 + \frac{1}{2}\Delta_{\frac{1}{2}} : \text{current belief} = \frac{1}{2}, \\ \frac{1}{2}\Delta_1 : \text{current belief} = 0, \end{cases}$$

$$P_B = \begin{cases} \frac{1}{2}\Delta_0 : \text{current belief} = 1, \\ \Delta_0 + \frac{1}{2}\Delta_{\frac{1}{2}} : \text{current belief} = \frac{1}{2}, \\ \Delta_0 + \Delta_{\frac{1}{2}} + \frac{1}{2}\Delta_1 : \text{current belief} = 0. \end{cases}$$

It should be noted that we have not included any exploration state as modelled in [74]. We choose instead to focus exclusively on the decision-making process that follows from the exploration of the swarm, simplifying the model by providing immediate feedback when the agent has updated their belief and chosen either A or B. From this abstraction, the absolute times to convergence are considerably reduced, thus enabling multiple runs of both simulation and embodied experiments. For example, a physical experiment on the Kilobot platform takes roughly 4 minutes, dramatically less than the 90 minutes required for each experiment in [74]. This also increases the number of experiments that can be run in each charging cycle, enabling us to deploy a swarm of 400 Kilobots for the experiments that follow.

4.3 Kilobot swarms

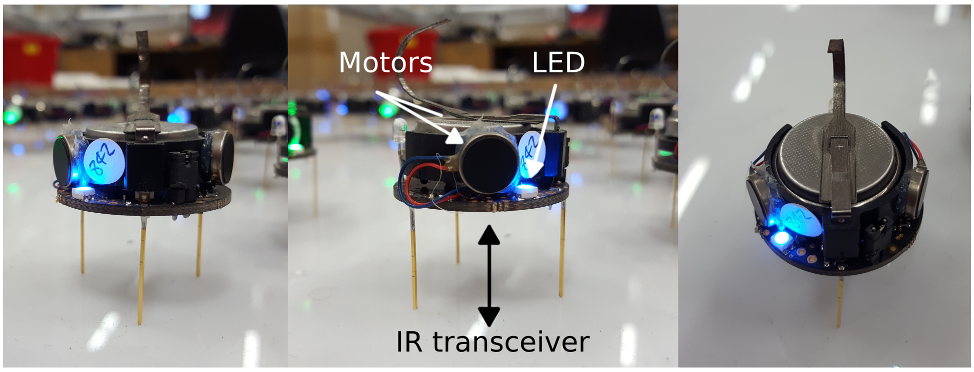


Figure 4.2: A Kilobot robot with an LED light for signalling, an underside IR transceiver for communication and side-mounted motors for movement.

Here we use Kilobots as a robotic platform for studying swarm decision-making. These are tri-pedal robots 33mm in diameter and 34mm tall, specifically designed to interact in large collectives, or 'swarms' [62]. Each Kilobot is an independent unit possessing, amongst other features, two motors providing left/right orientation and forward motion, an RGB LED indicator for signalling to an observer (e.g. an overhead camera) and an infrared transceiver (see Figure 4.2). Kilobots have a communication range of approximately 10 cm, over which they can send and receive messages of up to 9 bytes in length. However, the simulator allows for communication radii exceeding this limit and we exploit this feature to explore a range of communication radii r between 0 and 20 cm. Given that the number of Kilobots and the size of the arena are fixed, r can serve as a proxy to allow us to vary the density of Kilobots involved in the updating process. We consider this effect in Section 4.4 (Figure 4.8). Alternatively, by varying r we can also study directly the robustness of the two algorithms to different constraints on communications as might be relevant to different robotic platforms.

For both the simulation (Section 4.4) and embodied experiments (Section 4.6), a swarm of 400 Kilobots are deployed in a square 1.2m^2 arena. Whilst in the signalling state S, Kilobots move randomly[1] by either turning left or right, moving forward, or remaining stationary, i.e. at each time one of these 4 options is chosen with equal probability. At initialisation Kilobots are distributed randomly across the arena but then, as a result of random motion, they may collide and cluster together. Simulations are implemented according to the state transition diagram shown in Figure 4.1.

As described in Section 4.2 the experiments only model the mixing and information sharing part of the decision process, and the Kilobots do not visit specific physical locations or take other actions on the basis of their current beliefs. Instead, feedback is received immediately on the basis of their latest choice. While this is clearly a simplification, we believe that it still allows us to explore properties of the decision-making algorithms thanks to the reduction in run-time which allows us to repeat experiments multiple times.

[1]Due to the use of uncalibrated Kilobots, movement speed varies across the population.

4.4 Simulation experiments

In this section we describe experiments in which the Kilobots are simulated in a virtual environment and interact at random, iteratively updating their beliefs using the rules described in Section 4.2, in order to form a consensus about which is the best choice, A or B. Here we assume that choice A is of higher quality than choice B with respective quality values $\rho_A = 9$ and $\rho_B = 7$. These values represent two choices which are of similar quality but where A is slightly preferable to B.

4.4.1 Kilobot simulator

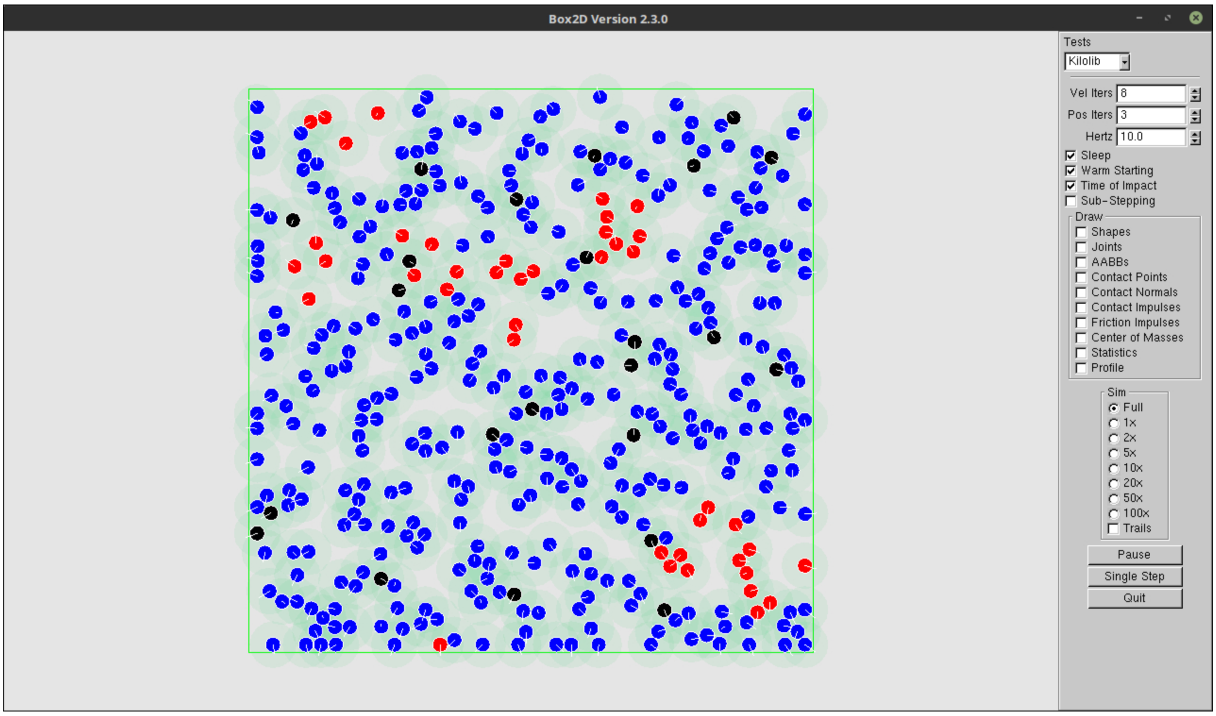


Figure 4.3: The Kilobot simulation environment used for all simulation experiments [35].

We employ a Kilobot simulation environment shown in Figure 4.3 which captures many of the physical properties of a Kilobot swarm including motion, direction, collisions, and communication between robots. The simulator is built on the open-source Box2D physics engine[2] and also implements the same API[3] as used on the actual robots which makes it easier to transfer code from simulations to the real world. In effect, the simulator allows for the development of a testing environment where you set the arena dimensions, swarm size, and where you may also alter properties of the robots if desired. You can then run repeated experiments using the same experimental setting each time. For example, in this section we explore communication radii which far exceed the capabilities of the physical Kilobots, but we do so in order to study the effect that an increased communication radius has on the convergence properties of the swarm. We then implement both the three-valued model and the weighted voter model in simulation

[2] `http://www.box2d.org/`
[3] https://www.kilobotics.com/docs/index.html

where we can stream data from each Kilobot during the experiments to output files. We can then analyse run-time performance of the models with much greater accuracy than we could with the physical Kilobots given that LED indicators are the only straight-forward method of data extraction; accomplished for large numbers via video processing.

The **black** Kilobots are in the updating state U. The **red** Kilobots are signalling for choice B while the **blue** Kilobots are signalling for choice A. In this particular experiment we see pockets of Kilobots that remain in favour of choice B despite the majority of the population having shifted in favour of choice A. The **green** covering the majority of the arena indicates the area in which signalling Kilobots are broadcasting (this effect is more apparent with Kilobots at the edges of the arena where the circular range of communication can be seen more clearly).

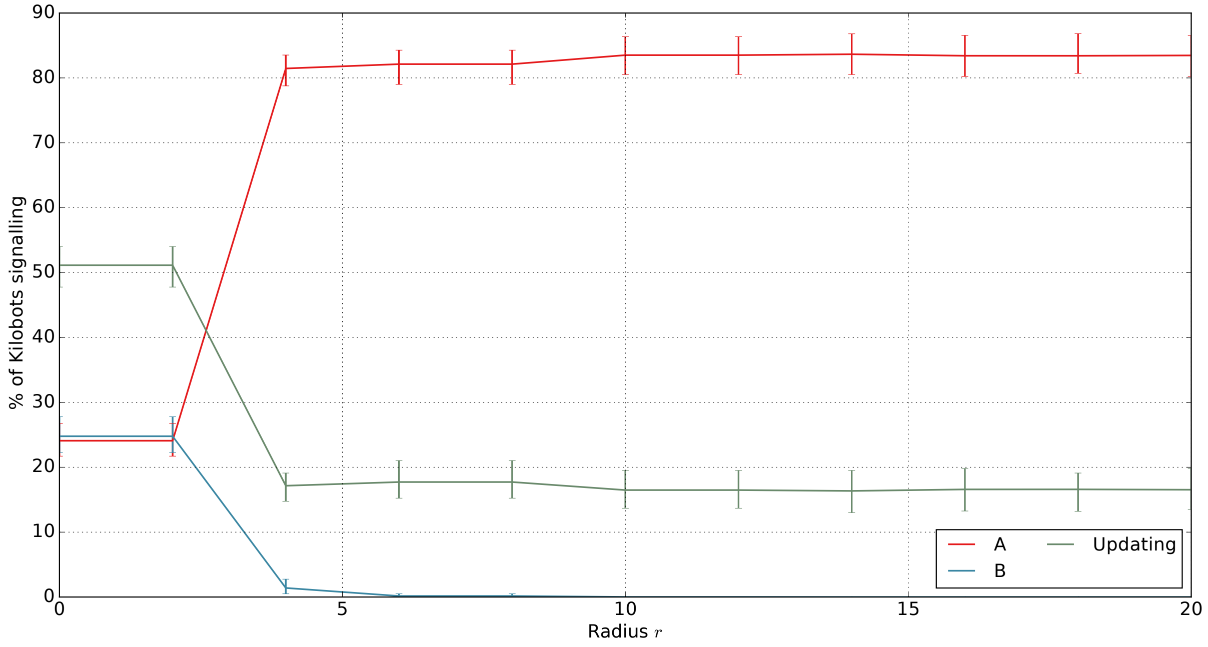
4.4.2 Results

In this section we present results for both the weighted voter model [73] and the three-valued model, comparing their convergence to consensus for different communication radii. Results are averaged over 50 independent runs, each of which terminate after 1 000 iterations; we found this to be a sufficient number of iterations for the system to reliably reach a steady state in which a consensus is formed. We begin by considering consensus in swarms of stationary Kilobots where robots are placed uniformly throughout the simulated arena. Here we aim to study the effect of motion on the consensus dynamics of the swarm by comparing the resulting decisions of stationary Kilobots with those undergoing random motion.

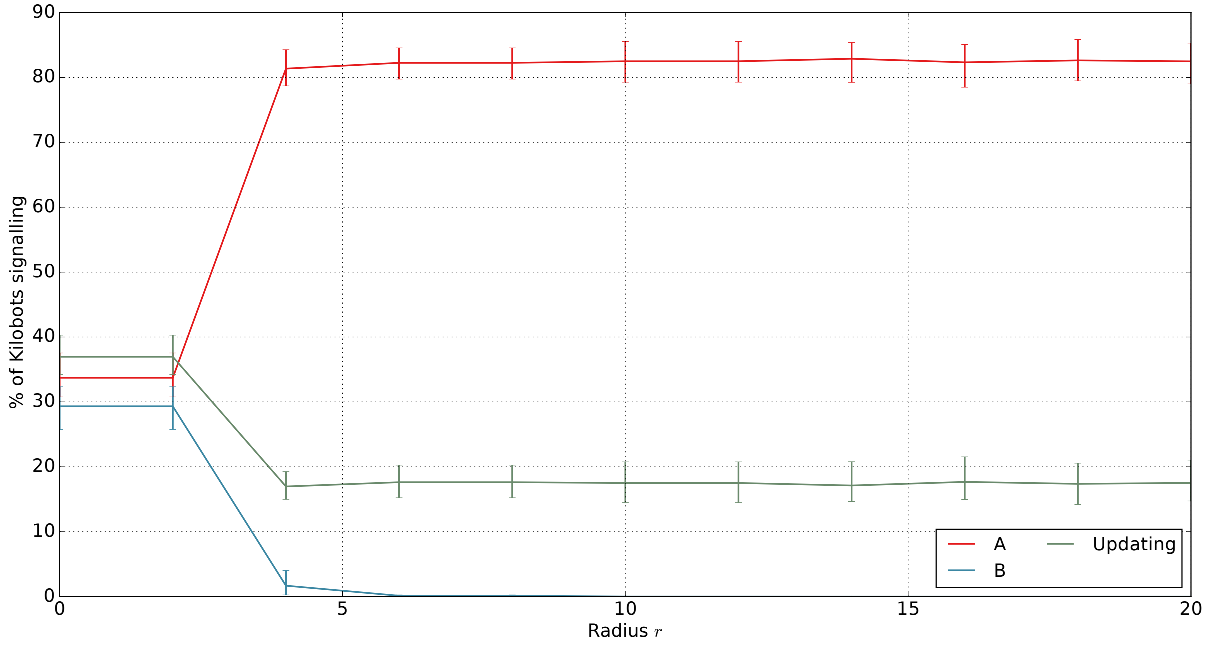
Stationary Kilobots

Figures 4.4a and 4.4b show the percentages of Kilobots in the signalling and updating states after 1 000 iterations for a range of communication radii, for both the weighted voter and three-valued models respectively. In this figure, the lines labelled A and B respectively refer to the percentage of Kilobots who are currently in the signalling state having previously chosen A or B prior to entering that state. For $r < 4$ cm we see that the population remains equally split between choices A and B; this is due to a combination of the Kilobots being stationary during the decision-making process and an insufficient communication radius that prevents Kilobots from being able to send/receive messages from their initial positions. Without a sufficiently large radius of communication, Kilobots in the updating state are therefore unable to receive messages from those in the signalling state, and so no updating takes place. The resulting effect is seen more clearly in Figure 4.5, where 0 messages are received by the population until the communication radius is extended to 4 cm, at which point Kilobots begin receiving 2 messages on average. It should therefore be expected that the initial positioning of stationary Kilobots would affect results for different communication radii r, but that this might be avoided by introducing random motion as in the following section.

For $r \geq 4$ cm we see an immediate shift in convergence of the swarm as the population adapts to the increase in communication. Although only 2 messages on average are being received by Kilobots in the updating state, it is sufficient for the population to converge in favour of choice A. For both the weighted voter model (Figure 4.4a) and the three valued model (Figure 4.4b) this corresponds to a partial consensus of more than 80% of Kilobots choosing to signal for choice A and around 2% signalling for choice B, on average, with the remaining Kilobots in the updating state. The extent of convergence continues to increase as r increases, reaching a population-wide consensus in choice A on average for $r \geq 6$ cm for both models.



(a) Weighted voter model.



(b) Three-valued model.

Figure 4.4: Percentage of stationary Kilobots signalling for choices A and B at steady state for different communication radii r.

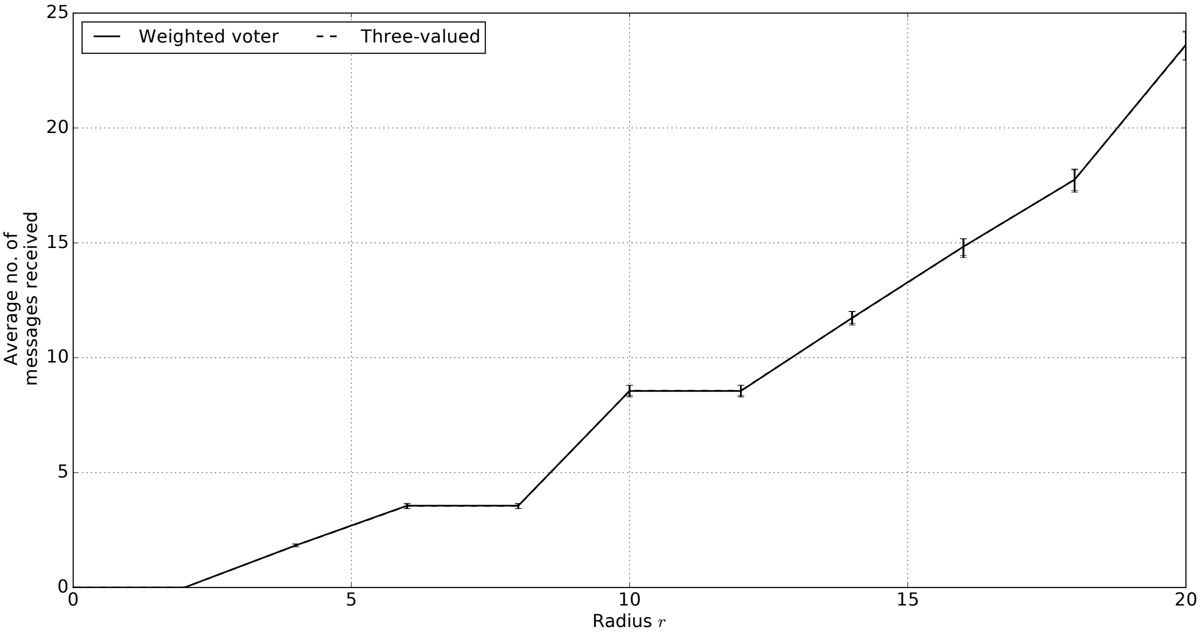


Figure 4.5: Average number of messages received by the Kilobots at steady state for different communication radii r.

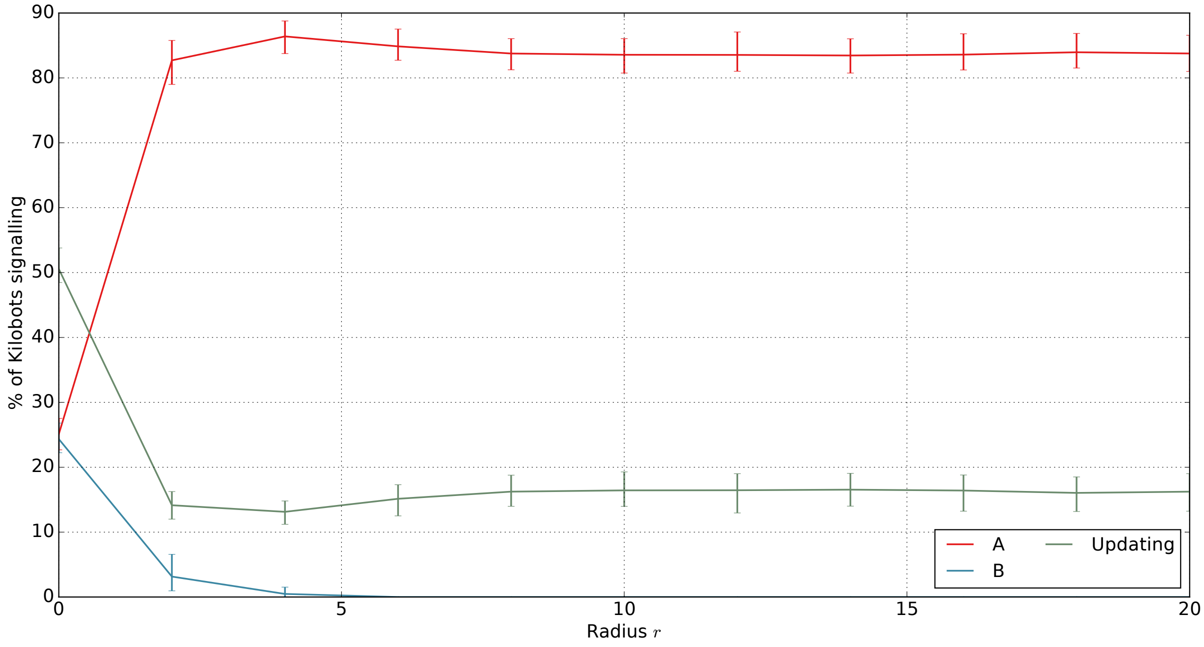
Moving Kilobots

Following the results for experiments where Kilobots remain stationary, we now study experiments in which Kilobots perform a random walk through their environment. They do so by selecting one of four states of motion: stationary; move forward; turn left; and turn right. Each robot then selects one state uniformly at random from these four states at each iteration. As in Figure 4.4, Figure 4.6 shows the percentages of Kilobots in the signalling and updating states after 1 000 iterations for a range of communication radii, for both the weighted voter model and the three-valued model. As before, the lines labelled A and B respectively refer to the percentage of Kilobots who are currently in the signalling state having previously chosen A or B prior to entering that state. For $r \geq 5$ cm we see that a clear majority have chosen A for both the weighted voter and the three-valued models.

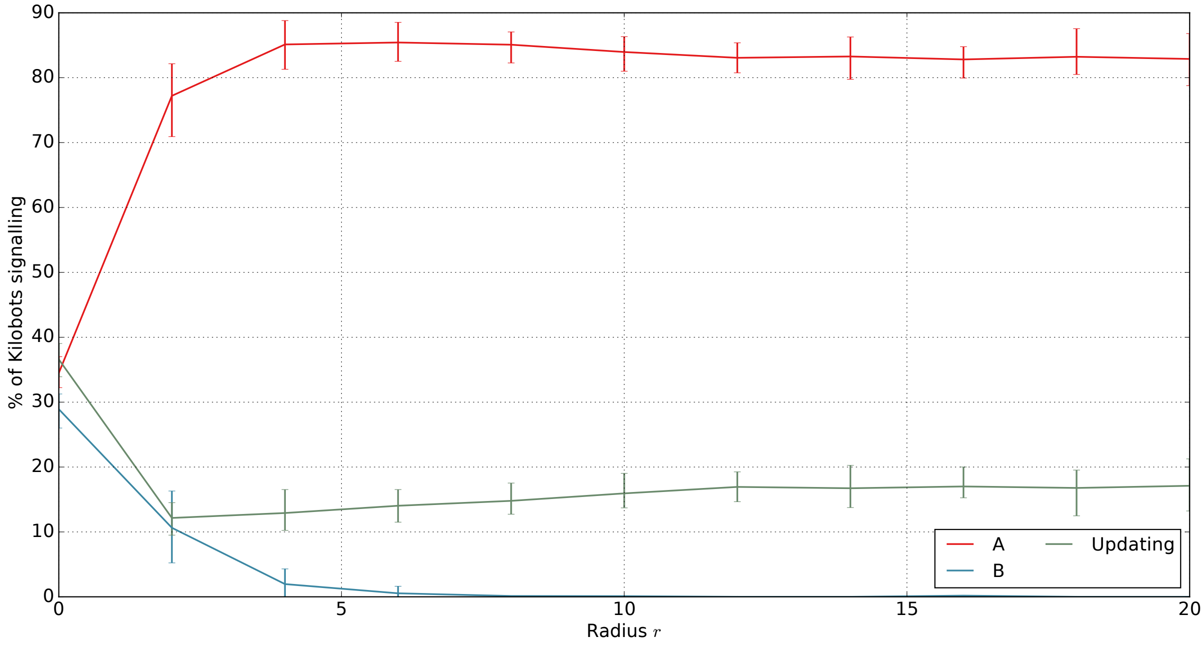
A more direct way of measuring convergence is to evaluate the average belief state of the Kilobots. In a population of k individuals we define the average belief state in a given iteration as follows: Let B_i denote the belief state of agent i for that iteration, then;

$$\text{average belief state} = \frac{1}{k} \sum_{i=1}^{k} B_i$$

This corresponds to a weighted average of 0 and 1 in the weighted voter model and of 0, $\frac{1}{2}$ and 1 in the three-valued model. For the weighted model there is a direct relationship between the average belief state and the percentage of the population choosing either A or B. This is because an agent chooses $A(B)$ if and only if their belief state is $1(0)$. For the three-valued model, however, the relationship between these two measures is less direct since, while an agent will definitely choose A when in belief state 1, they may also choose A (with probability 0.5) when in the intermediate belief state $\frac{1}{2}$. Hence, on average we would expect the percentage of agents choosing A to be proportional to the number of agents in belief state 1 plus 50% of the number in belief state $\frac{1}{2}$.



(a) Weighted voter model.



(b) Three-valued model.

Figure 4.6: Percentage of Kilobots in motion signalling for choices A and B at steady state for different communication radii r.

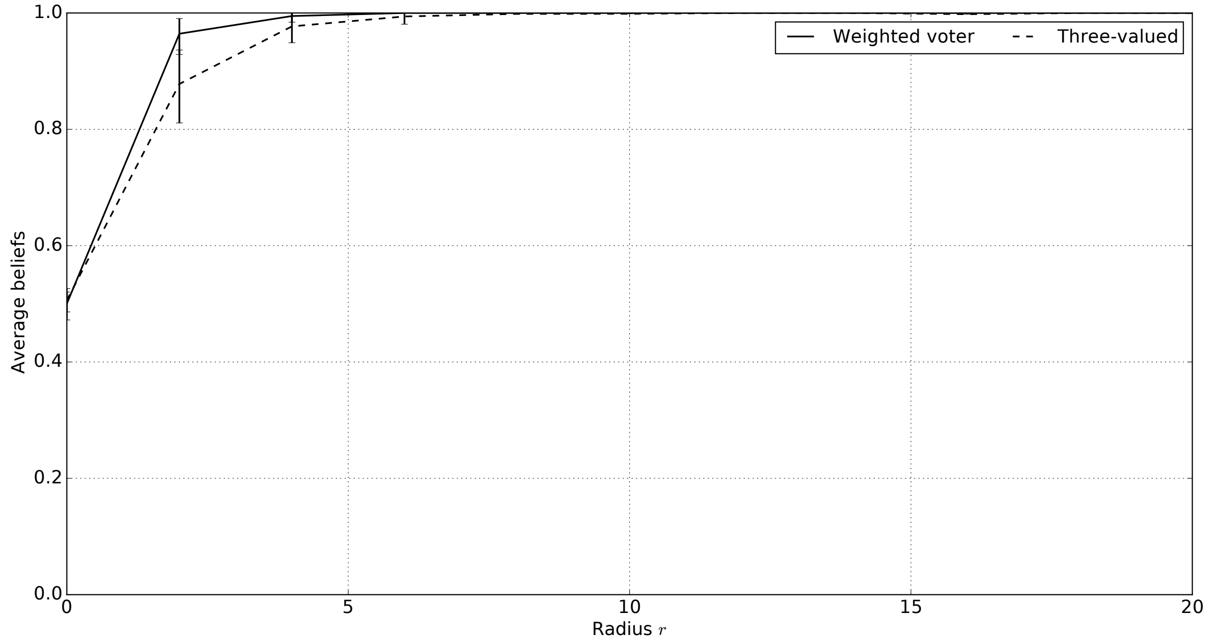


Figure 4.7: Average belief states at steady state for different communication radii r.

The average belief states are shown in Figure 4.7 where for $r \geq 5$ cm we can see that both models result in almost all Kilobots adopting the belief state 1 after 1 000 iterations. More precisely, for $r = 10$ cm the average belief states for the weighted voter model and the three-valued model after 1 000 iterations are 1.00 and 0.99 respectively. It is interesting to note that for the three-valued model the intermediate truth state is also totally abandoned, suggesting that there is convergence to total certainty that A is the best choice. This convergence to non-vague belief states reflects the kinds of convergence seen for the models studied in Chapters 2 and 3. The average number of messages per unit time received by each Kilobot in the updating state as a function of communication radius is shown in Figure 4.8. Notice that for $r = 5$ cm Kilobots receive just under 2 messages per unit time suggesting that both algorithms are robust to a relatively low population density of Kilobots.

If we measure quality of convergence either by the percentage choosing A or by the average belief state, then Figures 4.7 and 4.9 all suggest that convergence to A is slightly better for the weighted voter model than for the three-valued model, although the difference is very small for $r \geq 5$ cm. Furthermore, speed of convergence also appears to be faster for the weighted voter model. For example, Figure 4.9 shows the trajectory of average beliefs against iterations for both models when the communication radius is 10 cm. In this case the weighted voter model converges after about 200 iterations, while the three-valued model needs around 600 iterations to converge.

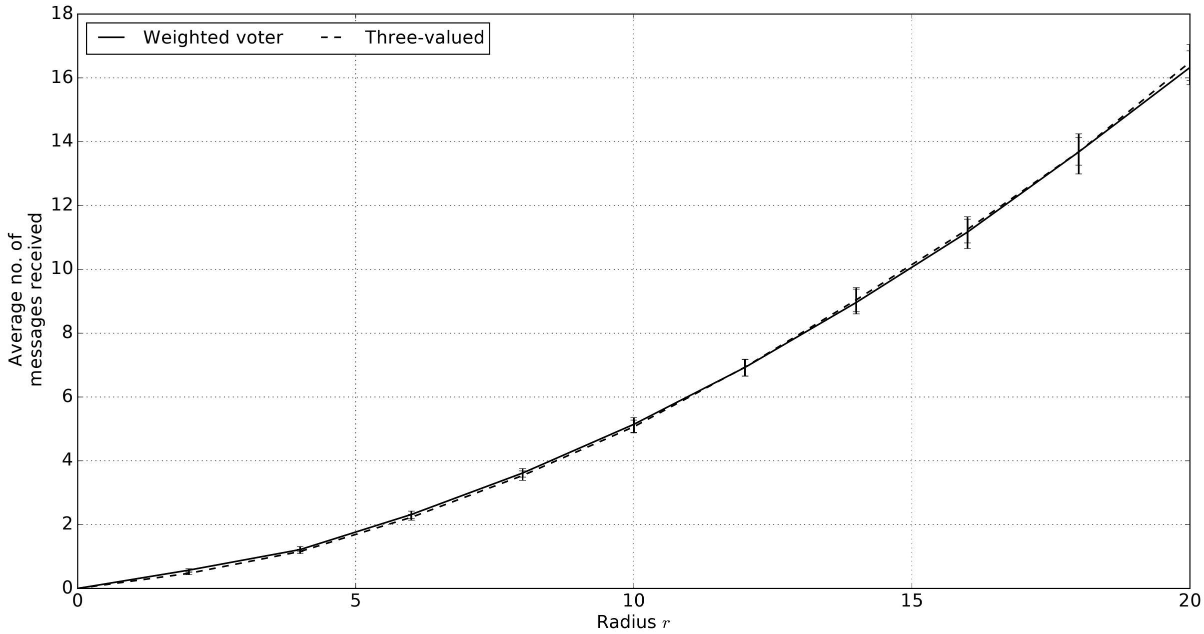


Figure 4.8: Average number of messages received by the Kilobots at steady state for different communication radii r.

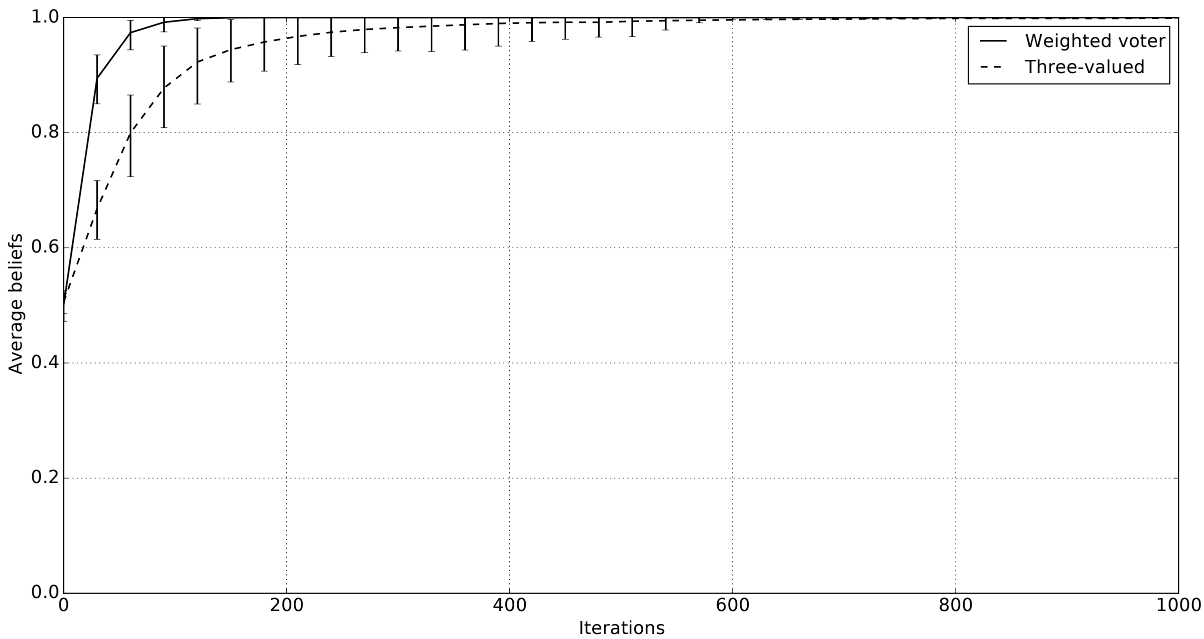


Figure 4.9: Average belief states against iterations for a communication radii r of 10 cm.

4.5 Robustness experiments

We now use the simulation environment to investigate the robustness of both models to the presence of malfunctioning Kilobots amongst the population. We know that, typically, distributed systems provide additional robustness in decision-making and similar decentralised processes when compared with their centralised counterparts. However, distributed systems are still subject to several types of malfunction that can occur at the individual level and may propagate throughout the population, so it is crucial that systems be robust to common types of malfunction. Given that we do not include exploration as part of the decision-making process, we choose to model malfunction occurring in the signalling/updating states.

In the updated model, we assume that a certain percentage λ of the Kilobots malfunction by selecting their beliefs at random. This allows us to simulate malfunction arising from either a signalling error, where a malfunctioning Kilobot is unable to signal their belief correctly and so appears to be signalling at random, or from an updating error, where a Kilobot incorrectly updates their internal beliefs based upon the received messages from signalling Kilobots.

4.5.1 Adapting the model to the presence of malfunctioning agents

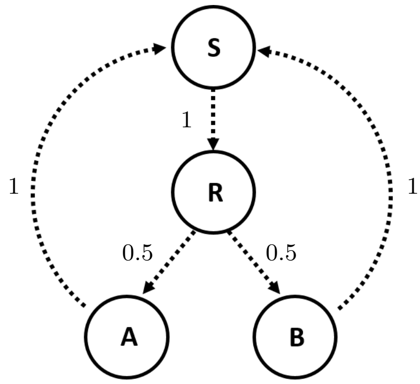


Figure 4.10: State transition diagram for malfunctioning Kilobots. R denotes randomize beliefs.

In the state transition diagram in Figure 4.10, R refers to a state in which the Kilobot simply selects its new belief state at random by picking uniformly from $\{0, 1\}$ in the case of the weighted voter model and from $\{0, \frac{1}{2}, 1\}$ for the three-valued model. Consequently, for both models there is then a probability of 0.5 that they will choose either A or B and receive the associated feedback value. As for functioning Kilobots, malfunctioning Kilobots then enter the signalling state and remain there for time ρ_A or ρ_B depending on their latest choice. We have adopted this particular model of malfunction as one which is likely to disrupt convergence to the desired belief state, by broadcasting randomized belief states to functioning Kilobots when the latter are updating their beliefs.

4.5.2 Results

Figure 4.11 shows the average belief states after 1 000 iterations for the weighted voter and the three-valued model respectively, with different percentages of malfunctioning agents (i.e.

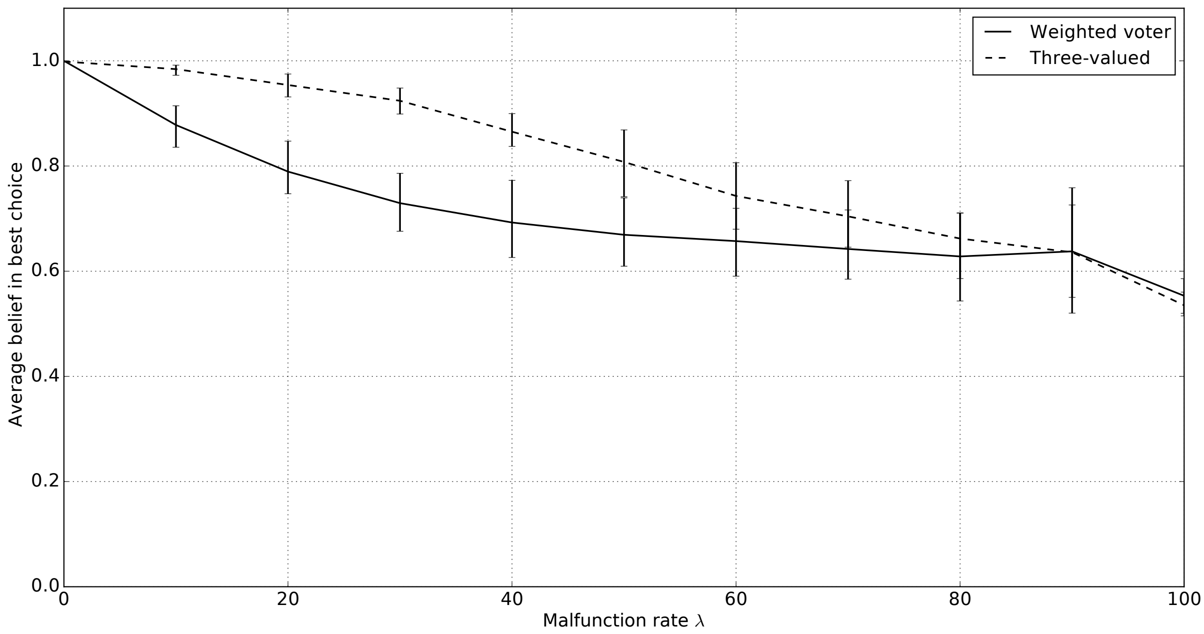


Figure 4.11: Average belief states against malfunction rates λ for a communication radii r of 10 cm.

$\lambda \in [0, 100]$) for a communication radius of 10 cm. Here the belief states are averaged across functioning Kilobots only[4], as the population will never appear to fully converge while the randomly-signalling, malfunctioning agents are included. From these figures it is apparent that the three-valued model is more fault tolerant than the weighted voter model in that it achieves average belief state values closer to 1 for each of the values of λ. For example, given a communication radius of 10 cm and assuming that 10% of the population is malfunctioning, then the three-valued model converges to an average belief state of 0.99 in the highest-value choice while the weighted voter model converges to an average belief state of 0.87. Indeed, even if 50% of the population is malfunctioning then the three-valued model still converges to an average belief of 0.83 while the average belief of the weighted voter model drops to 0.67. This is most likely due to the consensus operator providing a sort of 'buffer' between the two distinct choices (A and B) via the intermediate state. In the weighted voter model, even when a large majority of the population believes choice A to be the best, a malfunctioning agent can still cause another agent to immediately signal for choice B, and in doing so, increase the likelihood of additional agents switching to signal for choice B. In the three-valued model, however, if an agent is signalling for choice A, then it would require that they pick an agent signalling for choice B twice in a row before they also begin signalling for choice B. Of course, given the population's preference for choice A, it is more likely that an agent in the intermediate state will revert back to signalling for choice A than for choice B. The third truth state therefore slows the transition between the two choices enough that the malfunctioning agents do not affect the functioning population as severely as in the weighted voter model.

[4]Except for the case where $\lambda = 100\%$, in which results are averaged across all Kilobots due to the lack of any functioning Kilobots present in the population.

4.6 Kilobot experiments

Figure 4.12: 1.2 m^2Arena used for experiments in which Kilobots interact while moving amongst one another.

We now describe a series of experiments conducted on actual swarms of 400 Kilobots which follow the same template as the simulation studies in section 4. Figure 4.12 shows the 1.2 m^2 arena used. Note that it has a smooth and reflective surface so as to allow good communication between Kilobots and to enable motion. During the experiments each Kilobot in the signalling state displays a coloured light using its LED to indicate its most recent choice; blue for A and red for B. A video was made of every experiment and analysed using standard image processing algorithms (OpenCV) to identify the different coloured lights and to determine a time series of the percentage of A and B choices made. Each experiment was run independently 10 times with mean and percentiles (10% and 90%) then being determined. These are shown in Figures 4.13 and 4.14 with error bars indicating the 10^{th} and 90^{th} percentiles.

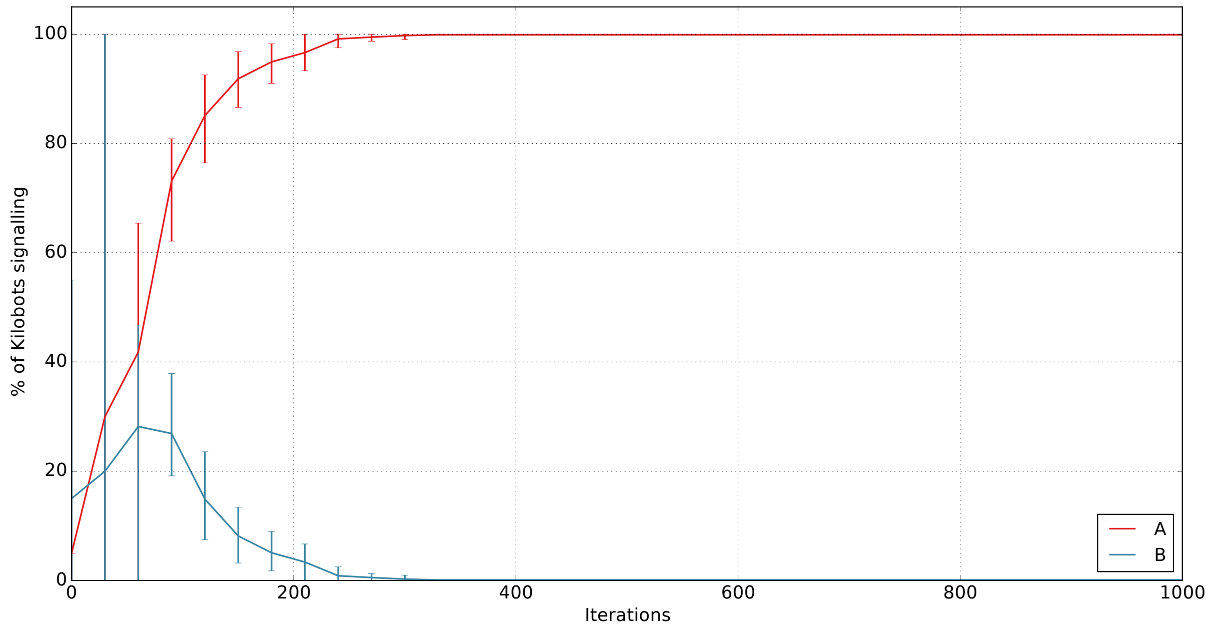
4.6.1 Experimental procedure

An overhead controller (OHC) was used to upload programs and initialisation instructions to each Kilobot. This resulted in non-uniform starting times across the population, leading to high variance in the results for the first 60 iterations. There are approximately 4 iterations per second, so that an experiment conducted over 1 000 iterations lasts just over 4 minutes.

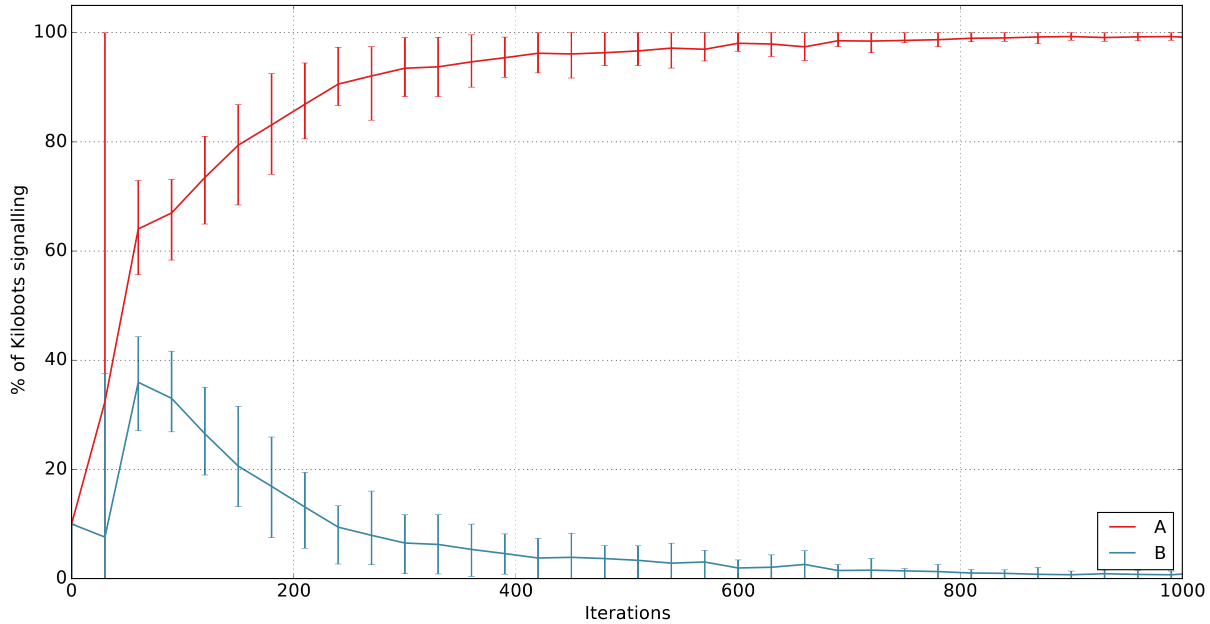
4.6.2 Results

Figure 4.13a shows the percentage of Kilobots signalling A or B as a function of time for the weighted voter model. In this case we can see that the swarm converges on choice A after approximately 240 iterations. In contrast, the three-valued model only fully converges to A after 800 iterations as can be seen in Figure 4.13b. Hence, as is consistent with the simulation studies we see that the weighted voter model significantly outperforms the three-valued model in terms of speed of convergence. However, after 1 000 iterations the level of convergence is the same for both models.

We also conducted experiments to test how fault-tolerant the two models were to the presence of malfunctioning Kilobots in the population. Here we introduced faulty Kilobots which malfunctioned according to the state transition diagram in Figure 4.10 and which made up $\lambda = 10\%$ of the population. As in the simulation experiments, the Kilobots signalling for each choice are recorded as a percentage of the functioning individuals only. Figure 4.14a shows the percentage of functioning signalling Kilobots which have chosen A and B, as a function of time, for the weighted voter model. After 1 000 iterations we see that 86.5% of the functioning Kilobots have chosen A. In contrast, Figure 4.14b shows that the three-valued model still maintains almost total convergence to A, notwithstanding the 10% of Kilobots that are malfunctioning, with 99.7% of functioning Kilobots choosing A after 1 000 iterations. On the other hand, Figure 4.14a shows that the weighted voter model achieves steady-state after about 120 iterations, while from Figure 4.14b we can see that the three-valued model requires around 700 iterations to achieve steady-state. Hence, as is consistent with the simulation experiments in section 4, these results suggest that while the weighted voter model converges more quickly than the three-valued model, the latter is much more fault tolerant than the former.

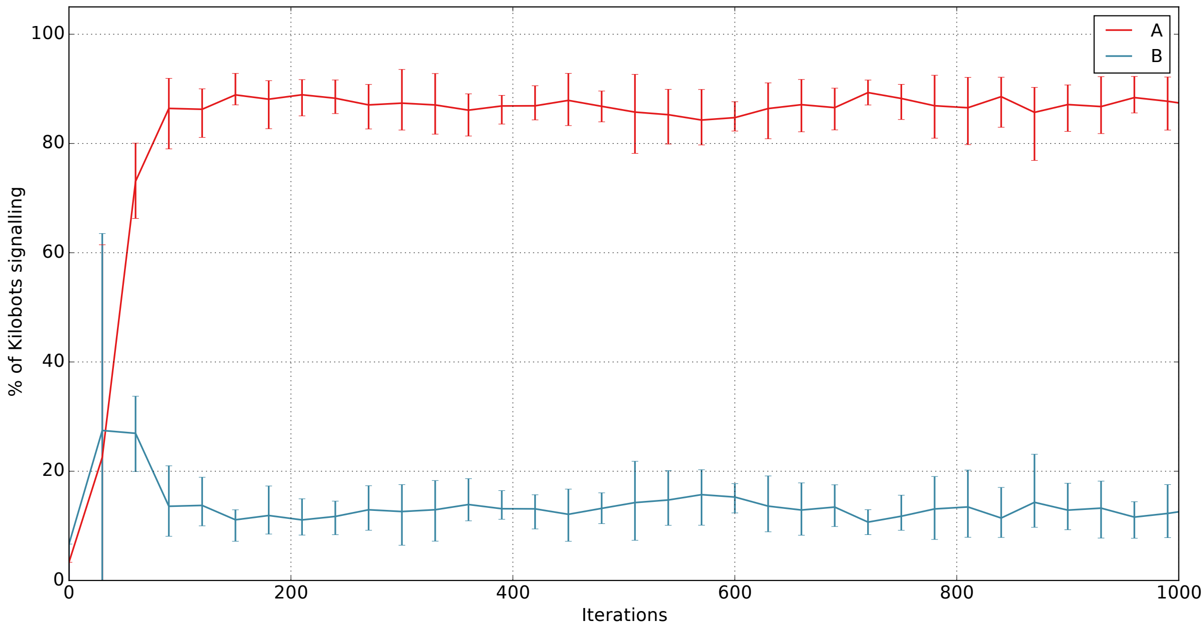


(a) Weighted voter model.

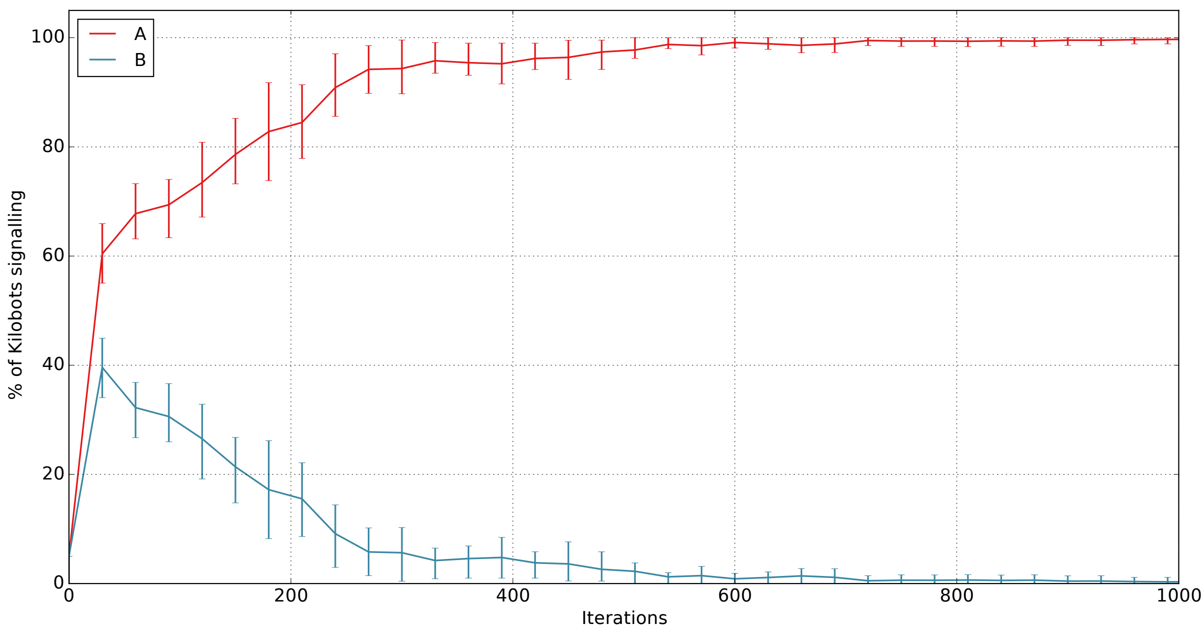


(b) Three-valued model.

Figure 4.13: Percentage of Kilobots signalling for choices A and B against iterations for a communication radii $r = 10$ cm.



(a) Weighted voter model.



(b) Three-valued model.

Figure 4.14: Percentage of Kilobots signalling for choices A and B against iterations for a communication radii $r = 10$ cm and malfunction rate $\lambda = 10\%$.

4.7 Best-of-n for $n > 2$

Much of the existing literature on the best-of-n problem for swarms concerns the $n = 2$ case, and so far we have dealt exclusively with this case in the interest of making a direct comparison between the weighted voter model and our three-valued approach. However, while we can directly extend the weighted voter model to the $n > 2$ case, it is less immediately clear how best to extend the three-valued model.

In this section, we therefore extend the weighted voter model and propose a possible extension of the three-valued model to the $n > 2$ case, before presenting simulation experiments for the $n = 3$ and $n = 5$ cases. We examine consensus formation for both models again with varying communication radii r, as well as comparing the robustness of both models to the presence of malfunctioning Kilobots. An important discussion then follows regarding the quality values assigned to each choice for different values of n, and how these changes are reflected in the performance of the models to reach a consensus.

4.7.1 Extending the three-valued model for $n > 2$

For the models presented in Chapters 2 and 3, we applied the consensus operator from Definition 2.5 to all of the propositional variables on which the pair of agents were trying to reach consensus. In doing so, we assumed that the propositional variables were independent, such that the truth value of one proposition did not affect the truth value of another. By definition of the best-of-n problem, however, this assumption does not hold; there may only be *one* option that is considered to be the 'best' choice, while the rest consequently must be considered to be measurably worse than the best choice. For the $n = 2$ case we modelled the belief state of an agent using a single variable, where a belief state of 0 represented 'B is the best choice' and 1 as 'A is the best choice', with the belief state of $\frac{1}{2}$ meaning 'I have no preference between A and B'. As we extend the model to the $n > 2$ case, the use of a single belief state is no longer plausible. Therefore, in order to apply the consensus operator to multiple belief states, we must adopt a an alternative representation.

One natural approach is to define belief states as n-dimensional vectors in $\{0, \frac{1}{2}, 1\}^n$, so that, for example, the belief state $< 0, \frac{1}{2}, \frac{1}{2}, 0, \ldots, 0 >$ is interpreted as meaning that choices 2 and 3 are believed to be better than all the other choices, but that there is no preference between them. The updating operator in Table 4.1 could then be applied independently to each dimension of the relevant belief states. For example, updating $< 0, \frac{1}{2}, \frac{1}{2}, 0, \ldots, 0 >$ given the signalled state $< \frac{1}{2}, \frac{1}{2}, 0, \ldots, 0 >$ results in the updating agent adopting the new state $< 0, \frac{1}{2}, 0, \ldots, 0 >$. However, the latter is a belief state in which, although the agent has ruled out all except the second choice, they still remain uncertain that this is the best choice. In effect they are not taking account of the fact that in the best-of-n problem the n choices are assumed to be exhaustive. Our approach is then to incorporate a form of normalisation into the model so that, for example, $< 0, \frac{1}{2}, 0, \ldots, 0 >$ is normalised to $< 0, 1, 0, \ldots, 0 >$. Similarly, if there is more than one choice with a belief state of 1, then those choices are normalised to $\frac{1}{2}$ with all others set to 0, e.g. $< 1, 1, \frac{1}{2}, \ldots, \frac{1}{2} >$ is normalised to $< \frac{1}{2}, \frac{1}{2}, 0, \ldots, 0 >$. An all-0 belief state

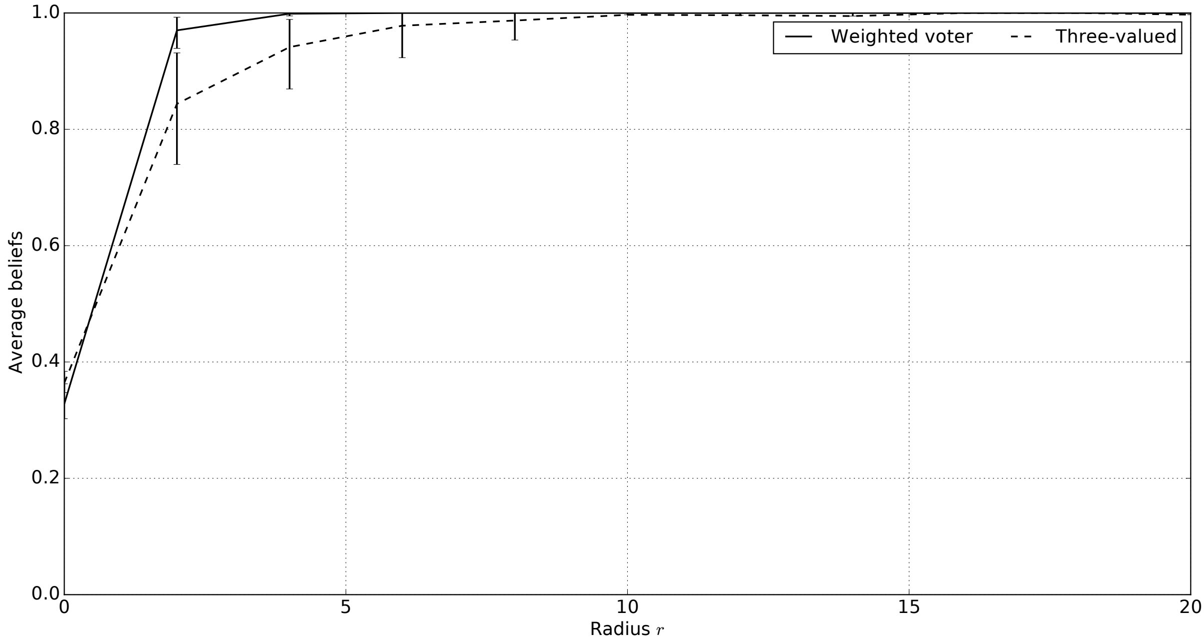


Figure 4.15: Average belief states at steady state for different communication radii r with $n = 3$ and $\delta_\rho = 3$.

$< 0, \ldots, 0 >$ is normalised to $< \frac{1}{2}, \ldots, \frac{1}{2} >$. More generally, the normalisation rules assume that there can only be one choice believed to be the *best* choice i.e. with a belief state of 1, and that to believe none of the choices are the best is to believe that all choices are borderline.

4.7.2 Results

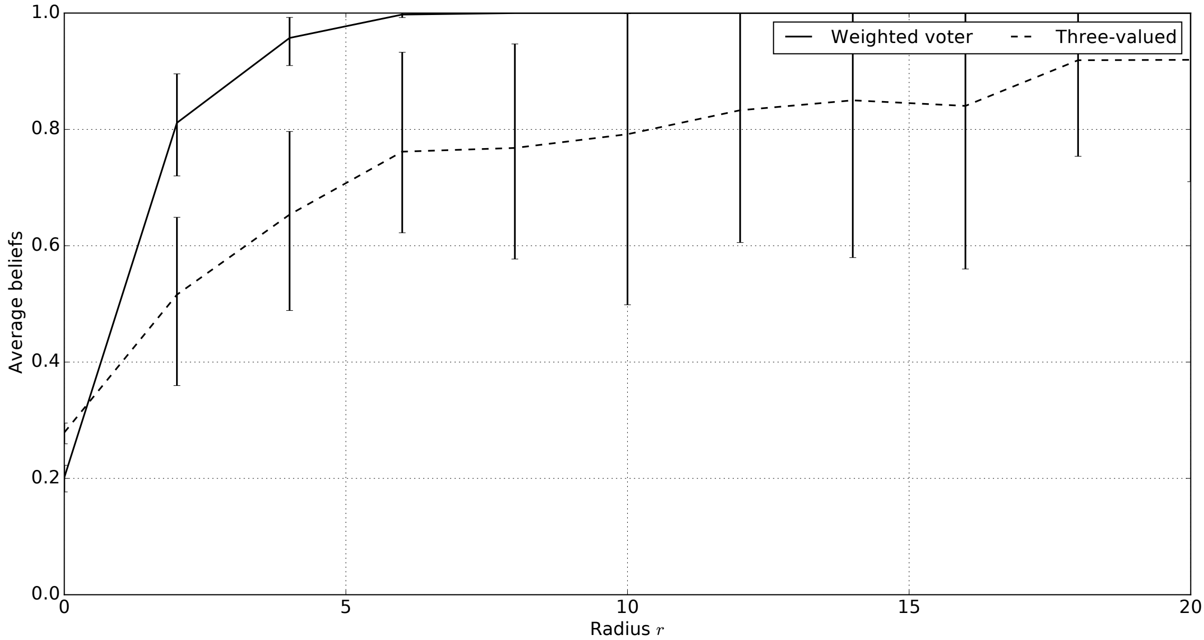
Using this approach we now present results for the $n = 3$ and $n = 5$ cases using the simulation environment. For $n = 3$ we assume that the choices are A, B and C with quality values $\rho_A = 11$, $\rho_B = 8$, $\rho_C = 5$. Notice that compared to the $n = 2$ case we have increased the difference in quality values between each choice to 3 rather than 2; we shall denote this quality interval by δ_ρ. We found that while the previous quality interval was sufficient for Kilobots to differentiate between two choices effectively, for the $n = 3$ case, a δ_ρ of 3 was required such that the Kilobots were once again able to reach a consensus.

Figure 4.15 shows the average belief states for the highest-value choice for both the weighted voter and three-valued models. For a communication radius $r = 10$ cm we see that both models have effectively converged on the best choice, with an average belief value of 1.00 and 0.99 for the weighted voter model and the three-valued model respectively. The performance of these two models for the $n = 3$ case with $\delta_\rho = 3$ is very similar to the $n = 2$ case presented in Figure 4.7, with the weighted voter model performing just the same, while the three-valued model requires a slightly larger radius of communication to fully converge. In Figure 4.7 the three-valued model averages a belief state of 0.99 for $r > 6$ cm, while in Figure 4.15 we see that it doesn't quite achieve the same level of consensus until $r \geq 10$ cm. It is likely that matching performance would be observed if the experiment runtime were to be extended beyond 1 000 iterations.

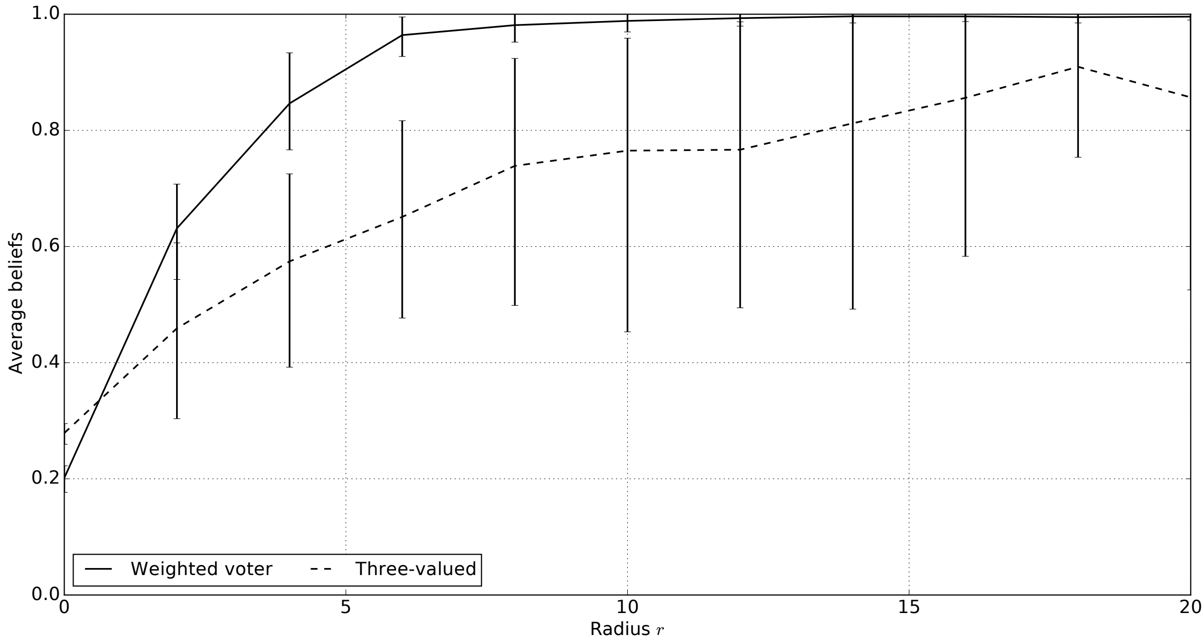
For the $n = 5$ case, we consider two quality assignments for the choices A, B, C, D and E.

Figure 4.16 shows the average belief states for both models with quality values $\rho_A = 25$, $\rho_B = 20$, $\rho_C = 15$, $\rho_D = 10$ and $\rho_E = 5$ for $\delta_\rho = 5$, and with quality values $\rho_A = 33$, $\rho_B = 26$, $\rho_C = 19$, $\rho_D = 12$ and $\rho_E = 5$ for $\delta_\rho = 7$. In Figure 4.16a, we see that by matching the quality interval to the number of choices such that $\delta_\rho = n$, the weighted voter model eventually converges the highest-value choice averaging a belief state of 1.0 for a communication radii $r \geq 6$ cm, while the three-valued model averages a belief just above 0.9.

Preliminary results indicated that as the number of choices n increases, so too must the quality interval δ_ρ. As such, we ran the same experiments for the $n = 5$ case for $\delta_\rho = 7$ and present the results in Figure 4.16b. Given a wider spread of quality values, it can be expected that the Kilobots would more easily distinguish between choices considering the increased differences in signalling time. This is because high quality choices are being signalled for by Kilobots for an increased period of time compared to choices of lesser quality. However for $\delta_\rho = 7$ the weighted voter model is unable to converge fully to an average belief state of 1.0 in the highest quality choice, while the three-valued model performs noticeably poorer than for $\delta_\rho = 5$ across the full range of communication radii r. As previously noted, this is likely to be the result of an insufficient runtime for experiments which have been limited to 1 000 iterations for the purpose of timely decision-making. Increasing the number of iterations beyond 1 000 would likely lead to eventual convergence of both models for $\delta_\rho = 7$, but as δ_ρ increases so does the required length of the experiments due to the nature of the signalling periods forming the feedback mechanism upon which the models rely in order to form consensus. The larger the values of δ_ρ, the longer the time required to reach a clear decision as increased signalling time will reduce the number of consensus operations occurring in the population if the length of the experiment's runtime is not extended accordingly. This leads to further implications for both models in dealing with larger numbers of n as we have seen so far that increasing n must be accompanied by a proportional increase in δ_ρ.



(a) $\delta_\rho = 5$.



(b) $\delta_\rho = 7$.

Figure 4.16: Average belief states at steady state for different communication radii r with $n = 5$ and $\delta_\rho \in \{5, 7\}$.

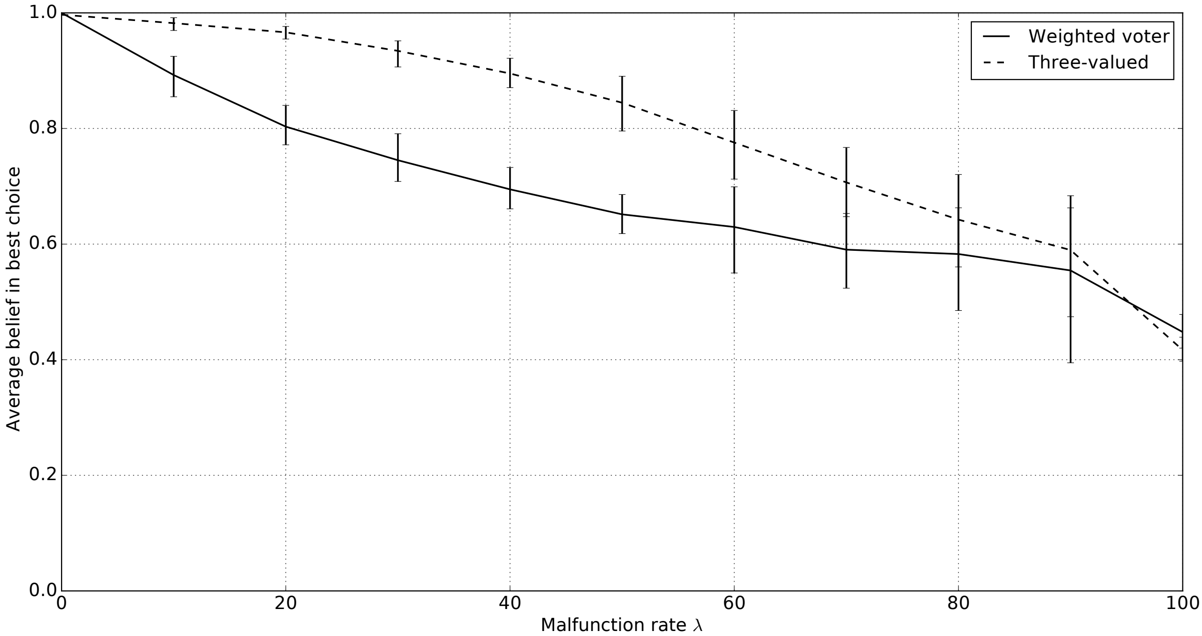
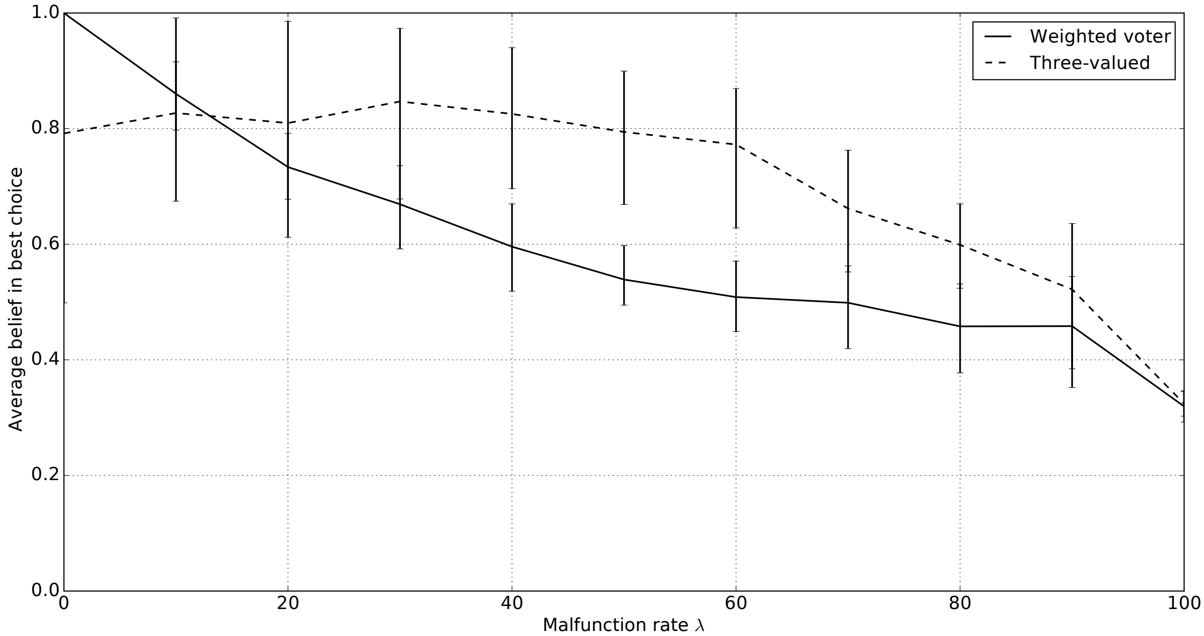


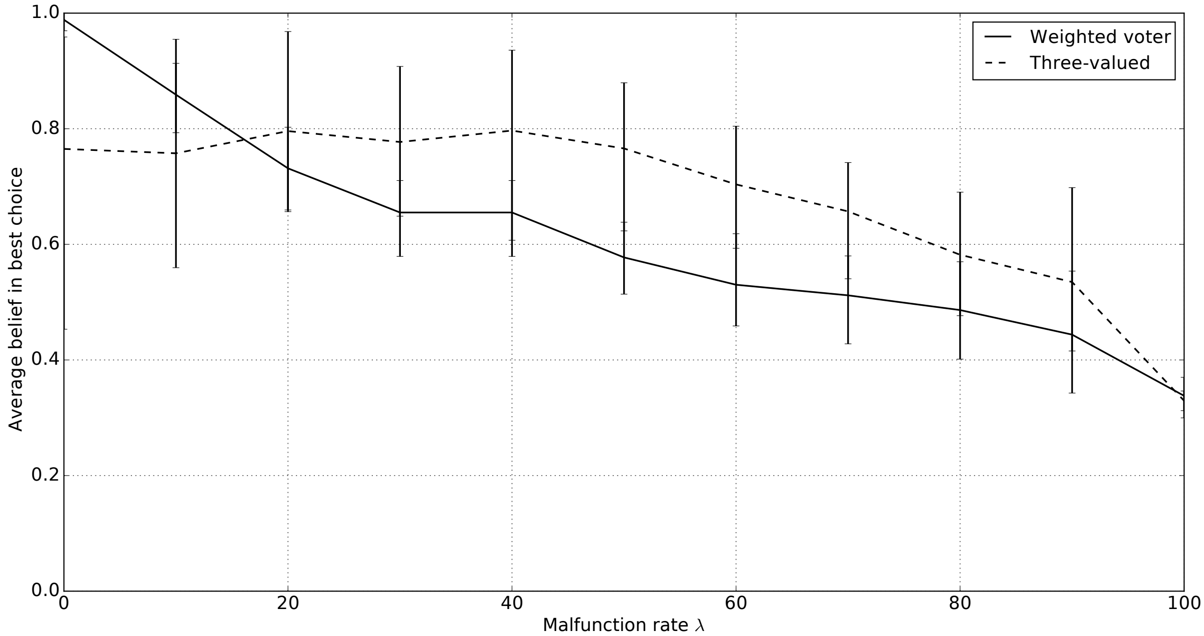
Figure 4.17: Average belief states against malfunction rates λ for a communication radii r of 10 cm with $n = 3$ and $\delta_\rho = 3$.

Robustness

Figures 4.17 and 4.18 show the average belief values for both algorithms plotted against the percentage of malfunctioning Kilobots $\lambda \in [0, 100]$. Overall, in all cases the three-valued model is more robust to malfunction than the weighted voter model. Although, as can be seen in Figure 4.18 the three-valued model performs worse for lower malfunction rates where $\lambda \leq 15\%$. This may be a result of reduced overall convergence of the three-valued model as the number of choices, n, increases. Further research is required in order to explore this effect more fully and, in general, to provide a more extensive analysis of the $n > 2$ case. In particular, analysis of the role that the quality interval δ_ρ plays in the dynamics of consensus requires further work as we believe that increasing the number of choices requires a similar increase in the difference between the associated quality values. This would lead to undesirably long signalling times and drastically slow the decision-making process; this may be catastrophic for time-sensitive applications such as search and rescue operations. These results likely make the case that an alternate feedback mechanism should be devised for the $n > 100$ case to avoid the signalling state altogether, perhaps via exploitation of Kilobot memory or message buffer lengths, but this requires additional research.



(a) $\delta_\rho = 5$.



(b) $\delta_\rho = 7$.

Figure 4.18: Average belief states at steady state against malfunction rates λ for a communication radii r of 10 cm with $n = 5$ and $\delta_\rho \in \{5, 7\}$.

4.8 Conclusions

In this chapter we have proposed a three-valued model for belief updating in the best-of-n distributed decision-making problem, and compared it to the weighted voter model. We have applied elements of our work from Chapters 2 and 3 on the consensus operators in multi-agent systems to a robotic system both in simulation and practice. Experiments were conducted using a realistic simulation environment, as well as on actual Kilobot swarms of 400 robots which provided an affordable distributed system in which group decisions can be made via a combination of environmental sensing and communication between individuals. We have analysed each model's performance in respect to speed of convergence and have focussed primarily on robustness of the swarm to individual malfunction or error as a measure of suitability of models to be applied in a real-world setting. The results from both simulation and embodied experiments show that the three-valued model is more robust to the presence of malfunction in the population than the weighted voter model. However, we have seen that the weighted voter model has the advantage of converging more quickly to the best choice, and we believe this speed of convergence is closely linked with the lack of robustness shown by the weighted voter model. As the three-valued model's intermediate belief state slows convergence to a decision, an individual's ability to make accurate decisions may be less affected by malfunction given the ability to resample from the population before becoming completely committed to an alternative choice.

We note that in both models, belief updating is based on the belief state of only one signalling agent. This property may be advantageous in scenarios where there is either very limited communication or low swarm density. Nonetheless, future work on robustness should consider decision algorithms which take account of larger samples of belief states drawn from the signalling agents within an individual's radius of communication. This could include majority rule models as studied in [74] and [75], as well as probabilistic pooling operators of the kind reviewed in [20]. One might hypothesise that by taking account of a larger sample of signalling agents, models would tend to be more robust to noise, error and malfunction. However, this robustness still needs to be considered in a broader context which also takes account of speed and overall level of convergence. Typically, models for distributed consensus or decision-making focus on systems of limited capabilities (e.g. communication or sensing) but it is clear that even low-cost systems such as the Kilobots possess greater capabilities than are often accounted for, including for the models discussed in this chapter where the assumption that Kilobots forget previous choices (and associated qualities) is made. For example, on-board memory is often not taken into consideration, and neither is communication bandwidth, and yet both of these aspects of the systems could be fully exploited. It would be possible for robots to remember previous choices and their respective qualities, and to disseminate additional information that would aid in the decision-making process, particularly for the $n > 2$ case. Therefore another approach to distributed decision-making for the best-of-n problem could involve the communication of preferences between a subset of the n choices. This would be particularly helpful when n becomes quite large (≥ 100 choices) as preferences between choices could be aggregated (exploiting the memory available to each individual Kilobot) to discard choices which are rarely signalled for.

In our experiments we have investigated robustness to the presence of a particular type of malfunctioning agent, in which error results from a proportion of the population continually selecting their beliefs at random rather than as part of the belief updating process. Clearly there are other models of error which should also be studied. For example, we might consider the errors resulting from some agents constantly broadcasting the same fixed beliefs or when agents maliciously broadcast the 'wrong' belief. Furthermore, while we have focussed on the case where there is a fixed proportion of malfunctioning or erroneous agents, and all other agents are error-free, it is also important to consider noise resulting from generic errors or sensing and processing limitations to which all agents are equally susceptible.

CHAPTER 5

From Propositions to Compound Sentences

In previous chapters we considered consensus in multi-agent and robotic systems, where the underlying values were propositional variables intended to represent statements about the state of the world. Adopting a third truth value then allows us to capture borderline cases inherent to vague statements, e.g. 'it is borderline raining/not raining'. It is clear, however, that propositional variables are not sufficient to express the rich complexity of opinions and beliefs about which agents may desire or need to reach consensus. Consider conditional beliefs such as 'if it is raining, then the ground will be wet and vision will be impaired'. These kinds of beliefs are common in natural language since it identifies relationships between events and their consequences. Hence, representing and reaching consensus about compound statements so as to come to a shared position on causal relationships is central to decision-making. For example, referring to the search and rescue application introduced in Section 1.2, it is imperative that the operatives be able to draw conclusions about the accessibility of areas based on their perceived state of the world. If the operatives know that it is raining in an area, then they will be able to draw the conclusion that the area will be more dangerous to enter, perhaps due to visibility issues or the increased likelihood of the ground being unstable. Similarly, the operatives may wish to reach a consensus about whether the accessibility of area a_1 implies that areas a_2 and a_3 are also accessible, perhaps via area a_1.

In this chapter we adapt the model for multi-agent consensus introduced in Chapter 2 to accommodate truth assignments on the compound sentences of a propositional language. Unlike in previous models, agents form consensus on the truth assignments at the sentence level, and adopt a new underlying valuation which is consistent with this newly adopted truth assignment. We then carry out simulation experiments and analyse some preliminary results, before proposing an alternative model which seeks to address constraints and limitations of forming consensus directly on the truth assignments on sentences, rather than on propositional variables.

5.1 Related work

As discussed in earlier chapters, much of the opinion dynamics [18, 19, 31, 32, 69] literature is primarily concerned with reaching a consensus about the value of a given variable, typically a real number in $[0, 1]$ or $[-1, 1]$ and an agent's belief corresponds to a real value, e.g. the probability of an event. The primary focus of opinion dynamics is then on the way in which opinions change and evolve over time in multi-agent systems. More specifically, the area is concerned with how agents should update their own beliefs based on the beliefs of other agents in the system in order to achieve certain macro-level properties such as convergence.

More recently we have seen studies of belief formation in multi-agent systems where beliefs are defined on some propositional language. An extension of the Hegselmann-Krause model presented in [61] transitions from representing agents' beliefs numerically to formulating them as theories in some finite propositional language. Belief revision games (BGRs) are introduced in [64] in which agents update their beliefs in an iterative manner, incorporating the beliefs of their peers at each iteration. In this model, different belief revision operators are then proposed for merging groups of beliefs. Other studies of belief formation include [29] and [7, 8]. In particular, [29] proposes a model of iterative belief formation and diffusion, where agents' beliefs are Boolean assignments on the variables of a propositional language and each agent determines a network of influence in the form of a directed graph. Agents update their beliefs iteratively by aggregating the beliefs of the other agents in their network of influence. Multiple aggregation procedures are then investigated, including majority rule and distance-based belief merging. Convergence is studied as a property of the networked structure adopted for a given population, independent of the initial beliefs.

In the recent works of Cholvy [7, 8], a similar model of iterative belief diffusion is proposed in which agents are influenced by a network of other agents, and beliefs of all such influencers are aggregated and used to update the influenced agent's beliefs. In this case beliefs are modelled as Boolean propositional formulae and, according to Cholvy, may even express a level of uncertainty i.e. for the formula $(p_1 \vee p_2) \wedge r$ the disjunction allows for an agent to express uncertainty about which proposition of p_1 or p_2, if not both, is true. We are doubtful, however, of whether this representation is really capable of capturing uncertainty. This work differs from the models proposed here in several important ways. Firstly, agents' beliefs are not represented by propositional formulae; rather, they are represented by Kleene valuations on the underlying propositions, which naturally lead to truth assignments on the sentences about which agents are attempting to form consensus. These sentences are preselected for experiments so that we can investigate the dynamics of the population in relation to the underlying Kleene valuations, as different sentences are satisfied by a varying number of underlying Kleene valuations. Secondly, beliefs are three-valued, rather than Boolean. Where other proposed models might rely on majority rule and other such aggregation procedures, we rely on exploiting this third truth value to improve consensus formation between pairs of agents and, in the second proposed model, to ease transitioning between conflicting states such that agents do not unnecessarily come to adopt completely new beliefs which are inconsistent with their current beliefs. Thirdly, we do not impose an arbitrary influence relation on agents such that their beliefs are more drastically

		φ		
	$\rightarrow$	1	$\frac{1}{2}$	0
θ	1	1	$\frac{1}{2}$	0
	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
	0	1	1	1

Table 5.1: Truth table for Kleene's strong implication of $\theta \rightarrow \varphi$.

altered by those considered of greater influence, and less affected by others. We instead examine consensus according to an inconsistency threshold to study the convergence of the population when we limit the extent to which conflicting agents can combine beliefs, as well as when agent interactions are unconstrained.

5.2 Model

Overview. We propose a model for consensus of compound sentences in which agents' valuations on the underlying propositions remain private. Instead, agents share the truth values they believe most accurately reflect the true state of the world regarding a set of compound sentences. Agents combine their truth values on a set of sentences Θ without revealing their underlying valuations on the propositions. Each agent then selects the valuation in the set of consensus valuations that is the most similar to their own. As a result, agents form consensus directly on the sentence truth values, but not necessarily on the underlying valuations.

5.2.1 A combination operator for compound sentences

We consider a simple language $\mathcal{L}$ based on Kleene's strong three-valued logic, with propositional variables $\mathcal{P} = \{p_1, ..., p_n\}$ and connectives $\neg$, $\vee$, $\wedge$ and $\rightarrow$. Here we are referring explicitly to Kleene's strong implication, such that $\theta \rightarrow \varphi \equiv \neg\theta \vee \varphi$. Let $S\mathcal{L}$ denote the sentences of $\mathcal{L}$ formed by recursive application of the logical connectives to the propositional variables in the usual manner. A Kleene valuation is then the allocation of truth values 0 (false), $\frac{1}{2}$ (borderline) and 1 (true) to the sentences of $\mathcal{L}$ as defined in Definition 2.1. The truth tables for Kleene's strong three-valued logic can be found in Tables 2.1 and 5.1. We reuse notation from Chapter 2 to represent a Kleene valuation $\mathbf{v}$ by its associated *orthopair* [41], (P, N), where $P = \{p_i \in \mathcal{P} : \mathbf{v}(p_i) = 1\}$ and $N = \{p_i \in \mathcal{P} : \mathbf{v}(p_i) = 0\}$. Notice that $P \cap N = \emptyset$ and that $(P \cup N)^c$ corresponds to the set of borderline propositional variables. We also use the same consensus operator $\odot$ from earlier chapters as defined in Table 2.4 for combining a pair of truth values $t_1, t_2 \in \{0, \frac{1}{2}, 1\}$.

We now extend the consensus operator to vectors of truth values on the sentences of $\mathcal{L}$. Given a set of sentences $\Theta = \{\theta_1, ..., \theta_k\}$ for $\Theta \subseteq S\mathcal{L}$, then a truth assignment on Θ is denoted by $\vec{t} \in \{0, \frac{1}{2}, 1\}^k$ where the i^{th} element of $\vec{t}$ is the truth value of θ_i for $i = 1, ..., k$. For a pair of truth assignments $\vec{t_1}, \vec{t_2}$ on Θ, we can then apply the consensus operator such that $\vec{t_1} \odot \vec{t_2}$ is given by

$$(t_{1,1},...,t_{1,k}) \odot (t_{2,1},...,t_{2,k}) = (t_{1,1} \odot t_{2,1},...,t_{1,k} \odot t_{2,k}).$$

Also, let $\mathbb{V}$ be the set of all Kleene valuations on $\mathcal{L}$. Then $\mathbb{V}_{\vec{t}} = \{\mathbf{v} \in \mathbb{V} : \mathbf{v}(\theta_i) = t_i, i = 1,...,k\}$ is the set of Kleene valuations consistent with the truth assignment $\vec{t}$ on Θ.

Given that consensus then takes place at the level of the truth assignments on the sentences Θ, for each of which it is possible that multiple Kleene valuations $\mathbf{v} \in \mathbb{V}$ produce the same truth assignment on Θ, we propose that agents would then adopt the corresponding underlying Kleene valuation that was the most similar to their currently held belief. To this end, we now introduce a similarity measure on $\mathbb{V}$.

Definition 5.1. A measure of similarity

A similarity measure between two Kleene valuations is a function $S : \mathbb{V}^2 \to [0,1]$ such that $\forall \mathbf{v}, \mathbf{v}' \in \mathbb{V}$:

$$S(\mathbf{v}, \mathbf{v}') = \frac{1}{n} \sum_{i=1}^{n} 1 - |\mathbf{v}(p_i) - \mathbf{v}'(p_i)|$$

Then, for a pair of agents with initial valuations $\mathbf{v}_1$, $\mathbf{v}_2$ and a consensus truth assignment $\vec{t_1} \odot \vec{t_2}$ adopted by both agents, $\mathbf{v}_1$ and $\mathbf{v}_2$ are replaced with new valuations $\mathbf{v}_1', \mathbf{v}_2' \in \mathbb{V}_{\vec{t_1} \odot \vec{t_2}}$ given by

$$\mathbf{v}_1' = arg\ max\ \{S(\mathbf{v}_1, \mathbf{v}) : \mathbf{v} \in \mathbb{V}_{\vec{t_1} \odot \vec{t_2}}\}$$

and

$$\mathbf{v}_2' = arg\ max\ \{S(\mathbf{v}_2, \mathbf{v}) : \mathbf{v} \in \mathbb{V}_{\vec{t_1} \odot \vec{t_2}}\}.$$

In this model, it is possible that $\mathbb{V}_{\vec{t_1} \odot \vec{t_2}} = \emptyset$, in which case there is no obvious resolution and so we simply skip the current iteration so that the selected pair of agents do not form consensus.

We now introduce a measure of inconsistency quantifying direct conflict between two truth assignments $\vec{t_1}$ and $\vec{t_2}$ as follows:

Definition 5.2. A measure of inconsistency

The degree of inconsistency between two truth assignments $\vec{t_1}, \vec{t_2}$ on the set of sentences $\Theta = \{\theta_1,...,\theta_k\}$ is the proportion of truth values in direct conflict between the two truth assignments, expressed as a function $I(\vec{t_1}, \vec{t_2}) \to [0,1]$, and is given by

$$I(\vec{t_1}, \vec{t_2}) = \frac{|\{j \in \{1,...,k\} : |t_{1,j} - t_{2,j}| = 1\}|}{k}.$$

We will employ this measure to study the resulting convergence properties of the model under varying restrictions on the level of inconsistency between pairs of agents. If, for a pair of agents, $I(\vec{t_1}, \vec{t_2}) > \gamma$ for an inconsistency threshold $\gamma \in [0,1]$, then the consensus operator is not applied and both agents retain their current beliefs. If, however, $I(\vec{t_1}, \vec{t_2}) \leq \gamma$ then the

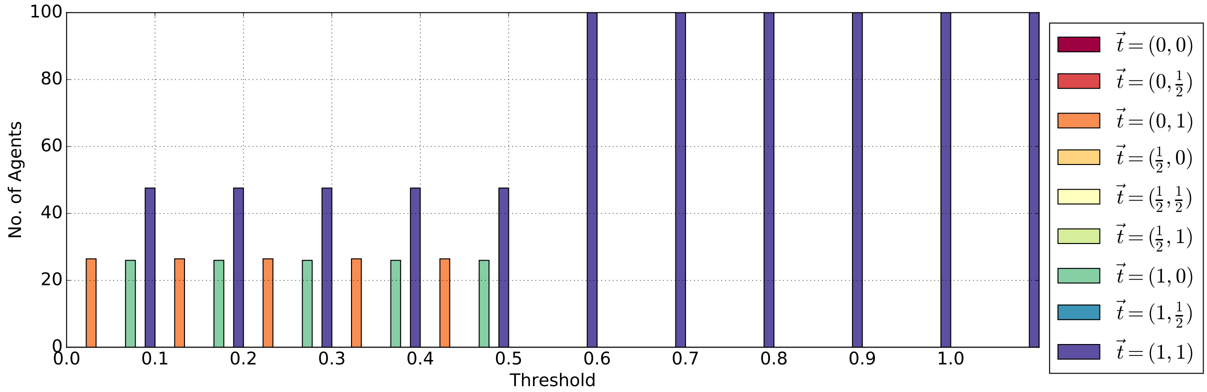


Figure 5.1: Number of agents with truth assignments on the sentences $\Theta = \{p_1 \rightarrow p_2, p_1 \rightarrow \neg p_2\}$.

consensus operator is applied and both agents adopt the resulting truth assignment $\vec{t}_1 \odot \vec{t}_2$ as well as updating their underlying beliefs. Notice that an inconsistency threshold $\gamma = 1$ means that every pair of agents chosen from the population will combine, given that by definition the inconsistency measure cannot exceed 1. Conversely, an inconsistency threshold $\gamma = 0$ would therefore allow only the most consistent pairs of agents to form consensus for $I(\vec{t}_1, \vec{t}_2) \leq \gamma$. In other words, only when the truth assignments on the sentences are either exactly the same, or one of the truth assignments assigns a borderline truth value $\frac{1}{2}$ while the other assigns either 1 or 0.

5.3 Simulation experiments for sentence-level consensus

We now illustrate this approach by running a number of simulation experiments in which agents aim to reach consensus on a set of sentences Θ; these sentences have been chosen to highlight interesting behaviours exhibited by the proposed models. We set a fixed limit of 1 000[1] iterations for each experiment, unless stated otherwise, and average results over 100 independent runs. Initial beliefs of agents are distributed uniformly at random across $\mathbb{V}$; we realise that this naturally generates a bias in favour of truth assignments with a greater number of associated valuations, however we felt that the most natural approach to belief initialisation was to initialise agents' internal beliefs, which consequently lead to truth assignments on the sentences.

While agents openly broadcast their truth assignments on Θ during the consensus process, their valuations on the propositions remain private. Given the consensus set of valuations $\mathbb{V}_{\vec{t}_1 \odot \vec{t}_2}$, agents adopt the most similar valuation to their currently held beliefs, with each agent remaining unaware of which valuation $\mathbf{v}' \in \mathbb{V}_{\vec{t}_1 \odot \vec{t}_2}$ has been adopted by the other.

5.3.1 Simulation results for $\Theta = \{p_1 \rightarrow p_2, p_1 \rightarrow \neg p_2\}$

Figures 5.1 and 5.2 are histograms showing the number of agents with varying truth assignments on the sentences $\Theta = \{p_1 \rightarrow p_2, p_1 \rightarrow \neg p_2\}$, and varying valuations on $\mathcal{P} = \{p_1, p_2\}$, respectively, at steady state plotted against inconsistency threshold γ. Results are averaged across the 100

[1]Preliminary experiments had shown this was more than sufficient to allow a population of 100 agents to reach consensus.

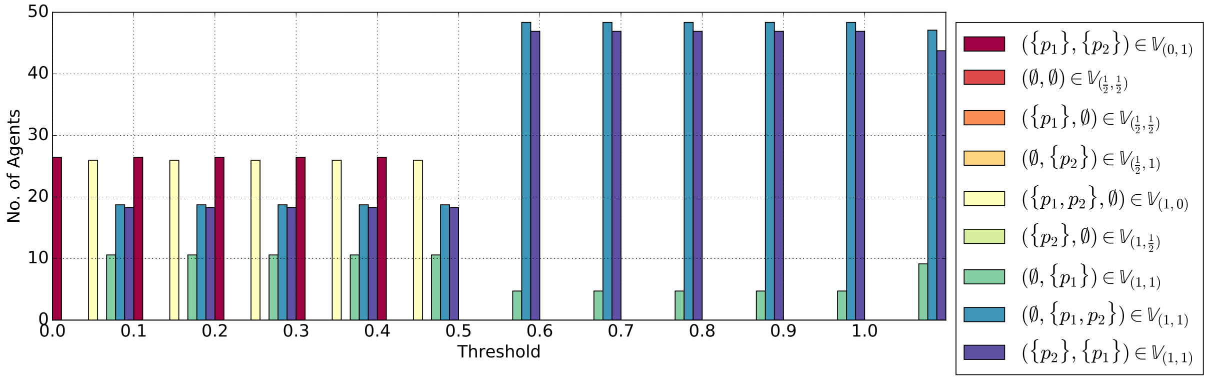


Figure 5.2: Number of agents with valuations on $\mathcal{L}$ for sentences $\Theta = \{p_1 \to p_2, p_1 \to \neg p_2\}$.

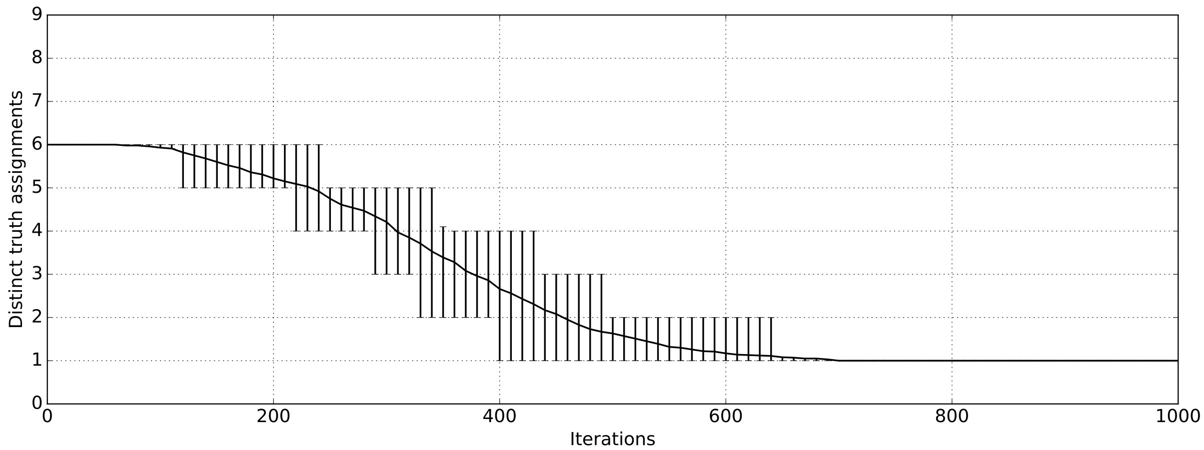


Figure 5.3: Number of distinct truth assignments on the sentences $\Theta = \{p_1 \to p_2, p_1 \to \neg p_2\}$ against iterations for an inconsistency threshold $\gamma = 0.5$.

independent runs, such that if all 100 agents have the truth assignment $(1, 1)$ averaged across all runs, then the population always converges on this truth assignment. From this, it is clear from Figure 5.1 that the population does converge on a single truth assignment on Θ, with the population forming consensus on the truth assignment $(1, 1)$ for $\gamma \geq 0.5$. However, Figure 5.2 provides a more detailed insight of the resulting consensus and from this we can see that, despite having reached an agreement on Θ, on average the population has not in fact converged on a single underlying valuation on the propositional variables. Instead, the majority of the population are effectively split between the two most precise valuations (i.e. admitting no borderline cases) on the propositional variables, and with a minority believing that p_2 is borderline. Even for an inconsistency threshold $\gamma = 1.0$ where all randomly selected pairs of agents combine to form consensus, the population still fails to converge to a single valuation on the propositional variables.

Figure 5.3 shows the number of distinct truth assignments on Θ as a trajectory plotted against iterations for $\gamma = 0.5$, providing a more detailed picture of the system's convergence. From this we can see that the population converges to a single truth assignment on Θ after just 600 iterations. There are three valuations consistent with the truth assignment $(1, 1)$ and every other possible truth assignment is associated with a set of Kleene valuations $\mathbb{V}_{\vec{t}}$ with lower

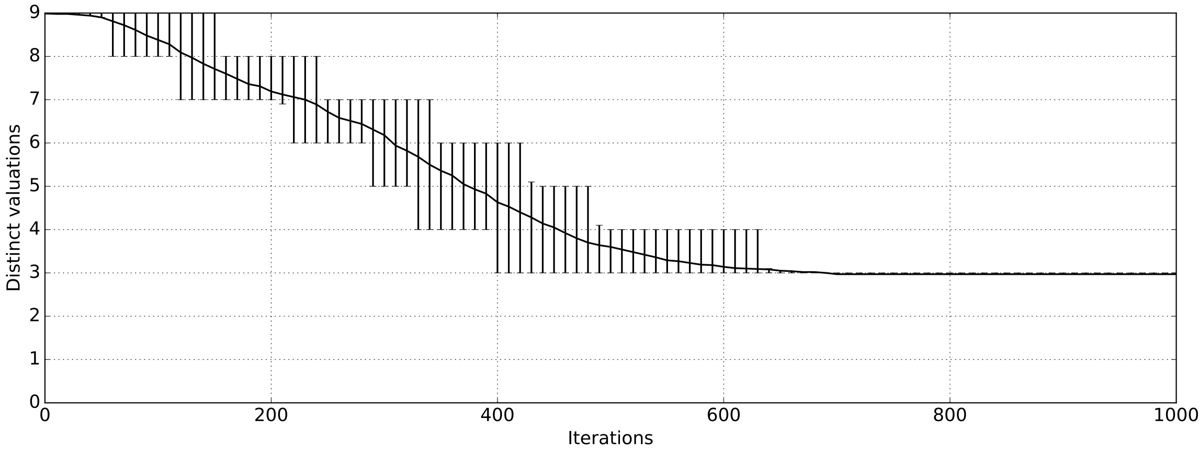


Figure 5.4: Number of distinct valuations on $\mathcal{L}$ against iterations for an inconsistency threshold $\gamma = 0.5$.

cardinality. Therefore we might consider if convergence at the sentence level corresponds to convergence at the propositional level. In this regard, Figure 5.4 shows the number of distinct valuations as a trajectory against iterations. From this, we can confirm that the population does not converge to a single valuation, despite converging to a single truth assignment, and instead converges to the three valuations in $\mathbb{V}_{(1,1)}$. Hence, Figure 5.2 is indicative of the system-level convergence at steady state for a typical run.

Given that consensus occurs at random, the primary factor we believe to be driving convergence to these three valuations appears to be that the set $\mathbb{V}_{(1,1)}$ is the set with maximal cardinality for the chosen Θ. That is, we hypothesise that the model favours the truth assignment $\vec{t^*}$ such that

$$\vec{t^*} = arg\ max\{\mid \mathbb{V}_{\vec{t}} \mid : \vec{t} \in \{0, \frac{1}{2}, 1\}^k\}.$$

where $\mathbb{V}_{\vec{t^*}}$ is the set with maximal cardinality, and therefore $\vec{t^*}$ is the maximal truth assignment. This effect is highlighted further for different sentences in Θ. More specifically, we suggest that since beliefs are initialised by selecting a valuation $\mathbf{v}$ uniformly at random from the set of all Kleene valuations $\mathbb{V}$ on $\mathcal{L}$, there is an inherent bias in favour of truth assignments with a greater number of corresponding valuations. Indeed this certainly appears to be the case for convergence results from this example. However, preliminary studies where agents' beliefs are initially selected uniformly at random across the truth assignments, with valuations then being assigned randomly from $\mathbb{V}_{\vec{t}}$, show that convergence to $(1, 1)$ still occurs.

5.3.2 Simulation results for $\Theta = \{p_1 \wedge p_2, \neg p_1 \wedge \neg p_2\}$

We now present experimental results for $\Theta = \{p_1 \wedge p_2, p_1 \wedge \neg p_2\}$. Figure 5.5 shows the average number of agents with varying truth assignments on the sentences Θ plotted against the inconsistency threshold γ. In these experiments, the population converges to the truth assignment $(0, 0)$ on Θ. $\mathbb{V}_{(0,0)}$ contains two completely precise valuations represented by the orthopairs $(\{p_1\}, \{p_2\})$ and $(\{p_2\}, \{p_2\})$, and there is close to uniform division between these two valuations at steady state for $\gamma \geq 0.5$ (see Figure 5.8).

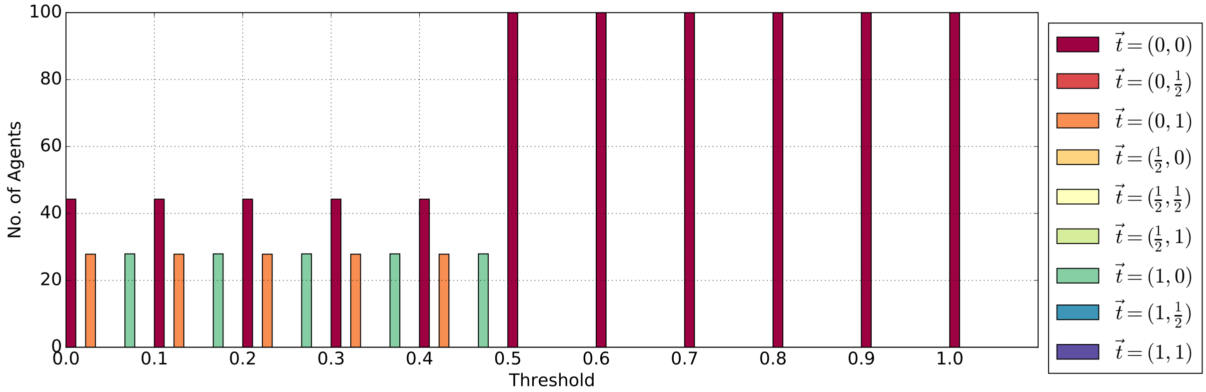


Figure 5.5: Number of agents with truth assignments on the sentences $\Theta = \{p_1 \wedge p_2, \neg p_1 \wedge \neg p_2\}$.

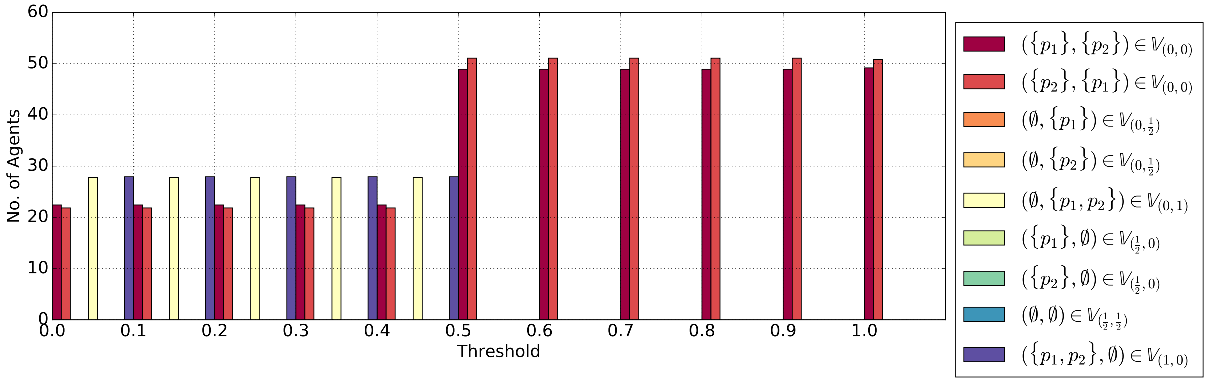


Figure 5.6: Number of agents with valuations on $\mathcal{L}$ for sentences $\Theta = \{p_1 \wedge p_2, \neg p_1 \wedge \neg p_2\}$.

Interestingly, the truth assignments $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$ are also each consistent with two valuations on p_1 and p_2. For $\mathbb{V}_{(0,\frac{1}{2})}$, the associated valuations are $(\emptyset, \{p\})$ and $(\emptyset, \{q\})$, and for $\mathbb{V}_{(\frac{1}{2},0)}$ they are $(\{p\}, \emptyset)$ and $(\{q\}, \emptyset)$. Note, however, that these valuations are more vague than either of those in $\mathbb{V}_{(0,0)}$. This is again consistent with the hypothesis that the population converges to the most precise truth assignment $\vec{t}$ on Θ when there is more than one truth assignment $\vec{t^*}$, where $\mathbb{V}_{\vec{t^*}}$ has maximal cardinality. In other words, when two or more sets have the same cardinality, and that their cardinality is larger than the cardinality of any other set, then the most precise truth assignment is favoured. This aligns with our expectations in relation to the properties of the consensus operator for which precise truth values dominate borderline truth values.

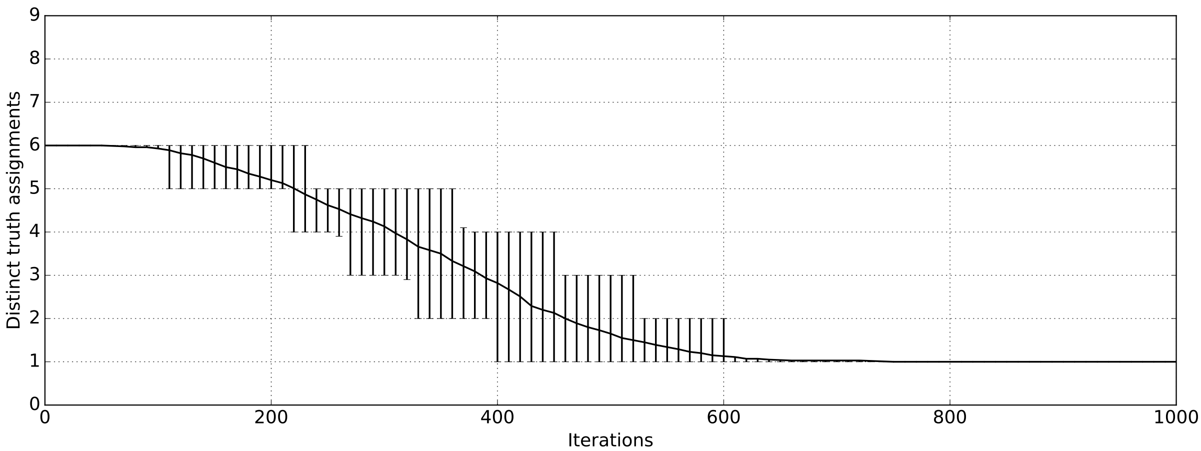


Figure 5.7: Number of distinct truth assignments on the sentences $\Theta = \{p_1 \wedge p_2, \neg p_1 \wedge \neg p_2\}$ against iterations for an inconsistency threshold $\gamma = 0.5$.

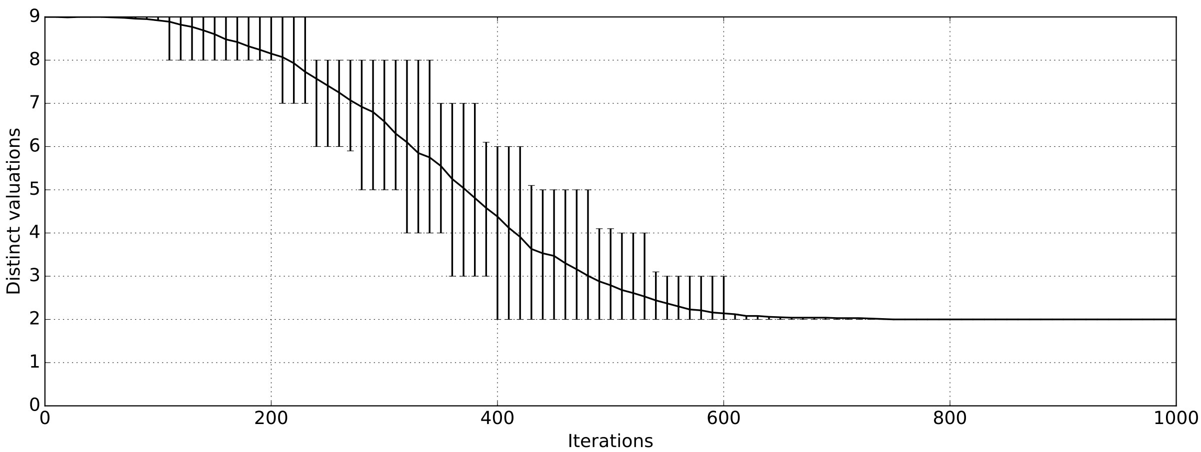


Figure 5.8: Number of distinct Kleene valuations on $\mathcal{L}$ against iterations for an inconsistency threshold $\gamma = 0.5$.

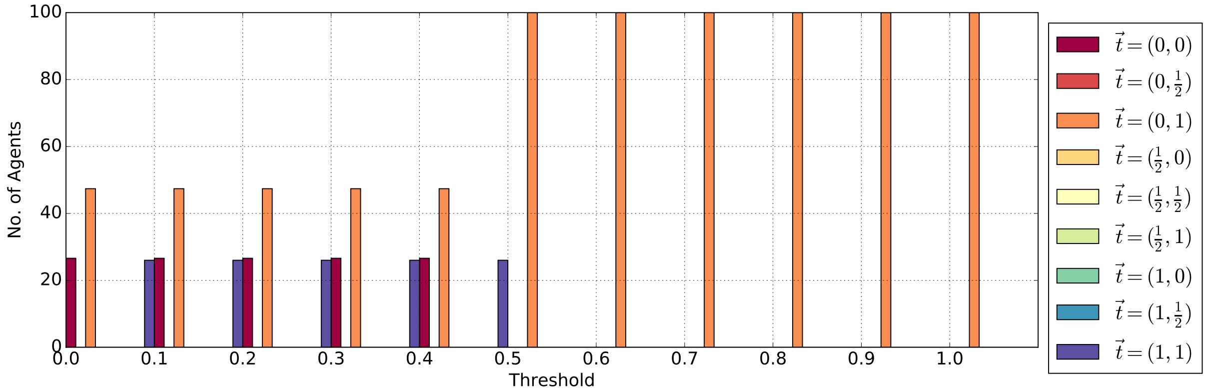


Figure 5.9: Number of agents with truth assignments on the sentences $\Theta = \{p_1 \wedge p_2, p_1 \vee p_2\}$.

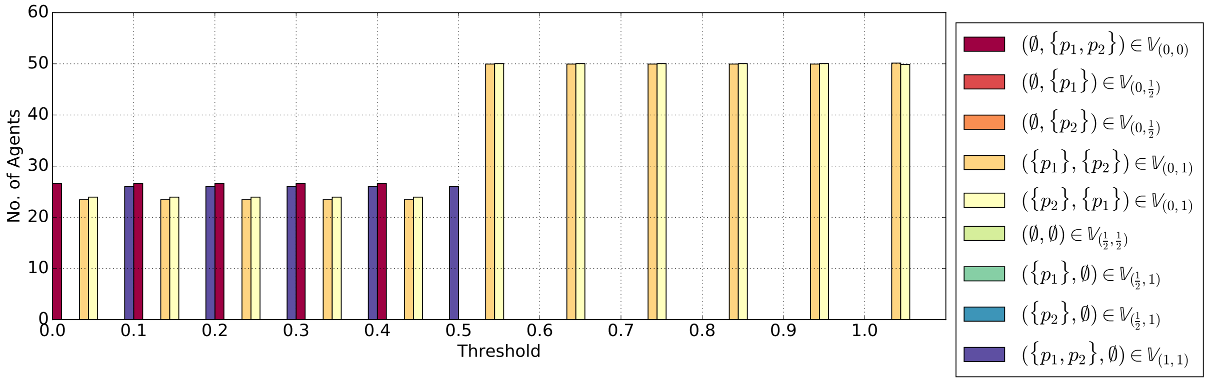


Figure 5.10: Number of agents with valuations on $\mathcal{L}$ for sentences $\Theta = \{p_1 \wedge p_2, p_1 \vee p_2\}$.

5.3.3 Simulation results for $\Theta = \{p_1 \wedge p_2, p_1 \vee p_2\}$

Figures 5.9 and 5.10 show similar results to those in Sections 5.3.1 and 5.3.2. For the sentences $\Theta = \{p_1 \wedge p_2, p_1 \vee p_2\}$, there are three possible truth assignments each consistent with two underlying valuations on the propositional variables. These are $(0, \frac{1}{2})$, $(0, 1)$ and $(\frac{1}{2}, 1)$. As in Sections 5.3.1 and 5.3.2, we see from Figure 5.9 that the population converges to the most precise truth assignment, in this case corresponding to $(0, 1)$. Furthermore, from Figure 5.10 we can see that the population is then split evenly between two precise valuations which are consistent with this assignment i.e. $(\{p_1\}, \{p_2\})$ and $(\{p_2\}, \{p_1\})$. Furthermore, Figures 5.11 and 5.12 display similar convergence behaviour to that shown in Sections 5.3.1 and 5.3.2.

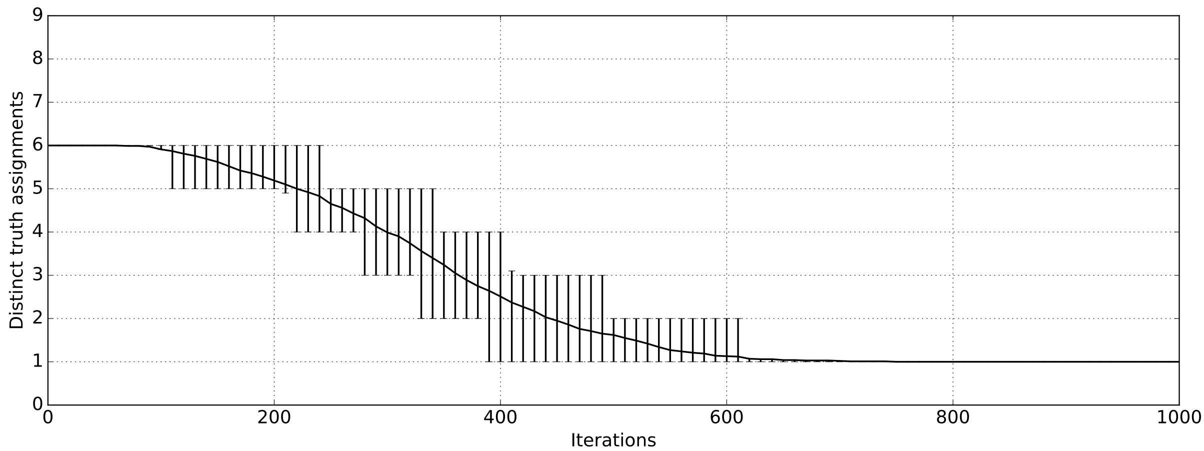


Figure 5.11: Number of distinct truth assignments on the sentences $\Theta = \{p_1 \wedge p_2, p_1 \vee p_2\}$ against iterations for an inconsistency threshold $\gamma = 0.5$.

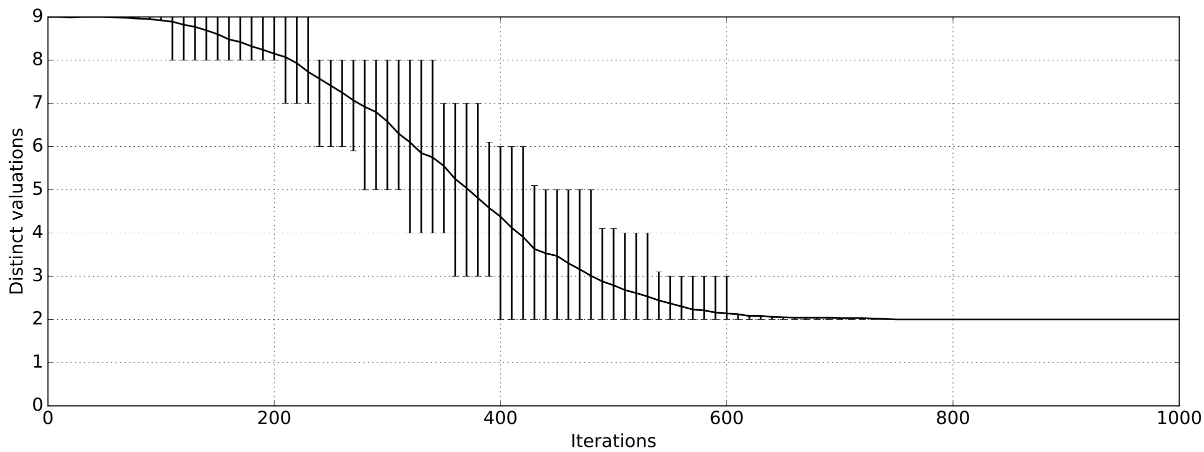


Figure 5.12: Number of distinct Kleene valuations on $\mathcal{L}$ against iterations for an inconsistency threshold $\gamma = 0.5$.

5.3.4 Simulation results for $\Theta = \{p_1 \to p_2, p_1 \to \neg p_2, \neg p_1 \to p_2, \neg p_1 \to \neg p_2\}$

We now present results for a larger set of four sentences $\Theta = \{p_1 \to p_2, p_1 \to \neg p_2, \neg p_1 \to p_2, \neg p_1 \to \neg p_2\}$. For simplicity, we omit the truth assignments figure shown in the previous sections and focus instead on Figure 5.13 in which the number of agents at steady state with varying valuations on p_1 and p_2 is given. Here, the resulting convergence behaviour is somewhat different from previous sections. We must note here that due to the size of Θ in relation to $\mathcal{P}$, some issues arise as a direct result of the application of the consensus operator under this model, which focuses on forming consensus between truth assignments on Θ as opposed to valuations on the underlying propositions. When applying the consensus operator at the propositional level, the resulting consensus valuation naturally corresponded with an overlying truth assignment on the sentences in Θ, by Definition 2.1 as a valuation on any sentence θ in $S\mathcal{L}$ is completely characterised by its values on $\mathcal{P}$. In this model, however, we apply the consensus operator directly on the truth assignments on Θ. For some sets of sentences, this can lead to inconsistencies in the consensus process as we mentioned earlier in Section 5.2. Specifically, it is possible that $\mathbb{V}_{\vec{t}_1 \odot \vec{t}_2} = \emptyset$ and such inconsistencies become quite frequent under the current model for the chosen Θ. As such, the current model must be adapted slightly.

Experiments are now run for 4 000 iterations, rather than 1 000, due to the increased convergence time. This is due to the inconsistent consensus between pairs of agents whose resulting consensus set $\mathbb{V}_{\vec{t}_1 \odot \vec{t}_2} = \emptyset$. For example, there are two valid truth assignments, $(0, 1, 1, 1)$ and $(1, 1, 1, 0)$ which, when combined via the consensus operator (see Definition 2.5), produce the truth assignment $(\frac{1}{2}, 1, 1, \frac{1}{2})$ which has no corresponding valuation on the propositions. This kind of combination is invalid under the set of sentences in Θ and leads to an inconsistent combination of beliefs between agents. In such a case, we opt to not apply the consensus operator for the chosen pair of agents and continue on to the next iteration in the simulation. Therefore, we need to increase the number of iterations and, as a result, the number of times agents are selected at random to form consensus. In the previous experiments, there also exist truth assignments without corresponding valuations, but such invalid truth assignments could not result from the application of the consensus operator because of the types of sentences in Θ. It would seem that both the size of Θ and the kinds of sentences $\theta \in \Theta$ determine the ability to apply the consensus operator at the sentence level; directly on pairs of truth assignments. We present an alternative model in Section 5.4 which addresses this issue while providing a reasonably intuitive approach to consensus at the valuation level.

Unlike in the experiments described in the previous sections, all truth assignments for these experiments correspond to a single valuation only; a motivating factor when choosing Θ to further study our model. That is, $|\mathbb{V}_{\vec{t}}| = 1$ for $\Theta = \{p_1 \to p_2, p_1 \to \neg p_2, \neg p_1 \to p_2, \neg p_1 \to \neg p_2\}$ and any $\vec{t}$. In Figure 5.13 we observe quite different convergence behaviour compared to $\Theta = \{p_1 \to p_2$ and $p_1 \to \neg p_2\}$. There is no longer a dominant truth assignment with a majority of corresponding valuations. In fact, simply from Figure 5.13 it is not clear that the population converges to any kind of consensus. However, we can clarify the situation by looking at Figures 5.14 and 5.15, which show trajectories for the number of distinct truth assignments and valuations, respectively. Here we see that there is clearly convergence in the population, but

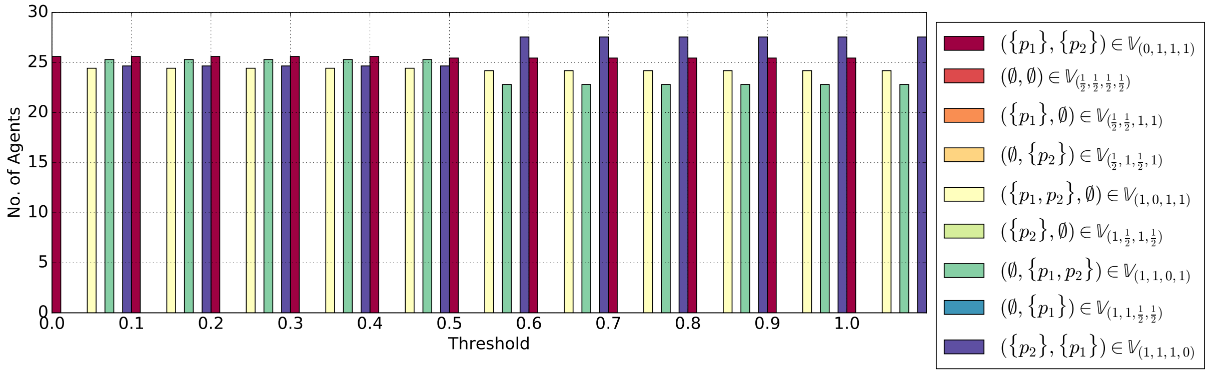


Figure 5.13: Number of agents with valuations on $\mathcal{L}$ for sentences $\Theta = \{p_1 \rightarrow p_2, p_1 \rightarrow \neg p_2, \neg p_1 \rightarrow p_2, \neg p_1 \rightarrow \neg p_2\}$.

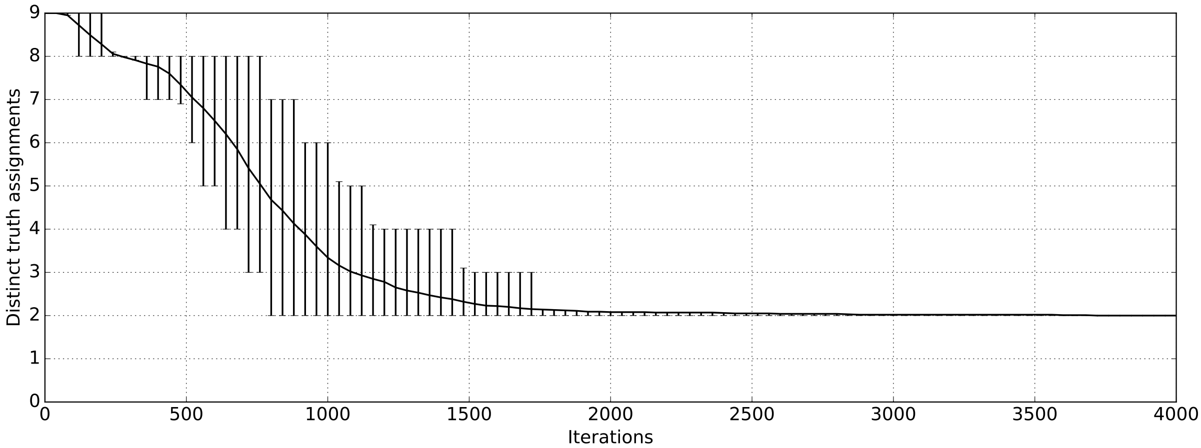


Figure 5.14: Number of distinct truth assignments on the sentences $\Theta = \{p_1 \rightarrow p_2, p_1 \rightarrow \neg p_2, \neg p_1 \rightarrow p_2, \neg p_1 \rightarrow \neg p_2\}$ against iterations for an inconsistency threshold $\gamma = 0.5$.

it remains split between two truth assignments and, similarly, between two underlying valuations. The four dominant truth assignments are naturally the most precise, just as we have seen in all previous experiments, and their corresponding valuations are equally precise in line with our expectations. In light of our earlier comments, we suggest that a possible explanation for why the population is converging to two distinct truth assignments, rather than a single assignment, is as follows. Given the inconsistent combinations that can result from applying the consensus operator at the sentence level (i.e. when $\mathbb{V}_{\vec{t_1} \odot \vec{t_2}} = \emptyset$), it is likely that skipping the consensus of these kinds of pairs is resulting in a split population without the possibility to merge these minority groups of agents. This can happen with the previous example, where agents with an associated truth assignment of $(0, 1, 1, 1)$ attempt to form consensus with agents with an associated truth assignment of $(1, 1, 1, 0)$. The resulting consensus is the set $\mathbb{V}_{(\frac{1}{2},1,1,\frac{1}{2})} = \emptyset$. These two truth assignments are associated with two of the four total precise valuations on p_1 and p_2, namely $(\{p_1\}, \{p_2\}) \in \mathbb{V}_{(0,1,1,1)}$ and $(\{p_1\}, \{p_2\}) \in \mathbb{V}_{(1,1,1,0)}$. Similarly, the combination of the truth assignments $(1, 0, 1, 1)$ and $(1, 1, 0, 1)$ results in the set $\mathbb{V}_{(1,\frac{1}{2},\frac{1}{2},1)} = \emptyset$.

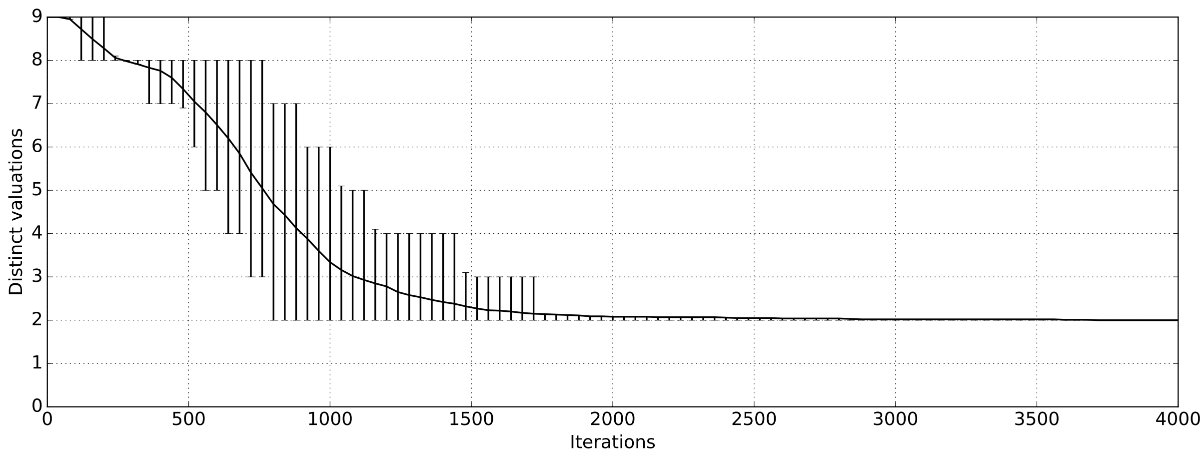


Figure 5.15: Number of distinct Kleene valuations on $\mathcal{L}$ against iterations for an inconsistency threshold $\gamma = 0.5$.

5.3.5 Summary

It appears, from these preliminary results, that forming consensus at the sentence level by applying the consensus operator to the truth assignments on Θ produces fairly intuitive results for a reduced set of simple sentences. For the example sentences Θ where $|\Theta| = 2$ we see that our approach results in consensus at the sentence level, but that it may not result in consensus at the level of the valuations. We also saw that when $|\Theta| > |\mathcal{P}|$ (for the set of sentences Θ studied here), there are an increased number of possible consensus combinations which lead to an inconsistent truth assignment such that $\mathbb{V}_{\vec{t_1} \odot \vec{t_2}} = \emptyset$.

Perhaps more significantly, it is evident that the relative cardinality of $\mathbb{V}_{\vec{t}}$ (e.g. $\left|\mathbb{V}_{\vec{t}}\right|$), for different truth assignments $\vec{t}$ on Θ, has an impact on the convergence of the system. Specifically, we hypothesised that the population tends to converge to the truth assignment $\vec{t^*}$ such that the corresponding set of valuations has maximal cardinality, denoted by $\mathbb{V}_{\vec{t^*}}$ in Section 5.3.1. Furthermore, there tends to be convergence to the most precise truth assignments at both the sentence and valuation levels; particularly when there is more than one truth assignment $\vec{t^*}$ on Θ.

5.4 A consistent sentence-level consensus model for compound sentences

In Section 5.2 we considered a scenario in which agents communicated their truth assignments on the sentences of Θ while keeping their chosen underlying valuations private from one another. The idea is that this reflects a system in which agents are intent on forming consensus on certain compound sentences but are not principally concerned with forming consensus on their underlying beliefs (the valuations assigning truth values to the propositional variables). In this context, we proposed a consensus model in which pairs of agents initially applied the consensus operator to the truth assignments of the compound sentences to generate a new agreed truth assignment. The two agents then adopted the valuation on the propositional variables which was consistent with the new truth assignment but which was most similar to their previous beliefs; that is, most similar to their own previously held beliefs such that the two agents may not have adopted the same valuation as a result. The intuition here is that the two agents aim to make the minimal changes to their beliefs in order to reach consensus on the sentences that are important to them.

However, there is an alternative approach which, while also adopting the paradigm of minimal belief change, applies the consensus operator directly to valuations on propositional variables. More specifically, this second sentence-level consensus model would work as follows. The two agents broadcast their respective truth assignments on Θ; $\vec{t_1}$ and $\vec{t_2}$. Agent 1 then considers all valuations consistent with $\vec{t_2}$, i.e. $\mathbb{V}_{\vec{t_2}}$, and similarly agent 2 considers all valuations consistent with $\vec{t_1}$, i.e. $\mathbb{V}_{\vec{t_1}}$. Each agent then identifies the valuation most similar to their current beliefs but which is consistent with the other's truth assignment. They then apply the consensus operator to their current valuation and this 'most similar' valuation. More formally, agent 1 identifies

$$\mathbf{v}_1' = arg\ max\ \{S(\mathbf{v}_1, \mathbf{v}) : \mathbf{v} \in \mathbb{V}_{\vec{t_2}}\}$$

and then adopts the new valuation $\mathbf{v}_1 \odot \mathbf{v}_1'$. Similarly, agent 2 identifies

$$\mathbf{v}_2' = arg\ max\ \{S(\mathbf{v}_2, \mathbf{v}) : \mathbf{v} \in \mathbb{V}_{\vec{t_1}}\}$$

and then adopts the new valuation $\mathbf{v}_2 \odot \mathbf{v}_2'$. Note that there is no reason in general that, when $\vec{t_1} \neq \vec{t_2}$, $\mathbf{v}_1 \odot \mathbf{v}_1'$ and $\mathbf{v}_2 \odot \mathbf{v}_2'$ should correspond to the same valuation and furthermore, that they should generate the same truth assignment on Θ. In other words, at the agent-level this approach does not force a consensus. This is in contrast to the approach discussed in Section 5.2. In the following sections, however, we present a number of simulation experiments suggesting that this second approach results in the population nonetheless converging to consensus regarding the truth values of the sentences in Θ.

5.5 Simulation experiments for the consistent sentence-level consensus model

We use the same experimental set-up as in Section 5.3. That is, for a population of 100 agents we iteratively apply the process described in Section 5.4 for a total of 1 000 iterations, unless stated otherwise. We continue to adopt the bounded rationality requirement that the chosen pair of agents are sufficiently consistent according to an inconsistency threshold $\gamma \in [0, 1]$. Results are averaged over 100 independent runs.

5.5.1 Simulation results for $\Theta = \{p_1 \rightarrow p_2, p_1 \rightarrow \neg p_2\}$

Figures 5.16 and 5.17 show the average number of agents with truth assignments on the sentences in Θ and the average number of agents with valuations on the propositions p_1 and p_2, respectively. Both figures show results at steady state for different inconsistency thresholds γ after 3 000 iterations. We immediately notice that in contrast to the first model there is no longer clear

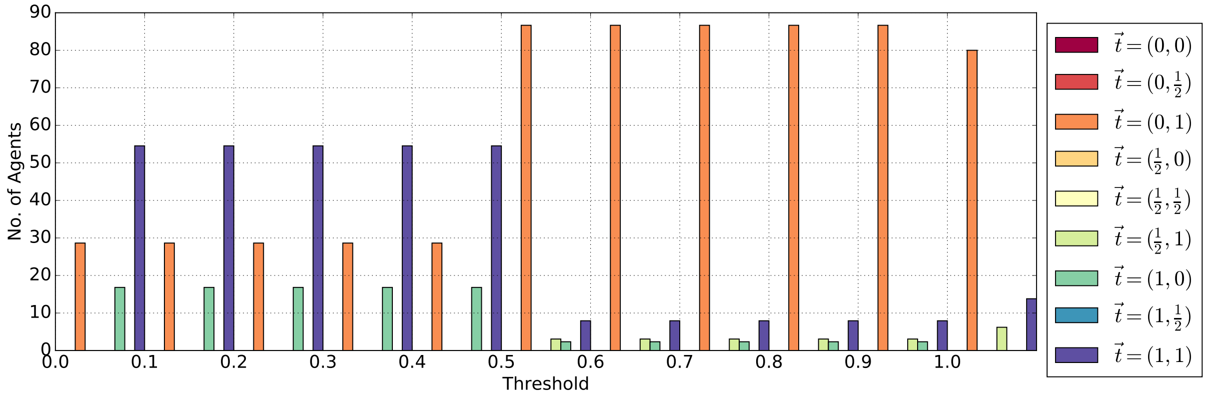


Figure 5.16: Number of agents with truth assignments on the sentences $\Theta = \{p_1 \to p_2, p_1 \to \neg p_2\}$.

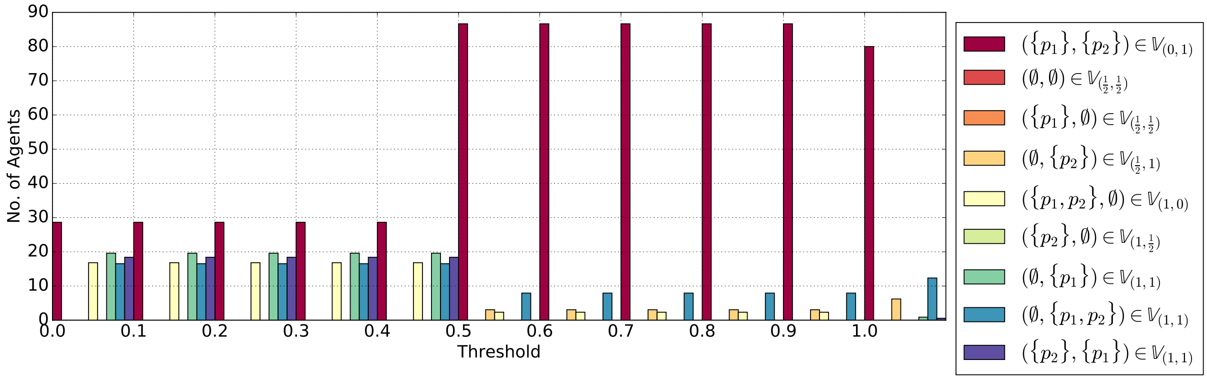


Figure 5.17: Number of agents with valuations on $\mathcal{L}$ for sentences $\Theta = \{p_1 \to p_2, p_1 \to \neg p_2\}$.

convergence to a single truth assignment, as seen in Figure 5.16. We do, however, see over 80% of the population adopting the same truth assignment at steady state for $\gamma \geq 0.5$. Figure 5.17 also suggests much stronger convergence at the valuation level in comparison to the results of the first model shown in Section 5.3.1.

To further study the convergence of this new model, we now study trajectory results for $\gamma = 1.0$ in order to gain a complete picture of the convergence properties of the current model, for the case where all randomly selected pairs of agents combine their beliefs. Figure 5.18 shows the average number of distinct truth assignments on Θ against iterations. In this new model, we can see that on average the population converges to 1.35 distinct truth assignments at steady state. This suggests that, while a majority of the time the population converges to a single truth assignment on Θ, there are others in which this is not the case. This is also true for extended tests in which we ran experiments beyond the 3 000 iterations shown here. Similarly, in Figure 5.19 the population converges to an average of 1.35 distinct valuations at steady state.

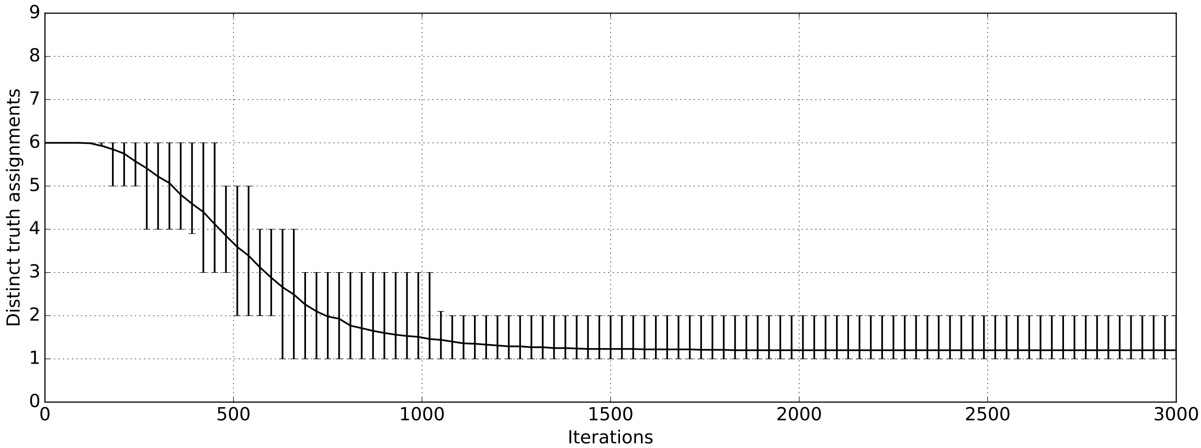


Figure 5.18: Number of distinct truth assignments on the sentences $\Theta = \{p_1 \rightarrow p_2, p_1 \rightarrow \neg p_2\}$ against iterations for an inconsistency threshold $\gamma = 1.0$.

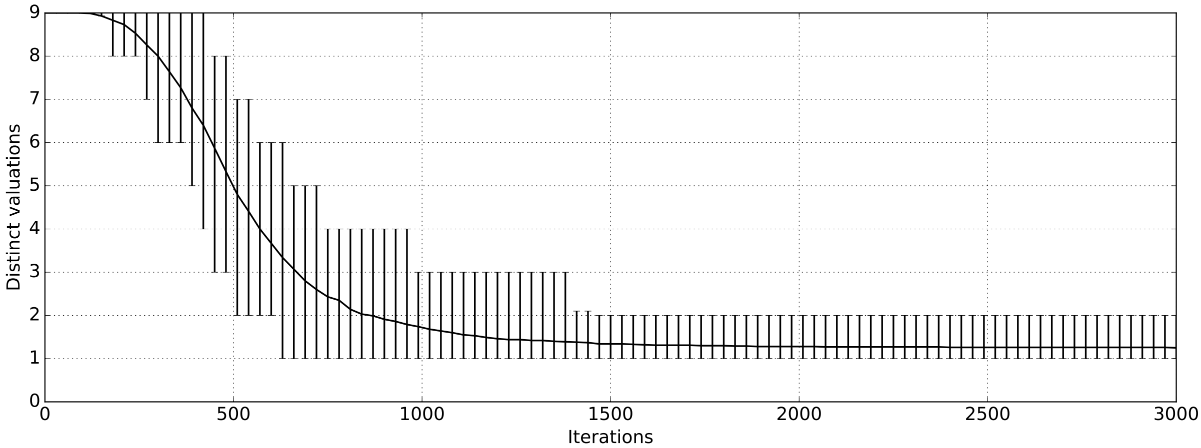


Figure 5.19: Number of distinct Kleene valuations on $\mathcal{L}$ against iterations for an inconsistency threshold $\gamma = 1.0$.

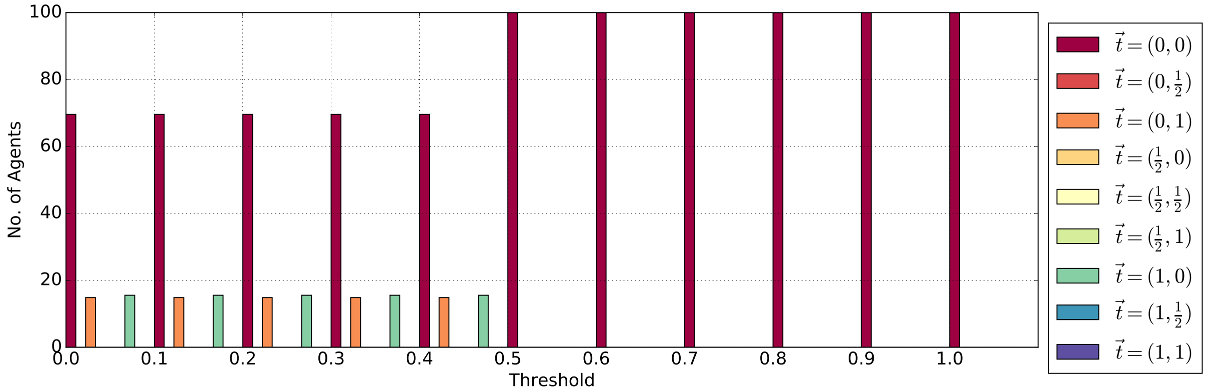


Figure 5.20: Number of agents with truth assignments on the sentences $\Theta = \{p_1 \wedge p_2, \neg p_1 \wedge \neg p_2\}$.

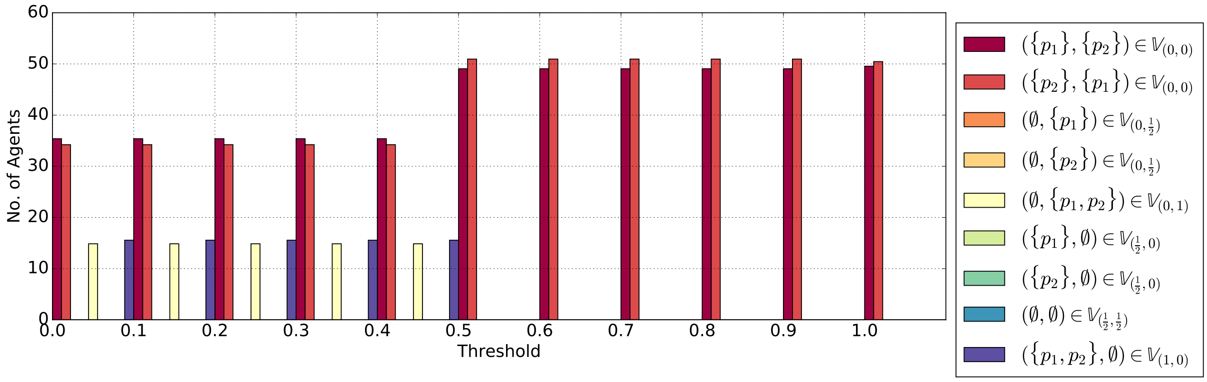


Figure 5.21: Number of agents with valuations on $\mathcal{L}$ for sentences $\Theta = \{p_1 \wedge p_2, \neg p_1 \wedge \neg p_2\}$.

5.5.2 Simulation results for $\Theta = \{p_1 \wedge p_2, \neg p_1 \wedge \neg p_2\}$

For the sentences $\Theta = \{p_1 \wedge p_2, \neg p_1 \wedge \neg p_2\}$ in Θ we see that Figures 5.20 and 5.21 differ very little under the new model, when compared with the results presented in Section 5.3.2. This is perhaps less surprising than for the previous set of sentences $\Theta = \{p_1 \rightarrow p_2, p_1 \rightarrow \neg p_2\}$ for which the majority of the population converged to a completely different truth assignment when compared with the previous model. Figure 5.20 shows the population converging to the same truth assignment $(0, 0)$ as seen in Figure 5.5. Indeed even for $\gamma < 0.5$ the system exhibits almost identical convergence behaviour, favouring $(0, 0)$ with small minorities adopting either of the truth assignments $(0, 1)$ or $(1, 0)$. When we look specifically at Figure 5.21 showing the average number of agents with valuations on p_1 and p_2, we see that for this set Θ, the population continues to converge on the truth assignment with the largest set of corresponding valuations i.e. that $\left|\mathbb{V}_{(0,0)}\right|$ is maximal amongst the precise truth assignments and, therefore, contains precise valuations for $\mathbf{v} \in \mathbb{V}_{(0,0)}$.

To identify whether the population does in fact reach a consensus on a single underlying valuation, and is simply ambivalent about either of the valuations $\mathbf{v} \in \mathbb{V}_{(0,0)}$, or whether the system remains split between both valuations, we examine the system's convergence more deeply in Figures 5.22 and 5.23. Specifically, in Figure 5.22 we see the number of distinct truth assignments on the sentences, on average, as a trajectory against iterations and for $\gamma = 1.0$. The system converges fully after just under 700 iterations, reaching a consensus about the truth assignment

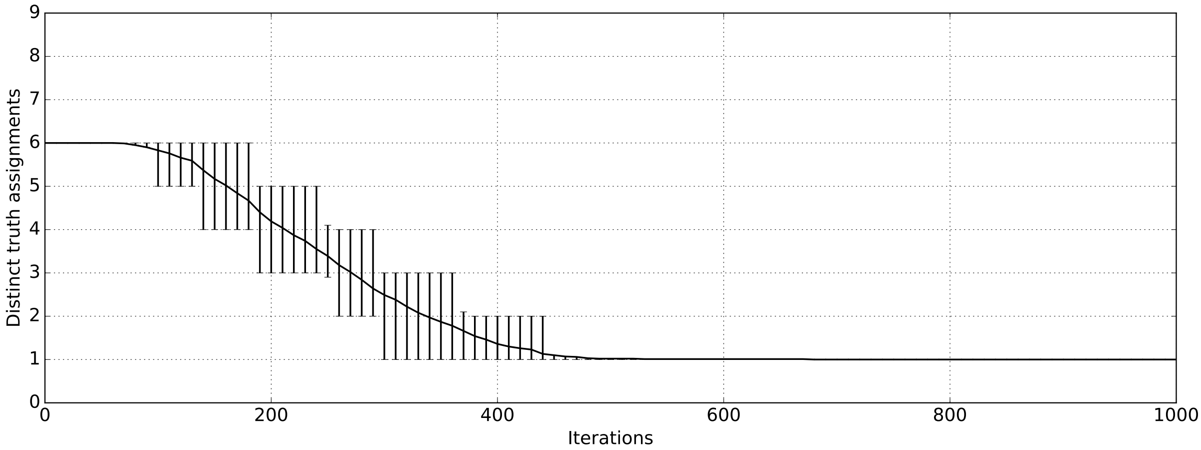


Figure 5.22: Number of distinct truth assignments on the sentences $\Theta = \{p_1 \wedge p_2, \neg p_1 \wedge \neg p_2\}$ against iterations for an inconsistency threshold $\gamma = 1.0$.

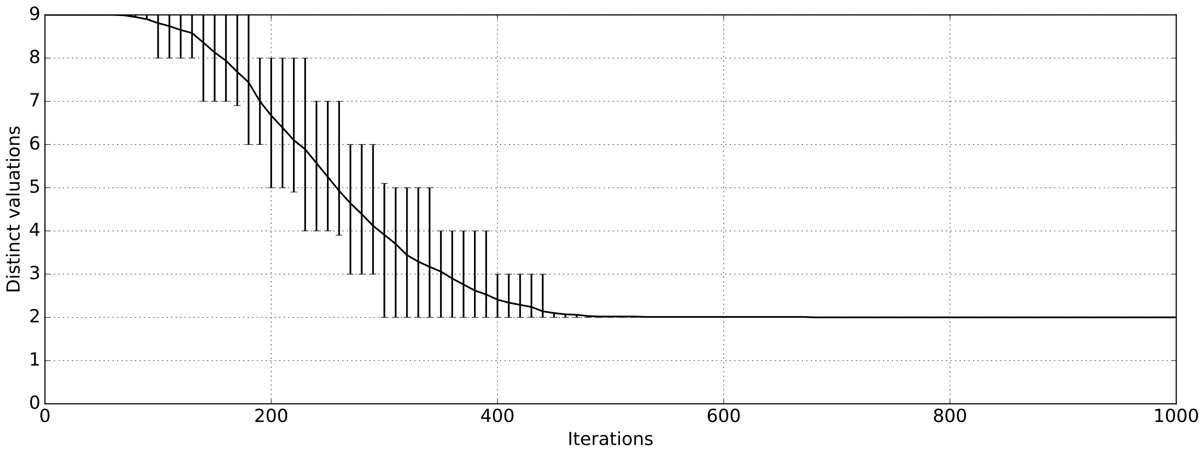


Figure 5.23: Number of distinct Kleene valuations on $\mathcal{L}$ against iterations for an inconsistency threshold $\gamma = 1.0$.

$(0, 0)$. In Figure 5.23, however, we can see that the system fails to form consensus at the valuation level, instead forming a split convergence on $(\{p_1\}, \{p_2\})$ and $(\{p_2\}, \{p_2\})$ with near-equal likelihood. This confirms that, under the new model, convergence for this set of sentences Θ is unchanged. Moreover it appears that convergence occurs in roughly the same amount of time for both models. This was certainly not the case for the previous set of sentences Θ. As such, we now turn our attention to a new set of sentences and assess whether convergence under this new model exhibits similar convergence properties as observed under the previous model.

5.5.3 Simulation results for $\Theta = \{p_1 \wedge p_2, p_1 \vee p_2\}$

In Figures 5.24 and 5.25 we see the average number of agents with associated truth assignments and valuations, respectively, at steady state. As for the previous set of sentences, with $\Theta = \{p_1 \wedge p_2, p_1 \vee p_2\}$ we see that the new model does perform almost identically under this new model as it did using the initial model introduced. Despite a change in application of the consensus operator, combining valuations rather than truth assignments, it would appear that

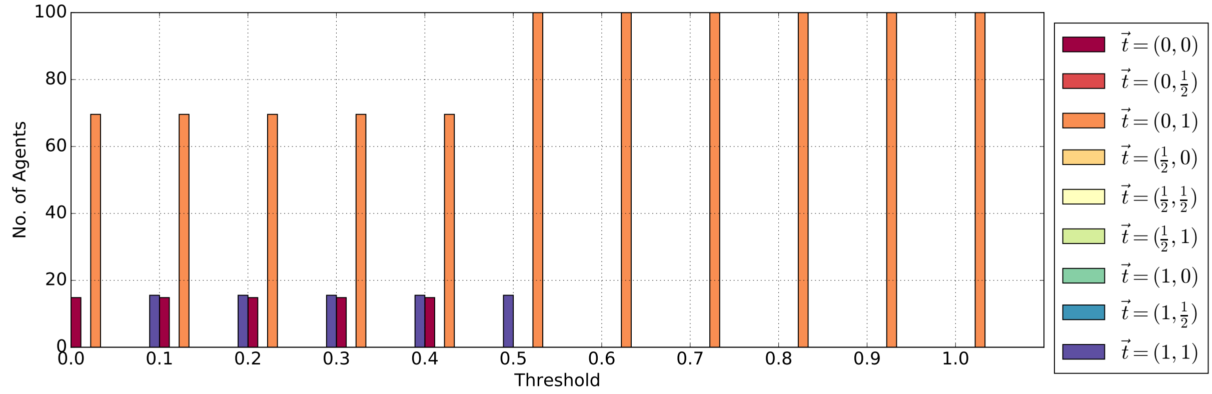


Figure 5.24: Number of agents with truth assignments on the sentences $\Theta = \{p_1 \wedge p_2, p_1 \vee p_2\}$.

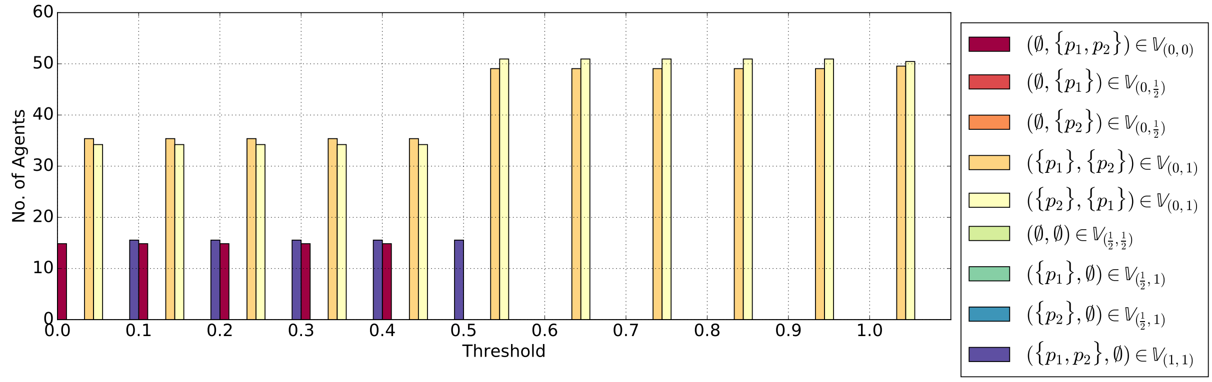


Figure 5.25: Number of agents with valuations on $\mathcal{L}$ for sentences $\Theta = \{p_1 \wedge p_2, p_1 \vee p_2\}$.

certain sets of sentences Θ behave very similarly under both models, while other sentences in Θ behave quite differently. In Figure 5.24 we see complete convergence to a single truth assignment $(0, 1)$ for $\gamma \geq 0.5$ just as was shown in Figure 5.9 from Section 5.3.3. For Figure 5.25 showing the number of agents with associated valuations as their beliefs, we again see that there is an equal split in the population at steady state between the two precise valuations associated with the truth assignment $(0, 1)$, those being $(\{p_1\}, \{p_2\})$ and $(\{p_2\}, \{p_2\})$. So it would seem that there is still an inherent bias in the population to gravitate towards the more precise truth assignment corresponding to the maximal set, where $\left|\mathbb{V}_{(0,1)}\right| = 2$ is maximal set, while $|\mathbb{V}_{0,0}| = |\mathbb{V}_{1,1}| = 1$ and $|\mathbb{V}_{1,0}| = 0$.

The trajectories of these experiments for $\gamma = 1.0$ are presented in Figures 5.26 and 5.27, showing the number of distinct truth assignments on Θ and distinct valuations on p_1 and p_2, respectively. When looking at these results, it is again clear that convergence to steady state occurs in under 700 iterations, slightly faster than under the initial model shown in figure 5.11. The population is quick to converge on the truth assignment $(0, 1)$ and both valuations with equal likelihood. Consensus at the sentence level is therefore achieved, but not at the underlying propositional level.

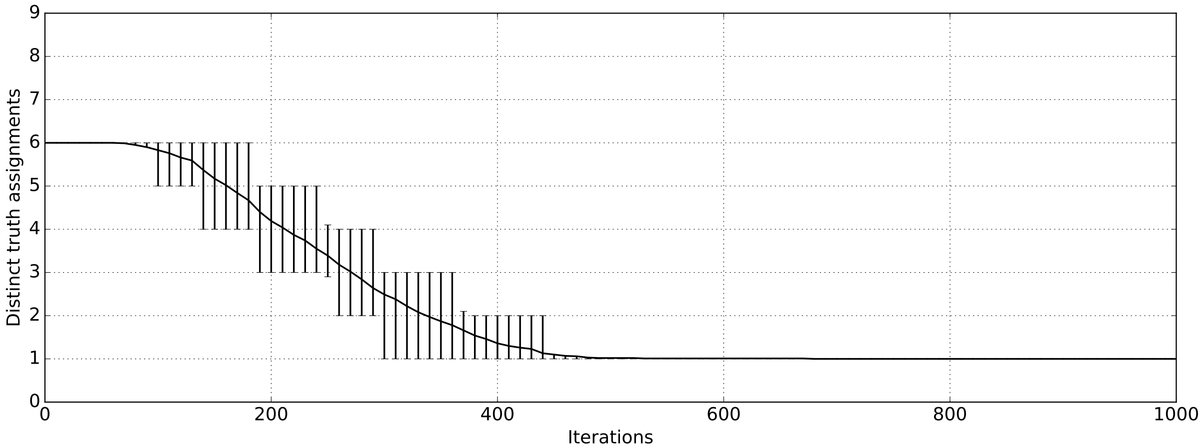


Figure 5.26: Number of distinct truth assignments on the sentences $\Theta = \{p_1 \wedge p_2, p_1 \vee p_2\}$ against iterations for an inconsistency threshold $\gamma = 1.0$.

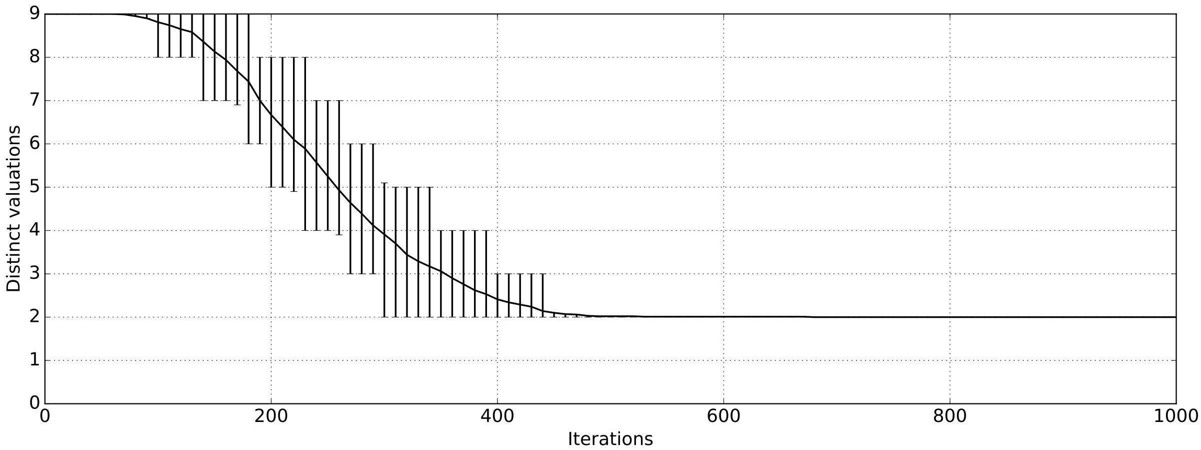


Figure 5.27: Number of distinct Kleene valuations on $\mathcal{L}$ against iterations for an inconsistency threshold $\gamma = 1.0$.

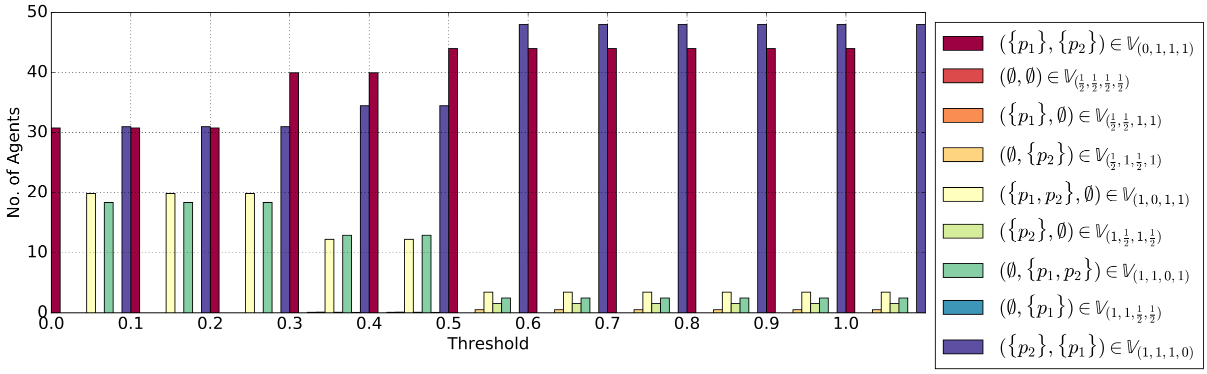


Figure 5.28: Number of agents with valuations on $\mathcal{L}$ for sentences $\Theta = \{p_1 \to p_2, p_1 \to \neg p_2, \neg p_1 \to p_2, \neg p_1 \to \neg p_2\}$.

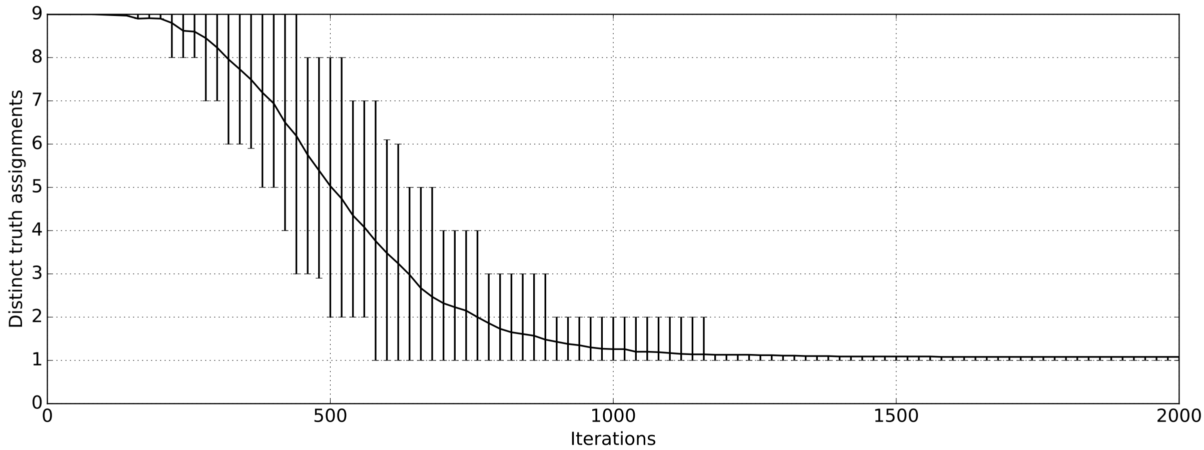


Figure 5.29: Number of distinct truth assignments on the sentences $\Theta = \{p_1 \to p_2, p_1 \to \neg p_2, \neg p_1 \to p_2, \neg p_1 \to \neg p_2\}$ against iterations for an inconsistency threshold $\gamma = 1.0$.

5.5.4 Simulation results for $\Theta = \{p_1 \to p_2, p_1 \to \neg p_2, \neg p_1 \to p_2, \neg p_1 \to \neg p_2\}$

Under the previous two sets of sentences Θ, both the first model and the new model behaved almost identically. However, for the initial set of sentences $\Theta = \{p_1 \to p_2, p_1 \to \neg p_2\}$ this was not the case. Therefore, as in Section 5.3, we now consider $\Theta = \{p_1 \to p_2,\ p_1 \to \neg p_2,\ \neg p_1 \to p_2,\ \neg p_1 \to \neg p_2\}$. As with the initial model's results, we consider only the valuation results at steady state given that the number of truth assignment combinations far exceeds our ability to present useful results in a similar format. For the purpose of highlighting the model's convergence under this set of sentences Θ, however, Figure 5.28 is more than sufficient. These experiments are extended to run for 2 000 iterations.

We see that the new model exhibits much stronger convergence than previously seen under the initial model. There were 4 valuations remaining at steady state across all values of γ under the initial model in Figure 5.13, and furthermore the population always converged to a single valuation with equal likelihood across all 4 precise valuations. While under this new model, however, we see that as γ increases, the population increasingly favours just two valuations above the other remaining valuations in the population. These are $(\{p_1\}, \{p_2\})$ and $(\{p_2\}, \{p_2\})$, the same valuations converged to for the previous two sets of sentences. This was perhaps not

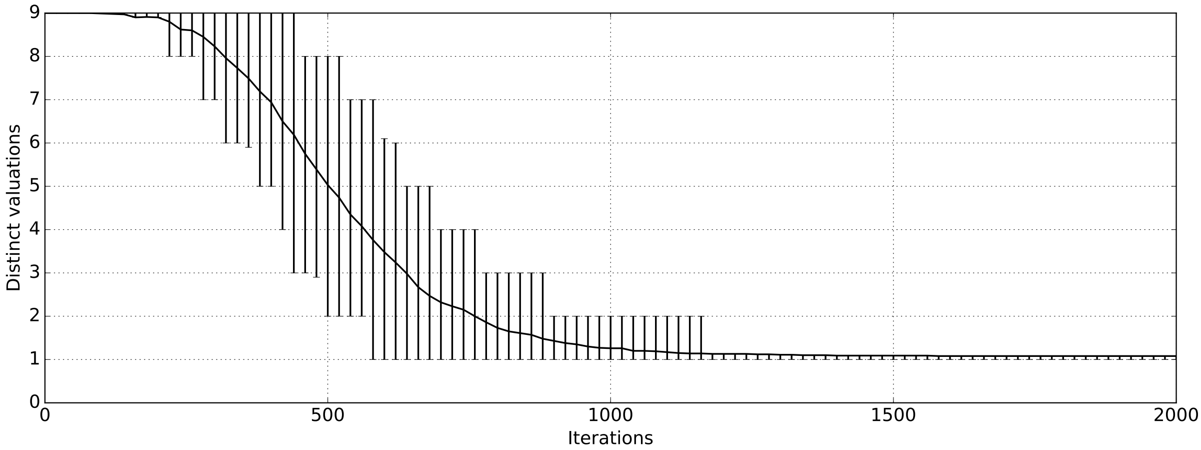


Figure 5.30: Number of distinct Kleene valuations on $\mathcal{L}$ against iterations for an inconsistency threshold $\gamma = 1.0$.

too surprising for the two previous sets as Θ, because these two valuations also reflected the maximal sets of truth assignments for which both the truth assignments and their corresponding valuations were precise (i.e. non-vague). The more surprising aspect of this model's convergence, then, is that these valuations are the most dissimilar from one another if one applies the similarity measure of Definition 5.1 to them. The precise valuations corresponding to equally precise truth assignments, that are no longer present in the final consensus of the population, include $(\{p_1, p_2\}, \emptyset)$ and $(\emptyset, \{p_1, p_2\})$ corresponding to the truth assignments $(1, 0, 1, 1)$ and $(1, 1, 0, 1)$, respectively. Both are equally as precise at both the sentence and valuation levels, yet under this new model of belief updating, are rarely present at steady state.

In Figures 5.29 and 5.30 we can see this more clearly, where it is observed that, for an inconsistency threshold $\gamma = 1.0$, the number of distinct Kleene valuations present in the population decreases to just above 1 after approximately 1600 iterations. This suggests that for a large majority of runs the population does converge to a single valuation, achieving consensus, and that for these runs the resulting valuation is in the set $\{(\{p_1\}, \{p_2\}), (\{p_2\}, \{p_2\})\}$ with a slight preference for $(\{p_2\}, \{p_2\})$. In a small number of runs, the model fails to fully converge to a single valuation and therefore a single truth assignment. This is true for experiments running for 4 000 iterations, where no further convergence to a single truth assignment nor valuation occurs.

5.6 Conclusions

Through simulation studies, we have highlighted several important properties of the proposed model of multi-agent consensus for compound sentences and how it differs from previous models of consensus restricted to propositional variables i.e. that convergence is not guaranteed for all sets of sentences Θ. In particular, we see that convergence at the sentence level does not guarantee convergence at the propositional level. We also note that convergence appears to favour the maximal set of valuations corresponding to a truth assignment that is precise at the sentence level, and that when no majority exists amongst precise truth assignments the population converges seemingly at random. We have also highlighted potential inconsistencies that can occur as a result of forming consensus as the sentence level, directly on the truth assignments on a set of sentences Θ, and have therefore proposed an alternative model that seeks to address this, while behaving in a similar manner as before, under the assumption that underlying valuations are not publicly shared amongst agents

Our initial model in Section 5.2 proved to be initially promising by showing strong convergence at the sentence level, achieving consensus on a single truth assignment in three of the four experiments studied. We also felt that this would be the most intuitive, broadly speaking, for a system in which agents are concerned with forming consensus at the sentence level and not necessarily concerned with forming consensus on the propositions. However, due to the issues highlighted for more complex sets of compound sentences, where truth assignment combinations can lead to inconsistencies, it is clear that this form of consensus will not scale well with the number of sentences, nor with the increase in complexity of sentences.

In Section 5.4 we introduced a more familiar model of consensus where the consensus operator is applied to the underlying valuations. Inconsistencies arising in the initial model provided sufficient justification to approach consensus in a similar manner as in Section 4. We referred to this model as belief updating, distinguishing this asymmetric process of updating agents' beliefs from that of symmetric consensus formation between pairs of agents, where both agents adopt the same consensus valuation. Under this new model, some experiments performed similarly in terms of convergence; both at the sentence level and at the propositional level. For other experiments, convergence was much improved over the initial model. In particular, for sentences $\Theta = \{p_1 \to p_2, p_1 \to \neg p_2, \neg p_1 \to p_2, \neg p_1 \to \neg p_2\}$ we saw convergence to a smaller subset of truth assignments, with the population forming consensus on a single truth assignment and valuation for the majority of the runs, while sometimes remaining split between two truth assignments in others.

Further analysis of the belief updating model is required, including studying the model with increased numbers of propositional variables and sentences to determine what kind of consensus, if any, is achieved. However, we believe that this model presents a promising basis for consensus of compound sentences and, given previous extensions of the consensus operator, we believe that it may be possible to combine it with a probabilistic model of uncertainty to allow agents to express both vagueness and uncertainty in their beliefs. In comparison to the related work of Cholvy [7, 8], we believe that by allowing agents to combine at random and limiting interactions only by their relative inconsistency, we avoid a seemingly arbitrary preference ordering being assigned to the population for each agent. We also favour pairwise interactions in allowing beliefs to change more naturally over time, as we feel this is more intuitive as opposed to attempting to aggregate a large set of beliefs at once and allows for the beliefs of others to also change over time.

CHAPTER 6

Conclusions

Reaching a consensus is often a precursor to distributed decision-making and is therefore an important aspect of multi-agent systems. In this thesis, we have argued that a natural route to consensus is to exploit vagueness in propositions by adopting a more vague interpretation of the underlying concepts about which two (or more) agents disagree. To this end, we have introduced a third (intermediate) truth state to enable an agreement between agents with opposing or conflicting opinions which, for Boolean propositions, would be inconsistent. We have then studied this model in both a multi-agent and a swarm robotics setting, and we have tried to highlight both the strengths and the limitations of this approach in the context of distributed decision-making.

In Chapter 2 we developed a three-valued model of consensus in a multi-agent setting, where a large population of agents sought to reach an agreement about a shared set of relevant propositions. By iteratively applying a pairwise consensus operator, we showed that the population of agents reached consensus on a single, shared opinion which was completely precise (i.e. admitting no borderline cases). We also introduced the notion of payoff in which during the consensus process, agent selection would be biased in favour of those agents whose opinions more accurately reflected the 'true' state of the world. In this case, the population converged to a more accurate opinion when compared with a Boolean version of our consensus model. An emergent property resulting from adopting a third truth value was that robustness to the presence of malfunctioning agents was improved when compared with a Boolean model, as detailed in Chapter 4.

As well as agents holding vague opinions, we have argued that agents also need to be able to express uncertainty about the underlying state of the world. In Chapter 3 we have extended the model from Chapter 2 so as to take account of epistemic uncertainty. In simulation experiments we showed that the population converged to a belief that was completely certain, as well as precise. Using this model, we explored how the process of consensus formation improved evidence dissemination in a multi-agent setting. We compared a model in which agents randomly received direct evidence at a given rate, and contrasted this evidence-only approach with a combination of the same evidence rate with random pairwise consensus. The results confirmed that the combination of consensus formation and evidential updating outperformed the evidence-only model.

In Chapter 4 we applied the three-valued approach to consensus of Chapter 2 to distributed decision-making in swarm robotics, specifically for a swarm of 400 Kilobot robots. We adapted our three-valued model and applied it to the 'best-of-n' problem; a decision problem in which agents must reach an agreement about which is the best of n possible choices. Here, rather than pairwise consensus, agents were split between signalling and updating states, which allowed for multiple agents to update their beliefs per iteration. In this context, we compared our approach to the weighted voter model [73], a Boolean model of asymmetric belief-updating also studied for Kilobot swarms [74], and conducted both simulation and physical experiments for both. We demonstrated that the three-valued model converged with similarly high accuracy as the weighted voter model and, although it was slower to reach a consensus, it proved to be more robust to the presence of malfunctioning agents.

Finally, in Chapter 5 we argued that consensus is not always required at the level of the propositional variables in the language. Instead, agents may need to reach an agreement about a set of compound sentences e.g. the antecedents of a set of decision-rules, which are central to decision-making. We show that it is possible to extend the three-valued approach to compound sentences and to reach a consensus at the level of the sentences while not necessarily forcing agents to reach an agreement at the level of the underlying propositional variables.

The research in this thesis has suggested a number of avenues of future research. We now summarise several of these in no particular order.

We proposed a three-valued model for group consensus in Chapter 2 in an attempt to speed up consensus formation in large agent populations. From these results, it is clear that the iterative pairwise combinations of the group's opinions, while natural, is a relatively ad hoc extension of a model ideally suited to pairwise combinations. The resulting model led to greatly improved convergence speeds, but the accuracy of the model suffered as a consequence. The pairwise model, while accurate, is much slower to converge. It would therefore be interesting to try to identify a group-wide consensus operator which leads to increased speeds of convergence without affecting the overall accuracy of the model. In Chapter 4 we considered an asymmetric model of belief-updating in which, for a pair of interacting agents, only one agent adopts a new opinion. While this model also led to faster convergence for large agent populations, and showed great accuracy in the context of the 'best-of-n' problem, the agents cannot be said to be forming 'consensus', as only one of the agents alters their opinion. Nevertheless, the work of this chapter may inspire alternative models for increasing the speed of consensus while maintaining a high level of accuracy.

In this thesis, consensus has been defined as convergence to a single, shared belief state. However, there are scenarios in which total unanimity may not be desirable. For example, in an environmental disaster it may be necessary for a swarm of robots to make a collective decision about which source of pollution they should prioritise for treatment. Alternatively, the swarm may need to identify multiple sources and divide themselves into smaller groups so that each group can be allocated to a source of pollution. These two scenarios were categorised by Brambilla et al. [3] as *consensus achievement* and *task allocation*, respectively. However, we would argue that both scenarios require the robots to reach a consensus regarding the state of the world they inhabit, and then involve making collective decisions based upon this consensus. Therefore, it would be interesting to see the development of an alternative consensus operator, which would enable the swarm to form a *proportional* consensus where the sizes of the minority groups are proportional to the desirability of each outcome or choice.

While consensus is important when making distributed decisions, the resulting consensus is often unalterable once achieved. For example, the introduction of a minority group of agents would be unsuccessful in disseminating potentially relevant, more up-to-date information to the swarm as the majority group would simply overwhelm the minority group. Furthermore, the environment is often changing as a consequence of the agents' actions and also due to external factors. Consequently, it would be interesting to develop models that either reach only partial consensus, allowing agents to adapt to an influx of new opinions, or models of consensus which allow agents to 'forget' or to revert back to a more uncertain or unbiased state in which they are able to incorporate new information, received either from other agents or directly from their environment. Indeed, it may be necessary to separate different levels of evidence, such that direct evidence, perhaps received via sensory modalities from the environment, has a higher weighting than indirect evidence received from other agents.

From the work described in Chapter 4, it is clear that much of the literature concerning the best-of-n problem is concerned primarily with the $n = 2$ case. From our preliminary work on extending both the weighted voter model and the three-valued model, it became apparent that scaling the models to larger values of n is potentially problematic. We have highlighted how increasing the size of n seems to require a similar increase in the variation of quality values, to enable the population to continue to select the best choice. Clearly, then, as n increases, so too must the time agents spend in the signalling state. This will lead to much slower convergence times which, depending on the intended applications of the swarm, may be detrimental to their performance. We would therefore like to suggest further work which considers scaling these consensus algorithms to larger values of n, and perhaps proposals for alternative methods for consensus. A potential solution is to adopt beliefs in the form of preferences, in place of single-choice beliefs. Agents would then be able to communicate more information in the form of an incomplete (total/partial) ordering on the n choices, or possibly a subset of the n choices. Rather than requiring discrete quality values for each choice, agents would be able to infer choice qualities relative to their position in the orderings, and this could therefore avoid the need for varying signalling durations entirely. Of course, the new consensus operator would need to be capable of merging possibly conflicting preference orderings.

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