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Hölder Dimension with Applications to Hyperbolic Groups

Hölder Dimension with Applications to Hyperbolic Groups

By

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Abstract

We introduce the 'Hölder dimension' of a metric space as the infimum of Hausdorff dimensions of Hölder equivalent spaces. This definition is a Hölder variant of the definition of conformal dimension, the study of which serves as inspiration for some of our work.

We present two main results, along with supporting results and examples. Firstly, in Chapter 4, we show that capacity dimension is an upper bound for the Hölder dimension of any compact, doubling metric space with finite capacity dimension. Consequently, for locally self-similar metric spaces, Hölder dimension is equal to topological dimension. We explore some caveats of this result in Chapter 5 where we give examples of Hölder dimension equalling topological dimension but only as a strict infimum, not a minimum, and an example of a space, which is not locally self-similar that has Hölder dimension strictly greater than its topological dimension. Secondly, in Chapter 6, we explain how to reduce understanding the Hölder dimension of boundaries of hyperbolic groups to understanding the Hölder dimension of the boundaries of hyperbolic groups with connected boundary.

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Beauty lies in the eyes of the bi-Hölder.

Author's declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

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Chapter 1

Introduction

A nice and motivating example to which the content of this thesis applies is that of hyperbolic groups, which are the main focus of Chapter 6, or more generally Gromov hyperbolic metric spaces. These are spaces that exhibit a coarse notion of negative curvature called ‘Gromov hyperbolicity’, which was introduced by Gromov in [Gro87] (see Definition 6.3.4). Such a definition that approximates an old concept in a new setting is useful because it means we can draw inspiration from the tools and techniques used to study the classical notion to build results in our new setting. One such concept that translates over particularly well is that of ‘boundaries’. That is, Gromov hyperbolic spaces have a ‘Gromov boundary’ in a similar way to how classical hyperbolic spaces have a ‘boundary at infinity’.

The Gromov boundary of a Gromov hyperbolic space is a bounded, complete metric space whose ‘fine-scale analysis’ corresponds to the ‘large-scale geometry’ of the Gromov hyperbolic space. This has been made formal thanks to contributions from Paulin [Pau96], Elek [Ele97], Bonk-Schramm [BS00], and Bourdon-Pajot [BP03]. One can interpret “large-scale geometry” and “fine-scale analysis” in a few different ways. In Geometric Group Theory one usually interprets “large-scale geometry” as the information encapsulated by the ‘quasi-isometry’ class of a given space. Under this correspondence we are led to interpret “fine-scale analysis” as the information encapsulated by the ‘quasi-symmetry’ class of the given space. Quasi-symmetries are homeomorphisms with some control on metric distortion, and are defined in Section 2.1. They are a generalisation of quasi-conformal maps to general metric spaces that don’t necessarily have a smooth structure. In [Hei01, Chapter 10], Heinonen describes them loosely as approximately preserving the shape of objects without much control on distortion of size, in comparison to bi-Lipschitz maps which distort both the shape and size of an object by a bounded amount.

So we want to understand the quasi-symmetric data of bounded, complete metric spaces, but this is exceptionally difficult in such a general setting. However, we can make the goal more attainable by restricting to nicer spaces. In particular, if we are considering the boundary of a hyperbolic group, then the metric wildness of this space is constrained. We derive from boundaries of hyperbolic groups some restrictions, which are ‘reasonable’ because of their natural occurrence in these spaces. For instance, boundaries of hyperbolic groups are compact, and the action of the group on itself allows one to show that its boundary exhibits a weak form of self-similarity known as “locally self-similar”, see Definition 4.2.6. Together, these properties suggest that we should be able to employ tools from Fractal Geometry to understand boundary-like spaces.

A powerful tool in Fractal Geometry is the notion of dimension. Broadly, these are concepts that give a sense of how large a space is, and they are most useful when they are invariant under an important equivalence of spaces. For instance, a common notion of dimension is ‘Hausdorff dimension’, which essentially measures how the ‘volume’ of subsets of a space scales with the diameter of those subsets. Intuitively, the area of a disc in $\mathbb{R}^2$ grows like the square of its diameter so $\mathbb{R}^2$ has Hausdorff dimension 2, and the area of a ball in $\mathbb{R}^3$ grows like the cube of its diameter so $\mathbb{R}^3$ has Hausdorff dimension 3. More generally, Hausdorff dimension can take any non-negative real value. Hausdorff dimension is an invariant of bi-Lipschitz equivalence, so, in particular, $\mathbb{R}^2$ and $\mathbb{R}^3$ are not bi-Lipschitz equivalent. Of course, there are other ways of seeing this, for instance they are not even topologically equivalent, but the concept of distinguishing spaces up to a given equivalence by an invariant notion of dimension is one that applies extremely broadly.

Alas, Hausdorff dimension is not an invariant of quasi-symmetric equivalence. However, we can tailor Hausdorff dimension to quasi-symmetric equivalence by considering how Hausdorff dimension can change under quasi-symmetric equivalence. That is, for a given space X, consider the infimum of the Hausdorff dimensions of spaces that are quasi-symmetrically equivalent to X, and call this infimum the *conformal dimension* of X, denoted $\mathrm{Confdim}(X)$. As we have chosen a single value for each quasi-symmetric equivalence class, we have artificially forced conformal dimension to be an invariant of quasi-symmetric equivalence. Conformal dimension was defined by Pansu in [Pan89b] and originally used to study rank one symmetric spaces, and is now also used to study boundaries of hyperbolic groups, and other fractal metric spaces, see [MT10].

A priori, one might be concerned with how much information we have lost by reducing to the conformal dimension, a non-negative real number, but this interpretation belies the power of this tool. For instance, conformal dimension is a finer invariant than topological dimension, but just knowing the topological dimension of the boundary of a group can already tell you some of its algebraic structure. An example of this is the following theorem, which can be found in [KB02, Theorem 8.1].

Theorem 1.0.1 ([Gro87], [CDP90], [GdLH90])**.** *Let Γ be a non-elementary hyperbolic group. Then the following are equivalent.*

- *The boundary of* Γ *has topological dimension* 0.
- Γ *is virtually free*
- *The boundary of* Γ *is homeomorphic to the Cantor set.*

The restriction "non-elementary" simply rules out the cases when Γ is finite and has empty boundary, or is virtually $\mathbb{Z}$ and has boundary equal to a pair of points. Here, knowing the topological dimension of the boundary is 0 gives you enough information to quite clearly understand Γ, however there are many situations where topological dimension is too coarse to usefully distinguish boundaries. In such situations, we look to conformal dimension as it can often distinguish what topological dimension cannot. An example of this, which will be discussed in more detail in Section 3.2, is the rank one symmetric spaces of non-compact type. These are hyperbolic spaces that have topological spheres of varying dimension as their boundaries, but with different metric structures. For example, 4-dimensional real hyperbolic space, $H^4_{\mathbb{R}}$, and 2-dimensional complex hyperbolic space, $H^2_{\mathbb{C}}$, both have topological 3-spheres as their boundary, so cannot be distinguished by the topology of their boundaries, but $\mathrm{Confdim}(\partial_\infty H^4_{\mathbb{R}}) = 3$ whereas $\mathrm{Confdim}(\partial_\infty H^2_{\mathbb{C}}) = 4$, see [MT10, Theorem 3.3.3]. This means that $\partial_\infty H^4_{\mathbb{R}}$ is not quasi-symmetric to $\partial_\infty H^2_{\mathbb{C}}$ and therefore $H^4_{\mathbb{R}}$ is not quasi-isometric to $H^2_{\mathbb{C}}$. In particular, this means that no group that acts 'geometrically' on $H^4_{\mathbb{R}}$ can be quasi-isometric to a group acting geometrically on $H^2_{\mathbb{C}}$.

Now we have some motivation to study the conformal dimension of boundary-like metric spaces, but this task is still quite difficult. The correspondence between the quasi-isometric data of Gromov hyperbolic spaces and the quasi-symmetric data of bounded, complete metric spaces gives perfect exchange of information, so, in a sense, understanding boundaries up to quasi-symmetric equivalence is just as hard as understanding Gromov hyperbolic spaces up to quasi-isometry. The reason one studies groups up to quasi-isometry in Geometric Group Theory in the first place is because one really wants to understand groups up to isomorphism, but this is incredibly difficult so one sacrifices some information for the easier task of understanding groups up to quasi-isometry. The huge strides made in understanding infinite groups via this strategy justifies this weakening of an equivalence as a valid approach. There are some potential candidates for weakening quasi-isometric equivalence further, for example sublinear biLipschitz equivalence (SBE) which was introduced by Cornulier, see [Cor17], and worked on by Pallier, see [Pal18] and [Pal19], but if we instead think about weakening quasi-symmetric equivalence, we find another natural candidate:

For bounded, uniformly perfect metric spaces, of which boundaries of non-elementary hyperbolic groups are examples, quasi-symmetric equivalences are also Hölder equivalences, see Lemma 2.1.4. Therefore, we can attempt to make the task of understanding boundary-like metric spaces up to quasi-symmetric equivalence easier by understanding their Hölder structure. Recall that a homeomorphism $f\colon X \to Y$, between metric spaces (X, d_X) and (Y, d_Y), is a

(λ, α, β)-*bi-Hölder* homeomorphism for $\lambda \geq 1$ and $\alpha, \beta > 0$ if, for any $x_1, x_2 \in X$,

$$\frac{1}{\lambda} d_X(x_1, x_2)^\alpha \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2)^\beta.$$

If such a map exists, X and Y are said to be *Hölder equivalent*.

In [Cor17, Theorem 1.7], Cornulier showed that SBEs induce Hölder equivalence on boundaries. This fact inspired Ilya Kapovich, in June 2017, to ask what happens when, in the definition of conformal dimension, one replaces quasi-symmetric maps with bi-Hölder maps. This led me to define a Hölder parallel of conformal dimension, which I introduced in [Col19, Definition 1.6]:

Definition 1.0.2. Let X be a metric space. Define the *Hölder dimension* of X, denoted $\text{Höldim}(X)$, by

$$\text{Höldim}(X) := \inf\{\dim_H(Y) \mid Y \text{ Hölder equivalent to } X\},$$

where $\dim_H(Y)$ denotes the Hausdorff dimension of the metric space Y. We say that X *attains* its Hölder dimension if there exists a space Y Hölder equivalent to X such that $\dim_H(Y) = \text{Höldim}(X)$.

Fundamentally, this thesis attempts to address the broad question:

Question 1.0.3. What can be said about the Hölder dimension of a boundary-like metric space?

There are some obvious initial observations one can make. For instance, if $f \colon X \to Y$ is a (λ, α, β)-bi-Hölder homeomorphism, one obtains the following bounds relating the Hausdorff dimensions of X and Y:

$$\frac{\dim_H(X)}{\alpha} \leq \dim_H(Y) \leq \frac{\dim_H(X)}{\beta}. \tag{1.0.3.1}$$

We can also find bounds by relating Hölder dimension to other common notions of dimension. For instance, in [Szp37, Theorem 2], Szpilrajn shows that the important classical notion of 'topological dimension', see Definition 2.2.2, is a lower bound for Hausdorff dimension. Therefore topological dimension is also a lower bound for Hölder dimension.

For bounded, uniformly perfect metric spaces, the Hölder equivalence class of a metric space includes the space's quasi-symmetric equivalence class, see Lemma 2.1.4, meaning that, under these weak assumptions, Hölder dimension is a lower bound for conformal dimension. Therefore, we see that, for any bounded, uniformly perfect space X,

$$\dim_T(X) \leq \text{Höldim}(X) \leq \text{Confdim}(X) \leq \dim_H(X). \tag{1.0.3.2}$$

Uniform perfectness is a condition that restricts the wildness of the isolation of subsets, see Definition 2.1.5. Examples include connected spaces, as no subset is isolated from the rest of the space, and boundaries of non-elementary hyperbolic spaces, where any isolation is controlled by the 'local self-similarity'.

Szpilrajn further proves that any metric space X is topologically equivalent to a space whose Hausdorff dimension is equal to the topological dimension of X, see Theorem 4.1.3. In particular, this means that if we infimize Hausdorff dimension over topological equivalence, then we are left with topological dimension. Not only that, but this infimum is actually a minimum. Szpilrajn's result makes sense if one interprets the act of infimizing Hausdorff dimension over an equivalence as "forgetting" all structure on X that is not preserved by that equivalence. Thus, infimizing over topological equivalence should leave us with a notion that is depends only on topological structure, such as topological dimension. Chapter 4 extends Szpilrajn's result and concludes that, among other results, for 'locally self-similar' metric spaces, one does not need the full wildness of topological equivalences to lower the Hausdorff dimension to be arbitrarily close to the topological dimension, instead it is enough to use Hölder equivalences. That is,

Corollary 1.0.4 (Corollary 4.1.2)**.** *If X is a compact, locally self-similar metric space, then X has Hölder dimension equal to its topological dimension.*

However, the Hölder dimension is not always attained as it was for topological dimension. For instance, the 1/3-Cantor set, see Example 3.1.1, has Hölder dimension 0 but every Hölder equivalent space has strictly positive Hausdorff dimension by (1.0.3.1).

In Chapter 2, we give some background that is common to the entire thesis, and delegate other background to the relevant chapters.

In Chapter 3, we present some initial observations about Hölder dimension that one can make without too much effort. For instance, we explain how existing results combine to give that Hölder dimension inherits the gap that conformal dimension has between 0 and 1, in the case that the given space is bounded. That is,

Theorem 1.0.5 (Theorem 3.3.5)**.** *The Hölder dimension of any bounded, separable metric space is either zero or at least one.*

In Chapter 4, we give an upper bound for the Hölder dimension of metric spaces that have some finiteness properties. In particular, we prove the following corollary.

Corollary 1.0.6 (Corollary 4.1.5)**.** *If X is a compact, doubling metric space with capacity dimension n, then X has Hölder dimension at most n.*

Capacity dimension is similar to topological dimension but with added metric control, see Definition 4.2.5. This upper bound is a consequence of Theorem 4.1.4, which we prove by constructing bi-Hölder embeddings of the given metric space, X, into ℓ^2, an infinite-dimensional separable Hilbert space, with the Hausdorff dimension of the image bounded above by a value arbitrarily close to the capacity dimension of X. To build these maps, we use the finiteness conditions imposed on X to approximate it by nice covers at a sequence of scales. We then use these covers to build maps that approximate X in ℓ^2 by simplicial complexes. From these nice covers, the approximating maps inherit metric control and their images inherit dimension control. These approximation maps limit to the aforementioned bi-Hölder embedding of X into ℓ^2.

Chapter 5 utilizes Cantor sets to build examples that illustrate the sharpness of the main result of Chapter 4. Namely, Theorem 5.1.1 gives a family of examples that show that, for each $n \in \mathbb{N}$, there exists a space with Hölder dimension n that does not attain its Hölder dimension; and Theorem 5.1.2 gives an example of a non-self-similar Cantor set with differing Hölder and topological dimensions. The proof of Theorem 5.1.1 draws inspiration from an important technique in the study of conformal dimension for producing lower bounds, refined by Bourdon in [Bou95, Lemma 1.6] from ideas formulated by Pansu in [Pan89a, Proposition 2.9] and [Pan89b, Lemma 6.3]. The example constructed for Theorem 5.1.2 draws inspiration from [Hak06] in which Hakobyan concludes there exist Cantor sets of Hausdorff dimension 1 that are minimal for conformal dimension. We construct a Cantor set that is a positive proportion of the unit interval $[0, 1]$ and has gaps that shrink faster than any power. Being a positive proportion of the unit interval means the Cantor set has Hausdorff dimension equal to 1, and the gaps shrinking faster than any power means this property cannot be disrupted by a Hölder map.

Chapter 6 gives a structural result for the Hölder dimension of boundaries of hyperbolic groups, which relates the Hölder dimension of the boundary of a hyperbolic group to the Hölder dimension of the components of that boundary. The main theorem, Theorem 6.1.1, of this chapter reduces the classification of the Hölder dimension of boundaries of hyperbolic groups to the one-ended case. In Section 6.9, we explain how it derives from the following:

Theorem 1.0.7 (Theorem 6.2.1)**.** *Let G_1 and G_2 be infinite hyperbolic groups that are not both virtually cyclic, and let $\Gamma = G_1 * G_2$. Then,*

$$\text{Höldim}\,(\partial_\infty \Gamma) = \max\{\text{Höldim}\,(\partial_\infty G_1)\,, \text{Höldim}\,(\partial_\infty G_2)\}.$$

Further, $\partial_\infty \Gamma$ attains its Hölder dimension if and only if $\partial_\infty G_i$ attains its Hölder dimension for each $i = 1, 2$ such that $\text{Höldim}(\partial_\infty G_i) = \text{Höldim}(\partial_\infty \Gamma)$.

Here, and throughout, $\partial_\infty G$ denotes the 'Gromov boundary' of a group G, see Section 6.3. The core idea is to use Hölder data of $\partial_\infty G_1$ and $\partial_\infty G_2$ along with the 'tree-of-spaces' structure that Γ exhibits from this free-product decomposition to build a new hyperbolic space whose boundary is bi-Hölder to $\partial_\infty \Gamma$. In building this space, we use concepts developed by Bonk and Schramm in [BS00].

Finally, in Chapter 7, we motivate and discuss some potential future directions of research regarding Hölder dimension.

CHAPTER 2

Metric Spaces and Dimension

This chapter provides standard background that is used throughout this thesis. We have delegated chapter-specific background to background sections within their respective chapters, see sections 4.2, 5.2, and 6.3.

2.1 Types of fine metric control

In this section, we present some standard definitions pertaining to the fine-scale analysis of metric spaces.

Definition 2.1.1 ([Mat95, Chapter 7])**.** For metric spaces (X, d_X) and (Y, d_Y), a map $f\colon X \to Y$ is called λ-*Lipschitz continuous*, for $\lambda \geq 1$, if, for any $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2).$$

If, in addition, for any $x_1, x_2 \in X$,

$$\frac{1}{\lambda} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)),$$

then f is a λ-*bi-Lipschitz embedding*. Finally, if, in addition, f is surjective, then f is a λ-*bi-Lipschitz* homeomorphism.

We will primarily concern ourselves with a form of metric control known as Hölder equivalence. Hölder continuity is prevalent in functional analysis, analysis on metric spaces, partial differential equations, and many other areas of mathematics. Furthermore, the kinds of metric spaces we will be interested in, like boundaries of hyperbolic groups, have a self-similarity property, see Definition 4.2.6, that behaves nicely under this kind of metric distortion. We use the following notation for Hölder maps.

Definition 2.1.2 ([Mat95, Chapter 7])**.** For metric spaces (X, d_X) and (Y, d_Y), a map $f\colon X \to Y$ is called (λ, β)-*Hölder continuous*, for $\lambda \geq 1$ and $\beta > 0$, if, for any $x_1, x_2 \in X$,

$$d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2)^\beta.$$

If, in addition, there exists $\alpha > 0$ such that, for any $x_1, x_2 \in X$,

$$\frac{1}{\lambda} d_X(x_1, x_2)^\alpha \leq d_Y(f(x_1), f(x_2)),$$

then f is a (λ, α, β)-*bi-Hölder embedding*. Finally, if, in addition, f is surjective, then f is a (λ, α, β)-*bi-Hölder* homeomorphism. If such a homeomorphism exists, then X and Y are said to be *Hölder equivalent*.

Definition 2.1.3 ([Hei01, Chapter 10])**.** Let $(X, d_X), (Y, d_Y)$ be metric spaces. A homeomorphism $f\colon X \to Y$ is called η-*quasi-symmetric* for $\eta\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ a homeomorphism, if for any $a, b, x \in X$ and $t > 0$,

$$d_X(x, a) \leq t d_X(x, b) \implies d_Y(f(x), f(a)) \leq \eta(t) d_Y(f(x), f(b)).$$

Under mild assumptions, quasi-symmetric homeomorphisms are Hölder homeomorphisms.

Lemma 2.1.4. *Let X be a bounded, uniformly perfect metric space. If $f\colon X \to Y$ is a quasi-symmetric homeomorphism to a metric space Y, then f is also a bi-Hölder homeomorphism.*

Definition 2.1.5 ([Hei01, Chapter 11])**.** A metric space (X, d) is *uniformly perfect* if there exists a constant $C \geq 1$ so that, for each $x \in X$ and $r > 0$, the set $B(x, r) \setminus B(x, r/C)$ is nonempty whenever the set $X \setminus B(x, r)$ is nonempty. Here, $B(x, r) \coloneqq \{y \in X \mid d(x, y) < r\}$.

Proof of Lemma 2.1.4. Note Y is also bounded and uniformly perfect, as boundedness and uniform perfectness are invariants of quasi-symmetric equivalence (for example, see [Hei01, Proposition 10.8] for the invariance of boundedness, and [MT10, Theorem 1.3.4] for the invariance of uniform perfectness). Quasi-symmetric embeddings on uniformly perfect spaces are Hölder continuous on bounded sets, see [Hei01, Corollary 11.5]. Thus f and its inverse are Hölder continuous, so f is a bi-Hölder homeomorphism. □

2.2 Notions of dimension

There are many different notions of dimension that come in many different flavours. However, the notions that we shall consider are all fundamentally concerned with covering a space by sets of a controlled diameter. This style of dimension is common because, heuristically, a cover can be thought of as an approximation of the covered space, and 'smaller' spaces are easier to approximate than 'larger' spaces. This section gives background on two well known variants of dimension that shall be used throughout this thesis.

We start with the classical notion of topological dimension. There are many versions of topological dimension that are equivalent under weak assumptions. The version of topological dimension we present is also sometimes referred to as "covering dimension".

Let X be a topological space. The following definitions are from [Mun00, Section 50].

Definition 2.2.1. A collection $\mathcal{A}$ of subsets of the space X is said to have *multiplicity* m if some point of X lies in m elements of $\mathcal{A}$, and no point of X lies in more than m elements of $\mathcal{A}$.

Given a collection $\mathcal{A}$, a collection $\mathcal{B}$ is said to *refine* $\mathcal{A}$, or to be a *refinement* for $\mathcal{A}$, if for each element B of $\mathcal{B}$, there is an element A of $\mathcal{A}$ such that $B \subset A$.

A collection $\mathcal{A}$ of subsets of a space X is said to *cover* X, or to be a *covering* of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called a *open covering* of X if its elements are open subsets of X.

These definitions culminate in the following topological notion of dimension.

Definition 2.2.2. The *topological dimension* of X, denoted $\dim_T(X)$, is defined to be the smallest integer m such that, for every open covering $\mathcal{A}$ of X, there exists an open covering $\mathcal{B}$ that refines $\mathcal{A}$ and has multiplicity at most $m + 1$.

Topological dimension is extremely useful as a basic notion of dimension and as a framework from which to build other notions of dimension. However, topological dimension is only sensitive to the underlying topology of a metric space. If we want to understand metric structures, then we will need a notion of dimension that incorporates metric data. There are a few ways to do this. Later, in Section 4.2 we shall see a notion called "capacity dimension", which is similar to topological dimension but enforces metric constraints by restricting covers to only contain subsets of a given scale. Because of this similarity in definition, topological and capacity dimension are both notions of dimension that exclusively take values in the integers. This conforms well to our intuitive notion of dimension, but is a fairly unnecessary restriction as we can formalise non-integer notions of dimension too. One common such notion is that of Hausdorff dimension, which we present below and can be found [Mat95, Chapter 4].

Definition 2.2.3. For any $0 \leq q < \infty$, the *Hausdorff q-measure* of a metric space X, denoted $\mathcal{H}^q(X)$, is given by

$$\mathcal{H}^q(X) \coloneqq \lim_{\delta \downarrow 0} \inf \left\{ \sum_{i \in I} \operatorname{diam}(U_i)^q \mid I \text{ countable},\ X \subseteq \bigcup_{i \in I} U_i,\ \operatorname{diam}(U_i) \leq \delta \right\}.$$

The *Hausdorff dimension* of X, denoted $\dim_H(X)$, is defined as

$$\dim_H(X) \coloneqq \inf \{ q \mid \mathcal{H}^q(X) < \infty \}.$$

Equivalently,

$$\dim_H(X) = \sup \{ q \mid \mathcal{H}^q(X) > 0 \}.$$

The motivation for these definitions of Hausdorff dimension, and the fact of their equivalence comes from the following properties of Hausdorff measures.

Theorem 2.2.4 ([Mat95, Theorem 4.7])**.** *For any $0 \leq s < t < \infty$ and metric space X,*

- $\mathcal{H}^s(X) < \infty$ *implies* $\mathcal{H}^t(X) = 0$*,*
- $\mathcal{H}^t(X) > 0$ *implies* $\mathcal{H}^s(X) = \infty$*.*

This means that there is a unique value $0 \leq q < \infty$ such that $\mathcal{H}^s(X) = \infty$ for all $s < q$ and $\mathcal{H}^t(X) = 0$ for all $t > q$. This critical value q is the Hausdorff dimension of X.

Example 2.2.5. For these to be sensible notions of dimension, we would hope that, for any $n \in \mathbb{N}$, n-dimensional Euclidean space, $\mathbb{R}^n$ equipped with the usual ℓ^2 metric, has dimension n. Thankfully, this is true; $\mathbb{R}^n$ has both topological dimension n and Hausdorff dimension n.

We have to look to slightly more exotic examples to find Hausdorff dimension deviating from topological dimension. For example, the standard 1/3-Cantor set, which will be discussed in Section 5.2, has topological dimension 0 and Hausdorff dimension $\log(2)/\log(3) \approx 0.63$.

It is simple to show that topological dimension is an invariant of topological equivalence, as one might expect. However, it is less clear which kinds of equivalence preserve Hausdorff dimension if any. Topological equivalence is far too flexible to preserve Hausdorff dimension. For instance, for any non-negative real number r, one can find a space of Hausdorff dimension r that is topologically equivalent to a Cantor set. It is, however, possible to find equivalences that do preserve Hausdorff dimension. Perhaps the most common example is bi-Lipschitz equivalence. By examination of the definitions, one can see that distorting a space by a bi-Lipschitz equivalence might change the Hausdorff q-measure of the space, but not whether that q-measure is zero, positive and finite, or infinite. Therefore, the Hausdorff dimension remains unaffected by the bi-Lipschitz distortion. Knowing which equivalences preserve a given dimension can be a powerful tool for the purpose of distinguishing spaces.

Chapter 3

Initial observations on Hölder dimension

This chapter is dedicated to providing some results for Hölder dimension that do not require too much effort. The aim is to answer some of the first questions that one might ask about Hölder dimension, and to give some intuition about the kinds of situations that can occur, before delving into more involved results.

3.1 Basic examples

There is a subtlety to Hölder dimension alluded to by the "attained" component in its definition, Definition 1.0.2. As Hölder dimension is an infimum, there is a question left open: "Is this infimum attained and in fact a minimum?" This section presents two examples that illustrate that both possibilities can occur.

Example 3.1.1. The 1/3-Cantor set, C, see Section 5.2, is a subset of the unit interval $[0,1]$ obtained from an iterative process of removing intervals until one is left with a totally disconnected subspace. These intervals are removed in a regular fashion giving the space a strong self-similarity. It is a compact, uniformly perfect metric space with topological dimension 0 and Hausdorff dimension $\log(2)/\log(3)$. As C is bounded and uniformly perfect, conformal dimension is an upper bound for Hölder dimension, so we get that it has Hölder dimension 0 directly from it having conformal dimension 0, for example see [MT10, Example 2.2.3.(2)]. We can give a direct proof of this by constructing bi-Hölder homeomorphisms between different Cantor sets that are contained in the unit interval $[0,1]$. In the following, we shall use the notation from Subsection 5.2 to describe these Cantor sets.

For $\sigma \in (0,1)$, consider the middle σ-Cantor set, denoted C_σ, where, for each $n \geq 0$ and $1 \leq i \leq 2^n$, $J_{n+1,i}$ is chosen to have diameter $\sigma \operatorname{diam}(I_{n,i})$. For clarity, and so that we can compare middle-interval Cantor sets with different parameters, for every $n \geq 0$ and $1 \leq i \leq 2^n$, denote $I_{n,i}$ by $I^\sigma_{n,i}$ and $J_{n+1,i}$ by $J^\sigma_{n+1,i}$. Note that, with this description, the 1/3-Cantor set, C, is denoted $C_{1/3}$.

There is a natural bijection between any two Cantor sets constructed this way. First, observe that there is a bijection between elements of C_σ and sequences of nested intervals $I^\sigma_{0,1} \supset \cdots \supset I^\sigma_{n,i_n} \supset \ldots$. Indeed, for any $x \in C_\sigma$, there exists a unique sequence of nested intervals $I^\sigma_{0,1} \supset \cdots \supset I^\sigma_{n,i_n} \supset \ldots$ that all contain x, and for any such nested sequence, there exists a unique element of C_σ that is contained in each interval. For any $\tau \in (0,1)$, this bijection extends to a bijection between C_σ and C_τ as, for any $n \geq 0$, $1 \leq i \leq 2^n$, and $1 \leq j \leq 2^{n+1}$, $I^\sigma_{n,i}$ contains $I^\sigma_{n+1,j}$ if and only if $I^\tau_{n,i}$ contains $I^\tau_{n+1,j}$. Let $F\colon C_\sigma \to C_\tau$ be this bijection, $\alpha = \log(2/(1-\tau))/\log(2/(1-\sigma))$, and $\lambda = \max\{(1/\sigma)^\alpha, 1/\tau\}$. We shall now prove that F is a $(\lambda, \alpha, \alpha)$-bi-Hölder homeomorphism.

For any $x, y \in C_\sigma$, let $N \in \mathbb{N}$ be maximal such that there exists $1 \leq i \leq 2^N$ such that $I^\sigma_{N,i}$ contains both x and y, and such that, for any $1 \leq j \leq 2^{N+1}$, $I^\sigma_{N+1,j}$ does not contain both x and y. Passing through the above bijection we see $F(x), F(y) \in I^\tau_{N,i}$, and $F(x)$ and $F(y)$ are not both contained in $I^\tau_{N+1,j}$ for any $1 \leq j \leq 2^{N+1}$. It is easy to check that, for any $n \geq 0$, $1 \leq j \leq 2^n$, $\operatorname{diam}(I^\sigma_{n,j}) = ((1-\sigma)/2)^n$ and $\operatorname{diam}(J^\sigma_{n+1,j}) = \sigma((1-\sigma)/2)^n$, and $\operatorname{diam}(I^\tau_{n,j}) = ((1-\tau)/2)^n$ and $\operatorname{diam}(J^\tau_{n+1,j}) = \tau((1-\tau)/2)^n$. Noting that x and y do not lie in any single interval of the form $I^\sigma_{N+1,j}$, but do lie in $I^\sigma_{N,i}$, we see that they must be separated by $J^\sigma_{N+1,i}$. Thus, we may observe the following helpful bounds

$$\sigma((1-\sigma)/2)^N \leq d(x,y) \leq ((1-\sigma)/2)^N.$$

Similarly,

$$\tau((1-\tau)/2)^N \leq d(F(x),F(y)) \leq ((1-\tau)/2)^N.$$

Combining these inequalities yields

$$\tau d(x,y)^\alpha \leq d(F(x),F(y)) \leq \left(\frac{1}{\sigma}\right)^\alpha d(x,y)^\alpha,$$

confirming that F is $(\lambda, \alpha, \alpha)$-bi-Hölder as desired.

One can check that $\dim_H(C_\tau) = \log(2)/\log(2/(1-\tau))$, and thus $\dim_H(C_\tau) \to 0$ as $\tau \to 1$. Recalling that $C = C_{1/3}$, we see that C has Hölder dimension 0.

However, this does not tell us anything about the attainment of the Hölder dimension of C. We can see that C does not attain its Hölder dimension from the trivial bound (1.0.3.1) on distortion of Hausdorff dimension noted in Chapter 1.

Example 3.1.2. For any $n \in \mathbb{N}$, n-dimensional Euclidean space $\mathbb{R}^n$ has topological and Hausdorff dimension n. As topological dimension is a lower bound for Hausdorff dimension by [Szp37, Theorem 2], no space Hölder equivalent to $\mathbb{R}^n$ can have Hausdorff dimension strictly less than n, so $\mathbb{R}^n$ has Hölder dimension n and is attained.

3.2 Boundaries of rank one symmetric spaces of non-compact type

An important class of examples for conformal dimension are the classical rank 1 symmetric spaces of non-compact type. These are a class of (Gromov) hyperbolic spaces that can be classified and are, for each $n \geq 2$, n-dimensional hyperbolic $\mathbb{K}$ space $H^n_{\mathbb{K}}$, where $\mathbb{K}$ is one of the division algebras the real line $\mathbb{R}$, the complex numbers $\mathbb{C}$, or the quaternions $\mathbb{H}$, as well as the Cayley hyperbolic plane $H^2_{\mathbb{O}}$. The study of their geometry was one of the first motivations for Pansu introducing conformal dimension in [Pan89b]. In this section, we recount what is known about the conformal dimension of the boundaries of these spaces and explain how existing results can be used to calculate their Hölder dimensions.

The following proposition can be found in [MT10, Section 3.3]. The conformal dimension aspect of the proposition is due to Pansu and is implicit in [Pan89a].

Proposition 3.2.1. *For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and any $n \geq 2$, or $K = \mathbb{O}$ and $n = 2$, let $k = \dim_{\mathbb{R}} \mathbb{K}$, then $\partial_\infty H^n_{\mathbb{K}}$ is topologically a sphere of dimension $kn - 1$, $\mathbb{S}^{kn-1}$. Further, there exists a visual metric on $\partial_\infty H^n_{\mathbb{K}}$, called the Carnot-Carathéodory metric, such that $\partial_\infty H^n_{\mathbb{K}}$ has Hausdorff and conformal dimension $kn + k - 2$ when equipped with this metric.*

Thus, for two different $H^n_{\mathbb{K}}$ and $H^{n'}_{\mathbb{K}'}$, the boundary spheres are not quasi-symmetric even when they have the same topological dimension. However, they are Hölder equivalent if they have the same topological dimension.

Theorem 3.2.2. *The boundaries of two rank one symmetric spaces of non-compact type are Hölder equivalent if and only if they are topologically equivalent. In particular,*

$$\text{Höldim}(\partial_\infty H^n_{\mathbb{K}}) = kn - 1,$$

and is attained.

Proof. Such boundaries have a visual metric that is a Carnot-Carathéodory metric as discussed in [Pan89b], which is equivalent to the usual Euclidean metric on the sphere by a homeomorphism that is locally bi-Hölder by [NSW85, Proposition 1.1]. Finally, as the boundary is compact, we see that this map is actually a bi-Hölder map. □

For an alternative direct proof of the local bi-Hölderness in the real, complex, or quaternion cases see [MT10, Section 3.3].

3.3 Hölder dimension does not assume values between 0 and 1

In [Kov06], Kovalev proves that conformal dimension cannot take values strictly between 0 and 1. In this section, we explain how the Hölder dimension of bounded spaces inherits this property from Kovalev's results.

Theorem 3.3.1 ([Kov06, Theorem 1.2])**.** *Let V be a real separable Banach space, and let $E \subset V$ be a set such that $\dim_H(E) < 1$. For any $\epsilon > 0$, there exists a quasi-symmetric homeomorphism $f : V \to V$ such that $\dim_H(f(E)) \leq \epsilon$.*

The following can be found in [Ban55, Chap. XI, Sec. 8].

Theorem 3.3.2 (Banach-Féchet-Mazur)**.** *Every separable metric space can be isometrically embedded into $C[0, 1]$, the space of continuous real-valued functions on $[0, 1]$ with the supremum norm.*

Combining these results allows Kovalev to conclude

Theorem 3.3.3 ([Kov06, Theorem 1.1])**.** *The conformal dimension of any metric space is either zero or at least one.*

We do not need to restrict to separable metric spaces, despite the restriction to them in Theorem 3.3.2, because any non-separable metric space has infinite Hausdorff dimension. Further, separability is a topological invariant. Thus, the conformal dimension of any non-separable metric space is infinite too.

Kovalev is saying more by proving the existence of ambient self maps of $C[0, 1]$ as this allows us to use the following.

Theorem 3.3.4 ([Hei01, Corollary 11.5])**.** *Quasi-symmetric embeddings on uniformly perfect spaces are Hölder continuous on bounded sets.*

We now may conclude the following result:

Theorem 3.3.5. *The Hölder dimension of any bounded metric space is either zero or at least one.*

Proof. Similar to Theorem 3.3.3, we can ignore non-separable metric spaces as they have infinite Hausdorff dimension, and therefore infinite Hölder dimension. Let X be a bounded, separable metric space, then the Banach-Féchet-Mazur theorem tells us that X isometrically embeds into $C[0, 1]$, so we can think of X as a subset of $C[0, 1]$. Now, if $\dim_H(X) < 1$, for any $\epsilon > 0$, Theorem 3.3.1 says there is a quasi-symmetric homeomorphism f of $C[0, 1]$ to itself such that $\dim_H(f(X)) \leq \epsilon$. As $C[0, 1]$ is uniformly perfect, and X is bounded, f is Hölder continuous on X. Further, quasi-symmetric homeomorphisms send bounded sets to bounded sets, see [Hei01, Proposition 10.8], so $f(X)$ is a bounded subset of $C[0, 1]$. Also, quasi-symmetries have quasi-symmetric inverses, so f^{-1} is quasi-symmetric. Therefore, $f^{-1}(X)$ is Hölder continuous on $f(X)$. Therefore, X is bi-Hölder homeomorphic to $f(X)$ by f as desired. ☐

To directly use Kovalev's work, we have had to add the assumption that our space is bounded. However, it is not clear whether a different approach could do away with this boundedness restriction.

Question 3.3.6. Does there exist an unbounded metric space with Hölder dimension strictly between 0 and 1?

I suspect the answer is still "no". However, proving so would probably require techniques of which I am unaware.

Chapter 4

Minimising Hausdorff Dimension Under Hölder Equivalence

This chapter is primarily taken from [Col19], Sections 1 through 5, with slight edits to make it compatible with this thesis.

4.1 Introduction

For this chapter, our initial motivation was to study how the Hausdorff dimension of boundaries of hyperbolic groups can vary under Hölder equivalences, prompted by a question posed by Ilya Kapovich in June 2017. In particular, what can be said about their 'Hölder dimension':

Definition 4.1.1 (Definition 1.0.2)**.** Let X be a metric space. Define the *Hölder dimension* of X, denoted $\mathrm{H\ddot{o}ldim}(X)$, by

$$\mathrm{H\ddot{o}ldim}(X) \coloneqq \inf\{\dim_H(Y) \mid Y \text{ Hölder equivalent to } X\},$$

where $\dim_H(Y)$ denotes the Hausdorff dimension of the metric space Y.

Our methods apply to much more general metric spaces than boundaries of hyperbolic groups. One such example is the following corollary of our main theorem on the Hölder dimension of locally self-similar metric spaces.

Corollary 4.1.2. *If X is a compact, locally self-similar metric space, then X has Hölder dimension equal to its topological dimension.*

Locally self-similar spaces are ones where small scales are similar to the whole space in a uniform way (see Definition 4.2.6). As one might expect, it is easy to show that this property is exhibited by classical fractals like the Koch snowflake curve, the Sierpiński carpet, and the

1/3-Cantor set. These spaces are built from 'tiles' at a geometric sequence of scales that are rescaled copies of the whole space in such a way that any ball in one of these spaces lies in a tile, or a union of adjacent tiles, of comparable size, so can be rescaled to be a copy of a large ball. However, 'locally self-similar' is more general than this kind of rigid self-similarity. Indeed, this behaviour is exhibited by boundaries of hyperbolic groups, see [BL07, Proposition 6.2].

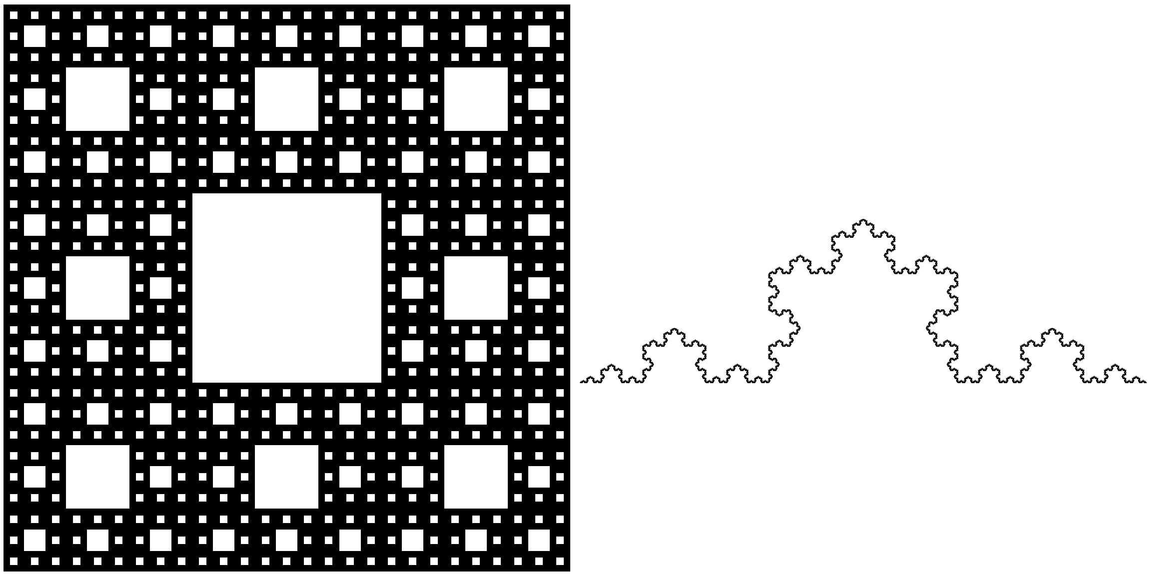

Figure 4.1: Finite approximations of the standard Sierpiński carpet (left) and the von Koch curve (right).

For context for our main theorem, we discuss the concept of an 'Embedding Theorem', which one may encounter within metric geometry. This is usually interpreted to be an answer to the question: "Can one 'faithfully depict' a space, X, as a subspace of a 'nice' space?". This type of question has been around for a while and one of the original meanings of it breaks down as 'faithfully depict' meaning topological embedding and 'nice' meaning Euclidean. The following result is an example of such a theorem, and can be found in [HW48, Theorem VII 5].

Theorem 4.1.3 (Szpilrajn, Eilenberg [Szp37, Theorem 3])**.** *Let X be a separable metric space of topological dimension at most n, then there exists an embedding $f\colon X \to I^{2n+1}$, which induces a homeomorphism between X and $f(X)$, such that $f(X)$ has Hausdorff dimension at most n. Here I denotes the unit interval* $[0,1] \subset \mathbb{R}$.

Theorem 4.1.3 is still an interesting statement even without the added dimension control on the image, but we include this strengthening because it makes it obvious that our main result is a Hölder-parallel of this one. The following is the main theorem of this chapter and the 'embedding theorem' version of its content.

Theorem 4.1.4. *Let X be a compact, N-doubling metric space with non-zero diameter that has capacity dimension n with coefficient σ, then, for any $q > n$, there exist constants $\mu = \mu(n, q, \sigma, N) > 0$, $\alpha = \alpha(n, q, \sigma, N) \geq 1$, and $0 < \beta = \beta(n, q, \sigma, N) \leq 1$, and a map $f \colon X \to \ell^2$ such that, for any $x, y \in X$,*

$$\frac{1}{\mu \operatorname{diam}(X)^{\alpha}} d(x,y)^{\alpha} \leq d(f(x), f(y)) \leq \frac{\mu}{\operatorname{diam}(X)^{\beta}} d(x,y)^{\beta},$$

and the image of f has Hausdorff q-measure at most 4^q, and therefore Hausdorff dimension at most q.

As an intermediate corollary, we have the following.

Corollary 4.1.5. *If X is a compact, doubling metric space with capacity dimension n, then X has Hölder dimension at most n.*

The space ℓ^2 is the space of square-summable, real-valued sequences with the norm $|(z_i)_{i \in \mathbb{N}}| = \left(\sum_{i \in \mathbb{N}} z_i^2\right)^{\frac{1}{2}}$. We give definitions of the conditions imposed on X in Section 4.2. For now, the reader can think of 'capacity dimension' as providing controlled open covers of X at small scales, and 'doubling' as giving an upper bound on the number of elements in these controlled covers.

In comparison to Theorem 4.1.3, we upgrade the embedding to one with bi-Hölder metric control, but at the expense of the generality of X and the finite-dimensionality of the range.

The idea of the proof is that these conditions on X allow us to approximate X, at a sequence of scales, by open covers with good properties. We then transfer these approximations over into ℓ^2 using maps with simplicial ranges, which inherit strong metric and dimension control from the properties imposed on these covers. Finally, we take the limit of these approximating maps to get an embedding, and check it has the desired metric and dimension control.

Corollary 4.1.2 follows from Theorem 4.1.4 because doubling and capacity dimension equal to topological dimension are both consequences of a compact metric space being locally self-similar.

In Theorem 4.2.10, we will show that this inequality between Hölder dimension and capacity dimension cannot be upgraded to an equality under these constraints on X.

In Section 4.2, we explain the various restrictions that we impose on our metric spaces and how they relate to each other. In particular, we explain, in Lemma 4.2.9, how to combine the notions of 'doubling' and 'capacity dimension' to ensure that one can cover a given space at any scale with a controlled number of subsets only at that scale, and with control on overlapping of subsets.

In Section 4.3, we define a sequence of maps that will approximate our given space to arbitrarily fine scales. Then, in Section 4.4 we show that the choices made when defining these approximating maps give them desirable properties. In particular, they do not stretch or squash distances too much on the scale they are approximating, they are a Cauchy sequence so will converge to a limit function, and their images are nice enough that we can easily bound the Hausdorff dimension of the image of the limit function.

Finally, in Section 4.5, we examine this limit function, proving that it is a bi-Hölder embedding whose image has Hausdorff dimension bounded by the capacity dimension of the domain. We end this section by explaining how corollaries 4.1.2 and 4.1.5 follow from Theorem 4.1.4, in Subsection 4.5.6.

4.2 Capacity dimension and metric spaces

A major component of the proof of Theorem 4.1.4 is approximating X by sequentially finer covers. However, not any old haphazard covers will do; we will need them to have quite a bit of structure. In this section, we delve into some background on how to cover metric spaces with open sets, and how concepts like capacity dimension and doubling allow us to take covers with more structure. We define capacity dimension and doubling, and provide a lemma that combines these properties to prove the existence of covers with especially useful properties.

Capacity dimension, which can be found in [BL07], is an integer notion of dimension that strengthens topological dimension in metric spaces by imposing metric constraints.

Definition 4.2.1. The *mesh* of a covering $\mathcal{U}$ is the supremum of the diameters of elements of $\mathcal{U}$.

$$\mathrm{mesh}(\mathcal{U}) \coloneqq \sup\{\mathrm{diam}(U) \mid U \in \mathcal{U}\}.$$

Definition 4.2.2. A covering $\mathcal{U}$ is said to be *coloured* if it is the union of $m \geq 1$ disjoint families, $\mathcal{U} = \cup_{i\in I}\mathcal{U}_i$, for some indexing set I of size m, with the property that, for any $i \in I$, if $U, V \in \mathcal{U}_i$ are distinct, then $U \cap V = \emptyset$. In this case we also say that $\mathcal{U}$ is m-coloured.

Note that an m-coloured covering $\mathcal{U}$ has multiplicity, see Definition 2.2.1, at most m. Indeed, if some members of $\mathcal{U}$ have non-empty intersection, then they must each lie in different families of which there are m, and, therefore, at most m can intersect non-trivially.

Definition 4.2.3. Let $\mathcal{U}$ be a family of open subsets in a metric space X that cover $A \subset X$. Given $x \in A$, we let

$$\mathcal{L}(\mathcal{U}, x) \coloneqq \sup\{d(x, X \setminus U) \mid U \in \mathcal{U}\}$$

be the *Lebesgue number of* $\mathcal{U}$ *at* x, $\mathcal{L}(\mathcal{U}) = \inf_{x\in A}\mathcal{L}(\mathcal{U}, x)$ be the *Lebesgue number of the covering* $\mathcal{U}$ *of* A.

We give the definition of Lebesgue number from [BL07] as this is the definition Buyalo and Lebadeva use when giving Definition 4.2.5. This definition is a little opaque, but, luckily, the reader need only concern themselves with the following key fact about Lebesgue number.

Lemma 4.2.4. *If* $\mathcal{U}$ *is a finite cover for* $A \subset X$*, then, for every* $x \in A$*, the open ball* $B(x, r)$ *in* X *of radius* $r \leq \mathcal{L}(\mathcal{U})$ *centred at* x *is contained in some element of the cover* $\mathcal{U}$*.*

Definition 4.2.5. The *capacity dimension* of a metric space X is the minimal integer $n \geq 0$ with the following property: There is a constant $\sigma' \in (0,1)$ such that for every sufficiently small $\delta > 0$ there exists an $(n+1)$-coloured open covering $\mathcal{U}'$ of X with $\mathrm{mesh}(\mathcal{U}') \leq \delta$ and $\mathcal{L}(\mathcal{U}') \geq \sigma'\delta$. We say that X has capacity dimension n with coefficient σ'.

Buyalo and Lebedeva then proceed to give some conditions for which one can assume that capacity dimension and topological dimension are equal.

Definition 4.2.6. A metric space (X,d) is *locally self-similar* if there exists $\lambda \geq 1$ such that for every sufficiently large $R > 1$ and every $A \subseteq X$ with $\mathrm{diam}(A) \leq \Lambda_0/R$, where $\Lambda_0 = \min\{1, \mathrm{diam}(X)/\lambda\}$, there is an embedding

$$f\colon A \to X$$

such that, for all $z_1, z_2 \in A$,

$$Rd(z_1,z_2)/\lambda \leq d(f(z_1),f(z_2)) \leq \lambda R d(z_1,z_2).$$

In other words, f is a λ-bi-Lipschitz homeomorphism from (A, Rd), the subspace A with a rescaled metric, to its image in (X,d).

The following theorem is Corollary 1.2 in [BL07].

Theorem 4.2.7. *The capacity dimension of every compact, locally self-similar metric space X is finite and coincides with its topological dimension.*

We will, in fact, not need the full strength of local self-similarity for our main theorem, but it is helpful to see here that a space having finite capacity dimension is not a particularly unreasonable assumption.

Finally, we need control on how many elements are in these covers. To this end, we introduce the concept of a doubling metric space, which can be found in [Hei01, Chapter 10].

Definition 4.2.8. A metric space (X,d) is *doubling* if there exists a constant $N < \infty$ such that, for any $x \in X$ and $r > 0$, there exists a cover of the closed ball $\overline{B(x,r)} = \{y \in X \mid d(x,y) \leq r\}$ by at most N balls of radius $r/2$. In particular, we say X is N-doubling.

Lemma 4.2.9. *Suppose X is a finite diameter, N-doubling metric space of capacity dimension n with coefficient σ', then there is a constant $\sigma = \sigma'/4 \in (0,1)$ such that, for every sufficiently small $\delta > 0$, there exists an $(n+1)$-coloured open covering $\mathcal{U}$ of X with mesh$(\mathcal{U}) \leq \delta$, $\mathcal{L}(\mathcal{U}) \geq \sigma\delta$, and*

$$|\mathcal{U}| \leq N^{\log_2(2\,\mathrm{diam}(X)/\sigma\delta)}.$$

Proof. If X has diameter 0, then, for any $\delta > 0$, $\{X\}$ is a cover of X which satisfies the conditions of the lemma, so we may assume $\operatorname{diam}(X) > 0$. Given that X is N-doubling, we can cover X by at most N balls of radius $\operatorname{diam}(X)/2$. Each of these balls can be covered by at most N balls of radius $\operatorname{diam}(X)/4$, so X can be covered by at most N^2 balls of radius $\operatorname{diam}(X)/4$. Continuing inductively, we see that X can be covered by at most N^k balls of radius $\operatorname{diam}(X)/2^k$ for any $k \in \mathbb{N}$. Let $\delta > 0$ be sufficiently small as in Definition 4.2.5 of capacity dimension, and fix $k \in \mathbb{N}$ to be the unique positive integer such that

$$\frac{\operatorname{diam}(X)}{2^k} < \frac{\sigma'\delta}{4} \leq \frac{\operatorname{diam}(X)}{2^{k-1}},$$

equivalently,

$$\frac{\sigma'\delta}{8} \leq \frac{\operatorname{diam}(X)}{2^k} < \frac{\sigma'\delta}{4}.$$

Rearranging, we see k satisfies

$$\log_2\left(\frac{4\operatorname{diam}(X)}{\sigma'\delta}\right) < k \leq \log_2\left(\frac{8\operatorname{diam}(X)}{\sigma'\delta}\right).$$

Now, let $\mathcal{U}'$ be a cover for X of mesh at most δ provided by the definition of capacity dimension. In particular, $\mathcal{L}(\mathcal{U}') \geq \sigma'\delta$, so for any $x \in X$, $B(x, \sigma'\delta/2) \subset U$ for some $U \in \mathcal{U}'$. Consider also a cover $\mathcal{D}$ of X by balls of radius $\sigma'\delta/4$. By N-doubling and the above, we can assume,

$$|\mathcal{D}| \leq N^k \leq N^{\log_2(8\operatorname{diam}(X)/\sigma'\delta)}.$$

Say $\mathcal{D} = \{B(x_j, \sigma'\delta/4)\}_{j=1}^{N^k}$, then for each j, using the Lebesgue number of $\mathcal{U}'$ we can pick a $U_j \in \mathcal{U}'$ such that $B(x_j, \sigma'\delta/4) \subseteq B(x_j, \sigma'\delta/2) \subseteq U_j$. Now, note that $\mathcal{U} := \{U_j\}_{j=1}^{N^k}$ is also an open covering of mesh at most δ, $(n+1)$-coloured, but now with $|\mathcal{U}|$ at most N^k. We are almost there, but we might have decreased the Lebesgue number by removing elements from $\mathcal{U}'$. To fix this, note that $B(x_j, \sigma'\delta/4)$ is still an open cover for X, so, for any $x \in X$ there exists a j such that $x \in B(x_j, \sigma'\delta/4)$. Consequently, $B(x, \sigma'\delta/4) \subseteq B(x_j, \sigma'\delta/2)$ by the triangle inequality, and therefore $B(x, \sigma'\delta/4) \subseteq U_j$. Hence, $\mathcal{L}(\mathcal{U}) \geq \sigma'\delta/4$, and taking $\sigma := \sigma'/4$, we get the desired result that $\mathcal{U}$ has capacity dimension properties plus

$$|\mathcal{U}| \leq N^k \leq N^{\log_2(2\operatorname{diam}(X)/\sigma\delta)}.$$

□

4.2.1 Hölder dimension can be less than capacity dimension

One of the main results of this Chapter, Corollary 4.1.5, bounds Hölder dimension above by capacity dimension for compact, doubling spaces with finite capacity dimension. A priori, it is not clear whether Hölder dimension is actually just capacity dimension in disguise or if there is actually a distinction between the two notions. This subsection provides a simple example to illustrate that, even for these restricted spaces, capacity dimension is not also a lower bound for Hölder dimension. This will verify that the inequality between capacity dimension and Hölder dimension in Corollary 4.1.5 cannot be upgraded to an equality. The content of this subsection is taken primarily from [Col19, Section 9].

Theorem 4.2.10. *The subspace* $\{0\} \cup \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}$ *is a compact, doubling metric space with capacity dimension* 1*, but Hölder dimension* 0*.*

The idea of Theorem 4.2.10 is that it only takes a countable number of points spaced out poorly to force capacity dimension to increase beyond 0, but a countable number of points has Hausdorff dimension 0.

Proof of Theorem 4.2.10 Note that X is compact because it is a closed and bounded subspace of $\mathbb{R}$, and X is doubling because it is a subspace of $\mathbb{R}$, which is doubling.

We can easily verify that X has Hölder dimension 0. Indeed, as countable collections of points have Hausdorff dimension 0, X has Hausdorff dimension 0. Further, Hölder dimension is non-negative and the Hausdorff dimension of X is an upper bound for its Hölder dimension, so X has Hölder dimension 0.

To see that X has capacity dimension 1, observe that X has capacity dimension at most 1 as it is a subspace of $\mathbb{R}$, which has capacity dimension 1. We now prove X has capacity dimension at least 1 by proving that it does not have capacity dimension 0.

For a contradiction, assume that X has capacity dimension 0 with coefficient σ. Take $n \in \mathbb{N}$ such that $n > \max\{2/\sigma, 2\}$, and so that $2/(\sigma n(n-1)) > 0$ is sufficiently small as to apply the definition of capacity dimension 0 with $\delta \coloneqq 2/(\sigma n(n-1))$. Let $\mathcal{U}$ be an open cover of X with mesh at most δ as per X having capacity dimension 0 with coefficient σ. For any $m \geq n$, we calculate

$$d\left(\frac{1}{m}, \frac{1}{m-1}\right) = \frac{m-(m-1)}{m(m-1)} = \frac{1}{m(m-1)} \leq \frac{1}{n(n-1)}.$$

By observing that $1/n(n-1) < 2/(n(n-1)) = \sigma\delta$, we see that, for every $m \geq n$, $\{1/m, 1/(m-1)\} \subseteq U_m$, for some $U_m \in \mathcal{U}$, by the Lebesgue number property of X having capacity dimension 0 with coefficient σ. From the multiplicity 1 restriction on $\mathcal{U}$, for any $U, V \in \mathcal{U}$, if $U \cap V \neq \emptyset$, then $U = V$. Hence, $U_m = U_{m+1}$ for every $m \geq n$ as $1/m \in U_m \cap U_{m+1}$. Inductively, $1/m \in U_n$ for every $m \geq n-1$. Also, $0 \in U_n$, as $n(n-1) \geq n-1$ and $d(0, 1/(n(n-1))) < \sigma\delta$, so there exists U_0 in $\mathcal{U}$ containing 0 and $1/(n(n-1))$, but $1/(n(n-1)) \in U_n$ so $U_0 = U_n$ too. Therefore, $\text{diam}(U_n) \geq d(0, 1/(n-1)) = 1/(n-1) > 2/(\sigma n(n-1)) = \delta$ as $2/(\sigma n) < 1$, by definition of n. This contradicts the diameter constraint on elements of $\mathcal{U}$. Therefore, no such $\mathcal{U}$ exists for any σ, and X cannot have capacity dimension 0. □

4.3 Construction of the approximating maps

We now begin the proof of Theorem 4.1.4. We start in Subsection 4.3.1 by defining a sequence of scales, approximating X by nice covers at each of these scales, and then building functions, in Subsection 4.3.2, that translate these approximating covers over into approximations of X in ℓ^2. We will then proceed to, in Section 4.4, show some useful properties of these approximating functions.

4.3.1 Approximating X by nice covers at controlled scales

Assume X is a compact, doubling metric space that has capacity dimension n. For now, we prove Theorem 4.1.4 assuming X has diameter equal to 1.

Let σ be as in Lemma 4.2.9, and let $L = L(n, \sigma)$ be defined by

$$L \coloneqq \frac{128(n+1)^2}{\sigma^2}. \tag{4.3.0.1}$$

Later, we will show that certain maps are locally Lipschitz, and L will appear in the corresponding Lipschitz constants.

Let $q > n$. We will construct a space that is Hölder equivalent to X and has Hausdorff dimension at most q.

Note 4.3.1. The 'N' in N-doubling, see Definition 4.2.8, is really only an upper bound; if $N' \geq N$ and X is N-doubling, then X is N'-doubling too. Similarly, if $\mathcal{U}$ is a cover as in Lemma 4.2.9, then for every $N' \geq N$

$$|\mathcal{U}| \leq (N')^{\log_2(2\operatorname{diam}(X)/\sigma\delta)},$$

because the exponent is positive considering that $\delta \leq \operatorname{diam}(X)$ and $0 < \sigma < 1$.

Throughout, it will be convenient to assume N is much larger than other constants. Indeed, as n, q, σ are all such that replacing N with a larger value has no effect on them, we can assume N is arbitrarily large with respect to n, q, and σ. The exact value of $N = N(n, q, \sigma)$ is determined by Lemma 4.5.8 later.

Let $\delta_0 = 1$, $\epsilon_0 = 1$, and inductively define $\epsilon_{i+1} = \epsilon_{i+1}(n, q, \sigma, N, \delta_i)$ by

$$\epsilon_{i+1} \coloneqq \frac{1}{8}\left((8\sqrt{n})^n N^{(n+2)\log_2(2/\sigma\delta_i)}\right)^{\frac{-1}{q-n}} \tag{4.3.1.1}$$

and $\delta_{i+1} = \delta_{i+1}(L, \delta_i, \epsilon_i, \epsilon_{i+1})$ by

$$\delta_{i+1} \coloneqq \frac{1}{L}\frac{\delta_i}{\epsilon_i}\epsilon_{i+1}, \tag{4.3.1.2}$$

for $i \geq 0$.

In Subsection 4.3.2 we will construct a sequence of maps from X to ℓ^2 dependent on these sequences $(\epsilon_j)_{j\in\mathbb{N}}$ and $(\delta_j)_{j\in\mathbb{N}}$. One should think of $(\epsilon_j)_{j\in\mathbb{N}}$ and $(\delta_j)_{j\in\mathbb{N}}$ as partitioning distances in ℓ^2 and X, respectively, into different 'scales'. The maps will approximate X at scale δ_i by simplices at scale ϵ_i in ℓ^2. Our exact choices of these ϵ_i and δ_i are made to give us good control on how the maps distort distance. For now, we note some properties of these two sequences as they will be integral to the overall proof.

Lemma 4.3.2. *The sequences $(\delta_j)_{j\in\mathbb{N}}$ and $(\epsilon_j)_{j\in\mathbb{N}}$ satisfy*

$$\delta_i = \left(\frac{1}{L}\right)^i \epsilon_i, \text{ for all } i \geq 0.$$

Proof. Inductively apply the recurrence relation and note you have a telescoping product. Finally, recall that $\epsilon_0 = \delta_0 = 1$. □

Lemma 4.3.3. *The sequence* $(\epsilon_j)_{j\in\mathbb{N}}$ *satisfies*

$$\frac{\epsilon_{i+1}}{\epsilon_i} \le \frac{1}{L}, \text{ for all } i \ge 0.$$

In particular, throughout, we will also require $(\epsilon_j)_{j\in\mathbb{N}}$ to be such that

$$\frac{\epsilon_{i+1}}{\epsilon_i} \le \frac{1}{8\sqrt{2(n+1)}} \le \frac{1}{2},$$

which is a consequence of Lemma 4.3.3 by the definition of L given in (4.3.0.1).

Proof. If $i \ge 1$ then by substituting out δ_i using Lemma 4.3.2 in the expression of ϵ_{i+1} given in (4.3.1.1), we see that

$$\begin{aligned}
\epsilon_{i+1} &= \frac{1}{8}\left((8\sqrt{n})^n N^{(n+2)\log_2\left(2L^i/\sigma\epsilon_i\right)}\right)^{\frac{-1}{q-n}}, \\
&= \frac{1}{8}\left(\frac{1}{B_1 N^{B_2} N^{iB_3}}\right) N^{\frac{n+2}{q-n}\log_2(\epsilon_i)}, \\
&= \frac{1}{8}\left(\frac{1}{B_1 N^{B_2} N^{iB_3}}\right) \epsilon_i^{\frac{n+2}{q-n}\log_2(N)}, \qquad (4.3.3.1)
\end{aligned}$$

where

$$B_1 \coloneqq B_1(n,q) \coloneqq (8\sqrt{n})^{\frac{n}{q-n}} > 0, \qquad (4.3.3.2)$$

$$B_2 \coloneqq B_2(n,q,\sigma) \coloneqq \frac{n+2}{q-n}\log_2\left(\frac{2}{\sigma}\right) > 0, \qquad (4.3.3.3)$$

$$B_3 \coloneqq B_3(n,q,L) \coloneqq \frac{n+2}{q-n}\log_2(L) > 0. \qquad (4.3.3.4)$$

The statements that B_2 and B_3 are positive come from $0 < \sigma < 1$ for B_2, and $L > 1$ for B_3.

Similarly, we can simplify the expression for ϵ_1 from (4.3.1.1) without substituting out δ_0 but instead recalling that $\delta_0 = \epsilon_0 = 1$ to get

$$\epsilon_1 = \frac{1}{8}\left(\frac{1}{B_1 N^{B_2}}\right)\epsilon_0^{\frac{n+2}{q-n}\log_2(N)},$$

which extends (4.3.3.1) to include the $i = 0$ case.

As $B_3 > 0$ and $N \ge 1$, we know that $N^{B_3} \ge 1$. Furthermore, by Note 4.3.1, we can assume

$$\frac{n+2}{q-n}\log_2(N) \ge 1$$

and

$$B_1 N^{B_2} N^{iB_3} \ge B_1 N^{B_2} \ge L,$$

for all $i \ge 0$. Now, as (4.3.3.1) holds for all $i \ge 0$ and $\epsilon_0 = 1$, we can observe that, inductively, $\epsilon_i \le 1$ and therefore $\epsilon_{i+1} \le \frac{1}{L}\epsilon_i$. □

By combining Lemma 4.3.3, Note 4.3.1, and the definition of δ_i in (4.3.1.2), we see that we can assume that δ_i is sufficiently small as to apply Lemma 4.2.9 with $\delta = \delta_i$ for all $i \geq 1$.

Let $\mathcal{U}_{i+1}$ be a cover of X, for all $i \geq 0$, as in Lemma 4.2.9 with $\mathrm{mesh}(\mathcal{U}_{i+1}) \leq \delta_{i+1}$. In particular, $\mathcal{L}(\mathcal{U}_{i+1}) \geq \sigma\delta_{i+1}$, $\mathcal{U}_{i+1}$ has multiplicity at most $n+1$, and, as we are assuming $\mathrm{diam}(X) = 1$,

$$|\mathcal{U}_{i+1}| \leq N^{\log_2(2/\sigma\delta_{i+1})}. \tag{4.3.3.5}$$

Note 4.3.4. These open sets are collections of points that have distances at most δ_{i+1} from each other, and, by Lemma 4.2.4, any collection of points with distances bounded strictly above by $\sigma\delta_{i+1}$ lies in one of these open sets. Therefore, one could interpret such a covering as approximating X by objects of roughly the scale δ_{i+1}.

Assumption 4.3.5. Later, in the proof of Proposition 4.4.1, we will assume that these covers, $(\mathcal{U}_j)_{j\in\mathbb{N}_{\geq 1}}$, satisfy a kind of 'non-redundancy' property: For any $\mathcal{U}_{i+1}$, $i \geq 0$, and any distinct $U, V \in \mathcal{U}_{i+1}$, we have that $U \nsubseteq V$. This assumption is justified as if there exists a pair of distinct elements $U, V \in \mathcal{U}_{i+1}$ such that $U \subseteq V$, then $\mathcal{U}_{i+1} \setminus \{U\}$ is still an open cover of X with mesh at most δ_{i+1}, Lebesgue number at least $\sigma\delta_{i+1}$, multiplicity at most $n+1$, and $|\mathcal{U}_{i+1}| \leq N^{\log_2(2/\sigma\delta_{i+1})}$. Therefore, replacing $\mathcal{U}_{i+1}$ with a cover that has had all of the 'redundant' elements removed in this way gives us a new cover with all the same desired properties.

4.3.2 The construction

Recall that ℓ^2 is the space of square-summable, real-valued sequences with the norm $|(z_i)_{i\in\mathbb{N}}| = \left(\sum_{i\in\mathbb{N}} z_i^2\right)^{1/2}$. Let $f_0 \colon X \to \ell^2$ be the constant zero map; $f_0(x) = (0,0,0,\dots)$ for all $x \in X$. We now inductively define a sequence of maps $(f_j)_{j\in\mathbb{N}}$ that will approximate X in ℓ^2 to progressively finer scales.

Suppose you have a map $f_i \colon X \to \ell^2$ such that the image of f_i is contained in

$$\{(z_1,\dots,z_{m_i},0,0,\dots) \mid z_i \in \mathbb{R}\},$$

for some m_i. In other words, $f_i(X)$ is contained in a particular finite-dimensional linear subspace of ℓ^2. Note that this condition does indeed hold for f_0 as $f_0(X) = \{(0,0,\dots)\}$ is of this form.

Order $\mathcal{U}_{i+1} = \{U_1, U_2, \dots, U_{|\mathcal{U}_{i+1}|}\}$ and for each $1 \leq k \leq |\mathcal{U}_{i+1}|$, pick $x_k \in U_k$. Then define,

$$p_k \coloneqq f_i(x_k) + \frac{\epsilon_{i+1}}{2} e_{m_i+k}, \tag{4.3.5.1}$$

where $e_j = (\delta_{k,j})_{k\in\mathbb{N}}$, $\delta_{k,j} = 1$ if $k = j$ and 0 otherwise. In other words, if $f_i(x_k)$ has the form

$$(z_1, z_2, \dots, z_{m_i}, 0, 0, \dots),$$

then p_k has the form

$$(z_1, \dots, z_{m_i}, 0, \dots, 0, \epsilon_{i+1}/2, 0, 0, \dots),$$

where the $\epsilon_{i+1}/2$ is in the $(m_i + k)$-th co-ordinate.

We will sometimes write $p_U \coloneqq p_k$ if $U = U_k \in \mathcal{U}_{i+1}$.

Now, define $f_{i+1}\colon X \to \ell^2$ as follows

$$x \mapsto \frac{\sum_{k=1}^{|\mathcal{U}_{i+1}|} d(x, X \setminus U_k) p_k}{\sum_{k=1}^{|\mathcal{U}_{i+1}|} d(x, X \setminus U_k)}. \tag{4.3.5.2}$$

Note, for any $x \in X$, $f_{i+1}(x)$ is a (finite) linear combination of vectors contained in $\{(z_1, \ldots, z_{m_{i+1}}, 0, 0, \ldots) \mid z_i \in \mathbb{R}\}$, where $m_{i+1} = m_i + |\mathcal{U}_{i+1}|$, and therefore $f_{i+1}(x)$ is also contained in this set. This justifies that we can indefinitely continue this inductive definition of functions to get an infinite sequence $(f_j)_{j \in \mathbb{N}}$.

4.4 Properties of the approximating maps

In this section, we show that the approximating maps, f_i, are locally-Lipschitz, that points that are clearly distinct at scale δ_i are clearly distinct in the image by scale ϵ_i, which we refer to as a 'separation' property, and finally that we have good control of the q-measure of the images, $f_i(X)$.

4.4.1 Locally Lipschitz

The functions f_i have been chosen so that they do not stretch distances too far. More precisely,

Proposition 4.4.1. *For all $i \geq 0$, f_i is locally Lipschitz. In particular, if $d(x, y) < \sigma\delta_i$, then $d(f_i(x), f_i(y)) \leq \frac{L}{2}\frac{\epsilon_i}{\delta_i} d(x, y)$, where L is defined in* (4.3.0.1).

We actually only require the following corollary.

Corollary 4.4.2. *For all $i \geq 0$,*

$$d(x, y) \leq \delta_{i+1} \implies d(f_i(x), f_i(y)) \leq \epsilon_{i+1}/2. \tag{4.4.2.1}$$

Proof. The case of $i = 0$ trivially holds because $f_0(x) = f_0(y)$ for all $x, y \in X$. For $i \geq 1$, recall the definition of δ_{i+1} from (4.3.1.2),

$$\delta_{i+1} = \frac{1}{L}\frac{\delta_i}{\epsilon_i}\epsilon_{i+1} = \frac{\sigma^2}{128(n+1)^2}\frac{\delta_i}{\epsilon_i}\epsilon_{i+1},$$

and note that $\sigma < 1$, $128(n + 1)^2 \geq 1$, and $\epsilon_{i+1} \leq \epsilon_i$ from Lemma 4.3.3, so $\delta_{i+1} < \sigma\delta_i$, and therefore by Proposition 4.4.1,

$$d(x, y) \leq \delta_{i+1} \implies d(f_i(x), f_i(y)) \leq \epsilon_{i+1}/2. \qquad \square$$

First we need some lemmas. Throughout the following lemmas, we impose conditions so that open sets, U, are not the entirety of X; this is simply so that $d(x, X \setminus U)$ is well-defined for $x \in X$.

Lemma 4.4.3. *For X a metric space, $x, y \in X$, and $U \subsetneq X$ an open set, we have*

$$|d(y, X \setminus U) - d(x, X \setminus U)| \leq d(x, y).$$

Proof. For any $z \in X \setminus U$, by definition and then the triangle inequality, we have

$$d(y, X \setminus U) \leq d(y, z) \leq d(x, y) + d(x, z).$$

Hence,

$$d(x, X \setminus U) = \inf_{z \in X \setminus U} d(x, z) \geq d(y, X \setminus U) - d(x, y).$$

This argument was symmetric in x and y, so the lemma holds. □

Lemma 4.4.4. *Suppose X is a metric space, and $\mathcal{U}$ is an open cover of X with mesh at most $0 < \delta < \operatorname{diam}(X)$ such that, for any distinct $U, V \in \mathcal{U}$, $U \nsubseteq V$. For any $x \in X$, if there is an element U of $\mathcal{U}$ such that $d(x, X \setminus U) > 2\delta$, then $d(U, X \setminus U) > \delta$ and $U \cap V = \emptyset$ for any $V \in \mathcal{U} \setminus \{U\}$, in particular x lies exclusively in U.*

Proof. Observe that for any $x \in X$ and $U \subsetneq X$ open, then $d(x, X \setminus U)$ is non-zero if and only if $x \in U$. Therefore, as $\delta > 0$, if $x \in X$ and $U \in \mathcal{U}$ with $d(x, X \setminus U) > 2\delta$, then $x \in U$. We see that $d(U, X \setminus U) > \delta$ by applying the triangle inequality while bearing in mind that the mesh constraint on $\mathcal{U}$ means that, for any $y \in U$, $d(x, y) \leq \delta$. To see that U intersects no other element of $\mathcal{U}$, observe that, if, for some $V \in \mathcal{U}$, there exists $y \in U \cap V$, then, for any $v \in V$, $d(y, v) \leq \delta$ by the mesh constraint on $\mathcal{U}$, but $d(U, X \setminus U) > \delta$ so $v \in U$, meaning $V \subseteq U$. Finally, by the non-redundancy restriction on $\mathcal{U}$, which is also described in Assumption 4.3.5, this must mean $V = U$. □

The message that the reader should take away from this lemma is that if $d(x, X \setminus U)$ is much larger than the scale of the cover $\mathcal{U}$, then x lies in precisely one subset U and this U is isolated from the rest of X at this scale.

Lemma 4.4.5. *Suppose X is a metric space and $\mathcal{U}$ is a finite open cover of X with mesh at most $\delta < \operatorname{diam}(X)$, Lebesgue number at least ξ, and multiplicity at most $m + 1$, such that, for any distinct $U, V \in \mathcal{U}$, $V \nsubseteq U$. If $x \in X$ is such that $d(x, X \setminus U) \leq 2\delta$ for every $U \in \mathcal{U}$, then*

$$\xi \leq \left| \sum_{U \in \mathcal{U}} d(x, X \setminus U) \right| = \sum_{U \in \mathcal{U}} d(x, X \setminus U) \leq 2(m + 1)\delta.$$

Proof. By Lemma 4.2.4, $B(x, \xi) \subseteq V$ for some $V \in \mathcal{U}$, meaning $d(x, X \setminus V) \geq \xi$. All other terms in the sum are non-negative, so the left inequality indeed holds.

The middle equality is just the observation that each term in this sum is non-negative.

The right inequality is the combination of the observations; that $d(x, X \setminus U) > 0$ if and only if $x \in U$, that x can be in at most $m+1$ members of the cover $\mathcal{U}$ because $\mathcal{U}$ has multiplicity at most $m+1$, and that $d(x, X \setminus U)$ is at most 2δ for every $U \in \mathcal{U}$ by assumption. □

Lemma 4.4.6. *Suppose X is a metric space, and $\mathcal{U}$ is an open cover of X with multiplicity at most $m+1$, and the property that $U \neq X$, for any $U \in \mathcal{U}$. Then, for any $x, y \in X$,*

$$\left| \sum_{U \in \mathcal{U}} d(x, X \setminus U) - \sum_{V \in \mathcal{U}} d(y, X \setminus V) \right| \leq 2(m+1) d(x, y).$$

Proof. This follows from combining the triangle inequality, the observation that there can be at most $m+1$ non-zero contributions from x and at most $m+1$ non-zero contributions from y, and Lemma 4.4.3. □

Lemma 4.4.7. *For any real numbers a, b, c, d,*

$$|ab - cd| \leq |a||b-d| + |a-c||d|.$$

Proof. Observe $|ab - cd| = |ab - ad + ad - cd|$ and use the triangle inequality. □

Lemma 4.4.8. *Let $i \geq 0$, and let f_i, $\mathcal{U}_{i+1}$, and $\{p_U \mid U \in \mathcal{U}_{i+1}\}$ be as in the construction, given in Subsection 4.3.2. Further assume that f_i satisfies*

$$d(x, y) \leq \delta_{i+1} \implies d(f_i(x), f_i(y)) \leq \epsilon_{i+1}/2, \tag{4.4.8.1}$$

for $x, y \in X$. If $U, V \in \mathcal{U}_{i+1}$, such that $U \cap V \neq \emptyset$, then

$$|p_U - p_V| = d(p_U, p_V) \leq 2\epsilon_{i+1}.$$

Proof. Assume $x \in U \cap V$. By construction of f_{i+1}, (4.3.5.2), we have that there exists $x_U \in U$ and $x_V \in V$ such that $d(f_i(x_U), p_U) = \epsilon_{i+1}/2$ and $d(f_i(x_V), p_V) = \epsilon_{i+1}/2$. Also, by the assumption (4.4.8.1), and noting that $d(x, x_U), d(x, x_V) \leq \delta_{i+1}$, we get that

$$d(f_i(x), f_i(x_U)), d(f_i(x), f_i(x_V)) \leq \epsilon_{i+1}/2.$$

Combining this all together and using the triangle inequality, we get $|p_U - p_V| = d(p_U, p_V) \leq 2\epsilon_{i+1}$, the desired result. □

Proof of Proposition 4.4.1. We proceed by induction.

When $i = 0$, $f_0(x) = f_0(y)$ for any $x, y \in X$ so $d(f_0(x), f_0(y)) = 0 \leq \frac{L}{2}\frac{\epsilon_0}{\delta_0} d(x, y)$ trivially holds. Thus, the base case is verified.

When $i = j$, assume that if $x, y \in X$ are such that $d(x, y) < \sigma\delta_j$, then

$$d(f_j(x), f_j(y)) \leq \frac{L}{2}\frac{\epsilon_j}{\delta_j} d(x, y).$$

In particular, by the same argument given in the proof of Corollary 4.4.2, we get

$$d(x, y) \leq \delta_{j+1} \implies d(f_j(x), f_j(y)) \leq \epsilon_{j+1}/2. \tag{4.4.8.2}$$

In fact, this implication is all we will actually use from the induction hypothesis.

When $i = j + 1$, if we pick $x, y \in X$ with $d(x, y) < \sigma\delta_{j+1}$ and write out explicitly $d(f_{j+1}(x), f_{j+1}(y))$ from the definition of f_{j+1}, given in (4.3.5.2), we get

$$\left| \frac{\sum_{U \in \mathcal{U}_{j+1}} d(x, X \setminus U) p_U}{\sum_{U \in \mathcal{U}_{j+1}} d(x, X \setminus U)} - \frac{\sum_{U \in \mathcal{U}_{j+1}} d(y, X \setminus U) p_U}{\sum_{U \in \mathcal{U}_{j+1}} d(y, X \setminus U)} \right|. \tag{4.4.8.3}$$

If $d(x, X \setminus U) > 2\delta_{j+1}$ or $d(y, X \setminus U) > 2\delta_{j+1}$ for any $U \in \mathcal{U}_{j+1}$, then, by Lemma 4.4.4; x or y is exclusively in U, $U \cap V = \emptyset$ for all $V \in \mathcal{U}_{j+1} \setminus \{U\}$, and $d(U, X \setminus U) > \delta_{j+1}$. We can use Lemma 4.4.4 here because of the non-redundancy assumption on $\mathcal{U}_{j+1}$ justified in Assumption 4.3.5. As x and y are close, $d(x, y) < \sigma\delta_{j+1} \leq \delta_{j+1}$, we have that both x and y lie in U. Therefore both x and y lie exclusively in U, meaning $d(x, X \setminus U)$ and $d(y, X \setminus U)$ are the only non-zero terms within the sums in Expression (4.4.8.3). This means both $f_{j+1}(x)$ and $f_{j+1}(y)$ evaluate to p_U and, therefore, this distance is equal to zero, which trivially satisfies the desired bound.

Therefore, we may assume that both $d(x, X \setminus U) \leq 2\delta_{j+1}$ and $d(y, X \setminus U) \leq 2\delta_{j+1}$, for every $U \in \mathcal{U}_{j+1}$, allowing us to use Lemma 4.4.5 in the following manipulations of Expression (4.4.8.3).

Note that $\delta_{j+1} < 1 = \operatorname{diam}(X)$, which can be seen by combining Lemma 4.3.3, the definition of δ_{j+1} from (4.3.1.2), and $\epsilon_0 = 1$. Therefore, no element of $\mathcal{U}_{j+1}$ is the entirety of X, so $d(x, X \setminus U)$ is well defined, and we can apply lemmas 4.4.3 and 4.4.6 in the following.

Since $d(x, y) < \sigma\delta_{j+1} \leq \mathcal{L}(\mathcal{U}_{j+1})$, there exists, by Lemma 4.2.4, $U_0 \in \mathcal{U}_{j+1}$ containing $\{x, y\}$. Let p_0 be the vertex corresponding to U_0 as in the construction, given in (4.3.5.1).

$$(4.4.8.3) = \left| \frac{\sum_{U \in \mathcal{U}_{j+1}} d(x, X \setminus U) p_U}{\sum_{U \in \mathcal{U}_{j+1}} d(x, X \setminus U)} - p_0 + p_0 - \frac{\sum_{U \in \mathcal{U}_{j+1}} d(y, X \setminus U) p_U}{\sum_{U \in \mathcal{U}_{j+1}} d(y, X \setminus U)} \right|$$

Noting that

$$\frac{\sum_{U \in \mathcal{U}_{j+1}} d(x, X \setminus U) p_0}{\sum_{U \in \mathcal{U}_{j+1}} d(x, X \setminus U)} = p_0,$$

trivially, we get

$$= \left| \frac{\sum_{U \in \mathcal{U}_{j+1}} d(x, X \setminus U)(p_U - p_0)}{\sum_{U \in \mathcal{U}_{j+1}} d(x, X \setminus U)} - \frac{\sum_{U \in \mathcal{U}_{j+1}} d(y, X \setminus U)(p_U - p_0)}{\sum_{U \in \mathcal{U}_{j+1}} d(y, X \setminus U)} \right|.$$

Pulling out the denominators and using the lower bound from Lemma 4.4.5, we observe

$$\leq \frac{1}{\sigma^2 {\delta_{j+1}}^2} \left| \sum_{U \in \mathcal{U}_{j+1}} d(y, X \setminus U) \sum_{U \in \mathcal{U}_{j+1}} d(x, X \setminus U)(p_U - p_0) \right.$$
$$\left. - \sum_{U \in \mathcal{U}_{j+1}} d(x, X \setminus U) \sum_{U \in \mathcal{U}_{j+1}} d(y, X \setminus U)(p_U - p_0) \right| .$$

Using Lemma 4.4.7, we see

$$\leq \frac{1}{\sigma^2 {\delta_{j+1}}^2} \left(\left| \sum_{U \in \mathcal{U}_{j+1}} d(y, X \setminus U) \right| \left| \sum_{U \in \mathcal{U}_{j+1}} (d(x, X \setminus U) - d(y, X \setminus U))(p_U - p_0) \right| \right.$$
$$\left. + \left| \sum_{U \in \mathcal{U}_{j+1}} (d(y, X \setminus U) - d(x, X \setminus U)) \right| \left| \sum_{U \in \mathcal{U}_{j+1}} d(y, X \setminus U)(p_U - p_0) \right| \right) .$$

Now, using lemmas 4.4.5 and 4.4.6, we obtain

$$\leq \frac{1}{\sigma^2 {\delta_{j+1}}^2} \left(2(n+1)\delta_{j+1} \left| \sum_{U \in \mathcal{U}_{j+1}} (d(x, X \setminus U) - d(y, X \setminus U))(p_U - p_0) \right| \right.$$
$$\left. + 2(n+1) d(x,y) \left| \sum_{U \in \mathcal{U}_{j+1}} d(y, X \setminus U)(p_U - p_0) \right| \right) .$$

Using the triangle inequality

$$\leq \frac{1}{\sigma^2 {\delta_{j+1}}^2} \left(2(n+1)\delta_{j+1} \sum_{U \in \mathcal{U}_{j+1}} |(d(x, X \setminus U) - d(y, X \setminus U))| \, |(p_U - p_0)| \right.$$
$$\left. + 2(n+1) d(x,y) \sum_{U \in \mathcal{U}_{j+1}} |d(y, X \setminus U)| \, |(p_U - p_0)| \right) .$$

Note, for any $z \in X$, $d(z, X \setminus U)$ is non-zero if and only if $z \in U$, so the first sum has a non-zero contribution from $(p_U - p_0)$ only if $x \in U$ or $y \in U$. We assumed $x, y \in U_0$ so, in fact, we have a non-zero contribution from $(p_U - p_0)$ only if x or y is in $U \cap U_0$. Without loss of generality, assume $x \in U$. By Lemma 4.4.8 applied to f_j, we know $|p_U - p_0| = d(p_U, p_0) \leq 2\epsilon_{j+1}$. We can apply Lemma 4.4.8 here because f_j satisfies (4.4.8.2), which was obtained from the induction assumption. Similarly, in the second sum, $|p_U - p_0| \leq 2\epsilon_{j+1}$ whenever there is a non-zero contribution from $d(y, X \setminus U)$. So, we can continue to bound the above expression by

$$\leq \frac{1}{\sigma^2 {\delta_{j+1}}^2} \Bigg(2(n+1)\delta_{j+1} \sum_{U \in \mathcal{U}_{j+1}} |(d(x, X \setminus U) - d(y, X \setminus U))| \, 2\epsilon_{j+1} \\ + 2(n+1) d(x,y) \sum_{U \in \mathcal{U}_{j+1}} |d(y, X \setminus U)| \, 2\epsilon_{j+1} \Bigg).$$

Finally, by lemmas 4.4.3 and 4.4.5,

$$\leq \frac{1}{\sigma^2 {\delta_{j+1}}^2} \Big(2(n+1)\delta_{j+1} 2(n+1) d(x,y) 2\epsilon_{j+1} + 2(n+1) d(x,y) 2(n+1)\delta_{j+1} 2\epsilon_{j+1} \Big)$$

$$= \frac{16(n+1)^2 \epsilon_{j+1}}{\sigma^2 \delta_{j+1}} d(x,y) \leq \frac{L}{2} \frac{\epsilon_{j+1}}{\delta_{j+1}} d(x,y),$$

which completes the induction. □

4.4.2 Separation

In this subsection we show that the functions f_i have been chosen so that points distinguished by the cover $\mathcal{U}_i$ remain uniformly distinguished after applying f_i. More precisely,

Lemma 4.4.9. *For any $i \geq 0$ and $x, y \in X$,*

$$\delta_{i+1} < d(x,y) \implies \frac{\epsilon_{i+1}}{\sqrt{2(n+1)}} \leq d(f_{i+1}(x), f_{i+1}(y)).$$

First, we need a lemma.

Lemma 4.4.10. *The function $f \colon \mathbb{R}^m \to \mathbb{R}$ defined by*

$$f(x_1, x_2, \ldots, x_m) = \sum_{k=1}^{m} x_k^2,$$

for $(x_1, \ldots, x_m) \in \mathbb{R}^m$, restricted to $\{(x_1, \ldots, x_m) \mid \sum_{k=1}^m x_k = 1 \text{ and } x_j \geq 0 \text{ for all } j\}$ is minimised by $1/m$ at the point $x_j = 1/m$ for all j.

Proof. This is standard. For example, one could use Lagrange multipliers. □

Proof of Lemma 4.4.9. Recall that evaluating f_{i+1} at a point x is given by the sum

$$\frac{\sum_{k=1}^{|\mathcal{U}_{i+1}|} d(x, X \setminus U_k) p_k}{\sum_{k=1}^{|\mathcal{U}_{i+1}|} d(x, X \setminus U_k)},$$

as defined in the construction in (4.3.5.2). It is key to note that the k-th term of the sum is non-zero if and only if $x \in U_k$. Now suppose $x, y \in X$ such that $d(x,y) > \delta_{i+1}$. As we chose $\mathcal{U}_{i+1}$, via

Lemma 4.2.9, to have mesh at most δ_{i+1}, x and y cannot both lie in any single element of $\mathcal{U}_{i+1}$. Therefore, the non-zero p_k components associated to x are disjoint from the non-zero p_k components associated to y. Further, $f_{i+1}(x)$ will have the form $\sum_{k=1}^{m_i}(z_k e_k)+\sum_{k=1}^{|\mathcal{U}_{i+1}|}(\lambda_k(\epsilon_{i+1}/2)e_{m_i+k})$ where $z_k \in \mathbb{R}$, and $\sum_{k=1}^{|\mathcal{U}_{i+1}|}\lambda_k = 1$ with $\lambda_k \geq 0$ for all k and $\lambda_k > 0$ for at most $n+1$ distinct k. Similarly, $f_{i+1}(y)$ will have the form $\sum_{k=1}^{m_i}(w_k e_k) + \sum_{k=1}^{|\mathcal{U}_{i+1}|}(\mu_k(\epsilon_{i+1}/2)e_{m_i+k})$ where $w_k \in \mathbb{R}$, and $\sum_{k=1}^{|\mathcal{U}_{i+1}|}\mu_k = 1$ with $\mu_k \geq 0$ for all k and $\mu_k > 0$ for at most $n+1$ distinct k each of which is distinct from the set of k such that $\lambda_k > 0$. This gives us a nice form to the distance between $f_{i+1}(x)$ and $f_{i+1}(y)$, namely

$$\begin{aligned} d(f_{i+1}(x), f_{i+1}(y)) &= \sqrt{\sum_{k=1}^{m_i}(z_k - w_k)^2 + \sum_{k=1}^{|\mathcal{U}_{i+1}|}\left(\frac{\epsilon_{i+1}\lambda_k}{2}\right)^2 + \sum_{k=1}^{|\mathcal{U}_{i+1}|}\left(\frac{\epsilon_{i+1}\mu_k}{2}\right)^2}, \\ &\geq \frac{\epsilon_{i+1}}{2}\sqrt{\sum_{k=1}^{|\mathcal{U}_{i+1}|}\lambda_k^2 + \sum_{k=1}^{|\mathcal{U}_{i+1}|}\mu_k^2}. \end{aligned}$$

Now by Lemma 4.4.10, bearing in mind the conditions on the λ_k and μ_k sums, we get

$$d(f_{i+1}(x), f_{i+1}(y)) \geq \frac{\epsilon_{i+1}}{2}\sqrt{\frac{2}{(n+1)}} = \frac{\epsilon_{i+1}}{\sqrt{2(n+1)}}. \qquad \square$$

4.4.3 Bounding the q-measure of the image

In this subsection we show the functions f_i have been chosen so that we can easily bound the q-measure of their images.

In the following we will refer to 'simplices' containing the image of f_{i+1}. What we mean by this is that if you consider $x \in X$, there exist $U_{j_1}, \ldots U_{j_m} \in \mathcal{U}_{i+1}$, such that $x \in U_{j_k}$ for all k. Then $f_{i+1}(x)$ sits inside

$$f_{i+1}\left(\bigcap_{k=1}^{m} U_{j_k}\right) \subseteq [p_{j_1}, \ldots, p_{j_m}] \coloneqq \left\{\sum_{k=1}^{m}\lambda_k p_{j_k} \mid \lambda_k \geq 0 \text{ for all } k \text{ and } \sum_{k=1}^{m}\lambda_k = 1\right\}.$$

We say that $[p_{j_1}, \ldots, p_{j_m}]$ is a *simplex* containing $f_{i+1}(x)$. Due to our construction of f_i, the image of f_i is contained in an n-dimensional simplicial complex with simplices of diameter approximately ϵ_{i+1}. For any $q > n$, subspaces in $\mathbb{R}^n$ can be covered efficiently at all scales, in the sense of Hausdorff q-measure. Hence, our approach is to cover each component simplex efficiently using our knowledge of covering n-dimensional euclidean space, then union over all simplices to cover the complex. The upper bound on the number of elements in the cover $\mathcal{U}_{i+1}$ from Lemma 4.2.9 will give control on the number of simplices in this simplicial complex.

For each $i \geq 0$ define

$$\eta_{i+1} \coloneqq 8\epsilon_{i+2}. \tag{4.4.10.1}$$

This will be the scale within ℓ^2 at which we cover $f_{i+1}(X)$, the $(i+1)$-th approximation of X.

Lemma 4.4.11. *For each $i \geq 0$, there exists a cover, $\mathcal{V}_{i+1}$, of $f_{i+1}(X)$ with mesh at most $4\eta_{i+1}$, Lebesgue number at least η_{i+1} as a cover of $f_{i+1}(X) \subseteq \ell^2$, and*

$$\sum_{V \in \mathcal{V}_{i+1}} \operatorname{diam}(V)^q \leq 4^q.$$

The 'Lebesgue number' component on this lemma is present so that $\mathcal{V}_{i+1}$ also covers a small ℓ^2-neighbourhood of the image. Hence, $\mathcal{V}_{i+1}$ will also cover the image of functions similar to f_{i+1}. In particular, this will mean that the image of the pointwise limit of $(f_j)_{j\in\mathbb{N}}$ will be covered by $\mathcal{V}_{i+1}$, and this will give us a useful family of covers for computing the Hausdorff q-measure of this limit image.

Proof of Lemma 4.4.11. Following our strategy detailed above, we begin by explaining how to cover an individual simplex containing some of the image of f_{i+1}.

For each simplex, $\Delta = [v_0, \ldots, v_m]$ with $v_j \in \{p_U \mid U \in \mathcal{U}_{i+1}\}$ and $m \leq n$, by picking a base vertex of Δ, say v_0, and letting S be the span of the vectors $(v_j - v_0)$ in ℓ^2, we observe that Δ sits inside a translated copy of $\mathbb{R}^r$, say $(S + v_0)$, for some $r \leq m \leq n$. Lemma 4.4.8 tells us that the edge lengths of Δ are at most $2\epsilon_{i+1}$, hence we see that Δ is contained in an r-cube in $(S + v_0)$ of edge length $4\epsilon_{i+1}$ centred at v_0. To be precise, identify S with $\mathbb{R}^r$, let $R = [-2\epsilon_{i+1}, 2\epsilon_{i+1}]^r \subset S$, and then translate R so that it is centred on v_0; $(R + v_0)$ is the aforementioned r-cube. Consider the cover of R by subdividing R into r-cubes of edge length $\eta_{i+1}/\sqrt{n}$. Note each of these covering r-cubes has diameter $\sqrt{r}\,\eta_{i+1}/\sqrt{n} \leq \eta_{i+1}$. This requires at most $4\sqrt{n}\,\epsilon_{i+1}/\eta_{i+1} + 1$ subdivisions along each edge of R. One may observe that $4\sqrt{n}\,\epsilon_{i+1}/\eta_{i+1} \geq 1$ by combining Lemma 4.3.3 with the definition of η_{i+1} as $8\epsilon_{i+2}$, (4.4.10.1). Hence, we can simplify this to at most $8\sqrt{n}\,\epsilon_{i+1}/\eta_{i+1}$ subdivisions. Thus, to cover the whole r-cube R we need at most $(8\sqrt{n}\,\epsilon_{i+1}/\eta_{i+1})^r$ r-cubes of edge length $\eta_{i+1}/\sqrt{n}$. As $r \leq n$ and $4\sqrt{n}\,\epsilon_{i+1}/\eta_{i+1} \geq 1$, we can see that $(8\sqrt{n}\,\epsilon_{i+1}/\eta_{i+1})^r \leq (8\sqrt{n}\,\epsilon_{i+1}/\eta_{i+1})^n$. Now, if we do this for each simplex in the complex $f_{i+1}(X)$ and take the collection of all these covering hypercubes of diameter at most η_{i+1}, we get a cover, say $\mathcal{V}_{i+1}$, for $f_{i+1}(X)$.

We picked $\mathcal{U}_{i+1}$ from Lemma 4.2.9, so we know that

$$|\mathcal{U}_{i+1}| \leq N^{\log_2(2/\sigma\delta_{i+1})}.$$

We also know that every simplex is a choice of at most $(n+1)$ distinct elements of $\mathcal{U}_{i+1}$, which is at most a choice of $(n+1)$ elements of $\mathcal{U}_{i+1}$ with repetition. Hence, if we define Δ_{i+1} to be the set of all simplices, then

$$\begin{aligned} |\Delta_{i+1}| &\leq \sum_{m=0}^{n} |\mathcal{U}_{i+1}|^{m+1}, \\ &\leq |\mathcal{U}_{i+1}|^{n+2}, \\ &\leq N^{(n+2)\log_2(2/\sigma\delta_{i+1})}. \end{aligned} \tag{4.4.11.1}$$

We can, therefore, estimate the q-measure of the image by

$$\sum_{V\in\mathcal{V}_{i+1}} \operatorname{diam}(V)^q \le \sum_{V\in\mathcal{V}_{i+1}} \eta_{i+1}^q \le |\Delta_{i+1}|(8\sqrt{n}\,\epsilon_{i+1}/\eta_{i+1})^n {\eta_{i+1}}^q,$$

with the cover described above.

Noting that $\epsilon_{i+1} \le 1$, by Lemma 4.3.3 and $\epsilon_0 = 1$, we can again bound by

$$\sum_{V\in\mathcal{V}_{i+1}} \operatorname{diam}(V)^q \le |\Delta_{i+1}|(8\sqrt{n})^n \eta_{i+1}^{q-n}. \tag{4.4.11.2}$$

Plugging the definition of ϵ_{i+2}, (4.3.1.1), into the definition of η_{i+1}, (4.4.10.1), and combining with equations (4.4.11.1) and (4.4.11.2) we obtain

$$\sum_{V\in\mathcal{V}_{i+1}} \operatorname{diam}(V)^q \le 1.$$

Now we have a cover of the image of f_{i+1}, but we are lacking control on the Lebesgue number of $\mathcal{V}_{i+1}$ to ensure that $\mathcal{V}_{i+1}$ also covers a small ℓ^2-neighbourhood of the image. For each $V \in \mathcal{V}_{i+1}$, pick a point $y_V \in V$, then consider $V \subseteq \overline{B(y_V, \eta_{i+1})}$ and diam $\left(\overline{B(y_V, \eta_{i+1})}\right) \le 2\eta_{i+1}$ so we can replace V by $\overline{B(y_V, \eta_{i+1})}$ and still cover $f_{i+1}(X)$ without significantly affecting the value of the above sum. We can also double the radius of $\overline{B(y_V, \eta_{i+1})}$ without changing the value of the above sum by much, so if we consider a new cover $\mathcal{V}'_{i+1} = \{\overline{B(y_V, 2\eta_{i+1})} \mid V \in \mathcal{V}\}$. Then

$$\sum_{V'\in\mathcal{V}'_{i+1}} \operatorname{diam}(V')^q \le \sum_{V\in\mathcal{V}_{i+1}} \eta_{i+1}^q \le 4^q.$$

Now, we proceed to show that we have good control of the Lebesgue number of this modified cover. Indeed, for any $y \in f_{i+1}(X)$, $\mathcal{V}$ is a cover of $f_{i+1}(X)$, so there exists some $V \in \mathcal{V}$ such that $y \in V \subset \overline{B(y_V, \eta_{i+1})}$ and thus $B(y, \eta_{i+1}) \subset \overline{B(y_V, 2\eta_{i+1})} \in \mathcal{V}'_{i+1}$, so $\mathcal{L}(\mathcal{V}'_{i+1}) \ge \eta_{i+1}$. Therefore, as the mesh of $\mathcal{V}'_{i+1}$ is at most $4\eta_{i+1}$, we see that $\mathcal{V}'_{i+1}$ satisfies the requirements for our lemma. ☐

4.4.4 The sequence is Cauchy

In this section, we justify that we have good control on how close each map in the sequence of maps $(f_j)_{j\in\mathbb{N}}$ is to the previous map.

Lemma 4.4.12. *For any $i \ge 0$, $d(f_{i+1}, f_i) \le \epsilon_{i+1}$.*

The following proof is essentially from [HW48, page 59].

Proof. For $x \in X$, let $U_{j_1}, \ldots, U_{j_m}$ be the collection of elements of $\mathcal{U}_{i+1}$ that contain x. Recall from (4.3.5.1) that, for each $1 \le k \le m$, U_{j_k} is associated to a point, x_{j_k}, contained within. As both x and x_{j_k} are contained in U_{j_k}, and $\operatorname{diam}(U_{j_k}) \le \delta_{i+1}$, we see that $d(x, x_{j_k}) \le \delta_{i+1}$.

Further, if we apply Corollary 4.4.2, we deduce $d(f_i(x), f_i(x_{j_k})) \leq \epsilon_{i+1}/2$. By our choice of p_{j_k}, we know $d(f_i(x_{j_k}), p_{j_k}) = \epsilon_{i+1}/2$, and therefore $|p_{j_k} - f_i(x)| = d(f_i(x), p_{j_k}) \leq \epsilon_{i+1}$ follows by application of the triangle inequality. Hence,

$$\begin{aligned} d(f_{i+1}(x), f_i(x)) &= \left| \left(\frac{\sum_{k=1}^{|\mathcal{U}_{i+1}|} d(x, X \setminus U_k) p_k}{\sum_{k=1}^{|\mathcal{U}_{i+1}|} d(x, X \setminus U_k)} \right) - f_i(x) \right| \\ &= \left| \frac{\sum_{k=1}^{m} d(x, X \setminus U_{j_k})(p_{j_k} - f_i(x))}{\sum_{k=1}^{m} d(x, X \setminus U_{j_k})} \right| \\ &\leq \frac{\sum_{k=1}^{m} d(x, X \setminus U_{j_k}) |p_{j_k} - f_i(x)|}{\sum_{k=1}^{m} d(x, X \setminus U_{j_k})} \\ &\leq \frac{\sum_{k=1}^{m} d(x, X \setminus U_{j_k}) \epsilon_{i+1}}{\sum_{k=1}^{m} d(x, X \setminus U_{j_k})} \\ &= \epsilon_{i+1}. \end{aligned}$$

□

Along with our knowledge of the sequence $(\epsilon_j)_{j\in\mathbb{N}}$ from Lemma 4.3.3, we see that the sequence $(f_j)_{j\in\mathbb{N}}$ is Cauchy.

4.5 The bi-Hölder embedding

In this section we define the map f mentioned in our main theorem, Theorem 4.1.4, and verify it has the desired properties for the theorem.

4.5.1 The definition and basic properties of f

In this subsection we define the map f as the pointwise limit of the sequence $(f_j)_{j\in\mathbb{N}}$, prove its existence, and show how nice properties of f_i translate over to f.

Lemma 4.5.1. *The pointwise limit, f, of $\{f_j\}_{j\in\mathbb{N}}$ exists, and, for $i \geq 1$, $d(f, f_{i-1}) \leq 2\epsilon_i$.*

Proof. Recall, Lemma 4.4.12 gives

$$d(f_j, f_{j+1}) \leq \epsilon_{j+1}, \text{ for all } j,$$

and Lemma 4.3.3 gives

$$\epsilon_{j+1} \leq \frac{1}{L}\epsilon_j, \text{ for all } j,$$

and $L \geq 2$. If we combine these with the triangle inequality, we may observe that $(f_j(x))_{j\in\mathbb{N}}$ is a Cauchy sequence in ℓ^2 for every $x \in X$ and therefore, by the completeness of ℓ^2, has a limit. Define $f \colon X \to \ell^2$ by $f(x) = \lim_{j\to\infty} f_j(x)$ for $x \in X$. Now for each $x \in X$ and $i \geq 1$, $d(f(x), f_{i-1}(x)) = \lim_{j\to\infty} d(f_j(x), f_{i-1}(x))$, but, for $j \geq i$, by repeated use of the triangle inequality and noting that $\epsilon_{j+1} \leq \epsilon_j/L \leq \epsilon_j/2$, for all $j \geq 0$, we get

$$d(f_{i-1}(x), f_j(x)) \leq \sum_{k=i}^{j} d(f_{k-1}(x), f_k(x)) \leq \sum_{k=i}^{j} \epsilon_k \leq \sum_{k=0}^{j-i} \frac{\epsilon_i}{2^k} = \epsilon_i \left(2 - \frac{1}{2^{j-i}}\right).$$

Therefore, taking $j \to \infty$, we see that $d(f, f_{i-1}) \leq 2\epsilon_i$. □

We see in the following two lemmas that the control imposed on the approximating functions, f_i, roughly follows through to the limit.

Lemma 4.5.2. *For all $i \geq 0$,*

$$d(x,y) \leq \delta_i \implies d(f(x), f(y)) \leq \frac{9}{2}\epsilon_i.$$

Proof. When $i \geq 1$, using the triangle inequality combined with $d(f, f_{i-1}) \leq 2\epsilon_i$ from Lemma 4.5.1, and

$$d(x,y) \leq \delta_i \implies d(f_{i-1}(x), f_{i-1}(y)) \leq \frac{1}{2}\epsilon_i,$$

from Corollary 4.4.2, we get the desired

$$d(x,y) \leq \delta_i \implies d(f(x), f(y)) \leq \frac{9}{2}\epsilon_i.$$

For $i = 0$ we similarly use the triangle inequality and $d(f, f_0) \leq 2\epsilon_1 \leq \epsilon_0$, but instead of using Corollary 4.4.2 we use the definition of f_0 to get $d(f_0(x), f_0(y)) = 0$ for all $x, y \in X$. □

Lemma 4.5.3. *For all $i \geq 0$,*

$$\delta_{i+1} < d(x,y) \implies \frac{1}{2\sqrt{2(n+1)}}\epsilon_{i+1} \leq d(f(x), f(y)).$$

Proof. By Lemma 4.4.9, we have

$$\delta_{i+1} < d(x,y) \implies \frac{1}{\sqrt{2(n+1)}}\epsilon_{i+1} \leq d(f_{i+1}(x), f_{i+1}(y)).$$

From Lemma 4.5.1, we have $d(f, f_{i+1}) \leq 2\epsilon_{i+2}$. From Lemma 4.3.3 we have $\epsilon_{i+2} \leq \epsilon_{i+1}/L$, and note that $L \geq (8\sqrt{2(n+1)})$, so

$$\epsilon_{i+2} \leq \frac{1}{L}\epsilon_{i+1} \leq \frac{1}{8\sqrt{2(n+1)}}\epsilon_{i+1}.$$

Combining these facts and using the triangle inequality, we get, for any $x, y \in X$ such that $\delta_{i+1} < d(x,y)$,

$$\frac{1}{2\sqrt{2(n+1)}}\epsilon_{i+1} = \frac{1}{\sqrt{2(n+1)}}\epsilon_{i+1} - 2(2\frac{1}{8\sqrt{2(n+1)}}\epsilon_{i+1}) \leq d(f(x), f(y)).$$ □

4.5.2 The Hausdorff dimension of $f(X)$ is at most q

Proposition 4.5.4. *The image, $f(X)$, of f has Hausdorff dimension at most q, in particular $f(X)$ has q-Hausdorff measure at most 4^q.*

Proof. By Lemma 4.4.11, for every $i \geq 1$, we have a cover, $\mathcal{V}_i$, of the image $f_i(X)$, with mesh at most $4\eta_i$ and Lebesgue number, as a cover of $f_i(X) \subseteq \ell^2$, at least η_i such that

$$\sum_{V \in \mathcal{V}_i} \operatorname{diam}(V)^q \leq \sum_{V \in \mathcal{V}_i} (4\eta_i)^q \leq 4^q.$$

Plugging the definition of η_{i+1}, given in (4.4.10.1), into the result of Lemma 4.5.1 we find $d(f, f_i) \leq \eta_i/4$. Hence, for any $x \in X$, $f(x)$ lies in the ball $B(f_i(x), \eta_i/2)$, which is contained in an element of $\mathcal{V}_i$ by the Lebesgue number component of Lemma 4.4.11. Therefore, $\mathcal{V}_i$ is also a cover for $f(X)$.

Noting that $\eta_i \to 0$, because $\epsilon_i \to 0$ by Lemma 4.3.3, we see that the Hausdorff q-measure of $f(X)$ is at most 4^q, which is finite so the Hausdorff dimension of $f(X)$ is at most q. □

4.5.3 The map f is bi-Hölder onto its image

Proposition 4.5.5. *The map f is bi-Hölder onto its image, in particular there exists $Q = Q(n, q, N)$ and $\lambda = \lambda(n, q, N, \sigma)$ such that, for any $x, y \in X$,*

$$\frac{1}{\lambda} d(x, y)^{2Q} \leq d(f(x), f(y)) \leq \lambda d(x, y)^{\frac{1}{4Q}}. \tag{4.5.5.1}$$

Lemma 4.5.6. *For all $i \geq 0$,*

$$\epsilon_i \leq \frac{1}{L^i}.$$

Proof. Recall $\epsilon_0 = 1$ so applying Lemma 4.3.3 inductively, we get

$$\epsilon_i \leq \left(\frac{1}{L}\right)^i \epsilon_0 = \frac{1}{L^i}.$$ □

As we did in Lemma 4.3.3, it is convenient to simplify our definition of ϵ_{i+1} as given in (4.3.1.1). Recall

$$\epsilon_{i+1} = \frac{1}{8}\left((8\sqrt{n})^n N^{(n+2)\log_2(2/\sigma\delta_i)}\right)^{\frac{-1}{q-n}}.$$

Let $C = C(n, q, N, \sigma)$, be defined as

$$C := 8(8\sqrt{n})^{\frac{n}{q-n}} N^{\frac{n+2}{q-n}\log_2(2/\sigma)} = \frac{1}{\epsilon_1}, \tag{4.5.6.1}$$

and $Q = Q(n, q, N)$ as

$$Q := \frac{n+2}{q-n}\log_2(N), \tag{4.5.6.2}$$

and note that C, Q are large positive constants. So now we can write, for any $i \geq 0$,

$$\epsilon_{i+1} = \frac{1}{C}\delta_i^Q. \tag{4.5.6.3}$$

Lemma 4.5.7. *For all $i \geq 0$,*

$$\epsilon_i \leq (C\epsilon_{i+1})^{\frac{1}{2Q}}.$$

Proof. By plugging Lemma 4.3.2 into (4.5.6.3), we see

$$\epsilon_{i+1} = \frac{1}{C}\left(\frac{1}{L^i}\epsilon_i\right)^Q.$$

Using Lemma 4.5.6, we observe

$$\frac{1}{C}\epsilon_i^{2Q} \leq \frac{1}{C}\left(\frac{1}{L^i}\epsilon_i\right)^Q = \epsilon_{i+1}.$$

Rearranging, we get

$$\epsilon_i \leq (C\epsilon_{i+1})^{\frac{1}{2Q}},$$

because $C, Q \geq 1$ from Note 4.3.1, as desired. □

Proof of Proposition 4.5.5. As a consequence of Lemma 4.3.3 and (4.3.1.2) we see that $(\delta_j)_{j\in\mathbb{N}}$ limits to zero, this sequence starts at $\delta_0 = 1$ and thus for any $r \in (0, 1]$ there exists i such that

$$\delta_{i+1} < r \leq \delta_i.$$

Also, as $\operatorname{diam}(X) = 1$, for any $x, y \in X$, either $x = y$, or $d(x, y) \in (0, 1]$ and, therefore, there exists i such that

$$\delta_{i+1} < d(x, y) \leq \delta_i.$$

If $x = y$, then (4.5.5.1) trivially holds, so we assume $\delta_{i+1} < d(x, y) \leq \delta_i$. This implies that

$$\frac{1}{2\sqrt{2(n+1)}}\epsilon_{i+1} \leq d(f(x), f(y)) \leq \frac{9}{2}\epsilon_i,$$

by combining Lemma 4.5.2 and Lemma 4.5.3. We can weaken the bounds on $d(x, y)$ using Lemma 4.5.6 and Lemma 4.3.2, to get

$${\epsilon_{i+1}}^2 \leq \frac{1}{L^{i+1}}\epsilon_{i+1} = \delta_{i+1} < d(x, y),$$

and

$$d(x, y) \leq \delta_i = \frac{1}{L^i}\epsilon_i \leq \epsilon_i.$$

Now, utilising Lemma 4.5.7, we get

$$d(f(x), f(y)) \leq \frac{9}{2}\epsilon_i \leq \frac{9}{2}C^{\frac{1}{2Q}}({\epsilon_{i+1}}^2)^{\frac{1}{4Q}} \leq \frac{9}{2}C^{\frac{1}{2Q}}(d(x, y))^{\frac{1}{4Q}},$$

and

$$\frac{1}{2\sqrt{2(n+1)}\,C}d(x, y)^{2Q} \leq \frac{1}{2\sqrt{2(n+1)}\,C}{\epsilon_i}^{2Q} \leq \frac{1}{2\sqrt{2(n+1)}}\epsilon_{i+1} \leq d(f(x), f(y)).$$

Let $\lambda = \lambda(n, C, Q)$ be defined as

$$\lambda \coloneqq \max\left\{\frac{9}{2}C^{\frac{1}{2Q}}, 2\sqrt{2(n+1)}\,C\right\}.$$

Summarising, we have the desired bi-Hölder inequality for f

$$\frac{1}{\lambda}d(x, y)^{2Q} \leq d(f(x), f(y)) \leq \lambda d(x, y)^{\frac{1}{4Q}}.$$ □

4.5.4 Checking the consistency of choices relating to N

Throughout, we made some assumptions on the size of N. In this subsection, we summarise these assumptions and verify that they can be satisfied simultaneously.

Recall the following definitions of constants:

$$L = \frac{128(n+1)^2}{\sigma^2}, \tag{4.3.0.1}$$

$$\delta_1 = \frac{1}{L}\epsilon_1, \tag{4.3.1.2}$$

$$B_1 = (8\sqrt{n})^{\frac{n}{q-n}}, \tag{4.3.3.2}$$

$$B_2 = \frac{n+2}{q-n}\log_2\left(\frac{2}{\sigma}\right), \tag{4.3.3.3}$$

$$C = 8(8\sqrt{n})^{\frac{n}{q-n}} N^{\frac{n+2}{q-n}\log_2(2/\sigma)} = \frac{1}{\epsilon_1}, \tag{4.5.6.1}$$

$$Q = \frac{n+2}{q-n}\log_2(N). \tag{4.5.6.2}$$

Lemma 4.5.8. *For N sufficiently large, the following hold:*

$$\delta_1 \textit{ is sufficiently small to apply Lemma 4.2.9,}$$
$$C \geq 1,$$
$$Q \geq 1,$$
$$B_1 N^{B_2} \geq L.$$

Proof. Note that $n+2 \geq 1$ and $q-n > 0$ so $(n+2)/(q-n) > 0$. Hence, $Q \to \infty$ as $N \to \infty$. Further, $0 < \sigma < 1$ so $\log_2(2/\sigma) > 0$, and, therefore $C \to \infty$ as $N \to \infty$ as well.

Note that B_1 and B_2, are both positive and independent of N, hence $B_1 N^{B_2} \to \infty$ as $N \to \infty$. Observe that L is independent of N and therefore $B_1 N^{B_2} \geq L$ for N large enough. This independence between L and N also has the consequence that $\delta_1 = 1/(LC) \to 0$ as $N \to \infty$. □

4.5.5 Proof of Theorem 4.1.4

We now summarise the above to conclude the proof of Theorem 4.1.4.

Proof of Theorem 4.1.4. Suppose $\operatorname{diam}(X) = 1$. The f stated in the theorem is the pointwise limit of the sequence of functions $(f_j)_{j\in\mathbb{N}}$ as defined in Section 4.3. By Proposition 4.5.4, the Hausdorff q-measure of $f(X)$ is at most 4^q, and, by Proposition 4.5.5, the distance estimate holds with $\mu = \lambda$, $\alpha = 2Q$ and $\beta = 1/4Q$.

For any bounded metric space (X, d) with non-zero diameter, we can always rescale the metric and get another metric space with much the same properties but with diameter equal to 1. Indeed, note that $\left(X, \frac{1}{\operatorname{diam}(X)}d\right)$ is also a metric space that is compact if X is compact;

similarly, doubling and capacity dimension n with coefficient σ are unaffected by rescaling the metric. Let

$$\phi\colon (X,d) \to \left(X, \frac{1}{\operatorname{diam}(X)} d\right)$$

be the identity map on the set X. As $\left(X, \frac{1}{\operatorname{diam}(X)} d\right)$ has diameter equal to 1, from the above, we have a map $f\colon \left(X, \frac{1}{\operatorname{diam}(X)} d\right) \to \ell^2$ satisfying inequalities

$$\frac{1}{\mu}\left(\frac{1}{\operatorname{diam}(X)} d(x,y)\right)^{\alpha} \leq d_{\ell^2}(f(x), f(y)) \leq \mu \left(\frac{1}{\operatorname{diam}(X)} d(x,y)\right)^{\beta},$$

for some μ, α, β. Pull this back to (X,d) via ϕ to get $f \circ \phi\colon (X,d) \to \ell^2$ with

$$\frac{1}{\mu \operatorname{diam}(X)^{\alpha}} d(x,y)^{\alpha} \leq d_{\ell^2}(f \circ \phi(x), f \circ \phi(y)) \leq \frac{\mu}{\operatorname{diam}(X)^{\beta}} d(x,y)^{\beta}.$$

As ϕ is the identity map on the set X, the image of f does not change under composition with ϕ, $f \circ \phi(X) = f(X)$, and therefore the image of $f \circ \phi$ also has Hausdorff q-measure at most 4^q. ☐

4.5.6 Proofs of corollaries

Proof of Corollary 4.1.5. If X has zero diameter, then clearly X has capacity and Hölder dimension equal to 0, and the corollary holds. Otherwise, from Theorem 4.1.4, for any $q > n$, we can find a bi-Hölder map, $f\colon X \to \ell^2$, where the image has finite Hausdorff q-measure and, therefore, Hausdorff dimension at most q. Restricting the range of f to its image makes f a Hölder equivalence between X and $f(X)$, which gives $\text{Höldim}(X) \leq q$. Further, as $q > n$ was arbitrary, we conclude $\text{Höldim}(X) \leq n$. ☐

To relate Theorem 4.1.4 to Corollary 4.1.2 we need to understand local self-similarity better. The following lemma is well-known, we include a proof for completeness.

Lemma 4.5.9. *If (X,d) is a compact, locally self-similar metric space, then X is N-doubling, for some $N \in \mathbb{N}$.*

Proof. Let λ and Λ_0 be as in the definition of X being locally self-similar, see Definition 4.2.6. Further, let R be sufficiently large as to apply this local self-similarity of X. Let $\epsilon = \min\{\Lambda_0/4R, \Lambda_0/16\lambda\}$. Consider the cover of X, $\bigcup_{x\in X} B(x,\epsilon)$, by ϵ-balls about every point in X, and use compactness to find a finite subcover, say $\bigcup_{i=1}^{N} B(x_i,\epsilon)$. Let $x \in X$ and $r > 0$. If $r \geq \Lambda_0/2R$ then, as $\epsilon \leq \Lambda_0/4R \leq r/2$, $\{B(x_i, r/2)\}_{i=1}^{N}$ is a cover of $B(x,r)$, as it covers X, of at most N balls of half the radius. If $r < \Lambda_0/2R$, then $\Lambda_0/2r \geq R$, so $\Lambda_0/2r$ is sufficiently large to apply the local self-similarity of X. As $\operatorname{diam}(B(x,r)) \leq 2r \leq \Lambda_0/(\Lambda_0/2r)$, we can find a λ-bi-Lipschitz embedding, $f\colon (B(x,r), \frac{\Lambda_0}{2r} d) \to X$, by the local self-similarity of X. Note that as $\{B(x_i,\epsilon)\}_{i=1}^{N}$ covers X, it also covers $f(B(x,r))$, so we can pull this cover of the image back through f to get a cover for $B(x, r)$. Using the lower bound of the bi-Lipschitz inequality for f, we get that the diameter of the preimage of each element of this cover, considered as an element of (X, d), is at most $4\lambda r\epsilon/\Lambda_0$. If the preimage of a ball is empty, then it contributes nothing to covering $B(x, r)$ and we can ignore it. If not, pick $y_i \in f^{-1}(B(x_i, \epsilon))$ and note that $f^{-1}(B(x_i, \epsilon)) \subseteq B(y_i, 8\lambda r\epsilon/\Lambda_0) \subseteq X$. Hence, $\{B(y_i, 8\lambda r\epsilon/\Lambda_0)\}_i$ covers $B(x, r) \subseteq X$. As $\epsilon \leq \Lambda_0/16\lambda$, the radius of each of these balls is at most $r/2$. There are at most N centres, y_i, so at most N balls in this collection. Hence, $B(x, r)$ is covered by at most N balls of half the radius and we obtain the desired result. □

Proof of Corollary 4.1.2. From Lemma 4.5.9, X being locally self-similar implies X is doubling, so we can use Corollary 4.1.5 to obtain that X has Hölder dimension at most its capacity dimension. By Theorem 4.2.7, X has capacity dimension equal to its topological dimension therefore, the Hölder dimension of X is at most its topological dimension. However, topological dimension is always a lower bound for Hölder dimension by [Szp37, Theorem 2], so we have the reverse inequality too. Combining these inequalities, we arrive at the desired equality. □

Chapter 5

Non-Equivalence of Topological and Hölder Dimension

This chapter uses Cantor sets to construct examples that illustrate the sharpness of the main result of Chapter 4, Theorem 4.1.4. Its contents are taken from [Col19] Sections 6, 7, and 8, with a few minor changes to make it compatible with this thesis.

5.1 Introduction

One of the main results of Chapter 4 is Corollary 4.1.2, which says that topological dimension is equal to Hölder dimension for nice enough spaces. This chapter aims to illustrate that topological dimension is, in general, not equivalent to Hölder dimension. In particular, we give a family of self-similar examples where Hölder dimension is equal to topological dimension but is not attained, and we give an example of a non-self-similar space that has different topological and Hölder dimensions. Our examples draw inspiration from similar results in the study of conformal dimension. We explain how to deal with the new challenges that arise from the extra flexibility of Hölder equivalences.

In this chapter, we consider two examples that fundamentally depend on Cantor sets so, in Section 5.2, we set up some notation that will allow us to concretely talk about different kinds of Cantor sets contained in the unit interval $[0, 1]$.

In Section 5.3, we show that we can upgrade the basic example of the 1/3-Cantor set not attaining its Hölder dimension to an example of non-attained Hölder dimension at any positive integer.

Theorem 5.1.1. *Let $n \in \mathbb{N}$, $I^n = [0,1]^n$ be the unit hyper-cube in $\mathbb{R}^n$, C be the 1/3-Cantor set, and $X = C \times I^n$ their product equipped with the ℓ^2 metric. Let Y be a metric space that is*

Hölder equivalent to X. Then Y has Hausdorff dimension strictly greater than n. In particular, $C \times I^n$ has Hölder dimension n but does not attain it.

Essentially, this is because the family $\{\{x\} \times I^n \mid x \in C\}$ sitting inside $C \times I^n$ is 'spread out' and consists of 'big' copies of I^n. Such a family forces the Hausdorff dimension of $C \times I^n$ to be strictly more than n. These properties are sufficiently preserved by bi-Hölder maps so that any equivalent space also contains a 'spread out' family of 'big' copies of I^n, also forcing equivalent spaces to have Hausdorff dimension strictly greater than n.

This theorem is inspired by a technique for finding lower bounds for conformal dimension, using product-like structures. The core idea can be found in [Bou95, Lemma 1.6]; where Bourdon makes explicit ideas formulated by Pansu in [Pan89a, Proposition 2.9] and [Pan89b, Lemma 6.3]. Essentially, if, in a space, X, one has a spread-out family of uniformly long curves, then X is somewhat rigid with respect quasi-symmetric maps and we get a restriction on how much the Hausdorff dimension can be lowered under quasi-symmetric equivalence. In particular, this technique can be used to show that $C \times I$ is minimal for conformal dimension. Further, one can upgrade this result to $C \times I^n$ inductively by expressing it as $(C \times I^{n-1}) \times I$, and repeating the argument. The added flexibility of Hölder equivalences means this induction does not work for calculating the Hölder dimension of $C \times I^n$ and we have to work a bit harder.

In section 5.4, we discuss a non-self-similar Cantor set we constructed to illustrate the necessity of some kind of strengthening of topological dimension.

Theorem 5.1.2. *There exists a compact, doubling metric space with topological dimension* 0 *but Hölder dimension* 1*.*

Inspiration for this example comes from [Hak06] in which Hakobyan concludes that there exist Cantor sets of Hausdorff dimension 1 that are minimal for conformal dimension. The fundamental ideas underlying the example are; in the construction of a Cantor set, if one cuts out progressively smaller gaps in proportion to the scale they are cut from, then the Hausdorff dimension of the resulting Cantor set can be forced to be 1, and if one takes the ratio of gap-to-scale to grow faster than any power, then this property cannot be broken by passing through a bi-Hölder map.

5.2 Notation for Cantor sets

In this section, we provide notation for a standard construction of Cantor sets in the interval $[0, 1]$. Cantor sets are more general objects than what is presented here; a Cantor set is any space topologically equivalent to the 1/3-Cantor set, see [Wil04, Theorem 30.3], which shall be defined momentarily. They are a source of relatively simple examples that have proved useful in illustrating different aspects of Hölder dimension.

The following notation is an extension of that which can be found in [Mat95, Section 4.10].

Denote $I = I_{0,1} = [0,1]$. To construct the standard 1/3-Cantor set, cut the middle third, $J_{1,1} = (1/3, 2/3)$, from $I_{0,1}$ to get $I_{1,1} = [0, 1/3]$ and $I_{1,2} = [2/3, 1]$, then repeat by cutting out the middle third of both $I_{1,1}$ and $I_{1,2}$, and so on. The remaining set, after cutting out middle thirds ad infinitum in this manner, is the 1/3-Cantor set. More precisely, inductively define a collection of intervals $I_{n,i}$, for $n \in \mathbb{N}$ and $1 \leq i \leq 2^n$, by removing the middle open interval, $J_{n+1,i}$, of diameter $\frac{1}{3}\operatorname{diam}(I_{n,i})$ from $I_{n,i}$ to get two disjoint closed intervals $I_{n+1,2i-1}$ and $I_{n+1,2i}$. Taking the collection of points that, for every $n \geq 0$, lie in one of $I_{n,i}$ gives us a Cantor set, C,

$$C = \bigcap_{n=0}^{\infty} \bigcup_{i=1}^{2^n} I_{n,i}.$$

This Cantor set is known as the 1/3-Cantor set because of the constant ratio of diameters

$$\frac{\operatorname{diam}(J_{n+1,i})}{\operatorname{diam}(I_{n,i})} = \frac{1}{3}.$$

However, we could have chosen $J_{n+1,i}$ to be the middle open interval of diameter $\lambda \operatorname{diam}(I_{n,i})$ from $I_{n,i}$ for any $0 < \lambda < 1$ and this process would have produced another Cantor set with identical topological properties but potentially different metric ones.

We can go further and chose arbitrary diameters for $J_{n+1,i}$ for all $n \geq 0$, $1 \leq i \leq 2^n$ provided that $0 < \operatorname{diam}(J_{n+1,i}) < \operatorname{diam}(I_{n,i})$, and still obtain a Cantor set.

In the following, we will use the 1/3-Cantor set as an example of a locally self-similar space, and define a generalised Cantor set, using this notation, as a non-example of local self-similarity. The key to constructing the non-example of Theorem 5.1.2 will be to have, as n increases, the diameters of the gaps become progressively smaller proportions of the intervals they are cut from.

5.3 Product of a Cantor set and a hyper-cube

As discussed in the introduction of Chapter 4, the Hölder dimension of a space can be not attained, and we gave the 1/3-Cantor set as an example when the Hölder dimension was equal to 0 but does not attain it. In this section, we provide a family of examples of compact, locally self-similar spaces with Hölder dimension n, for any $n \geq 1$, none of which attain their Hölder dimension. Recall,

Theorem 5.3.1 (Theorem 5.1.1)**.** *Let $n \geq 1$, $I^n = [0,1]^n$ be the unit hyper-cube in $\mathbb{R}^n$, C be the 1/3-Cantor set, and $X = C \times I^n$ their product equipped with the ℓ^2 metric. Let Y be a metric space Hölder equivalent to X. Then Y has Hausdorff dimension strictly greater than n. In particular, $C \times I^n$ has Hölder dimension n but does not attain it.*

As mentioned in Section 5.1, we draw inspiration from an important method in the study of conformal dimension. In the $n = 1$ case, this method tells us that the Hausdorff dimension of

$C \times I$ cannot be lowered by quasi-symmetric equivalence and, therefore, $C \times I$ is minimal for conformal dimension, see [Pan89a]. Under Hölder equivalence we do not get something quite as strong, but from this method we can still derive that, for $C \times I$, the Hausdorff dimension of Hölder equivalent spaces can never be lowered to 1 (or below). Essentially, if one has a large family of curves in a space, X, that are spread out enough so that the Hausdorff dimension of X is greater that 1, and if these curves are still sufficiently spread out in a quantifiable way after applying, in our case, a bi-Hölder map, then the image of these curves still force the Hausdorff dimension of the image to be greater than 1. To then generalise this to $n \geq 2$ in the quasi-symmetric case, one considers $C \times I^n$ as $\left(C \times I^{n-1}\right) \times I$ and uses the same idea about curves. However, this does not work for Hölder equivalence, so we introduce some tools to prove that we can, instead, consider a large family of copies of I^n in a similar way. We will refer to any topological copy of I^n as a *hyper-curve*.

There are two key ingredients to this argument; we can find a family of hyper-curves, Γ, in Y such that:

1. The hyper-curves are uniformly 'big'. Formally, there exists a uniform, $B > 0$, lower bound away from zero such that for any hyper-curve $\gamma \in \Gamma$, the Hausdorff n-measure of γ as a subspace of Y is at least B. The importance of checking this 'big'-ness property is that it should mean that every hyper-curve substantially contributes to the (at least) n-dimensionality of Y, and if we have enough of them, then we should break-out beyond dimension n.

2. The hyper-curves are 'spread out'. We will quantify this by fitting a measure on Γ such that, uniformly, the measure of the set of hyper-curves intersecting a given subset of Y is bounded above polynomially by the diameter of the subset. Such an upper bound encapsulates the 'spread out' property, which can be seen by examining the opposite via the extreme of a Dirac mass on a single hyper-curve:

   Pick one of the hyper-curves $\tau_0 \in \Gamma$ and say a subset of Γ has measure 1 if it contains τ_0 and has measure 0 otherwise, then picking a point on τ_0 as the centre of a ball, we see that the measure of the hyper-curves intersecting this ball is 1, independent of the diameter of the ball. In other words, dumping all the measure's mass in one place does not satisfy this upper bound. This is a special case, but roughly, the upper bound dependent on diameter means you are not dropping too much mass in any one place. The importance of checking this 'spread out'-ness is that it should mean that Y has a lot of room in it, more than any space with Hausdorff dimension at most n.

For any $f : C \times I^n \to Y$ bi-Hölder homeomorphism to an equivalent space Y, there is a natural choice for a family of hyper-curves in Y obtained by pushing the fibres of C in $C \times I^n$ through f. Formally, let $\mathcal{C} = \{\gamma_x : I^n \to C \times I^n \mid x \in C\}$ where $\gamma_x : \underline{t} \mapsto (x, \underline{t})$, and

$\Gamma = \{f \circ \gamma_x \mid x \in C\}$. We proceed to show that elements of Γ are 'big' and 'spread out' as described above.

Firstly, we state precisely what we mean by curves being uniformly 'big'.

Lemma 5.3.2. *If $\gamma\colon I^n \to Y$ is a (λ, α, β)-bi-Hölder homeomorphism onto its image, then there exists a constant $B = B(\lambda, n) > 0$ such that the Hausdorff n-content of $\gamma(I^n)$ is at least B.*

For any real number $q > 0$, the *Hausdorff q-content* of a metric space X, denoted $\mathcal{H}^q_\infty$, is

$$\mathcal{H}^q_\infty(X) = \inf \sum_{U \in \mathcal{U}} (\mathrm{diam}(U))^q \,, \tag{5.3.2.1}$$

where the infimum is taken over all covers $\mathcal{U}$ of X by closed balls. This definition of Hausdorff content and the following facts about it can be found in [Hei01, Section 8.3]. Note the similarity in definition between Hausdorff content and Hausdorff measure, Definition 2.2.3; the main difference between them being that Hausdorff content lacks a diameter restriction on covers. It is easy to see that, for any $q > 0$,

$$\mathcal{H}^q_\infty \leq \mathcal{H}^q.$$

This means that the bound given in Lemma 5.3.2 is also a bound for Hausdorff q-measure. A key point of the proof of Lemma 5.3.2 will be that the q-content of I^n is strictly positive. This can be seen by a combination of the following facts:

- For any space X, $\mathcal{H}^q(X) = 0$ if and only if $\mathcal{H}^q_\infty(X) = 0$.
- For any $n \in \mathbb{N}$, $\mathcal{H}^n$ on $\mathbb{R}^n$ is comparable to Lebesgue measure.

To see that $\mathcal{H}^n_\infty(I^n)$ is positive, note that the Lebesgue measure of I^n is positive, so $\mathcal{H}^n(I^n)$ is positive, and, therefore, $\mathcal{H}^n_\infty(I^n)$ is positive. We summarise this in the following lemma.

Lemma 5.3.3. *For $n \geq 1$, the Hausdorff n-content of I^n is positive.*

We make one final observation, in the following lemma, about how q-content is distorted by Lipschitz maps, which follows easily from the definition.

Lemma 5.3.4. *Suppose $g\colon X \to Y$ is a λ-Lipschitz surjection between metric spaces X and Y. Then, for any $q > 0$, $\mathcal{H}^q_\infty(Y) \leq \lambda^q \mathcal{H}^q_\infty(X)$.*

Lemma 5.3.2 is useful to us because of the following observation. Recall, for any $\gamma \in \Gamma$, by definition, there exists $x \in C$ such that $\gamma = f \circ \gamma_x$. Note that, for any $x \in C$, γ_x is an isometry onto its image. Thus, as f is a (λ, α, β)-bi-Hölder homeomorphism, $\gamma\colon I^n \to Y$ is a (λ, α, β)-bi-Hölder homeomorphism onto its image.

We need to introduce the following machinery before we can prove Lemma 5.3.2.

Proposition 5.3.5. *For $n \geq 1$, let $F\colon I^n \to I^n$ be a continuous map such that $F(\partial I^n) \subseteq \partial I^n$ and the map $F|_{\partial I^n}\colon \partial I^n \to \partial I^n$ induces a non-trivial endomorphism on the reduced $(n-1)$-homology group of the boundary, $\tilde{H}_{n-1}(\partial I^n)$. Then F is surjective.*

Proof. First, note that the boundary map $F|_{\partial I^n}$ is surjective. If not, then there exists a point $z_0 \in \partial I^n \setminus F(\partial I^n)$ and we can factor $F|_{\partial I^n}$ through $\partial I^n \setminus \{z_0\}$ to find $F|_{\partial I^n}\colon \partial I^n \to \partial I^n$ is equivalent to the path

$$\partial I^n \xrightarrow{F} \partial I^n \setminus \{z_0\} \hookrightarrow \partial I^n.$$

However, $\partial I^n \setminus \{z_0\}$ deformation retracts to a point, and, therefore, has trivial reduced $(n-1)$-homology. Hence, this path, through $\partial I^n \setminus \{z_0\}$, implies that the endomorphism induced by $F|_{\partial I^n}$ on $\tilde{H}_{n-1}(\partial I^n)$ is trivial, contradicting our assumption.

Now, for a contradiction, suppose that there exists a point $y_0 \in I^n \setminus F(I^n)$. Note that this point must lie in the interior of I^n as F surjects onto the boundary by the above. As y_0 is interior, we have an inclusion $\iota\colon \partial I^n \hookrightarrow I^n \setminus \{y_0\}$, which has a retract $r\colon I^n \setminus \{y_0\} \to \partial I^n$ defined as follows. For any point $y \in I^n \setminus \{y_0\}$, let $r(y)$ be the point of intersection of ∂I^n and the straight line $l_y \coloneqq \{ty + (1-t)y_0 \mid t \geq 0\}$ that originates at y_0 and passes through y. Note, r restricts to the identity on the boundary, so $r \circ \iota$ is the identity map on ∂I^n. The composition $\iota \circ r$ is homotopic to the identity by the straight line homotopy $(y,t) \mapsto ty + (1-t)r(y)$. This map takes values in $I^n \setminus \{y_0\}$ because $y_0, y, r(y)$ are co-linear, in that order along the line segment $l_y \cap I^n$, and the points $ty + (1-t)r(y)$, for $0 \leq t \leq 1$, are contained in the segment of l_y containing y and $r(y)$, which does not contain y_0.

Hence, ι is a homotopy equivalence and we may deduce that $\iota_*\colon \tilde{H}_{n-1}(\partial I^n) \to \tilde{H}_{n-1}(I^n \setminus \{y_0\})$ is an isomorphism.

Observe that we have the following commutative diagram;

$$\begin{array}{ccc} I^n & \xrightarrow{\quad F \quad} & I^n \setminus \{y_0\} \\ \uparrow & & \uparrow \iota \\ \partial I^n & \xrightarrow[\quad F \quad]{} & \partial I^n \end{array}$$

which induces the following commutative diagram in the reduced $(n-1)$-homology;

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbb{Z} \\ \uparrow & & \uparrow \iota_* \\ \mathbb{Z} & \xrightarrow[F_*]{} & \mathbb{Z} \end{array}$$

Further, as F induces a non-trivial homomorphism

$$F_*\colon \tilde{H}_{n-1}(\partial I^n) \to \tilde{H}_{n-1}(\partial I^n)$$

and $\iota_*\colon \tilde{H}_{n-1}(\partial I^n) \to \tilde{H}_{n-1}(I^n \setminus \{y_0\})$ is an isomorphism, the commutative diagram in the homology above gives contradiction. □

Recall, for any $\gamma \in \Gamma$, $\gamma\colon I^n \to Y$ is a (λ, α, β)-bi-Hölder homeomorphism onto its image. To massage this set-up into one that can utilise Proposition 5.3.5 we introduce the following notation.

Let A_i be the 'axial' face of I^n defined to be the subset of I^n with value 0 in the i-th coordinate;

$$A_i \coloneqq I^{i-1} \times \{0\} \times I^{n-i}.$$

Each axial face has an 'opposite' face O_i defined to be the subset of I^n with value 1 in the i-th coordinate;

$$O_i \coloneqq I^{i-1} \times \{1\} \times I^{n-i}.$$

The idea is that γ transfers these faces over to Y as distorted versions of themselves, which we use to build a map from Y to the cube, and then compose with γ to obtain a self-map of the cube to which we can apply Proposition 5.3.5.

For each i, we define a map ϕ_i as a kind of projection in the i-th direction. Define $\phi_i\colon Y \to \mathbb{R}_{\geq 0}$ by

$$\phi_i(y) = \inf_{x \in A_i} d_Y(y, \gamma(x)),$$

for any $y \in Y$. More concisely written;

$$\phi_i(y) = d_Y(y, \gamma(A_i)).$$

Note, as ϕ_i is a distance function to a set, ϕ_i is 1-Lipschitz and thus also continuous.

We could now take the product of these maps to build a map to $\mathbb{R}^n$, but it is unclear what the image of this map will look like. Instead, we make a slight adjustment to these maps to make their product simpler. Define $\psi_i\colon Y \to [0,1]$ by capping ϕ_i at $1/\lambda$ and then rescaling to $[0,1]$. That is,

$$\psi_i(y) \coloneqq \lambda \min\{\phi_i(y), 1/\lambda\}.$$

Note that ψ_i inherits ϕ_i's Lipschitz-ness, but is now λ-Lipschitz for each i. Let $\Psi\colon Y \to I^n$ be the product of these maps; for any $y \in Y$ define

$$\Psi(y) \coloneqq (\psi_i(y))_{1 \leq i \leq n}, \tag{5.3.5.1}$$

and note the following lemma.

Lemma 5.3.6. *The product map Ψ is $\lambda\sqrt{n}$-Lipschitz.*

The product map Ψ has image in the unit hyper-cube, I^n, so we can reduce to studying continuous self-maps of the unit hyper-cube, with some desirable properties, by composing with γ. Define the continuous map $F\colon I^n \to I^n$ by

$$F \coloneqq \Psi \circ \gamma. \tag{5.3.6.1}$$

A useful observation to make is that the $(n-1)$-dimensional faces are mapped to themselves under F. Indeed, for any $x \in A_i$, $\gamma(x) \in \gamma(A_i)$ so $d_Y(\gamma(x), \gamma(A_i)) = 0$, and therefore, the i-th coordinate of $F(x)$ is 0, which characterises being an element of A_i. For $x \in O_i$, x is at least 1 away from every point in A_i, and therefore, by the lower bound of the bi-Hölder inequality for γ, $d_Y(\gamma(x_i), \gamma(A_i)) \geq 1/\lambda$. After applying the capping and rescaling, we see that the i-th component of $F(x)$ takes the value 1, thus determining it as an element of O_i. This observation will allow us to use the following lemma when studying F.

Lemma 5.3.7. *For any $n \geq 1$, let $F\colon I^n \to I^n$ be a continuous map such that, for all i, $F(A_i) \subseteq A_i$ and $F(O_i) \subseteq O_i$, then $F(\partial I^n) \subseteq \partial I^n$ and $F\big|_{\partial I^n}\colon \partial I^n \to \partial I^n$ is homotopic to the identity on ∂I^n.*

Proof. Firstly, $F\big|_{\partial I^n}$ has image in ∂I^n, because ∂I^n is covered by the $(n-1)$-dimensional faces and the $(n-1)$-dimensional faces are all contained in ∂I^n.

To see that $F\big|_{\partial I^n}$ is homotopic to the identity on ∂I^n, consider the map $H\colon \partial I^n \times I \to \partial I^n$ defined as

$$H(x,t) \coloneqq tF(x) + (1-t)x.$$

At face value, $H(x,0) = x$ and $H(x,1) = F(x)$, and H is continuous. However, it is not immediately obvious that the linear combination $tF(x) + (1-t)x$ is in ∂I^n for all $t \in [0,1]$ and not just in $\mathbb{R}^n$. However, for all i, F maps faces A_i and O_i to A_i and O_i respectively. Therefore, for any $x \in \partial I^n$, x lies in a face S, and thus, $F(x)$ also lies in S. The faces of I^n are convex, meaning that the straight line joining x and $F(x)$, namely $\{tF(x) + (1-t)x \mid t \in [0,1]\}$, also lies in the face S. Thus we can conclude that F restricted to the boundary is indeed homotopic to the identity. ☐

We can now combine Lemma 5.3.7 and Proposition 5.3.5 to get the following.

Lemma 5.3.8. *For $n \geq 1$, let $F\colon I^n \to I^n$ be a continuous map such that, for all i, $F(A_i) \subseteq A_i$ and $F(O_i) \subseteq O_i$. Then F is surjective.*

Proof. By Lemma 5.3.7, $F(\partial I^n) \subseteq \partial I^n$, and $F\big|_{\partial I^n}\colon \partial I^n \to \partial I^n$ is homotopic to the identity on ∂I^n and therefore induces the identity on $\tilde{H}_{n-1}(\partial I^n)$. However, ∂I^n is homotopic to $\mathbb{S}^{n-1}$, which has $\tilde{H}_{n-1}(\mathbb{S}^{n-1}) \cong \mathbb{Z}$, which is non-trivial. Hence, the endomorphism of $\tilde{H}_{n-1}(\partial I^n)$ induced by $F\big|_{\partial I^n}\colon \partial I^n \to \partial I^n$ must be non-trivial.

We have verified that F satisfies the conditions to apply Proposition 5.3.5 allowing us to conclude that F is surjective. ☐

We now have the requisite tools to prove Lemma 5.3.2.

Proof of Lemma 5.3.2. From the discussion earlier, we have an induced continuous map F, see (5.3.6.1), that maps faces to faces, and is, therefore, surjective by Lemma 5.3.8. This forces the product of projections Ψ, see (5.3.5.1), to be surjective as well. Also, $\Psi\colon Y \to I^n$ is $(\lambda\sqrt{n})$-Lipschitz by lemma 5.3.6. Thus, Lemma 5.3.4 allows us to observe that

$$\mathcal{H}^n_\infty(I^n) = \mathcal{H}^n_\infty(\Psi(Y)) \le (\lambda\sqrt{n})^n\, \mathcal{H}^n_\infty(Y).$$

This allows us to conclude that the Hausdorff n-content of Y is at least $\mathcal{H}^n_\infty(I^n)/(\lambda\sqrt{n})^n$, which is strictly positive by Lemma 5.3.3. □

Now, onto the 'spread out' property; Let μ_C be the probability measure on C defined as the weak* limit of measures on the covers $\bigcup_{j=1}^{2^i} I_{i,j}$, see Section 5.2, defined by letting the measure of each interval $I_{i,j}$ be $1/2^i$. This induces a measure, μ_Γ, on Γ by

$$\mu_\Gamma(S) \coloneqq \mu_C(\{x \in C \mid f \circ \gamma_x \in S\}), \tag{5.3.8.1}$$

for any $S \subseteq \Gamma$. We claim μ_Γ has a 'spread out' property in the form of the following lemma.

Lemma 5.3.9. *There exists $A > 0$ such that, for any $U \subseteq Y$,*

$$\mu_\Gamma\left(\{\gamma \in \Gamma \mid \gamma \cap U \neq \emptyset\}\right) \le A\,\mathrm{diam}(U)^{\log 2/\alpha\log 3}.$$

Proof. We start by noting a similar upper bound for μ_C. The measure μ_C on C is "Ahlfors $\log(2)/\log(3)$-regular", see [Fal86, Theorem 1.14], in particular, there exists $\nu > 0$ such that for any closed ball $\overline{B}$ in C,

$$\mu_C\left(\overline{B}\right) \le \nu\,\mathrm{diam}\left(\overline{B}\right)^{\log(2)/\log(3)}. \tag{5.3.9.1}$$

Let $\phi\colon C \times I^n \to C$, defined by $(x, \underline{t}) \mapsto x$, be the projection of $C \times I^n$ onto C, and observe that ϕ is 1-Lipschitz. Now, for any subset $U \subseteq Y$ pick $y \in U$; we can assume U is non-empty as the inequality trivially holds for empty U. Let $\mathcal{I}_U \coloneqq \{x \in C \mid f(\{x\} \times I^n) \cap U \neq \emptyset\} \subseteq C$, and note that

$$\mathcal{I}_U = \phi(f^{-1}(U)). \tag{5.3.9.2}$$

The 1-Lipschitz property of ϕ combined with the bi-Hölder inequalities for f tells us that

$$\mathrm{diam}(\mathcal{I}_U) \le \mathrm{diam}(f^{-1}(U)) \le (\lambda\,\mathrm{diam}(U))^{1/\alpha}. \tag{5.3.9.3}$$

Further, if $\overline{B}$ is the closed ball in C of radius $\mathrm{diam}(\mathcal{I}_U)$ centred at $\phi(f^{-1}(y))$, then $\mathcal{I}_U \subseteq \overline{B}$ and

$$\mathrm{diam}\left(\overline{B}\right) \le 2\,\mathrm{diam}(\mathcal{I}_U). \tag{5.3.9.4}$$

Therefore, we can conclude

$$\begin{aligned}
\mu_\Gamma\left(\{\gamma \in \Gamma \mid \gamma \cap U \neq \emptyset\}\right) &= \mu_C(\mathcal{I}_U) && \text{by (5.3.8.1) and (5.3.9.2),}\\
&\le \mu_C\left(\overline{B}\right) && \text{as } \mathcal{I}_U \subseteq B,\\
&\le \nu\,\mathrm{diam}\left(\overline{B}\right)^{\log 2/\log 3} && \text{by (5.3.9.1),}\\
&\le 2^{\log(2)/\log(3)}\nu\,\mathrm{diam}(\mathcal{I}_U)^{\log(2)/\log(3)} && \text{by (5.3.9.4),}\\
&\le A\,\mathrm{diam}(U)^{\log 2/\alpha\log 3} && \text{by (5.3.9.3),}
\end{aligned}$$

where $A = 2^{\log 2/\log 3}\nu\lambda^{\log 2/\alpha\log 3}$. □

We now know something about arbitrary decompositions of each $\gamma \in \Gamma$ via Lemma 5.3.2, and something about how decompositions of Y interact with Γ via Lemma 5.3.9. We introduce the following notation for indicator functions, as they will be useful for converting decompositions of Y to decompositions for $\gamma \in \Gamma$, which is how we shall link these two ideas. For any $U \subseteq Y$ and $\gamma \in \Gamma$, define

$$\mathbb{1}_U(\gamma) = \begin{cases} 1 & \text{if } \gamma \cap U \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

We now have sufficient tools to prove Theorem 5.3.1.

Proof of Theorem 5.3.1. Let $B > 0$ be as in Lemma 5.3.2. Observe, as μ_Γ is a probability measure,

$$B = \int_\Gamma B \, \mathrm{d}\mu_\Gamma(\gamma),$$

so, for any cover $\mathcal{U}$ of Y by closed balls, by Lemma 5.3.2,

$$\begin{aligned} &\leq \int_\Gamma \sum_{U \in \mathcal{U}} \operatorname{diam}(U)^n \mathbb{1}_U(\gamma) \, \mathrm{d}\mu_\Gamma(\gamma), \\ &= \sum_{U \in \mathcal{U}} \operatorname{diam}(U)^n \int_\Gamma \mathbb{1}_U(\gamma) \, \mathrm{d}\mu_\Gamma(\gamma), \end{aligned}$$

then, by Lemma 5.3.9, there exists some $A > 0$ such that,

$$\begin{aligned} &\leq \sum_{U \in \mathcal{U}} \operatorname{diam}(U)^n A \operatorname{diam}(U)^{\log 2/\alpha \log 3}, \\ &= A \sum_{U \in \mathcal{U}} \operatorname{diam}(U)^{n + \log 2/\alpha \log 3}. \end{aligned}$$

Hence,

$$\sum_{U \in \mathcal{U}} \operatorname{diam}(U)^{n + \log 2/\alpha \log 3} \geq \frac{B}{A} > 0,$$

and therefore, the Hausdorff $(n + \log 2/\alpha \log 3)$-content of Y is non-zero as $\mathcal{U}$ was arbitrary, which implies the Hausdorff $(n + \log 2/\alpha \log 3)$-measure of Y is non-zero. Therefore, the Hausdorff dimension of Y is at least $n + \log 2/\alpha \log 3 > n$.

To see that the Hölder dimension of $C \times I^n$ is equal to n, we use Corollary 4.1.5. Equally, one could also use Corollary 4.1.2 by showing that $C \times I^n$ is locally self-similar and has topological dimension n. The compactness of $C \times I^n$ comes from being a product of compact spaces. Considering $C \times I^n$ as a subspace of $\mathbb{R}^{n+1}$, we see that it is doubling directly from the doubling property of $\mathbb{R}^{n+1}$. Finally, $C \times I^n$ has capacity dimension n by Proposition 5.3.10 below. $\square$

We now present a proof of the capacity dimension result for $C \times I^n$ used above.

Proposition 5.3.10. *For $n \geq 1$, $C \times I^n$ has capacity dimension n.*

For this proposition, we utilise the following lemma, which is Theorem 9.5.1 in [BS07]. The interested reader should note that Buyalo and Schroeder refer to capacity dimension as "ℓ-dimension" in this source.

Lemma 5.3.11. *For any metric spaces X_1 and X_2, the capacity dimension of $X_1 \times X_2$ is at most the sum of the capacity dimensions of X_1 and X_2.*

Proof of Proposition 5.3.10. Note that $C \times I^n$ contains a copy of I^n as $\{0\} \times I^n$, and therefore has topological dimension at least n. As topological dimension is a lower bound to capacity dimension, we observe that $C \times I^n$ also has capacity dimension at least n. Therefore, using Lemma 5.3.11, it only remains to check that the capacity dimensions of C and I^n are at most 0 and n respectively.

For C, take any $0 < \delta \leq 1$ and let $n \in \mathbb{N}$ such that $1/3^n \leq \delta < 1/3^{n-1}$. The cover $\{I_{n,i} \cap C \mid 1 \leq i \leq 2^n\}$ has mesh at most $1/3^n \leq \delta$, multiplicity 1, and Lebesgue number at least $1/3^n > \delta/3$. Therefore, C has capacity dimension at most 0.

For I^n, we show that I has capacity dimension at most 1, then inductively use Lemma 5.3.11 to prove that I^n has capacity dimension at most n.

For I, take any $0 < \delta \leq 1$. The cover of I by balls of radius $\delta/2$ centred at $n\delta/2$, for $n \in \mathbb{N}$ and $0 \leq n \leq (2/\delta) + 1$, has mesh at most δ, multiplicity 2, and Lebesgue number at least $\delta/4$. Therefore, I has capacity dimension at most 1. □

5.4 Capacity dimension versus topological dimension

Theorem 4.1.4 shows that the Hölder dimension of a compact, doubling space is at most its capacity dimension. However, as topological dimension is a more commonly used notion of dimension, one could ask if Hölder dimension is, in fact, at most the space's topological dimension, extending the self-similar case. In this section, we provide an example of a compact, doubling space that has topological dimension 0 but Hölder dimension 1, proving that "capacity dimension" cannot be replaced with "topological dimension" in Theorem 4.1.4.

Theorem 5.4.1 (Theorem 5.1.2)**.** *Let X be the Cantor set defined in Section 5.2 where the diameter of the gaps,* $\operatorname{diam}(J_{n,i})$*, is taken to be $\frac{1}{10n^n}$ for all $n \geq 1$ and $1 \leq i \leq 2^{n-1}$. Then X has Hölder dimension equal to* 1*.*

As X is a Cantor set, it has topological dimension 0 and is compact. It is doubling as it is a subspace of $\mathbb{R}$, which is doubling. Therefore, to accomplish the goal stated above, we need only prove this theorem.

The main idea is, by making the gaps shrink fast enough that they cannot account for all the Hausdorff 1-measure in I, we have forced X to have positive 1-dimensional Hausdorff measure, and hence Hausdorff dimension 1. Furthermore, as the shrinking is faster than any fixed power of n, no Hölder equivalence can find an equivalent space Y without this 'fast-shrinking gap' property, meaning any equivalent space will also have Hausdorff dimension 1.

Our construction of X allows us to choose the diameters for the gaps, $J_{n,i}$, but leaves the diameters of $I_{n,i}$ implicit. Investigation of the construction gives us the following easy, but useful, bound.

Lemma 5.4.2.

$$\frac{1}{3^n} \leq \operatorname{diam}(I_{n,i}),$$

for every $n \in \mathbb{N}$ and $1 \leq i \leq 2^n$.

Proof. Let C be the 1/3-Cantor set as constructed in Section 5.2, with intervals $I^C_{n,i}$ and gaps $J^C_{n+1,i}$. Recall that $\operatorname{diam}(J^C_{n+1,i}) = 1/3^{n+1}$ for all $n \geq 0$ and $1 \leq i \leq 2^n$. Compare this with how we defined the gaps in X to have diameter $1/10(n+1)^{n+1}$ to see that we cut out at most the middle third of every interval in the construction of X;

$$\frac{1}{10(n+1)^{n+1}} \leq \frac{1}{3^{n+1}},$$

for all $n \geq 0$. Therefore, inductively we see that $\operatorname{diam}(I_{n,i}) \geq \operatorname{diam}(I^C_{n,i}) = 1/3^n$. □

It is helpful to note the following lemma, which is a consequence of exclusively cutting from the interior of intervals in the construction of X.

Lemma 5.4.3. *Endpoints of intervals $I_{n,i}$ lie in X.*

The following is [McS34, Corollary 1].

Lemma 5.4.4. *Let S be a subset of a metric space Z, and let $g\colon S \to \mathbb{R}$ be a real-valued Hölder continuous function. Then g can be extended to Z preserving the Hölder condition.*

We now have sufficient tools to prove the theorem.

Proof of Theorem 5.4.1. Note that

$$[0,1] \setminus X = \bigcup_{k,j} J_{k,j},$$

so, for any countable cover of X by closed balls, say $\mathcal{A} = \{A_i\}_{i\in\mathbb{N}}$, we can extend $\mathcal{A}$ to a cover of $[0,1]$ by closed balls by including $\overline{J_{k,j}}$, the closure of $J_{k,j}$, for each k, j. Note, each $J_{k,j}$ was an open ball, so its closure is indeed a closed ball, and $\operatorname{diam}\left(\overline{J_{k,j}}\right) = \operatorname{diam}(J_{k,j})$. Also, as $[0,1]$ is connected,

$$\sum_i \operatorname{diam}(A_i) + \sum_{k,j} \operatorname{diam}(J_{k,j}) = \sum_i \operatorname{diam}(A_i) + \sum_{k,j} \operatorname{diam}\left(\overline{J_{k,j}}\right) \geq \operatorname{diam}([0,1]) = 1,$$

but

$$\sum_{k,j} \operatorname{diam}(J_{k,j}) = \sum_{k=1}^{\infty} \sum_{j=1}^{2^{k-1}} \frac{1}{10k^k} = \frac{1}{10} \sum_{k=1}^{\infty} \frac{2^k}{k^k}.$$

For $k \geq 4$, $k^k \geq 4^k = 2^{2k}$ so we see

$$\sum_{k=1}^{\infty} \frac{2^k}{k^k} \leq \sum_{k=1}^{3} \frac{2^k}{k^k} + \sum_{k=4}^{\infty} \frac{2^k}{2^{2k}} \leq \sum_{k=1}^{3} \frac{2^k}{k^k} + \sum_{k=1}^{\infty} \frac{1}{2^k} \leq \sum_{k=1}^{3} \frac{2^k}{k^k} + 1 < 5 < \infty.$$

Hence, $\sum_{k,j} \operatorname{diam}(J_{k,j}) < 5/10 = 1/2$. Therefore, $\sum_i \operatorname{diam}(A_i) > 1/2$. Recall, from (5.3.2.1), $\mathcal{H}^1_\infty$ denotes the Hausdorff 1-content. The cover $\mathcal{A}$ was arbitrary so $\mathcal{H}^1(X) \geq \mathcal{H}^1_\infty(X) \geq 1/2 > 0$ and hence $\dim_H(X) \geq 1$, but $\dim_H(X) \leq \dim_H([0,1]) = 1$, so together we get that $\dim_H(X) = 1$.

Now consider $f\colon X \to Y$, a (λ, α, β) bi-Hölder homeomorphism between X and a metric space Y. We would like to prove that $\dim_H(Y) \geq 1$. To do this, let us reduce to working in $\mathbb{R}$ so that we can extend decompositions to decompositions of intervals, like in the above. Consider $\psi\colon Y \to \mathbb{R}$ defined by $y \mapsto d_Y(f(0), y)$. Note that ψ is a 1-Lipschitz map via the triangle inequality. Hence, we can preserve the upper bound of our Hölder inequality for f when we compose with ψ. That is, for any $z_1, z_2 \in X$

$$d_{\mathbb{R}}(\psi(f(z_1)), \psi(f(z_2))) \leq d_Y(f(z_1), f(z_2)) \leq \lambda d_X(z_1, z_2)^\beta. \tag{5.4.4.1}$$

Hence, by Lemma 5.4.4 there exists an extension, F, of $\psi \circ f$ to I that is (λ, β)-Hölder continuous too. This extension means we can derive information from the gaps, $J_{k,j}$, too, instead of just from the space X. For instance, for any j, k,

$$\operatorname{diam}(J_{k,j}) = \frac{1}{10k^k} \implies \operatorname{diam}(F(J_{k,j})) \leq \lambda \left(\frac{1}{10k^k} \right)^\beta.$$

We interpret this as the Hölder map, F, being unable to break the shrinking property of the gaps.

Consider $I_{k,1}$, which has width at least $1/3^k$ by Lemma 5.4.2. We know that the smaller endpoint of $I_{k,1}$ is 0, and let its larger endpoint be x, for some $x > 0$. By Lemma 5.4.3, both 0 and x lie in X and, therefore, F evaluates to $\psi \circ f$ on them as F is an extension of $\psi \circ f$. Hence,

$$\begin{aligned} |F(0) - F(x)| &= |\psi(f(0)) - \psi(f(x))|, \\ &= |0 - d_Y(f(0), f(x))|, \\ &= d_Y(f(0), f(x)) \geq \frac{1}{\lambda} d_X(0,x)^\alpha \geq \frac{1}{\lambda 3^{\alpha k}}. \end{aligned}$$

Thus, $F(I_{k,1})$ contains $\left[0, 1/(\lambda 3^{\alpha k})\right]$ as a subset, by the Intermediate Value Theorem.

Now, the gaps within $I_{k,1}$ are precisely $J_{i,j}$ where $i \geq k+1$ and $1 \leq j \leq 2^{i-(k+1)}$. Each $J_{i,j}$ has diameter $1/10i^i$, so the corresponding gap, $F(J_{i,j})$, in the image has diameter at most

$\lambda\left(1/(10i+1^{i+1})\right)^{\beta}$, by the Hölder continuity of F. Thus, the 1-measure of gaps in the image is controlled as follows;

$$\begin{aligned}\sum_{i,j}\operatorname{diam}(F(J_{i,j})) &\leq \sum_{n=k+1}^{\infty} 2^{n-(k+1)}\lambda\left(\frac{1}{10n^n}\right)^{\beta}, \\ &\leq \frac{\lambda}{2^{k+1}}\left(\frac{1}{10}\right)^{\beta}\sum_{n=k+1}^{\infty}\left(\frac{2}{(k+1)^{\beta}}\right)^{n}, \\ &= \frac{\lambda}{10^{\beta}2^{k+1}}\left(\frac{2}{(k+1)^{\beta}}\right)^{k+1}\frac{(k+1)^{\beta}}{(k+1)^{\beta}-2}, \\ &= \frac{\lambda}{10^{\beta}(k+1)^{\beta k}((k+1)^{\beta}-2)}, \\ &\leq \frac{\lambda}{10^{\beta}(k+1)^{\beta k}}. \end{aligned} \tag{5.4.4.2}$$

The last inequality holds for $(k+1)^{\beta}\geq 3$, which will be true for sufficiently large k, because $(k+1)^{\beta}\to\infty$ as $k\to\infty$. Note,

$$\left(\frac{(k+1)^{\beta}}{3^{\alpha}}\right)^{k}\to\infty,\ \text{as } k\to\infty.$$

Hence, by taking k sufficiently large, we may assume simultaneously;

- $(k+1)^{\beta}\geq 3$, which we just used above for (5.4.4.2), and
- $\left((k+1)^{\beta}/3^{\alpha}\right)^{k} > 4\lambda^2/10^{\beta}$.

The latter is important because it is equivalent to

$$\frac{\lambda}{10^{\beta}(k+1)^{\beta k}} < \frac{1}{4}\frac{1}{\lambda 3^{\alpha k}}, \tag{5.4.4.3}$$

which allows us to conclude that gaps cannot account for all the 1-measure in Y. More precisely, for any countable decomposition, $B=\{B_j\}_{j\in\mathbb{N}}$, of Y, define $A_j=f^{-1}(B_j)$ for all j, and let $A=\{A_j\}_{j\in\mathbb{N}}$ be the decomposition of X induced by pulling B back through f. Note that A is also a cover for $I_{k,1}\cap X$, and if we add in the gaps contained in $I_{k,1}$, then we have a cover for $I_{k,1}$. Explicitly, $A' \coloneqq A\cup\{J_{i,j}\mid i\geq k+1, 1\leq j\leq 2^{i-(k+1)}\}$ covers $I_{k,1}$. Hence, $F(A')$ covers $F(I_{k,1})$. For each $U\in F(A')$ non-empty, pick any $z\in U$ and note $U\subseteq\overline{B(z,\operatorname{diam}(U))}$ and $\operatorname{diam}\left(\overline{B(z,\operatorname{diam}(U))}\right)\leq 2\operatorname{diam}(U)$. Thus, we could replace $F(A')$ by a cover by closed balls with the cost of, at worst, doubling diameters. Therefore, as $F(I_{k,1})$ contains the connected space $\left[0,1/(\lambda 3^{\alpha k})\right]$,

$$\sum_j 2\operatorname{diam}(F(A_j))+\sum_{i,j}2\operatorname{diam}(F(J_{i,j}))\geq\operatorname{diam}\left(\left[0,1/(\lambda 3^{\alpha k})\right]\right)\geq\frac{1}{\lambda 3^{\alpha k}}.$$

From equations (5.4.4.2) and (5.4.4.3), for sufficiently large $k = k(\alpha, \beta, \lambda)$ independent of the decomposition B,

$$\sum_{J_{i,j} \subset I_{k,1}} \operatorname{diam}(F(J_{i,j})) < \frac{1}{4\lambda 3^{\alpha k}}.$$

Hence,

$$\sum_j \operatorname{diam}(F(A_j)) \geq \frac{1}{4\lambda 3^{\alpha k}} > 0,$$

for all decompositions B. Now, by definition and as $A_j \subseteq X$ for all j, $F(A_j) = \psi \circ f(A_j) = \psi(B_j)$, and $\operatorname{diam}(\psi(B_j)) \leq \operatorname{diam}(B_j)$, so

$$\sum_j \operatorname{diam}(B_j) \geq \sum_j \operatorname{diam}(F(A_j)) \geq \frac{1}{4\lambda 3^{\alpha k}}.$$

Hence, $\mathcal{H}^1(Y) \geq 1/4\lambda 3^{\alpha k}$ and $\dim_H(Y) \geq 1$. $\square$

CHAPTER 6

Hölder Dimension and Hyperbolic Groups with Disconnected Boundaries

6.1 Introduction

Boundaries of hyperbolic groups are compact by [GdLH90, Proposition 7.2.9], and locally self-similar by [BL07, Proposition 6.2]. Therefore, a consequence of Corollary 4.1.2 is that the Hölder dimension of a boundary of a hyperbolic group is equal to its topological dimension. However, as Hölder dimension is defined as an infimum over a class of metric spaces, a natural question left open is whether there exists a member of the class of Hölder equivalent spaces that realises this infimum as a minimum. That is, when do boundaries of hyperbolic groups attain their Hölder dimension?

In this chapter we provide a partial answer. In particular, we reduce to the case that the group in question is 1-ended by relating the Hölder dimension of the boundary of an infinite ended hyperbolic group to the Hölder dimension of its connected components. There are a few edge cases that make the exact statement of the theorem messy, but the fundamental concept is that infinite-ended hyperbolic groups are quasi-isometric to a free product of finitely many 1-ended hyperbolic groups, see Subsection 6.9.2, and the boundary of the infinite-ended hyperbolic group is comprised of the boundaries of the factors in this free product, and a 'small' subset that can be identified with the boundary of a tree. The boundaries of the factors determine the Hölder dimension and the attainment, and the small subset does not contribute in the Hölder dimension or attainment. The following is the main theorem of this chapter.

Theorem 6.1.1. *Suppose Γ is an infinite hyperbolic group that splits as a finite graph of groups $\mathcal{G}$ with finite edge groups. Let $G_1, G_2, \ldots, G_n$ represent all quasi-isometry types of the vertex groups of $\mathcal{G}$ that are infinite and not virtually cyclic. If $n \geq 1$, then* $\operatorname{Höldim} \partial_\infty \Gamma =$

$\max\{\text{Höldim}\,\partial_\infty G_i \mid 1 \leq i \leq n\}$ *and* $\partial_\infty \Gamma$ *attains its Hölder dimension if and only if* $\partial_\infty G_i$ *attains its Hölder dimension for each* i *such that* $\text{Höldim}\,\partial_\infty G_i = \text{Höldim}\,\partial_\infty \Gamma$.

If $n = 0$*, then* $\text{Höldim}(\partial_\infty \Gamma) = 0$ *and the Hölder dimension of* $\partial_\infty \Gamma$ *is attained if and only if* Γ *is virtually cyclic.*

When every vertex space in $\mathcal{G}$ is virtually cyclic (i.e. finite or virtually $\mathbb{Z}$), Γ is either virtually cyclic or virtually a non-abelian free group. The Hölder dimension of the boundary is 0 regardless, but the Hölder dimension of $\partial_\infty \Gamma$ is attained if Γ is virtually cyclic and not attained if it is virtually a non-abelian free group, and these are the only options as Γ is virtually free. This is explained in further detail in the proof of Proposition 6.2.7.

Theorem 6.1.1 is a corollary of the following theorem, plus some machinery that will be discussed later in Subsection 6.9.2.

Theorem 6.1.2. *Let* $\Gamma = G_1 * G_2 * \cdots * G_n$*, for* $n \geq 2$*, where* G_i *is infinite hyperbolic, for all* $1 \leq i \leq n$*, and not every* G_i *is virtually cyclic. Then*

$$\text{Höldim}\,\partial_\infty \Gamma = \max\{\text{Höldim}\,\partial_\infty G_i \mid 1 \leq i \leq n\}$$

and $\partial_\infty \Gamma$ *attains its Hölder dimension if and only if* $\partial_\infty G_i$ *attains its Hölder dimension for each* i *such that* $\text{Höldim}\,\partial_\infty G_i = \text{Höldim}\,\partial_\infty \Gamma$.

By [PW02] and some inductive arguments, which we delve into in Section 6.9, these theorems reduce to the case when Γ is a free product of two groups. Free products can be modelled geometrically by spaces with a 'tree-of-spaces' structure, see [SW79] for why they have this structure. In [MS15], Martin and Świątkowski use this tree-of-spaces structure to partition the boundary into topological copies of the boundaries of the factors of the free product and the boundary of the Bass-Serre tree. A consequence of their description is the following topological version of Theorem 6.1.1, which is a special case of [Dah03, Theorem 0.2].

Theorem 6.1.3. *Suppose* Γ *is an infinite hyperbolic group that splits as a finite graph of groups* $\mathcal{G}$ *with finite edge groups. Let* $G_1, G_2, \ldots, G_n$ *represent all quasi-isometry types of the vertex groups of* $\mathcal{G}$ *that are infinite and not virtually cyclic. If* $n \geq 1$*, then* $\dim_T \partial_\infty \Gamma = \max\{\dim_T \partial_\infty G_i \mid 1 \leq i \leq n\}$*, where* $\dim_T(X)$ *is the topological dimension of a space* X*, otherwise* Γ *is virtually free and* $\partial_\infty \Gamma$ *has topological dimension* 0*.*

This theorem is also a consequence of Theorem 6.1.1 and Corollary 4.1.2.

The following theorem is a conformal version of Theorem 6.1.1. This statement and a proof can be found in [CP11, Theorem 6.2].

Theorem 6.1.4. *If* Γ *is an infinite hyperbolic group with a finite graph of groups decomposition where the vertex groups are* $\{G_i\}$ *and the edge groups are finite, then*

$$\text{Confdim}\,\partial_\infty \Gamma = \max\{\text{Confdim}\,\partial_\infty G_i \mid G_i \text{ infinite}\},$$

where we declare $\max \emptyset = 0$.

Here Confdim(X) is the Ahlfors regular conformal dimension of a space X, the most commonly studied version of conformal dimension, which was introduced in [BP03]. Here the qualifier "Ahlfors regular" means that the infimum of Hausdorff dimensions is restricted to only 'Ahlfors regular' spaces that are quasi-symmetrically equivalent. Ahlfors regular spaces are ones where, if they have Hausdorff dimension q, the Hausdorff q-measure of any ball is comparable to the q-th power of its radius. Ahlfors regular conformal dimension agrees with the non-restricted version for the main examples we consider in this thesis, such as the 1/3-Cantor set, $\mathbb{R}^n$, and the boundaries of rank 1 symmetric spaces of non-compact type.

An extension of Theorem 6.1.4 to graphs of groups with virtually cyclic edge groups can be found in an upcoming paper of Carrasco and Mackay [CM20]. Carrasco and Mackay also present an attainment result:

Theorem 6.1.5. *Suppose* Γ *is a hyperbolic group, and we are given a graph of groups decomposition of* Γ *with vertex groups* $\{G_i\}$ *and all edge group virtually cyclic. Then the conformal dimension of* $\partial_\infty \Gamma$ *is attained if and only if either:*

- $\mathrm{Confdim}\, \partial_\infty \Gamma = 0$ *and* Γ *is virtually cyclic, or*
- $\mathrm{Confdim}\, \partial_\infty \Gamma = 1$ *and* Γ *is virtually cocompact Fuchsian, or*
- $\Gamma = G_i$ *for some vertex group with* $\partial_\infty G_i$ *attaining its conformal dimension*

$$\mathrm{Confdim}\, \partial_\infty G_i > 1.$$

A group is *cocompact Fuchsian* if it acts discretely by isometries on the hyperbolic plane with compact quotient. For example, the fundamental group of a surface of genus at least 2 is cocompact Fuchsian.

Comparing this theorem with Theorem 6.1.1, we can see that the attainment of conformal dimension and the attainment of Hölder dimension behave quite differently for boundaries of hyperbolic groups. For instance, in some sense, it is much harder for the boundary of such a composite group to attain its conformal dimension than it is to attain its Hölder dimension, as illustrated in the following example.

Example 6.1.6. Let G be the fundamental group of a surface of genus at least 2. Then G is an infinite Gromov hyperbolic group that is quasi-isometric to the real hyperbolic plane. Its boundary, therefore, is quasi-symmetric and Hölder equivalent to the unit circle $\mathbb{S}^1$, which has Hausdorff dimension equal to its topological dimension equal to 1. Thus $\partial_\infty G$ attains both its conformal and Hölder dimensions. Let $\Gamma = G * G$. Theorem 6.1.1 tells us that $\partial_\infty \Gamma$ has Hölder dimension 1 and attains it. Similarly, Theorem 6.1.4 tells us that $\partial_\infty \Gamma$ has conformal dimension equal to 1. However, Theorem 6.1.5 tells us that $\partial_\infty \Gamma$ attains its conformal dimension only if Γ is virtually cocompact Fuchsian. If Γ were virtually cocompact Fuchsian, then it would be quasi-isometric to the real hyperbolic plane and, therefore, have boundary topologically equivalent to a circle, which is not the case as $\partial_\infty \Gamma$ is not connected. Thus $\partial_\infty \Gamma$ does not attain its conformal dimension.

We now describe the main ideas used in the proof of Theorem 6.1.1. As mentioned above and in Section 6.9, we can reduce to the case when Γ can be expressed as a free product, $\Gamma = G_1 * G_2$, of two infinite hyperbolic groups, G_1 and G_2, that are not both virtually cyclic. Our understanding of Γ will come from its 'tree-of-spaces' structure. Essentially, there exists a geometric model for Γ that consists of infinitely many copies of Cayley graphs of G_1 and G_2 joined by edges so that if one collapses these Cayley graphs, called vertex spaces, to single points, then one is left with a tree, called the 'Bass-Serre tree'. The boundary of this geometric model can be partitioned, via this tree-of-spaces structure, into infinitely many copies of the boundaries of G_1 and G_2, which are the boundaries of vertex spaces, and the boundary of the Bass-Serre tree.

Now, let Z be a metric space that is Hölder equivalent to $\partial_\infty \Gamma$. We understand $\partial_\infty \Gamma$ because it is the boundary of a Gromov hyperbolic space, so it would be nice if Z was also the boundary of a Gromov hyperbolic space. Finding hyperbolic spaces that have prescribed boundaries is a well studied endeavour with contributions from Paulin [Pau96], Elek [Ele97], and Bonk-Schramm [BS00]. However, it is not clear when these spaces have a tree-of-spaces structure, or if it would be 'compatible' with the one Γ exhibits. Thankfully, it turns out we do not need to understand completely arbitrary spaces that are Hölder equivalent to $\partial_\infty \Gamma$ to understand its Hölder dimension. Instead, it will be enough to consider a small family of spaces that are the boundaries of Gromov hyperbolic spaces that we construct, in Section 6.5, from the tree-of-spaces structure of Γ, and Hölder data of $\partial_\infty G_1$ and $\partial_\infty G_2$. One can think of the construction as replacing the vertex spaces of the tree-of-spaces structure of Γ with new ones that encode Hölder data about $\partial_\infty G_1$ and $\partial_\infty G_2$. We use Bonk and Schramm's construction from [BS00, Section 7] to do this encoding. The key facts we require from their construction are summarised in Subsection 6.3.5.

Theorem 6.1.1 can be thought of as reducing the question of attainment of Hölder dimension for boundaries of hyperbolic groups to those that are 1-ended. This leaves us with an obvious question:

Question 6.1.7. When is the Hölder dimension of the boundary of a 1-ended hyperbolic group attained?

In Theorem 5.1.1, we showed that products of the standard 1/3-Cantor set and Euclidean spaces do not attain their Hölder dimension. Thus, I expect the answer to this question is not "always" as topological copies of these spaces can be seen sitting inside boundaries of certain hyperbolic groups. For instance, there exist boundaries of (1-ended) hyperbolic groups that are topologically equivalent to the 1/3-Sierpiński carpet, see Figure 4.1, which contains the Cartesian product of the 1/3-Cantor set and an interval. However, it is not clear if these boundaries are Hölder equivalent copies of the 1/3-Sierpiński carpet.

A key part of the construction in this chapter involves defining a map between two hyperbolic spaces whose boundaries are Hölder equivalent. For the purposes of this chapter, fully understanding this map was not necessary and proved challenging enough to ignore for the sake of progressing with the main result. However, within the context of the correspondences between maps on hyperbolic spaces and maps on their boundaries that Bonk and Schramm give in [BS00, Section 7], we are motivated to ask the following two questions.

Question 6.1.8. What kind of maps between hyperbolic spaces induce Hölder equivalences on their boundaries?

Question 6.1.9. For hyperbolic spaces X and Y, and $f\colon \partial_\infty X \to \partial_\infty Y$ a bi-Hölder homeomorphism, is f induced by a 'sensible' map $X \to Y$? What kind of metric control does such an induced map have?

As mentioned in Chapter 1, in [Cor17, Theorem 1.7], Cornulier showed that SBEs induce Hölder equivalence on boundaries. However, in [Pal18, Theorem 2], Pallier showed that rank one symmetric spaces of non-compact type are SBE equivalent only if they are homothetic. Not all rank one-symmetric spaces of non-compact type are homothetic, in particular 4 dimensional real hyperbolic space, $\mathbb{H}^4_{\mathbb{R}}$, and 2 dimensional complex hyperbolic space, $\mathbb{H}^2_{\mathbb{C}}$, are not homothetic, so are not SBE equivalent, but their boundaries are still Hölder equivalent, see Theorem 3.2.2. Therefore if there is a 'sensible' type of metric control between hyperbolic spaces that corresponds to Hölder equivalences on the boundary, it must be even wilder than, or of a different nature to, SBE in general.

Section 6.2 presents the main punchline with a summary of the key results proved throughout this chapter. In Section 6.3, we discuss some background required throughout this chapter. In Section 6.4, we describe a geometric model for a free product of two groups and explore how it emphasises the tree-of-spaces structure of the free product. In Section 6.5, we construct a Gromov hyperbolic space from this tree-of-spaces structure combined with Hölder data of the boundaries of the component groups. Sections 6.6, 6.7, and 6.8 delve into the calculations necessary to conclude the key results used in Section 6.2. Finally, Section 6.9 explains how we can conclude Theorem 6.1.1 from Theorem 6.2.1.

6.2 The key idea

Through well-established machinery, explained in Subsection 6.9.2, we can reduce the proof of Theorem 6.1.1 to establishing Theorem 6.1.2. Further, we can use the inductive argument, given in Subsection 6.9.1, to reduce to the following scenario.

Theorem 6.2.1. *Let G_1 and G_2 be infinite hyperbolic groups that are not both virtually cyclic, and let $\Gamma = G_1 * G_2$. Then,*

$$\text{Höldim}\,(\partial_\infty \Gamma) = \text{Höldim}\,(\partial_\infty G_1) \vee \text{Höldim}\,(\partial_\infty G_2)\,.$$

Further, $\partial_\infty \Gamma$ attains its Hölder dimension if and only if $\partial_\infty G_i$ attains its Hölder dimension for each $i = 1, 2$ such that $\text{Höldim}(\partial_\infty G_i) = \text{Höldim}(\partial_\infty \Gamma)$.

Here and throughout the chapter, we use the following notation.

Notation 6.2.2. Let $a \vee b$ denote the maximum of two numbers $a, b \in \mathbb{R}$, and $a \wedge b$ the minimum.

To talk about the boundary of Γ we need a geometric model for Γ. Usually we would take a Cayley graph for Γ with respect to a finite generating set to be a geometric model for Γ. However, we will take a slightly different space that accentuates the tree-like structure that Γ has from being expressed as a free product. We explain in Section 6.4 exactly which space we choose and why that choice is valid. For the sake of proving Theorem 6.2.1 the reader need only know that there exist geodesic Gromov hyperbolic spaces $\left(\tilde{X}^{(1)}_\Gamma, d_{\tilde{X}^{(1)}_\Gamma}\right)$, $\left(\tilde{X}^{(1)}_1, d_{\tilde{X}^{(1)}_1}\right)$, and $\left(\tilde{X}^{(1)}_2, d_{\tilde{X}^{(1)}_2}\right)$ such that:

- $\left(\tilde{X}^{(1)}_\Gamma, d_{\tilde{X}^{(1)}_\Gamma}\right)$ is a bounded degree graph that is quasi-isometric to any Cayley graph for Γ with respect to a finite generating set, so we take $\partial_\infty \Gamma = \partial_\infty \tilde{X}^{(1)}_\Gamma$,
- $\left(\tilde{X}^{(1)}_1, d_{\tilde{X}^{(1)}_1}\right)$ and $\left(\tilde{X}^{(1)}_2, d_{\tilde{X}^{(1)}_2}\right)$ are Cayley graphs for G_1 and G_2, respectively, with respect to finite generating sets, so we take $\partial_\infty G_1 = \partial_\infty \tilde{X}^{(1)}_1$ and $\partial_\infty G_2 = \partial_\infty \tilde{X}^{(1)}_2$,
- $\tilde{X}^{(1)}_\Gamma$ is equipped with a continuous surjection $\pi\colon \tilde{X}^{(1)}_\Gamma \to T$, from Proposition 6.4.1, to a tree T (the Bass-Serre tree of the decomposition $G_1 * G_2$) with vertex set $V(T)$,
- for every $v \in V(T)$, $X_v := \pi^{-1}(v)$ is equal to an isometric copy of $\left(\tilde{X}^{(1)}_1, d_{\tilde{X}^{(1)}_1}\right)$ or $\left(\tilde{X}^{(1)}_2, d_{\tilde{X}^{(1)}_2}\right)$,
- for every edge, e, in T, $\pi^{-1}(e)$ is equal to a single edge in $\tilde{X}^{(1)}_\Gamma$.

Essentially, $\tilde{X}^{(1)}_\Gamma$ is composed of isometric copies of $\tilde{X}^{(1)}_1$ and $\tilde{X}^{(1)}_2$ with single edges joining the copies into a 'tree-of-spaces' structure. This tree-of-spaces structure gives us a way of decomposing the boundary of $\tilde{X}^{(1)}_\Gamma$ into simpler pieces that are easier to manipulate.

For any geodesic ray, γ, in $\tilde{X}^{(1)}_\Gamma$, γ projects to a non-backtracking path in T. Therefore, either there exists some vertex $v \in V(T)$ such that $\gamma(t) \in \pi^{-1}(v)$ for all t sufficiently large, or γ keeps branching into new vertex-spaces forever and can be identified with a point in the boundary of T. This separates $\partial_\infty \tilde{X}_\Gamma^{(1)}$ into countably many copies of $\partial_\infty \tilde{X}_1^{(1)}$ and $\partial_\infty \tilde{X}_2^{(1)}$, and a set that can be identified with the boundary of T. Our assumption that not both G_1 and G_2 are virtually cyclic essentially means that the component of the boundary corresponding to the boundary of T is 'dimensionally small' with respect to the copies of $\partial_\infty \tilde{X}_1^{(1)}$ and $\partial_\infty \tilde{X}_2^{(1)}$.

Two main problems arise when proving Theorem 6.2.1. Firstly, the copies of $\partial_\infty \tilde{X}_1^{(1)}$ and $\partial_\infty \tilde{X}_2^{(1)}$ might not have minimal Hausdorff dimension among Hölder equivalent spaces and therefore $\partial_\infty \tilde{X}_\Gamma^{(1)}$ has large Hausdorff dimension just because we picked poorly when choosing geometric models. Secondly, the component of the boundary corresponding to the boundary of the tree T might just happen to have Hausdorff dimension that dominates the Hausdorff dimension of the copies of $\partial_\infty \tilde{X}_1^{(1)}$ and $\partial_\infty \tilde{X}_2^{(1)}$, again making $\partial_\infty \tilde{X}_\Gamma^{(1)}$ have a larger Hausdorff dimension than necessary.

We solve the first problem by building a new tree-of-spaces called W, where we have carefully replaced copies of $\tilde{X}_1^{(1)}$ and $\tilde{X}_2^{(1)}$ with new spaces such that their boundaries are uniformly Hölder equivalent to $\partial_\infty \tilde{X}_1^{(1)}$ and $\partial_\infty \tilde{X}_2^{(1)}$, respectively, and such that we have control on the Hausdorff dimension of the boundaries of these new spaces. We address the second problem by distorting the contribution of T to the metric on W to get a one-parameter family of metrics on W, (W, d_W^l) with $l \in \mathbb{R}_{\geq 1}$, with the property that the subset in the boundary of (W, d_W^l) that corresponds to the boundary of T has Hausdorff dimension that can be made arbitrarily small for l large enough. This distortion does not change the local metric of vertex spaces, only the distance between vertex spaces, so we do not affect the Hausdorff dimension of boundaries of vertex spaces. This second step is where we require the added freedom of using Hölder equivalence on the boundary instead of quasi-symmetric equivalence, which is why we can obtain a different result than the conformal dimension case (compare the attainment part of Theorem 6.1.1, and Theorem 6.1.5).

We now present sufficient facts to prove Theorem 6.2.1.

For $i = 1, 2$, let $\mathfrak{G}_i$ be a metric space bi-Hölder to $\partial_\infty \tilde{X}_i^{(1)}$. Then, for any $l \geq 1$, there exists a metric, Δ_l, on $\partial_\infty \tilde{X}_\Gamma^{(1)}$ that is Hölder equivalent to $d_{\partial_\infty \tilde{X}_\Gamma^{(1)}}$ such that:

Proposition 6.2.3 (Proposition 6.8.3)**.** *For any $l \geq 1$ and for each $v \in V(T)$, $(\partial_\infty X_v, \Delta_l)$ bi-Lipschitz embeds into $\mathfrak{G}_1$ or $\mathfrak{G}_2$.*

This sorts out the problem that $\partial_\infty X_v$ might have large Hausdorff dimension because we picked our geometric models poorly, as $\mathfrak{G}_i$ could be chosen arbitrarily in the Hölder equivalence class of $\partial_\infty G_i$. This means we can choose $\mathfrak{G}_i$ to have Hausdorff dimension arbitrarily close to $\mathrm{Höldim}\, \partial_\infty G_i$.

We use 'half-spaces', which are discussed in Subsection 6.8.2 and denoted H_v, to cover the portion of $\partial_\infty \tilde{X}_\Gamma^{(1)}$ that corresponds to the boundary of T to show that it does not dominate the copies of $\partial_\infty \tilde{X}_1^{(1)}$ and $\partial_\infty \tilde{X}_2^{(1)}$ with respect to Hausdorff dimension. To get control on this cover, it is helpful to partition vertices of T based on the metric structure of $\tilde{X}_\Gamma^{(1)}$ in the following way:

In Subsection 6.4.2, $\tilde{X}_\Gamma^{(1)}$ is given global base-point o, and, for each $v \in V(T)$, X_v is given a local base-point o_v that is the unique element of X_v that has minimal distance to o. For all $v \in V(T)$, $d_{\tilde{X}_\Gamma^{(1)}}(o, o_v) \in \mathbb{N}$, so we partition $V(T)$ by defining, for any $k \in \mathbb{N}$,

$$\mathcal{V}_k \coloneqq \{v \in V(T) \mid d_{\tilde{X}_\Gamma^{(1)}}(o, o_v) = k\},$$

and simplify some notation by defining

$$\mathcal{V}_{\leq k} \coloneqq \{v \in V(T) \mid d_{\tilde{X}_\Gamma^{(1)}}(o, o_v) \leq k\}.$$

These definitions now let us define the family of covers we will use to calculate the Hausdorff dimension of $(\partial_\infty \tilde{X}_\Gamma^{(1)}, \Delta_l)$:

Proposition 6.2.4 (Proposition 6.8.8)**.** *For any $r \in \mathbb{N}$,*

$$\mathcal{U} \coloneqq \{\partial_\infty X_v \mid v \in \mathcal{V}_{\leq r}\} \cup \bigcup_{k \geq r+1} \{\partial_\infty H_v \mid v \in \mathcal{V}_k\},$$

covers $\partial_\infty \tilde{X}_\Gamma^{(1)}$.

Why $\mathcal{U}$ is a cover is explained in Subsection 6.8.3. We need some control on these covers for them to be useful. We summarise sufficient control in the following propositions:

Proposition 6.2.5. *There exists $M \geq 1$ such that, for each $k \in \mathbb{N}$, $|\mathcal{V}_k| \leq M^k$.*

Proof. This is because $\tilde{X}_\Gamma^{(1)}$ is a graph with bounded degree. □

We prove in Proposition 6.8.10 the following upper bound on the diameter of the boundaries of half-spaces.

Proposition 6.2.6. *There exists $C, \tau > 0$ such that, for any $l \geq 1$ and for each $v \in V(T)$, the subspace $\partial_\infty H_v$ of $\left(\partial_\infty \tilde{X}_\Gamma^{(1)}, \Delta_l\right)$ has diameter at most*

$$\exp(C) \exp\left(-\frac{l}{\tau} d_{\tilde{X}_\Gamma^{(1)}}(o, o_v)\right),$$

Finally, we need to be able to rule out the special case that $\mathfrak{G}_1$ and $\mathfrak{G}_2$ both have Hausdorff dimension equal to zero, as then the subset of the boundary that corresponds to the boundary of T might have Hausdorff dimension that dominates the Hausdorff dimension of $\tilde{X}_\Gamma^{(1)}$. The following proposition, which we prove at the end of this section, allows us to do so:

Proposition 6.2.7. *Let Γ be an infinite hyperbolic group. Then* $\text{Höldim}(\partial_\infty \Gamma) = 0$ *if and only if Γ is virtually free. Moreover, $\partial_\infty \Gamma$ attains Hölder dimension* 0 *if and only if Γ is virtually cyclic.*

This is now sufficient information to prove the following:

Proposition 6.2.8. *For all l large enough,*

$$\dim_H \left(\partial_\infty \tilde{X}_\Gamma^{(1)}, \Delta_l\right) \leq \dim_H \mathfrak{G}_1 \vee \dim_H \mathfrak{G}_2.$$

Proof. Let $D = \dim_H \mathfrak{G}_1 \vee \dim_H \mathfrak{G}_2$. As G_1 and G_2 are infinite hyperbolic groups, their boundaries are non-empty. Thus, $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are non-empty, so both have Hausdorff dimension at least 0, and $D \geq 0$. If $D = 0$, then the Hölder dimensions of $\partial_\infty G_1$ and $\partial_\infty G_2$ are both 0 and are both attained. Therefore, by Proposition 6.2.7, G_1 and G_2 are virtually cyclic. This contradicts the assumption, stated in Theorem 6.2.1, that not both G_1 and G_2 were virtually cyclic. Thus, for the rest of this proof, we may assume that $D > 0$. Let M be as in Proposition 6.2.5, and C and τ be as in Proposition 6.2.6. Choose l sufficiently large so that

$$\frac{M}{\exp\left(\frac{lD}{\tau}\right)} \leq \frac{1}{2}\frac{1}{\exp\left(C(D+1)\right)}, \tag{6.2.8.1}$$

which is possible because $D, \tau > 0$.

Let $D < q \leq D + 1$ and $\epsilon > 0$. We will show that $\left(\partial_\infty \tilde{X}_\Gamma^{(1)}, \Delta_l\right)$ has Hausdorff q-measure at most ϵ. This is sufficient to prove that $\left(\partial_\infty \tilde{X}_\Gamma^{(1)}, \Delta_l\right)$ has Hausdorff dimension at most D, because, as ϵ is arbitrary, this implies the Hausdorff q-measure is zero, and therefore $\left(\partial_\infty \tilde{X}_\Gamma^{(1)}, \Delta_l\right)$ has Hausdorff dimension at most q. Further, q can be chosen arbitrarily close to D from above, meaning that $\left(\partial_\infty \tilde{X}_\Gamma^{(1)}, \Delta_l\right)$ cannot have Hausdorff dimension strictly greater than D. The additional assumption of $q \leq D + 1$ is included for a later technical reason, and does not affect this argument.

Let $d > 0$ and choose $r \in \mathbb{N}$ sufficiently large so that

$$2^{-r} \leq \epsilon/2, \tag{6.2.8.2}$$

and

$$\exp(C)\exp\left(-\frac{l}{\tau}(r+1)\right) \leq d, \tag{6.2.8.3}$$

which is possible because $\tau > 0$.

Proposition 6.2.5 tells us that there are finitely many $v \in \mathcal{V}_r$. By Proposition 6.2.3, each $(\partial_\infty X_v, \Delta_l)$ bi-Lipschitz embeds into $\mathfrak{G}_1$ or $\mathfrak{G}_2$, so has Hausdorff dimension at most D. Considering that the Hausdorff dimension of a finite union of spaces is the maximum of the Hausdorff dimensions of the individual spaces, we see that

$$\mathrm{Core}_r := \bigcup_{v \in \mathcal{V}_{\leq r}} \partial_\infty X_v$$

has Hausdorff dimension at most D. Thus we can find a cover $\mathcal{U}_r$ of Core_r such that $\mathrm{diam}(U) \leq d$ for every $U \in \mathcal{U}_r$ and

$$\sum_{U \in \mathcal{U}_r} \mathrm{diam}(U)^q < \frac{\epsilon}{2}. \tag{6.2.8.4}$$

Now, we have control on the Core_r part of the cover $\mathcal{U}$, but we still need to control the half-spaces contribution.

We would like to bound

$$\sum_{k\geq r+1}\sum_{v\in\mathcal{V}_k}\mathrm{diam}\left(\partial_\infty H_v\right)^q.$$

By Proposition 6.2.5,

$$\sum_{k\geq r+1}\sum_{v\in\mathcal{V}_k}\mathrm{diam}\left(\partial_\infty H_v\right)^q \leq \sum_{k\geq r+1} M^k \max_{v\in\mathcal{V}_k}\left\{\mathrm{diam}\left(\partial_\infty H_v\right)^q\right\},$$

by Proposition 6.2.6

$$\leq \sum_{k\geq r+1} M^k \exp(Cq)\exp\left(-\frac{l}{\tau}qk\right),$$

we now use the additional technical assumption that $q \leq D+1$

$$\leq \sum_{k\geq r+1} M^k \exp(C(D+1))\exp\left(-\frac{l}{\tau}Dk\right),$$

$$\leq \sum_{k\geq r+1}\left(\frac{M}{\exp\left(\frac{lD}{\tau}\right)}\right)^k \exp(C(D+1)),$$

by our choice of l (6.2.8.1)

$$\leq \sum_{k\geq r+1}\left(\frac{1}{2}\right)^k \exp(C(D+1))^{1-k},$$

as $C(D+1)\geq 0$ and $k\geq 1$

$$\leq \sum_{k\geq r+1}\left(\frac{1}{2}\right)^k,$$

$$=\left(\frac{1}{2}\right)^r, \tag{6.2.8.5}$$

by our choice of r (6.2.8.2)

$$\leq \frac{\epsilon}{2}. \tag{6.2.8.6}$$

Note that, by Proposition 6.2.6, the assumption (6.2.8.3) on r means that $\mathrm{diam}\left(\partial_\infty H_v\right)\leq d$ for any $v\in\mathcal{V}_k$ for $k\geq r+1$.

Combining (6.2.8.4) and (6.2.8.6), we see that we can find a countable cover $\mathcal{U}$, defined in Proposition 6.2.4, for $\left(\partial_\infty \tilde{X}_\Gamma^{(1)}, \Delta_l\right)$ such that every element of the cover has diameter at most d, and

$$\sum_{U\in\mathcal{U}}\mathrm{diam}(U)^q \leq \epsilon. \qquad \square$$

Corollary 6.2.9.

$$\text{Höldim}\,(\partial_\infty \Gamma) = \text{Höldim}\,(\partial_\infty G_1) \vee \text{Höldim}\,(\partial_\infty G_2)\,.$$

Proof. $\partial_\infty \left(\tilde{X}_\Gamma^{(1)}, d_{\tilde{X}_\Gamma^{(1)}}\right)$ contains bi-Lipschitz copies of $\partial_\infty G_i$, see Proposition 6.8.4, for each $i \in \{1,2\}$ in the form of $\partial_\infty X_v$, which is defined in Definition 6.8.1. Thus, any bi-Hölder homeomorphism of $\partial_\infty \tilde{X}_\Gamma^{(1)} \to Z$ restricts to a bi-Hölder homeomorphism $\partial_\infty X_v \to Z'$, for any $v \in V(T)$ and for some $Z' \subset Z$, which induces a bi-Hölder homeomorphism $\partial_\infty G_i \to Z'$. Thus, $\dim_H Z \geq \dim_H Z' \geq \text{Höldim}\,\partial_\infty G_i$, which implies that $\text{Höldim}(\partial_\infty \tilde{X}_\Gamma^{(1)}) \geq \text{Höldim}\,\partial_\infty G_i$ for each i. In other words,

$$\text{Höldim}(\partial_\infty \tilde{X}_\Gamma^{(1)}) \geq \text{Höldim}\,(\partial_\infty G_1) \vee \text{Höldim}\,(\partial_\infty G_2)\,.$$

A direct consequence of Proposition 6.2.8 is that the Hölder dimension of $\partial_\infty \tilde{X}_\Gamma^{(1)}$ is at most $\text{Höldim}(\partial_\infty G_1) \vee \text{Höldim}(\partial_\infty G_2)$. Therefore we get the desired result. □

Lemma 6.2.10. *$\partial_\infty \Gamma$ attains its Hölder dimension if and only if $\partial_\infty G_i$ attains its Hölder dimension for each $i = 1, 2$ such that* $\text{Höldim}(\partial_\infty G_i) = \text{Höldim}(\partial_\infty \Gamma)$.

Proof. To see that the 'if' direction:

Suppose that, amongst G_i such that $\text{Höldim}(\partial_\infty G_i) = D := \text{Höldim}(\partial_\infty G_1) \vee \text{Höldim}(\partial_\infty G_2)$, $\partial_\infty G_i$ attains its Hölder dimension. If $\partial_\infty G_i$ attains its Hölder dimension, let $\mathfrak{G}_i$ be a space realising that attainment, else let $\mathfrak{G}_i$ be a space with Hausdorff dimension at most D. Then Proposition 6.2.8 tells us that there exists a metric Δ_l on $\partial_\infty \tilde{X}_\Gamma^{(1)}$ bi-Hölder equivalent to $\partial_\infty \left(\tilde{X}_\Gamma^{(1)}, d_{\tilde{X}_\Gamma^{(1)}}\right)$ such that $\dim_H \left(\tilde{X}_\Gamma^{(1)}, \Delta_l\right) \leq \max\{\dim_H \mathfrak{G}_i \mid i \in \{1,2\}\} = D$. However, from Corollary 6.2.9, $\text{Höldim}(\partial_\infty \tilde{X}_\Gamma^{(1)}) = D$ so $\left(\tilde{X}_\Gamma^{(1)}, \Delta_l\right)$ is a space that realises the Hölder dimension of $\partial_\infty \tilde{X}_\Gamma^{(1)}$.

For the 'only if' direction, assume that $\partial_\infty \Gamma$ attains its Hölder dimension and let $f\colon \partial_\infty \tilde{X}_\Gamma^{(1)} \to Z$ be a bi-Hölder homeomorphism realising this attainment, for some space Z. Fix i such that $\text{Höldim}(\partial_\infty G_i) = \text{Höldim}(\partial_\infty \Gamma)$. We can find $v \in V(T)$ such that $X_v \subset \tilde{X}_\Gamma^{(1)}$ is an isometric copy of $\tilde{X}_i^{(1)}$. Therefore, restricting f to $\partial_\infty X_v$ and composing with the bi-Lipschitz homeomorphism, see Proposition 6.8.4, $\partial_\infty \tilde{X}_i^{(1)} \to \partial_\infty X_v$, induced by the isometry between $\tilde{X}_i^{(1)}$ and X_v, gives a bi-Hölder homeomorphism $f_i\colon \partial_\infty \tilde{X}_i^{(1)} \to Z_i$ for some $Z_i \subset Z$. Trivially, $\dim_H Z_i \leq \dim_H Z = \text{Höldim}(\partial_\infty \Gamma)$. However, Z_i is a space Hölder equivalent to $\partial_\infty \tilde{X}_i^{(1)}$, which has Hölder dimension equal to $\text{Höldim}(\partial_\infty \Gamma)$ by assumption. Thus, $\dim_H Z_i \geq \text{Höldim}(\partial_\infty \Gamma)$ too. Therefore, Z_i is a space realising the Hölder dimension of $\partial_\infty \tilde{X}_i^{(1)}$, and we can conclude that $\partial_\infty G_i$ does indeed attain its Hölder dimension. □

Finally, we can deduce Theorem 6.2.1.

Proof of Theorem 6.2.1. Corollary 6.2.9 gives the equality and Lemma 6.2.10 gives the attainment. □

We conclude this section by giving the following proof of Proposition 6.2.7.

Proof of Proposition 6.2.7. If Γ is an elementary hyperbolic group, then it is quasi-isometric to $\mathbb{Z}$, as we assumed Γ was not finite, and has boundary equal to a pair of points. Therefore, Γ is virtually cyclic, and thus is virtually free. Further, the Hausdorff dimension of a pair of points is zero regardless of metric, so $\partial_\infty\Gamma$ attains its Hölder dimension, which is zero.

If Γ is a non-elementary hyperbolic group, then the equivalence can be seen by the combination of the following two results:

- $\partial_\infty\Gamma$ has topological dimension zero if and only if Γ is virtually free, see [KB02, Theorem 8.1].
- The Hölder dimension of the boundary of a hyperbolic group is equal to its topological dimension, by combination of Corollary 4.1.2 and [BL07, Proposition 6.2].

Note, as Γ is non-elementary hyperbolic, it is finitely generated and not virtually cyclic, so if it is virtually free, it must contain a finite index subgroup isomorphic to F_n, the free group on n generators, for some $n \geq 2$. Therefore, if Γ has Hölder dimension zero, it must be quasi-isometric to F_n, which is quasi-isometric to F_2. Thus $\partial_\infty\Gamma$ is quasi-symmetric (see [GdLH90, Proposition 7.4.14]), and therefore bi-Hölder (by Lemma 2.1.4), to $\partial_\infty F_2$, which is a Cantor set with strictly positive Hausdorff dimension. One can easily show that if X and Y are metric spaces such that there exists $f \colon X \to Y$ a (λ, α, β)-bi-Hölder homeomorphism, then $\dim_H(Y) \geq \dim_H(X)/\alpha$. In particular, any space Hölder equivalent to one with strictly positive Hausdorff dimension also has strictly positive Hausdorff dimension. Therefore, any space Hölder equivalent to $\partial_\infty\Gamma$ has strictly positive Hausdorff dimension, which allows us to conclude that $\partial_\infty\Gamma$ does not attain its Hölder dimension in this case. □

6.3 Background

In this section, we present some background material concerning the contents of this chapter. Most of the section will likely be well-known to those with experience in Geometric Group Theory. Subsection 6.3.5 contains relatively niche material.

6.3.1 Coarse metric distortion

In this subsection we introduce concepts used when studying the large-scale geometry of Gromov hyperbolic spaces. The concepts of 'cobounded' and 'quasi-isometry' are standard, for example see [GdLH90, Chapter 5]. The concept of 'rough similarity' is less well-known, and can be found in [BS00], where Bonk and Schramm present it to allow for more nuanced control of large-scale distortion.

Let $\lambda \geq 1$ and $k \geq 0$, and let $f\colon X \to Y$ be a map between metric spaces X and Y. The image $f(X) \subseteq Y$ is said to be *k-cobounded* in Y if, for any $y \in Y$, there exists $x \in X$ such that $d(f(x), y) \leq k$. The map f is said to be a *(λ, k)-quasi-isometry* if $f(X)$ is k-cobounded in Y and

$$\frac{1}{\lambda} d(x_1, x_2) - k \leq d(f(x_1), f(x_2)) \leq \lambda d(x_1, x_2) + k,$$

for all $x_1, x_2 \in X$. The map f is said to be a *(λ, k)-rough similarity* if $f(X)$ is k-cobounded in Y and

$$\lambda d(x_1, x_2) - k \leq d(f(x_1), f(x_2)) \leq \lambda d(x_1, x_2) + k,$$

for all $x_1, x_2 \in X$. For both types of maps, if we define $X \sim Y$ if there exists a map of the chosen type from X to Y, then $\sim$ is an equivalence relation. If f satisfies one of the inequalities above, but $f(X)$ is not cobounded in Y then we give it the same name but add *embedding* to the end.

6.3.2 Some useful Hölder facts

In this subsection, we present two simple but helpful lemmas for working with Hölder maps.

It is easy to prove the following lemma.

Lemma 6.3.1. *Let X, Y be metric spaces, with X bounded. Let $f\colon X \to Y$ be a (μ, a, b)-bi-Hölder map for some $\mu \geq 1$ and $a, b > 0$. Then f is also a (λ, α, β)-bi-Hölder map for some $\lambda = \lambda(\mu, a, b, \operatorname{diam}(X)) \geq 1$, $\alpha = \alpha(a) \geq 1$, and $\beta = \beta(b) \leq 1$.*

Throughout, we will only care about Hölder maps on bounded spaces, so we will henceforth assume for a (λ, α, β)-bi-Hölder map, that $\lambda \geq 1$, $\alpha \geq 1$, and $\beta \leq 1$.

We will use the following:

Lemma 6.3.2. *For any $\alpha \geq 1$, $0 < \beta \leq 1$, and $t, s \in \mathbb{R}_{\geq 0}$, the following hold:*

$$t^\beta + s^\beta \leq 2^{1-\beta}(t+s)^\beta \tag{6.3.2.1}$$

$$(t+s)^\alpha \leq 2^{\alpha-1}(t^\alpha + s^\alpha). \tag{6.3.2.2}$$

Proof. Both inequalities are consequences of the Hölder inequality. Recall, if $p, q > 1$ are such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then, for any $(a_i)_{i=1}^n, (b_i)_{i=1}^n \subset \mathbb{R}$,

$$\sum_{i=1}^n |a_i b_i| \leq \left(\sum_{i=1}^n |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |b_i|^q\right)^{\frac{1}{q}}.$$

For $t, s \geq 0$, $0 < \beta < 1$, consider $n = 2$, $(a_i)_{i=1}^n = (t^\beta, s^\beta)$ and $(b_i)_{i=1}^n = (1,1)$, and $p = 1/\beta$, to see that

$$t^\beta + s^\beta \leq (t+s)^\beta 2^{1-\beta},$$

noting that $1/q = 1 - 1/p = 1 - \beta$.

For $t, s \geq 0$, $1 < \alpha$, consider $n = 2$, $(a_i)_{i=1}^n = (t, s)$ and $(b_i)_{i=1}^n = (1,1)$, and $p = \alpha$, to see that

$$t + s \leq (t^\alpha + s^\alpha)^{\frac{1}{\alpha}} 2^{1-\frac{1}{\alpha}},$$

thus,

$$(t+s)^\alpha \leq (t^\alpha + s^\alpha) 2^{\alpha - 1}.$$

□

6.3.3 Boundaries of hyperbolic spaces

In this subsection we present standard background on boundaries of hyperbolic spaces required throughout the chapter. The following definition can be found in [GdLH90, Definition 2.1.1].

Definition 6.3.3. Let X be a metric space equipped with a fixed base-point $o \in X$. The *Gromov product* of $x, y \in X$ with respect to o is

$$(x \mid y)_o := \frac{1}{2}\left(d(x,o) + d(y,o) - d(x,y)\right).$$

We present the following definition for a metric space to be 'hyperbolic'. This is one given by Gromov in [Gro87] and does not require the space to be geodesic, which is convenient, and is equivalent to Rips' thin triangles condition for geodesic metric spaces, up to changing δ. Recall that $a \vee b$ denotes the maximum of two real numbers $a, b \in \mathbb{R}$, and $a \wedge b$ denotes the minimum. The following definition can be found in [GdLH90, Definition 2.1.3].

Definition 6.3.4. A metric space X is said to be *δ-hyperbolic* if, for any $x, y, z \in X$ and any base-point $w \in X$,

$$(x \mid z)_w \geq (x \mid y)_w \wedge (y \mid z)_w - \delta. \qquad (6.3.4.1)$$

A finitely generated group is said to be *hyperbolic* if its Cayley graph, for some finite generating set, is δ-hyperbolic for some $\delta \geq 0$.

Note from [GdLH90, Theorem 5.2.12] Gromov hyperbolicity is invariant under quasi-isometries of geodesic metric spaces, so if one Cayley graph for a group is hyperbolic, they all are.

The following definitions can be found in [GdLH90, Chapter 7, Section 1].

Definition 6.3.5. Let X be δ-hyperbolic and $w \in X$. A sequence of points $\{x_i\} \subset X$ is said to *converge at infinity*, if

$$\lim_{i,j \to \infty} (x_i \mid x_j)_w = \infty.$$

Two sequences $\{x_i\}, \{y_i\}$ that converge at infinity are *equivalent*, if

$$\lim_{i,j\to\infty} (x_i \mid y_j)_w = \infty.$$

This defines an equivalence relation for sequences in X converging at infinity. Convergence to infinity and equivalence of sequences do not depend on the choice of base-point w because changing from base-point w to $p \in X$ corresponds to an additive error of $d(w,p)$.

The *Gromov boundary* $\partial_\infty X$ of X is defined as the set of equivalence classes of sequences converging at infinity.

Again, this is not the most widely known definition for the Gromov boundary as the one that considers equivalence classes of geodesic rays is probably more common. However, many of the spaces we shall consider will not be geodesic, so we shall need this extra generality. For a proper, geodesic metric space, this definition and the one using geodesic rays are equivalent, see [GdLH90, Proposition 7.4].

Note, this is only the setwise definition of the boundary. We can metrize it by extending Gromov products to the boundary in the following standard way, which can be found in [GdLH90, Chapter 7, Section 2]. Let X be a δ-hyperbolic space with base-point $o \in X$, and let $\xi, \eta \in \partial_\infty X$. Then,

$$(\xi \mid \eta)_o := \sup \liminf_{i,j\to\infty} (x_i \mid y_j)_o . \tag{6.3.5.1}$$

Here the supremum is taken over all representatives $\{x_i\} \in \xi, \{y_i\} \in \eta$.

The following helpful lemma is a combination of [GdLH90, Remark 7.8] and [Väi05, Lemma 5.6].

Lemma 6.3.6. *Let X be δ-hyperbolic metric space with base-point $o \in X$. For any $\xi, \eta \in \partial_\infty X$ and any $\{x_i\} \in \xi$, $\{y_i\} \in \eta$,*

$$(\xi \mid \eta)_o - 2\delta \leq \liminf_{i,j\to\infty} (x_i \mid y_j)_o \leq \limsup_{i,j\to\infty} (x_i \mid y_j)_o \leq (\xi \mid \eta)_o + 2\delta.$$

This lemma is important because it allows us to approximate Gromov products of points in $\partial_\infty X$ by Gromov products of points in X with good control on the error.

In some sense, the Gromov product of two sequences measures how far from the base-point the sequences diverge from each other. Intuitively, the further away from the base-point two sequences diverge, the closer they should be in the boundary. With this in mind, one might suggest that the distance between two points $z_1, z_2 \in \partial_\infty X$ should look like $a^{-(z_1|z_2)_w}$, for some base-point $w \in X$, and some exponential growth rate $a > 0$.

However, this is not necessarily a metric as it does not necessarily satisfy the triangle inequality. We can force the triangle inequality to be satisfied with the following important definition which will be used throughout this chapter. It can be found in [GdLH90, Chapter 7, Section 3].

Definition 6.3.7. Let X be a hyperbolic metric space with base-point w, and let $\epsilon > 0$. For any $z_1, z_2 \in \partial_\infty X$ define

$$d_{\epsilon,w}(z_1, z_2) = \inf \sum_{i=1}^{N} \exp(-\epsilon(x_{i-1}|x_i)_w),$$

where the infimum is taken over all finite sequences $z_1 = x_0, x_1, x_2, \ldots, x_N = z_2$, where $N \in \mathbb{N}$ and $x_i \in \partial_\infty X$, for each i. We say that $d_{\epsilon,w}$ is (a particular kind of) *visual metric* if there exists $\lambda \geq 1$ such that for every $z_1, z_2 \in \partial_\infty X$

$$\frac{1}{\lambda} \exp\left(-\epsilon(z_1|z_2)_w\right) \leq d_{\epsilon,w}(z_1, z_2) \leq \lambda \exp\left(-\epsilon(z_1|z_2)_w\right). \tag{6.3.7.1}$$

We say that $d_{\epsilon,w}$ is λ-visual on X.

In the context of these metrics, such a sequence is called a *chain* between z_1 and z_2.

Without the extra 'visual' condition, there is a chance that ϵ is too large and the infimum over all chains equals 0 for pairs of points that are not equal. With the 'visual' condition, we forbid this degeneracy and are, indeed, left with a metric.

Note that $d_{\epsilon,w}$ depends on the base-point we choose for our Gromov products, but only up to bi-Lipschitz equivalence so it doesn't affect whether or not $d_{\epsilon,w}$ is visual.

Equipped with a visual metric as above, the boundary $\partial_\infty X$ of a hyperbolic space X is always bounded as Gromov products take non-negative real values, and always complete (for example, see [Väi05, Proposition 5.31]). If, in addition, X is proper and geodesic, then $\partial_\infty X$ is compact, see [GdLH90, Proposition 7.2.9].

The following special case of [GdLH90, Proposition 7.3.10] tells us that visual metrics always exist for boundaries of Gromov hyperbolic spaces.

Proposition 6.3.8. *Suppose X is a δ-hyperbolic metric space, and let $x_0 \in X$. If $\epsilon > 0$ is such that* $\exp(\epsilon\delta) < \sqrt{2}$*, then d_{ϵ,x_0} is a visual metric on $\partial_\infty X$.*

Any two visual metrics on $\partial_\infty X$ are Hölder and quasi-symmetrically equivalent, see [MT10, Theorem 3.2.4]. Therefore, when we talk about the boundary of a Gromov hyperbolic space X, unless specified otherwise, we shall mean the set $\partial_\infty X$ equipped with a visual metric.

6.3.4 Boundaries of hyperbolic groups

We now explain how one extends the definition of a boundary of a Gromov hyperbolic space to define the boundary of a hyperbolic group.

Usually, "the boundary of a hyperbolic group" means the boundary of a Cayley graph for the group with respect to a finite generating set. As any two Cayley graphs for a group, with respect to finite generating sets, are not necessarily isometric but are guaranteed to be quasi-isometric, and these quasi-isometries induce quasi-symmetric equivalences in the boundary, boundaries of hyperbolic groups are only well-defined up to quasi-symmetric equivalence.

In this chapter, it will not be ideal to take a Cayley graph as a geometric model for one of the spaces we are working with, so we want to justify that a broader definition of the boundary of a group is acceptable. The following lemma explains why it is okay to take any 'nice' space that is quasi-isometric to a Cayley graph of the group.

Lemma 6.3.9. *Let* Γ *be a hyperbolic group, and* X *a Cayley graph for* Γ *with respect to a finite generating set. Let* Y *be any proper, geodesic Gromov hyperbolic space that is quasi-isometric to* X*, then* $\partial_\infty X$ *is Hölder equivalent and quasi-symmetrically equivalent to* $\partial_\infty Y$ *when both are equipped with visual metrics.*

Proof. Firstly, the degenerate cases: If Γ is elementary (finite or virtually cyclic), then the boundary of any such Y is either empty, if Γ is finite, or a pair of points, if Γ is infinite. Any two metrics on a pair of points are Hölder and quasi-symmetrically equivalent so the lemma holds in this case.

If Γ is non-elementary, then by [Coo93, Proposition 7.4] $\partial_\infty X$ is uniformly perfect. Further, the quasi-isometry between X and Y induces a Hölder and quasi-symmetric equivalence between their boundaries, see [Pau96]. □

This means that the boundary of any proper, geodesic Gromov hyperbolic metric space that is quasi-isometric to a Cayley graph for Γ will be in the same quasi-symmetry class as the boundary of that Cayley graph. Thus, we define the *boundary of* Γ, denoted $\partial_\infty\Gamma$, to be the boundary of an arbitrary proper, geodesic Gromov hyperbolic metric space, that is quasi-isometric to a Cayley graph for Γ with respect to a finite generating set, equipped with a visual metric.

As these induced quasi-symmetric equivalences are also Hölder equivalences, all such representative spaces for the boundary of Γ will have equal Hölder dimension and equivalent attainment or non-attainment. Therefore, this definition is compatible with the aim of this chapter.

6.3.5 The convex hull

In [BS00] Bonk and Schramm construct a Gromov hyperbolic metric space $\mathrm{Con}(Z)$ from a given bounded metric space (Z, d). They credit their construction as similar to one given by Gromov [Gro87, 1.8.A.(b)] and to a construction of Trotsenko and Väisälä [TV99]. They also say $\mathrm{Con}(Z)$ has properties analogous to the hyperbolic convex hull of a set in the boundary of real hyperbolic space. Thus, for convenience, we shall refer to $\mathrm{Con}(Z)$ as the convex hull of Z.

As we will not require the full generality of the results in [BS00], we give partial or paraphrased versions of Bonk and Schramm's results for brevity. Their paper is significantly more powerful than what is presented here, so we recommend the intrigued reader seek out the full details.

Let (Z, d) be a bounded metric space.

Here, and in the following we let $D(Z) \coloneqq \operatorname{diam}(Z)$, if $\operatorname{diam}(Z) > 0$, and $D(Z) \coloneqq 1$ if $\operatorname{diam}(Z) = 0$.

Define

$$\operatorname{Con}(Z) \coloneqq Z \times (0, D(Z)]. \tag{6.3.9.1}$$

Further, define $\rho\colon \operatorname{Con}(Z) \times \operatorname{Con}(Z) \to [0, \infty)$ by

$$\rho((z,h),(z',h')) = 2\log\left(\frac{d(z,z') + h \vee h'}{\sqrt{hh'}}\right), \tag{6.3.9.2}$$

for any $z, z' \in Z$ and $h, h' \in (0, D(Z)]$.

Theorem 6.3.10 ([BS00, Lemma 7.1 and Theorem 7.2])**.** *ρ is a metric on $\operatorname{Con}(Z)$ and $\operatorname{Con}(Z)$ is Gromov hyperbolic with respect to this metric.*

$\operatorname{Con}(Z)$ equipped with the metric ρ is called the *convex hull* of Z.

The motivation behind defining the convex hull is to be able to find, for an arbitrary bounded metric space Z, a Gromov hyperbolic space X for which Z is the boundary of X. The hope is to be able to convert analytic questions about Z into question about the large-scale geometry of X, and vice versa. The following paraphrasing of Theorem [BS00, Theorem 8.1] says that the boundary of $\operatorname{Con}(Z)$ shares the same Lipschitz data as Z.

Theorem 6.3.11. *Suppose (Z, d) is a complete bounded metric space and fix $x_0 = (z_0, h_0) \in \operatorname{Con}(Z)$, then d_{1,x_0} is a visual metric on $\partial_\infty \operatorname{Con}(Z)$ and $(\partial_\infty \operatorname{Con}(Z), d_{1,x_0})$ is $(\exp(A + 2\delta))$-bi-Lipschitz equivalent to (Z, d), where d_{1,x_0} is from Definition 6.3.7, $A = A(D(Z), h_0)$ is from Lemma 6.3.12 below, and $\operatorname{Con}(Z)$ is δ-hyperbolic.*

Later, in Subsection 6.4.4, we use results of Bonk and Schramm that say that if Z is already the boundary of a Gromov hyperbolic space X, then $\operatorname{Con}(Z)$ has the same large-scale geometry as X, and if two bounded metric spaces share the same fine-scale analysis, then their convex hulls share the same large-scale geometry.

As we shall be considering the Gromov boundary of convex hulls, we will need to understand their Gromov products. The following lemma can be found in the proof of [BS00, Theorem 8.1]

Lemma 6.3.12. *Let (Z, d) be a bounded metric space, $D = D(Z)$ as defined in Definition (6.3.9.1), $x = (z, h), x' = (z', h') \in \operatorname{Con}(Z)$, and $x_0 = (z_0, h_0) \in \operatorname{Con}(Z)$ be the base-point for Gromov products. Then there exists $A = A(D, h_0) \geq 0$ such that*

$$|(x \mid x')_{x_0} + \log\left(d(z,z') + h \vee h'\right)| \leq A.$$

In particular, if $D(Z) = 1$ and $h_0 = 1$, then $A = 2\log 2$.

6.4 Setup

Our goal is to prove Theorem 6.2.1. To do so, we first need geometric models for the groups in question so that we can talk about their boundaries.

Let G_1 and G_2 be infinite hyperbolic groups that are not both virtually cyclic, and let $\Gamma = G_1 * G_2$. Usually one would use a Cayley graph as a geometric model for Γ, as we shall do for G_1 and G_2, but Bass-Serre theory gives us another quasi-isometric space, which emphasises the free product structure of Γ. We summarise the important information about these geometric models in the following proposition:

Proposition 6.4.1. *Let G_1 and G_2 infinite hyperbolic groups that are not both virutally cyclic, and let $\Gamma = G_1 * G_2$. Then, there exists bounded degree graphs $\tilde{X}_\Gamma^{(1)}$, $\tilde{X}_1^{(1)}$, and $\tilde{X}_2^{(1)}$, a tree T, and a continuous surjection $\pi\colon \tilde{X}_\Gamma^{(1)} \to T$ such that the following hold:*

- *$\tilde{X}_\Gamma^{(1)}$ is quasi-isometric to Γ.*
- *$\tilde{X}_1^{(1)}$ and $\tilde{X}_2^{(1)}$ are Cayley graphs for G_1 and G_2, respectively.*
- *For any vertex v in T, $\pi^{-1}(v)$ is graph isomorphic to $\tilde{X}_1^{(1)}$ or $\tilde{X}_2^{(1)}$.*
- *For any edge e in T, $\pi^{-1}(e)$ is a single edge in $\tilde{X}_\Gamma^{(1)}$.*
- *$\tilde{X}_\Gamma^{(1)}$, $\tilde{X}_1^{(1)}$, and $\tilde{X}_2^{(1)}$ are geodesic Gromov hyperbolic spaces when equipped with the shortest path metric where every edge is assigned length 1.*

Notation 6.4.2. Let $d_{\tilde{X}_\Gamma^{(1)}}$ and d_T be the shortest path metrics on $\tilde{X}_\Gamma^{(1)}$ and T where every edge is assigned length 1, respectively. Let $V(T)$ be the vertex set of T and, for any $v \in V(T)$, we call the subspace $\pi^{-1}(v)$ of $\tilde{X}_\Gamma^{(1)}$ a *vertex space* and denote it by X_v. Let $\delta_X \geq 0$ be such that $\tilde{X}_1^{(1)}$, $\tilde{X}_2^{(1)}$, $\tilde{X}_\Gamma^{(1)}$ are δ_X-hyperbolic. When we specify $x \in \tilde{X}_\Gamma^{(1)}$ we shall mean an element of the vertex set of $x \in \tilde{X}_\Gamma^{(1)}$.

Proof of Proposition 6.4.1. For $i = 1, 2$, let X_i be a finite presentation complex for G_i. Such finite presentations exist because G_i are hyperbolic. Let X_Γ be the space obtained by joining X_1 and X_2 at their respective base-points by a copy of $I = [0, 1]$. Then, by a standard application of van Kampen's theorem, X_Γ has fundamental group isomorphic to Γ. Let $\tilde{X}_i$ be the universal cover of X_i, for $i = 1, 2, \Gamma$.

As X_1 X_2, X_Γ are CW complexes, $\tilde{X}_1$, $\tilde{X}_2$, and $\tilde{X}_\Gamma$ inherit cellular structures. Denote by $\tilde{X}_1^{(1)}$, $\tilde{X}_2^{(1)}$, and $\tilde{X}_\Gamma^{(1)}$ the 1-skeletons of $\tilde{X}_1$, $\tilde{X}_2$, and $\tilde{X}_\Gamma$, respectively. As X_1 and X_2 were taken to be finite presentation complexes for G_1 and G_2, $\tilde{X}_1^{(1)}$ and $\tilde{X}_2^{(1)}$ are Cayley graphs for G_1 and G_2 with respect to finite generating sets, respectively. Further, as G_1 and G_2 are hyperbolic groups, this means both $\tilde{X}_1^{(1)}$ and $\tilde{X}_2^{(1)}$ are Gromov hyperbolic when equipped with the shortest path metric. Each of $\tilde{X}_1^{(1)}$, $\tilde{X}_2^{(1)}$, and $\tilde{X}_\Gamma^{(1)}$ has bounded degree because the 1-skeletons of X_1,

X_2 and X_Γ are bounded degree graphs, respectively. They are all geodesic with respect to this metric as they are path connected and the metric is integer valued on pairs of vertices.

Noting that Γ is isomorphic to the group of deck transformations of $\tilde{X}_\Gamma$, We get an induced covering space action of Γ on $\tilde{X}_\Gamma^{(1)}$ with quotient equal to the 1-skeleton of X_Γ. A standard application of the Milnor-Schwarz lemma, which can be found in [BH99, I.8.19], allows one to prove that $\left(\tilde{X}_\Gamma^{(1)}, d_{\tilde{X}_\Gamma^{(1)}}\right)$ is quasi-isometric to Γ bestowed with the word metric with respect to any finite generating set. Therefore, $\tilde{X}_\Gamma^{(1)}$ is quasi-isometric to Γ. Further, as Γ is a hyperbolic group, $\tilde{X}_\Gamma^{(1)}$ is Gromov hyperbolic.

The following is a description of the structure of $\tilde{X}_\Gamma$, which has been taken from [SW79, Page 166] but adapted to our specific case. From this description we shall be able to infer the structure of $\tilde{X}_\Gamma^{(1)}$.

$\tilde{X}_\Gamma$ is a union of copies of the universal covers of X_1, X_2, and I. In $\tilde{X}_\Gamma$, for $i = 1, 2$, identify each copy of $\tilde{X}_i$ to a point giving a quotient space T with projection $\pi\colon \tilde{X}_\Gamma \to T$. Clearly T is a graph. Scott and Wall then proceed to build a map $j\colon T \to \tilde{X}_\Gamma$ such that $\pi \circ j$ is homotopic to the identity so that they can conclude that T is also connected and simply-connected, and hence a tree.

This tree T is commonly referred to as the Bass-Serre tree for this decomposition of Γ. It reveals a 'tree-of-spaces' structure to $\tilde{X}_\Gamma$. When we restrict to the 1-skeleton of $\tilde{X}_\Gamma$, this tree-of-spaces structure remains. That is, restricting to $\tilde{X}_\Gamma^{(1)}$, $\pi\colon \tilde{X}_\Gamma^{(1)} \to T$ is a continuous surjection such that, for any $v \in V(T)$, $\pi^{-1}(v)$ is a graph isomorphic copy of $\tilde{X}_1^{(1)}$ or $\tilde{X}_2^{(1)}$, and, for any edge e in T, $\pi^{-1}(e)$ is a single edge in $\tilde{X}_\Gamma^{(1)}$. □

6.4.1 Shortest path metric in a tree-of-spaces

To actually use the space $\tilde{X}_\Gamma^{(1)}$ it will be helpful to re-phrase the shortest-path metric bestowed upon it in a way that utilizes the tree-of-spaces structure.

For any vertices x_1, x_2 in $\tilde{X}_\Gamma^{(1)}$, note that $\pi(x_1)$ and $\pi(x_2)$ are both vertices in T. As T is a tree, there is a unique non-repeating sequence $\pi(x_1) = v_0, v_1, \ldots, v_n = \pi(x_2)$ corresponding to the geodesic path between $\pi(x_1)$ and $\pi(x_2)$ in T. Let e_i be the edge connecting v_{i-1} to v_i, and observe $\tilde{e}_i = \pi^{-1}(e)$ is the edge connecting $X_{v_{i-1}}$ to X_{v_i}. Let $z_{2,i-1}$ be the endpoint of $\tilde{e}_i$ in $X_{v_{i-1}}$, and $z_{1,i}$ the endpoint in X_{v_i}. Further, let $z_{1,0} = x_1$ and $z_{2,n} = x_2$.

Observe that any geodesic path, γ, from x_1 to x_2 in $\tilde{X}_\Gamma^{(1)}$ must project, under π, onto the geodesic path between $\pi(x)$ and $\pi(y)$. This means that γ must pass through the edge $(z_{2,i-1}, z_{1,i})$, for each $1 \leq i \leq n$, and, within each X_{v_i} for $0 \leq i \leq n$, γ restricts to a geodesic between $z_{1,i}$ and $z_{2,i}$ contained within X_{v_i}. Hence, we obtain the following distance formula for the distance between x_1 and x_2, calculated from distances localized to vertex spaces and the number of vertex spaces between x_1 and x_2.

$$d_{\tilde{X}_\Gamma^{(1)}}(x_1, x_2) = \sum_{i=0}^{n} \left(d_{X_{v_i}}(z_{1,i}, z_{2,i}) \right) + n. \tag{6.4.2.1}$$

Here, and throughout this chapter, for any $v \in V(T)$, d_{X_v} is the restriction of $d_{\tilde{X}_\Gamma^{(1)}}$ to X_v.

One should think of this as travelling within a vertex space until you reach an edge, traversing this edge, with length 1, into a new vertex space, continuing on through this new vertex space to another length 1 edge, and so on, until you reach the vertex space that contains your target and then travel through this copy to your target. The n added on the end is the length added by traversing n edges each of length 1. The tree-of-spaces structure means that this sequence of vertex spaces is unique.

6.4.2 Closest-point projections

An important consequence of the tree-of-spaces structure of $\tilde{X}_\Gamma$ is that points have canonical projections onto vertex spaces, which we call 'closest-point projections'. These projections give us a clean language with which to convert intuition about spaces with a tree-of-space structure into concrete results.

Lemma 6.4.3. *For any $x \in \tilde{X}_\Gamma^{(1)}$ and $v \in V(T)$, there is a unique point $x_v \in X_v$ such that*

$$d_{\tilde{X}_\Gamma^{(1)}}(x, x_v) = \min\{d_{\tilde{X}_\Gamma^{(1)}}(x, x') \mid x' \in X_v\}.$$

We call such an x_v the *closest-point projection* of x in X_v.

Proof. Given any vertex v in T and x in $\tilde{X}_\Gamma^{(1)}$, either $\pi(x) = v$ or not. If $\pi(x) = v$, then the closest-point projection of x in X_v is x. Otherwise, $\pi(x) \neq v$, and we can take a non-trivial geodesic path between $\pi(x)$ and v in T. Let e be the edge of this path with v as one of its endpoints, and let x_v be the endpoint of $\pi^{-1}(e)$ in X_v. Now, for any $x' \in X_v$, any path γ from x to x' in $\tilde{X}_\Gamma^{(1)}$ projects to a path from $\pi(x)$ to v in T. Note that, as T is a tree, removing e disconnects T into two connected components, one containing $\pi(x)$ and the other containing v, so $\pi(\gamma)$ must cross e. Hence, γ crosses $\pi^{-1}(e)$ and thus also passes through x_v. Therefore, $d_{\tilde{X}_\Gamma^{(1)}}(x, x') \geq d_{\tilde{X}_\Gamma^{(1)}}(x, x_v)$, and, as $x_v \in X_v$, we can conclude that x_v is the closest-point projection of x in X_v. □

This definition helps us phrase (6.4.2.1) with slicker language that will streamline later proofs.

Lemma 6.4.4. *For any $x_1, x_2 \in \tilde{X}_\Gamma^{(1)}$, let $\pi(x_1) = v_0, \ldots, v_n = \pi(x_2)$ be the unique non-repeating sequence of vertices in T corresponding to the geodesic path between $\pi(x_1)$ and $\pi(x_2)$. Let $z_{1,i}$ and $z_{2,i}$ be the closest-point projections of x_1 and x_2 in X_{v_i}, respectively. Then*

$$d_{\tilde{X}_\Gamma^{(1)}}(x_1, x_2) = \sum_{i=0}^{n} \Big(d_{X_{v_i}}(z_{1,i}, z_{2,i})\Big) + n.$$

For technical reasons later, it will be extremely important to consider and keep track of base-points. Closest point projections give us a convenient way of choosing nice base-points.

Definition 6.4.5. Fix $o \in \tilde{X}_\Gamma^{(1)}$ and, for each $v \in V(T)$, let $o_v \in \tilde{X}_\Gamma^{(1)}$ be the closest-point projection of o in X_v. We call o the *global* base-point for $\tilde{X}_\Gamma^{(1)}$, and o_v the *local* base-point for X_v.

For $i = 1, 2$, fix o_i a base-point for $\tilde{X}_i^{(1)}$. Recall, for each $v \in V(T)$, X_v is a graph isomorphic copy of $\tilde{X}_1^{(1)}$ or $\tilde{X}_2^{(1)}$. A priori, it is not clear that our choices of base-point for X_v and $\tilde{X}_i^{(1)}$ are compatible with this isomorphism. It is also not clear how these isomorphisms interact with the metric on $\tilde{X}_\Gamma^{(1)}$. The following lemma allows us to resolve these two potential issues.

Lemma 6.4.6. *For each $v \in V(T)$, there exists $i \in \{1, 2\}$ and an isometry*

$$\iota_v \colon \left(X_v, d_{\tilde{X}_\Gamma^{(1)}}\right) \to \left(\tilde{X}_i^{(1)}, d_{\tilde{X}_i^{(1)}}\right),$$

such that $\iota_v(o_v) = o_i$. Here, $d_{\tilde{X}_i^{(1)}}$ is the shortest path metric on $\tilde{X}_i^{(1)}$ where every edge has been assigned length 1.

Proof. Firstly, for any $v \in V(T)$, we can upgrade graph isomorphisms between $\left(X_v, d_{\tilde{X}_\Gamma^{(1)}}\right)$ and $\left(\tilde{X}_i^{(1)}, d_{\tilde{X}_i^{(1)}}\right)$ to isometries, because X_v is a convex subset of $\tilde{X}_\Gamma^{(1)}$. Indeed, if a path leaves X_v then it must cross an edge that projects to an edge in T. Therefore, to re-enter X_v, the path must at some point cross over this edge again, due to T being a tree. Clearly, such a path is not a geodesic. Hence, X_v is convex in $\tilde{X}_\Gamma^{(1)}$. This means the restriction of $d_{\tilde{X}_\Gamma^{(1)}}$ to X_v is the shortest path metric on X_v (one is only allowed to take paths that lie within X_v) where each edge is assigned length 1. Now there is no possibility that the ambient space $\tilde{X}_\Gamma^{(1)}$ allows for shortcuts and any graph isomorphism between X_v and $\tilde{X}_i^{(1)}$ is an isometry.

Note that, as $\tilde{X}_i^{(1)}$ is a Cayley graph for G_i, G_i acts transitively by isometries on the vertex set of $\left(\tilde{X}_i^{(1)}, d_{\tilde{X}_i^{(1)}}\right)$. We can compose any isometry $X_v \to \tilde{X}_i^{(1)}$ with the action of an element of G_i to edit it so that it sends o_v to o_i and is still an isometry. □

6.4.3 The plan for replacing vertex spaces

The boundary of Γ is essentially a collection of bi-Lipschitz copies of $\partial_\infty G_1$ and $\partial_\infty G_2$, plus some 'Cantor-like dust'. The copies of $\partial_\infty G_1$ and $\partial_\infty G_2$ are the boundaries of vertex spaces sitting inside $\tilde{X}_\Gamma^{(1)}$. The dust is there because geodesic rays do not have to end up in a single vertex space forever. Instead, they can keep jumping into new vertex spaces without ever settling into a single one. This dust is the main problem when attempting to understand the Hölder data of $\partial_\infty\Gamma$.

In this chapter we provide a method for upgrading Hölder data about $\partial_\infty G_1$ and $\partial_\infty G_2$ to Hölder data about $\partial_\infty\Gamma$. That is, given spaces $\mathfrak{G}_1$ and $\mathfrak{G}_2$ that are Hölder equivalent to $\partial_\infty G_1$ and $\partial_\infty G_2$, respectively, we describe a method of building a space from the tree-of-spaces structure, T, of $\tilde{X}_\Gamma^{(1)}$, $\mathfrak{G}_1$, and $\mathfrak{G}_2$ that extends the Hölder equivalences on $\partial_\infty G_1$ and $\partial_\infty G_2$ to an equivalence on the whole of $\partial_\infty \Gamma$.

We do this by finding hyperbolic spaces W_1 and W_2 such that, for $i = 1, 2$, $\partial_\infty W_i = \mathfrak{G}_i$, and replacing the vertex spaces in $\tilde{X}_\Gamma^{(1)}$ with copies of W_1 and W_2. To find W_1 and W_2, we look to [BS00, Section 7] where Bonk and Schramm provide us with the "the convex hull" of a space. As discussed above, they credit their construction as being similar to one given by Gromov [Gro87, 1.8.A.(b)] and to another of Trotsenko and Väisälä [TV99]. We presented key information about convex hulls in Subsection 6.3.5.

6.4.4 Replacing vertex spaces

In this subsection, we build the spaces and maps that we will use to replace the vertex spaces of $\tilde{X}_\Gamma^{(1)}$. There are quite a lot of objects to keep track of, so we have included Figure 6.5 to aid the reader.

As an artefact of our method for replacing vertex spaces, it will be helpful to fix metrics on $\partial_\infty \tilde{X}_\Gamma^{(1)}$, $\partial_\infty \tilde{X}_1^{(1)}$, and $\partial_\infty \tilde{X}_2^{(1)}$ rather than working with the whole class of equivalent metrics. Further, for Proposition 6.8.4 it will be helpful to bestow them with metrics that are easily comparable. We will do this by equipping them with visual chain metrics that all share the same parameter, as follows.

Recall from Notation 6.4.2 that $\tilde{X}_\Gamma^{(1)}$, $\tilde{X}_1^{(1)}$, and $\tilde{X}_2^{(1)}$ are all δ_X-hyperbolic. Thus, by Proposition 6.3.8, there exists $\epsilon_X > 0$ such that

$$\left(\tilde{X}_\Gamma^{(1)}, d_{\epsilon_X, o}\right), \ \left(\tilde{X}_1^{(1)}, d_{\epsilon_X, o_1}\right), \text{ and } \left(\tilde{X}_2^{(1)}, d_{\epsilon_X, o_2}\right) \text{ are all visual metrics,} \tag{6.4.6.1}$$

where $d_{\epsilon_X,*}$ is the chain metric defined in Definition 6.3.7. It will make calculations simpler if we can assume that $D_i \coloneqq D\left(\partial_\infty \tilde{X}_i^{(1)}, d_{\epsilon_X, o_i}\right) = 1$, where $D(\cdot)$ is defined in Definition (6.3.9.1). Regardless of the value of D_i, note that

$$D\left(\partial_\infty \tilde{X}_i^{(1)}, d_{\epsilon_X, o_i}/D_i\right) = 1. \tag{6.4.6.2}$$

Note that, $d_{\epsilon_X, o_i}/D_i$ is still a visual metric on $\partial_\infty \tilde{X}_i^{(1)}$. Thus, by [BS00, Theorem 8.2], there exists a (ϵ_X, k_i)-rough similarity $\phi_i \colon \tilde{X}_i^{(1)} \to \operatorname{Con}\left(\partial_\infty \tilde{X}_i^{(1)}, d_{\epsilon_X, o_i}/D_i\right)$, for some $k_i \geq 0$. To remove some subscripts, let $k = k_1 \vee k_2$, then ϕ_1 and ϕ_2 are both (ϵ_X, k)-rough similarities.

Continuing our care of base-points, fix $\xi_{o_i} \in \partial_\infty \tilde{X}_i^{(1)}$ and set $\operatorname{Con}\left(\partial_\infty \tilde{X}_i^{(1)}\right)$ to have base-point $(\xi_{o_i}, 1)$, which is a point in $\operatorname{Con}\left(\partial_\infty \tilde{X}_i^{(1)}\right)$ by (6.4.6.2). Now, adjust ϕ_i to $\Phi_i \colon \tilde{X}_i^{(1)} \to \operatorname{Con}\left(\partial_\infty \tilde{X}_i^{(1)}\right)$ by defining

$$\Phi_i(x) = \phi_i(x), \tag{6.4.6.3}$$

for all $o_i \neq x \in \tilde{X}_i^{(1)}$ and

$$\Phi_i(o_i) = (\xi_{o_i}, 1),$$

so that Φ_i sends the base-point of $\tilde{X}_i^{(1)}$ to the base-point of $\mathrm{Con}\left(\partial_\infty \tilde{X}_i^{(1)}\right)$. If $k' = k + \max\left\{ d_{\mathrm{Con}\left(\partial_\infty \tilde{X}_i^{(1)}\right)}(\phi_i(o_i), \Phi_i(o_i)) \mid i \in \{1,2\} \right\}$ then Φ_i is a (ϵ_X, k')-rough similarity.

For ease of notation, let, for $x \in \tilde{X}_i^{(1)}$, $\Phi_i(x) = (\xi_x, h_x)$, where $\xi_x \in \partial_\infty \tilde{X}_i^{(1)}$ and $h_x \in (0,1]$. For example, the above means $h_{o_i} = 1$.

For $i = 1, 2$, let $(\mathfrak{G}_i, d_{\mathfrak{G}_i})$ be a metric space such that there exists a $(\lambda_i, \alpha_i, \beta_i)$-bi-Hölder homeomorphism

$$f_i \colon \left(\partial_\infty \tilde{X}_i^{(1)}, d_{\epsilon_X, o_i}/D_i\right) \to \mathfrak{G}_i, \tag{6.4.6.4}$$

for some $\lambda_i \geq 1$, $\alpha_i \geq 1$, $\beta_i \leq 1$. To make later calculations simpler, without loss of generality, we may assume that $\mathfrak{G}_i$ has diameter 0 or 1, so that

$$D\left(\mathfrak{G}_i, d_{\mathfrak{G}_i}\right) = 1. \tag{6.4.6.5}$$

Again, let $\lambda = \lambda_1 \vee \lambda_2$, $\alpha = \alpha_1 \vee \alpha_2$, and $\beta = \beta_1 \wedge \beta_2$, so that both f_1 and f_2 are (λ, α, β)-bi-Hölder homeomorphisms.

Define $\bar{f}_i \colon \mathrm{Con}\left(\partial_\infty \tilde{X}_i^{(1)}, d_{\epsilon_X, o_i}/D_i\right) \to \mathrm{Con}(\mathfrak{G}_i)$ by

$$\bar{f}_i(\xi, h) = (f_i(\xi), h), \tag{6.4.6.6}$$

for all $(\xi, h) \in \mathrm{Con}\left(\partial_\infty \tilde{X}_i^{(1)}, d_{\epsilon_X, o_i}/D_i\right)$. This map is well defined because, from (6.4.6.2) and (6.4.6.5), we rescaled both $\partial_\infty \tilde{X}_i^{(1)}$ and $\mathfrak{G}_i$ so that $D\left(\partial_\infty \tilde{X}_i^{(1)}, d_{\epsilon_X, o_i}/D_i\right) = 1$ and $D(\mathfrak{G}_i) = 1$. Continuing our care with base-points, set the base-point of $\mathrm{Con}(\mathfrak{G}_i)$ to be $\mathfrak{o}_i := (f_i(\xi_{o_i}), 1)$. Observe that $\bar{f}_i((\xi_{o_i}, 1)) = (f_i(\xi_{o_i}), 1)$, so $\bar{f}_i$ preserves base-points.

The metric properties of this map are far from clear. For example, distances between points can become increasingly more distorted the further away these points are from the base-point. This is in stark contrast to quasi-isometries whose description of distortion control is independent of base-point. However, in Subsection 6.7.1 we will see that this map behaves well with respect to Gromov products, which is all we really need to understand when considering Gromov boundaries.

Define

$$F_v := \bar{f}_i \circ \Phi_i \circ \iota_v \colon X_v \to \mathrm{Con}(\mathfrak{G}_i). \tag{6.4.6.7}$$

Note, as each map in this composition sends the base-point of its domain to the base-point of its range, F_v sends o_v, the base-point of X_v, to $(f_i(\xi_{o_i}), 1)$, the base-point of $\mathrm{Con}(\mathfrak{G}_i)$. We dropped '$i$' from the notation for F_v because i is determined by v. This map, combined with the distance formula (6.4.2.1), is how we will replace each X_v in $\tilde{X}_\Gamma^{(1)}$ with a copy of $\mathrm{Con}(\mathfrak{G}_i)$, for appropriate $i \in \{1,2\}$.

6.5 Construction

We now explain how to use the maps F_v from Subsection 6.4.4 to replace the vertex spaces of $\tilde{X}_\Gamma^{(1)}$ with copies of $\mathrm{Con}(\mathfrak{G}_1)$ and $\mathrm{Con}(\mathfrak{G}_2)$.

For each vertex v in T, take a copy of $\left(\mathrm{Con}(\mathfrak{G}_i), d_{\mathrm{Con}(\mathfrak{G}_i)}\right)$ and call it (W_v, d_{W_v}), where i is such that $\pi^{-1}(v)$ is a copy of $\tilde{X}_i^{(1)}$. As a set, define

$$W := \bigsqcup_{v \in V(T)} W_v. \tag{6.5.0.1}$$

W comes with a couple of natural maps: We can combine the maps F_v to get a map from $\tilde{X}_\Gamma^{(1)}$ to W. That is, define

$$F(x) := F_{\pi(x)}(x) \text{ in } W_{\pi(x)}. \tag{6.5.0.2}$$

For a given $x \in \tilde{X}_\Gamma^{(1)}$ this map recognises x as an element of the vertex space $X_{\pi(x)}$ and maps it over to the copy of $\mathrm{Con}(\mathfrak{G}_i)$, for an appropriate i, corresponding to $\pi(x) \in V(T)$, using the map $F_{\pi(x)}$. We also have a natural map

$$\pi_W : W \to T \tag{6.5.0.3}$$

defined by $w \mapsto v$ precisely when $w \in W_v$. This is a direct parallel of $\pi \colon \tilde{X}_\Gamma^{(1)} \to T$, and will be how we bestow W with the same tree-of-spaces structure as $\tilde{X}_\Gamma^{(1)}$.

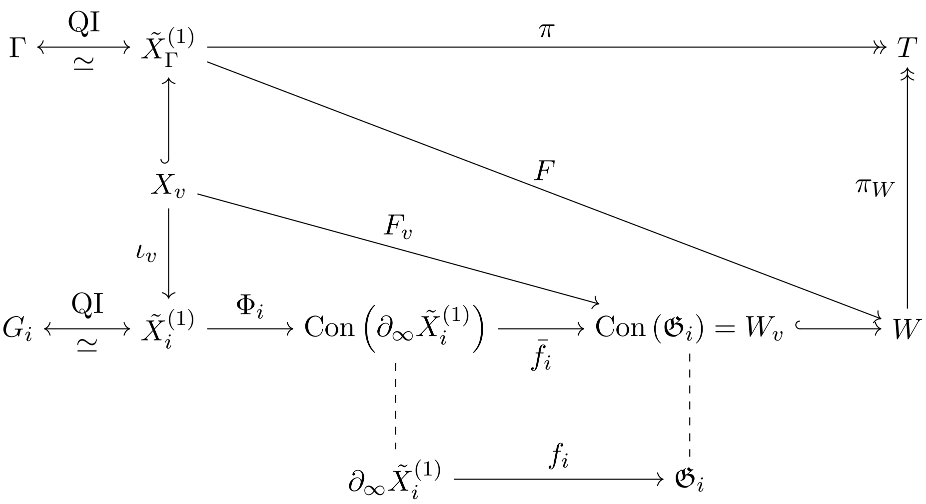

Figure 6.1: A commutative diagram summarising the spaces, and maps between them, used in the construction of W.

Motivated by the distance formula (6.4.2.1), we define a metric on W as follows. First, let $E(T)$ denote the set of edges of T, and let $L\colon E(T) \to \mathbb{R}$ be any function such that $L(e) > 0$ for every $e \in E(T)$. One can think of this as assigning a length to the edge $e = (v, v')$, which will dictate the distance between W_v and $W_{v'}$. Take any pair $w_1, w_2 \in W$. Note, both

$\pi_W(w_1)$ and $\pi_W(w_2)$ are vertices in T, As T is a tree, there is a unique non-repeating sequence $\pi_W(w_1) = v_0, v_1, \ldots, v_n = \pi_W(w_2)$ corresponding to the geodesic path between $\pi_W(w_1)$ and $\pi_W(w_2)$ in T. For $1 \leq j \leq n$, each edge $e_j = (v_{j-1}, v_j)$ in T corresponds to a unique edge in $\tilde{X}_\Gamma^{(1)}$ connecting $X_{v_{j-1}}$ to X_{v_j}. Let $x_{2,j-1} \in X_{v_{j-1}}$ and $x_{1,j} \in X_{v_j}$ be the endpoints of this edge. Let $z_{2,j-1} = F(x_{2,j-1})$ and $z_{1,j} = F(x_{1,j})$. Further, let $z_{1,0} = w_1$ and $z_{2,n} = w_2$. Define

$$d_{W,L}(w_1, w_2) = \sum_{i=0}^{n} \Big(d_{W_{v_i}}(z_{1,i}, z_{2,i}) \Big) + \sum_{i=1}^{n} L(e_i). \tag{6.5.0.4}$$

Here, we use the convention that the empty sum is equal to 0, which is only relevant if $\pi_W(w_1) = \pi_W(w_2)$, as then there are no edges in the geodesic path in T between $\pi_W(w_1)$ and $\pi_W(w_2)$.

Lemma 6.5.1. *$d_{W,L}$ is a metric on W.*

By defining a metric on W using paths in T, we incorporate the tree structure of T into W in a similar way as it appears in $\tilde{X}_\Gamma^{(1)}$.

Proof. Firstly $d_{W,L} \colon W \times W \to \mathbb{R}_{\geq 0}$ is a well-defined map, because it takes non-negative real values, and, for each pair (w_1, w_2), there is a unique sequence of vertices, $(v_i)_{i=0}^n$, and edges, $(e_i = (v_{i-1}, v_i))_{i=1}^n$, in T joining $\pi_W(w_1)$ to $\pi_W(w_2)$.

The formula is symmetric because each (local) metric $d_{W_{v_i}}$ is symmetric, and the unique non-repeating sequence of vertices and edges joining $\pi_W(w_1)$ to $\pi_W(w_2)$ is the same as the unique non-repeating sequence of vertices and edges joining $\pi_W(w_2)$ to $\pi_W(w_1)$ as they both correspond to the unique geodesic path between $\pi_W(w_1)$ and $\pi_W(w_2)$.

If $w_1 = w_2$, then $\pi_W(w_1) = \pi_W(w_2)$. Thus, if we attempt to calculate $d_{W,L}$ via (6.5.0.4), then we find

$$d_{W,L}(w_1, w_2) = d_{W_{\pi_W(w_1)}}(w_1, w_2) + 0,$$

but this equals 0 because $d_{W_{\pi_W(w_1)}}$ is a metric. Further, if $d_{W,L}(w_1, w_2) = 0$ for some $w_1, w_2 \in W$, then $n = 0$ in the distance formula (6.5.0.4). If not, then there would be a contribution from $L(e) > 0$ for some e, and, as each $d_{W_{v_i}}(z_{1,i}, z_{2,i})$ is non-negative, this would mean $d_{W,L}(w_1, w_2) > 0$, a contradiction. Thus $\pi_W(w_1) = \pi_W(w_2)$ and $d_{W_{\pi_W(w_1)}}(w_1, w_2) = 0$, which implies $w_1 = w_2$ as $d_{W_{\pi_W(w_1)}}$ is a metric.

To see that $d_{W,L}$ satisfies the triangle inequality, pick three (not necessarily distinct) points $w_1, w_2, w_3 \in W$. Checking whether or not the triangle inequality holds for w_1, w_2, and w_3 comes down to checking an inequality involving the distance formula (6.5.0.4) evaluated along the three sides of a geodesic triangle in T with corners $\pi_W(w_1)$, $\pi_W(w_2)$, and $\pi_W(w_3)$. As T is a tree, this triangle is a tripod. Running back and forth along the leaves of this tripod will result in cancellations in the formula. If $v \in V(T)$ is the vertex at the centre of this tripod, then checking the triangle inequality holds for $d_{W,L}$ for these three points reduces to checking the triangle inequality holding for three (possibly different) points in W_v, which it does because $d_{W,L}$ restricts to the metric d_{W_v} in W_v. Thus $d_{W,L}$ is indeed a metric on W. □

Again, we can interpret the metric on W by the 'closest-point projection' language introduced earlier in the context of the distance formula (6.4.2.1) for $\tilde{X}_\Gamma^{(1)}$. In (6.5.0.4), each $z_{j,i}$ is the closest-point projection of w_j in W_i. Note that, for any $x \in \tilde{X}_\Gamma^{(1)}$, if x_v is the closest-point projection of x in X_v, then $F(x_v)$ is the closest-point projection of $F(x)$ in W_v. This is a consequence of how we used T and the maps F_v, the localisations of F to vertex spaces, to build W.

Continuing our care of base-points, for each $v \in V(T)$, we have two potential definitions for the base-point of W_v: If we set the global base-point of W to be

$$\mathfrak{o} \coloneqq F(o),$$

then, following the definition of the local base-point, o_v, for X_v in $\tilde{X}_\Gamma^{(1)}$, the local base-point for W_v should be the closest-point projection of $\mathfrak{o}$ in W_v. From the above, the closest-point projection of $F(o)$ to W_v is $F(o_v)$. Another candidate is via the identification $W_v = \mathrm{Con}(\mathfrak{G}_i)$, for some $i = 1, 2$ as appropriate. As noted following the definition of F_v in (6.4.6.7), $F_v(o_v)$ is the base-point of $\mathrm{Con}(\mathfrak{G}_i)$. F restricts to F_v on X_v so these two definitions agree and defining the local base-point of W_v to be

$$\mathfrak{o}_v \coloneqq F(o_v)$$

is sensible.

The above essentially means F behaves well with respect to base-points. We summarise this in the following lemma.

Lemma 6.5.2. *For any $x \in \tilde{X}_\Gamma^{(1)}$, if x_v is the closest-point projection of x in X_v, then $F(x_v)$ is the closest-point projection of $F(x)$ in $W_v \subset (W, d_{W,L})$. In particular, $F(o_v) = F_v(o_v) = \mathfrak{o}_v$ for all $v \in V(T)$.*

6.6 Understanding the structure of W and its boundary

Understanding Gromov products of a hyperbolic space is, by definition, integral to understanding the metric on its boundary. In this section, we explain how the tree-of-spaces structure shared by $\tilde{X}_\Gamma^{(1)}$ and W will allow us to simplify Gromov products in both spaces and relate them to each other, so that we cannot only understand their boundaries, but compare them as well.

6.6.1 Evaluation of Gromov products

This subsection explains how the tree-of-spaces structure underlying $\tilde{X}_\Gamma^{(1)}$ and W allows us to simplify Gromov products using the important concept of 'evaluation'.

As we are working with a metric on W that depends on a choice of function L, we need to be clear what metric is being used when calculating a Gromov product. Thus, for the sake of clarity, we introduce the following notation.

Notation 6.6.1. For any $w, w', z \in (W, d_{W,L})$ we shall denote by $(w \mid w')^L_z$ the Gromov product of w and w' with respect to z.

Calculating a Gromov product essentially corresponds to a calculation with lengths of sides of a geodesic triangle. The tree-of-spaces structure of $\tilde{X}^{(1)}_\Gamma$ and W will restrict what kind of triangles can appear in these spaces, making Gromov products easier to calculate. Let Δ be a geodesic triangle in $\tilde{X}^{(1)}_\Gamma$ or W where one of the vertices is the global base-point, then $\pi(\Delta)$ or $\pi_W(\Delta)$ is a tripod in T, respectively. This tripod will have a central vertex, say $\hat{v}$. By examining the formula for the Gromov product, Definition 6.3.3, along with our distance formulas (6.4.2.1) and (6.5.0.4), we see that we can break down the calculation of a Gromov product in the whole space, $\tilde{X}^{(1)}_\Gamma$ or W, into two calculations: The distance between $o_{\hat{v}}$ or $\mathfrak{o}_{\hat{v}}$ and the global base-point of that space, o or $\mathfrak{o}_v$, respectively, and a Gromov product contained within a single vertex space. We formalise this below for W, but exactly the same proof also holds for $\tilde{X}^{(1)}_\Gamma$.

Lemma 6.6.2. *Fix $w, w' \in (W, d_{W,L})$, and suppose $w \in W_v$ and $w' \in W_{v'}$. Let $\hat{v}$ be the central vertex of the tripod with vertices $\pi_W(\mathfrak{o})$, v, and v' in T. Let $\hat{w}$ and $\hat{w}'$ be the closest-point projections of w and w' in $W_{\hat{v}}$ respectively. Then*

$$(w \mid w')^L_{\mathfrak{o}} = d_{W,L}(\mathfrak{o}, \mathfrak{o}_{\hat{v}}) + (\hat{w} \mid \hat{w}')^L_{\mathfrak{o}_{\hat{v}}},$$

Definition 6.6.3. In the situation hypothesised by Lemma 6.6.2, we say that w, w' *evaluates* in $W_{\hat{v}}$ by the points $\hat{w}$ and $\hat{w}'$.

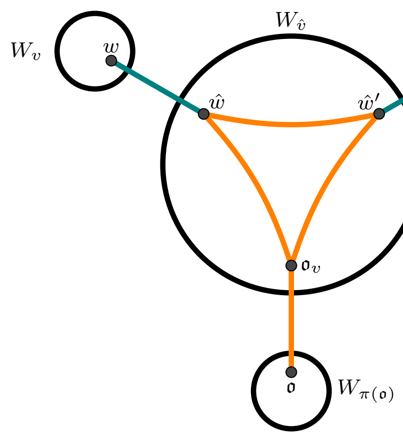

Figure 6.2: The Gromov product $(w, w')_{\mathfrak{o}}$ reduces to calculating $(\hat{w}, \hat{w}')_{\mathfrak{o}}$ (orange) as the contributions from $d_{W,L}(w, \hat{w})$ and $d_{W,L}(w', \hat{w}')$ (teal) end up cancelling.

Proof of Lemma 6.6.2. Recall,

$$(w \mid w')^L_{\mathfrak{o}} = \frac{1}{2}(d_{W,L}(\mathfrak{o}, w) + d_{W,L}(\mathfrak{o}, w') - d_{W,L}(w, w')).$$

Suppose $w \in W_v$ and $w' \in W_{v'}$, then v, v', and $\pi_W(\mathfrak{o})$ form the three points of a triangle in T. The sequences of vertices and edges in T needed to calculate $d_{W,L}(\mathfrak{o}, w)$, $d_{W,L}(\mathfrak{o}, w')$, and $d_{W,L}(w, w')$ via the distance formula (6.5.0.4) correspond to the sides of this triangle. However, T is a tree, so this triangle is a tripod, possibly degenerate. Regardless, this tripod has a unique 'central' vertex $\hat{v}$ that is the intersection of all three sides. Let $\hat{w}$ and $\hat{w}'$ be the closest-point projections of w and w' in $W_{\hat{v}}$ respectively. We can group terms in the expression, (6.5.0.4), of $d_{W,L}$ to rewrite distances between $\mathfrak{o}$, w, and w' in the following way:

$$\begin{aligned}
d_{W,L}(\mathfrak{o}, w) &= d_{W,L}(\mathfrak{o}, \mathfrak{o}_{\hat{v}}) + d_{W_{\hat{v}}}(\mathfrak{o}_{\hat{v}}, \hat{w}) + d_{W,L}(\hat{w}, w),\\
d_{W,L}(\mathfrak{o}, w') &= d_{W,L}(\mathfrak{o}, \mathfrak{o}_{\hat{v}}) + d_{W_{\hat{v}}}(\mathfrak{o}_{\hat{v}}, \hat{w}') + d_{W,L}(\hat{w}', w'),\\
d_{W,L}(w, w') &= d_{W,L}(w, \hat{w}) + d_{W_{\hat{v}}}(\hat{w}, \hat{w}') + d_{W,L}(\hat{w}', w').
\end{aligned}$$

Now, we can plug this into the definition of the Gromov product to see the effect of the tripod structure.

$$\begin{aligned}
(w \mid w')^L_{\mathfrak{o}} &= \frac{1}{2}(2d_{W,L}(\mathfrak{o}, \mathfrak{o}_{\hat{v}}) + d_{W_{\hat{v}}}(\mathfrak{o}_{\hat{v}}, \hat{w}) + d_{W_{\hat{v}}}(\mathfrak{o}_{\hat{v}}, \hat{w}') - d_{W_{\hat{v}}}(\hat{w}, \hat{w}')),\\
&= d_{W,L}(\mathfrak{o}, \mathfrak{o}_{\hat{v}}) + (\hat{w} \mid \hat{w}')^L_{\mathfrak{o}_{\hat{v}}}.
\end{aligned}$$

□

Similarly, the parallel result holds for $\tilde{X}^{(1)}_\Gamma$:

Lemma 6.6.4. *Fix $x, x' \in \tilde{X}^{(1)}_\Gamma$, and suppose $x \in X_v$ and $x' \in X_{v'}$. Let $\hat{v}$ be the central vertex of the tripod with vertices $\pi(o)$, v, and v' in T. Let $\hat{x}$ and $\hat{x}'$ be the closest-point projections of x and x' in $X_{\hat{v}}$ respectively. Then*

$$(x \mid x')_o = d_{\tilde{X}^{(1)}_\Gamma}(o, o_{\hat{v}}) + (\hat{x} \mid \hat{x}')_{o_{\hat{v}}}.$$

Immediately from the concept of evaluation, we can prove two lemmas that will be helpful when working with the chain metric given in Definition 6.3.7.

Lemma 6.6.5. *For any $w_1, w_2, w_3 \in (W, d_{W,L})$, either the pairs w_1, w_2 and w_2, w_3 both evaluate in the same W_v for some $v \in V(T)$, or $(w_1 \mid w_3)^L_{\mathfrak{o}}$ is equal to $(w_1 \mid w_2)^L_{\mathfrak{o}}$ or $(w_2 \mid w_3)^L_{\mathfrak{o}}$.*

Proof. Suppose $w_2 \in W_v$ and let $\pi_W(\mathfrak{o}) = v_0, v_1, \ldots, v_k = v$ be the unique geodesic path in T from $\pi_W(\mathfrak{o})$ to v. For $i, j \in \{1, 2, 3\}$, let $v_{i,j} \in V(T)$ be such that w_i, w_j evaluate in $W_{v_{i,j}}$. Then $v_{1,2}$ and $v_{2,3}$ feature in the list $(v_i)_{i=0}^k$, say as v_i and v_j, respectively. If w_1, w_2 and w_2, w_3 do not both evaluate in the same W_*, then $v_{1,2} \neq v_{2,3}$. Our set-up is symmetric in w_1 and w_3 so, without loss of generality, we may assume $d_T(\pi_W(\mathfrak{o}), v_{1,2}) < d_T(\pi_W(\mathfrak{o}), v_{2,3})$, and therefore

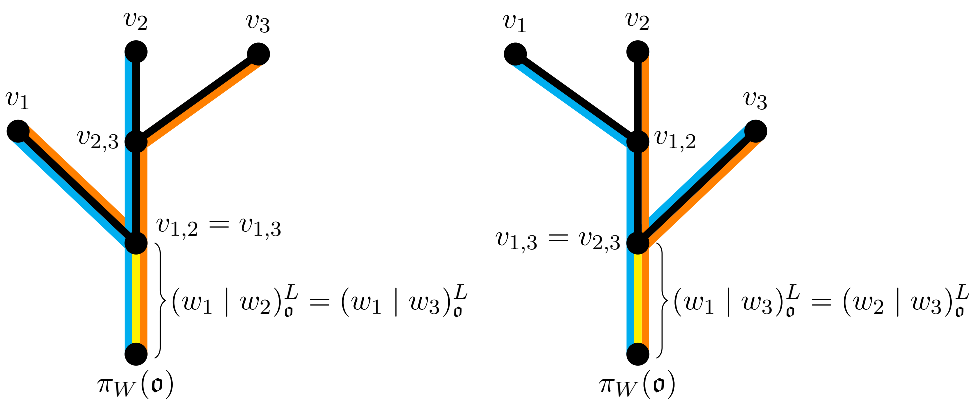

Figure 6.3: In both cases, the cyan (left colour) indicates the tripod $\pi_W(\mathfrak{o})v_1v_2$, the orange (right colour) indicates the tripod $\pi_W(\mathfrak{o})v_2v_3$, and the yellow (middle-coloured segment between $\pi_W(\mathfrak{o})$ and $v_{1,3}$) indicates the segment that corresponds to $(w_1 \mid w_3)^L_{\mathfrak{o}}$.

$i < j \le k$. Suppose $w_3 \in W_{v'}$ and let $\pi_W(\mathfrak{o}) = v'_0, \dots, v'_l = v'$ be the unique geodesic path in T from $\pi_W(\mathfrak{o})$ to v'. This path and the one from $\pi_W(\mathfrak{o})$ to v are the same up until $v_{2,3}$, because w_2, w_3 evaluate in $W_{v_{2,3}}$. Thus, $v_{2,3} = v_j = v'_j$ and $v_{1,2} = v_i = v'_i$, and $i < j \le l$. Therefore, $v_{1,2}$ is not equal to either v or v'. Also, as the geodesic path from $\pi_W(\mathfrak{o})$ to $\pi_W(w_1)$ diverges from the path between $\pi_W(\mathfrak{o})$ and v at $v_{1,2}$, this path must also diverge from the path between $\pi_W(\mathfrak{o})$ and v' at $v_{1,2}$ as the v and v' paths agree beyond $v_{1,2}$. This tells us that w_1, w_3 also evaluate in $W_{v_{1,2}}$.

Let e be the edge joining v_i to v_{i+1} in T, and let $\hat{w} = F(x)$, where $x \in \tilde{X}^{(1)}_\Gamma$ is the endpoint that is contained in X_{v_i} of the edge $\pi^{-1}(e)$. Now, e disconnects T into two components, one containing $\pi_W(\mathfrak{o})$, and the other containing both v and v', as these two vertices are joined by $v = v_k, \dots, v_j = v_{2,3} = v'_j, \dots, v'_l$, which does not pass through e. Hence, any path in T between $\pi_W(\mathfrak{o})$ and v or v' must pass through e. By examining the way we defined the metric $d_{W,L}$ on W, see (6.5.0.4), we see that the closest-point projection of both w_2 and w_3 in $W_{v_{1,2}}$ is $\hat{w}$. If the closest-point projection of w_1 to $W_{v_{1,2}}$ is $\hat{w}_1$, then Lemma 6.6.2 tells us that

$$(w_1 \mid w_3)^L_{\mathfrak{o}} = d_{W,L}(\mathfrak{o}, \mathfrak{o}_{v_{1,2}}) + (\hat{w}_1 \mid \hat{w})^L_{\mathfrak{o}_{v_{1,2}}} = (w_1 \mid w_2)^L_{\mathfrak{o}},$$

as desired.

We only obtained this equality because we assumed that $d_T(\pi_W(\mathfrak{o}), v_{1,2}) < d_T(\pi_W(\mathfrak{o}), v_{2,3})$. Instead, if we have assumed $d_T(\pi_W(\mathfrak{o}), v_{1,2}) > d_T(\pi_W(\mathfrak{o}), v_{2,3})$, then we would have obtained $(w_1 \mid w_3)^L_{\mathfrak{o}} = (w_2 \mid w_3)^L_{\mathfrak{o}}$. □

Lemma 6.6.6. *Let $w_0, w_1, \dots, w_k \in (W, d_{W,L})$, and suppose that v is a vertex of T such that w_{i-1}, w_i evaluate in W_v for all i. Let $\hat{w}_i$ be the closest-point projections of w_i to W_v for each i.*

Then,

$$(w_0 \mid w_k)^L_{\mathfrak{o}} \geq (\hat{w}_0 \mid \hat{w}_k)^L_{\mathfrak{o}}.$$

Proof. If $k = 0$ the statement is trivial, so instead suppose $k \geq 1$. Let $w_0 \in W_{v_{w_0}}$ and $w_k \in W_{v_{w_k}}$, and let $\pi_W(\mathfrak{o}) = v_0, v_1, \ldots, v_l = v_{w_0}$, $\pi_W(\mathfrak{o}) = v'_0, v'_1, \ldots, v'_m = v_{w_k}$ be geodesic paths from $\pi_W(\mathfrak{o})$ to v_{w_0} and v_{w_k} in T, respectively. As w_0, w_1 evaluate in W_v, v appears in the sequence (v_j). Similarly, as w_{k-1}, w_k evaluate in W_v, v appears in the sequence (v'_j). If $0 \leq i$ is maximal such that $v_i = v'_i$, then w_0, w_k evaluate in W_{v_i} as the maximality means v_i is the centre of the tripod $\pi_W(\mathfrak{o}), v_{w_0}, v_{w_k}$ in T. Suppose $v = v_j$ for some j, then $v = v'_j$ too, else there are two distinct geodesic paths from $\pi_W(\mathfrak{o})$ to v in T, contradicting that T is a tree. By the maximality of i, $j \leq i$. If $i = j$, then, as $\hat{w}_0$ and $\hat{w}_k$ are the closest-point-projections of w_0 and w_k, respectively, in the W_* that w_0, w_k evaluate in, using Lemma 6.6.2,

$$\begin{aligned}(w_0 \mid w_k)^L_{\mathfrak{o}} &= d_{W,L}(\mathfrak{o}, \mathfrak{o}_v) + (\hat{w}_0 \mid \hat{w}_k)^L_{\mathfrak{o}_v} \\ &= (\hat{w}_0 \mid \hat{w}_k)^L_{\mathfrak{o}}.\end{aligned}$$

Otherwise, $j < i$ and therefore $\hat{w}_0 = \hat{w}_k$. Therefore, $(\hat{w}_0 \mid \hat{w}_k)^L_{\mathfrak{o}} = d_{W,L}(\mathfrak{o}, \hat{w}_0) < d_{W,L}(\mathfrak{o}, \mathfrak{o}_{v_i}) \leq (w_0 \mid w_k)^L_{\mathfrak{o}}$. □

6.6.2 The hyperbolicity of W

In this subsection, we show W is Gromov hyperbolic independent of the choice of the edge length function L in (6.5.0.4).

From Theorem 6.3.10, $\mathrm{Con}(\mathfrak{G}_i)$ is hyperbolic for $i = 1, 2$. Let δ_i be the hyperbolicity constant of $\mathrm{Con}(\mathfrak{G}_i)$. Let

$$\delta_W \coloneqq \delta_1 \vee \delta_2, \tag{6.6.6.1}$$

and note that $\mathrm{Con}(\mathfrak{G}_i)$ is δ_W-hyperbolic. Intuitively, $(W, d_{W,L})$ should be δ_W-hyperbolic as we have just glued together copies $\mathrm{Con}(\mathfrak{G}_1)$ and $\mathrm{Con}(\mathfrak{G}_2)$ in a tree. In particular, $(W, d_{W,L})$ should be δ_W-hyperbolic independent of our choice for the function L because L only affects the tree component of W and trees are always 0-hyperbolic. Lemmas 6.6.5 and 6.6.6 allow us to make this intuition formal.

Lemma 6.6.7. $(W, d_{W,L})$ *is* δ_W*-hyperbolic.*

Proof. Recall that W is δ_W-hyperbolic if, for any $w_1, w_2, w_3 \in W$,

$$(w_1 \mid w_3)^L_{\mathfrak{o}} \geq (w_1 \mid w_2)^L_{\mathfrak{o}} \wedge (w_2 \mid w_3)^L_{\mathfrak{o}} - \delta_W.$$

For any $w_1, w_2, w_3 \in W$, suppose w_1, w_2 evaluate in $W_{v_{1,2}}$ and w_2, w_3 evaluate in $W_{v_{2,3}}$. Either $v_{1,2} = v_{2,3}$ or not. If $v_{1,2} = v_{2,3} = v$, then let $\hat{w}_i$ be the closest point projection of w_i in W_v.

Then lemmas 6.6.2 and 6.6.6 give us

$$\begin{aligned}(w_1 \mid w_3)^L_{\mathfrak{o}} &\geq (\hat{w}_1 \mid \hat{w}_3)^L_{\mathfrak{o}} \\ &= d_{W,L}(\mathfrak{o}, \mathfrak{o}_v) + (\hat{w}_1 \mid \hat{w}_3)^L_{\mathfrak{o}_v}\end{aligned}$$

because W_v is δ_W-hyperbolic from (6.6.6.1),

$$\begin{aligned}&\geq d_{W,L}(\mathfrak{o}, \mathfrak{o}_v) + (\hat{w}_1 \mid \hat{w}_2)^L_{\mathfrak{o}_v} \wedge (\hat{w}_2 \mid \hat{w}_3)^L_{\mathfrak{o}_v} - \delta_W \\ &= (w_1 \mid w_2)^L_{\mathfrak{o}} \wedge (w_2 \mid w_3)^L_{\mathfrak{o}} - \delta_W.\end{aligned}$$

Now, if $v_{1,2} \neq v_{2,3}$, then Lemma 6.6.5 tells us that $(w_1 \mid w_3)^L_{\mathfrak{o}}$ is equal to $(w_1 \mid w_2)^L_{\mathfrak{o}}$ or $(w_2 \mid w_3)^L_{\mathfrak{o}}$, and therefore that $(w_1 \mid w_3)^L_{\mathfrak{o}}$ is at least the minimum of $(w_1 \mid w_2)^L_{\mathfrak{o}}$ and $(w_2 \mid w_3)^L_{\mathfrak{o}}$. Noting that δ_W is non-negative completes the proof. □

The second case essentially only uses the tree structure on W, so it makes sense that we should be able to prove a kind of '0-hyperbolic' condition for this case. Each vertex in the tree is, however, only a δ_W-hyperbolic space, so overall we can only show W is δ_W-hyperbolic, and no better.

6.6.3 The metric on $\partial_\infty W$

Knowing that W is hyperbolic tells us W has a well-defined boundary that can be metrized by $d_{\epsilon,\mathfrak{o}}$, defined in Definition 6.3.7, for some $\epsilon > 0$, but it will be helpful to be quite careful about the metric we bestow to $\partial_\infty W$.

Again, Gromov products in W depend on our choice of L, so, for any $\xi, \xi' \in \partial_\infty (W, d_{W,L})$, we denote by $(\xi \mid \xi')^L_{\mathfrak{o}}$ the Gromov product of ξ and ξ' with respect to $\mathfrak{o}$.

Proposition 6.6.8. *The chain metric $d_{1,\mathfrak{o}}$ is visual on $\partial_\infty(W, d_{W,L})$. In particular, for any $\xi, \xi' \in \partial_\infty W$,*

$$\exp\left(-2(2\delta_W + 1 + 2\log 2)\right) \exp\left(-(\xi \mid \xi')^L_{\mathfrak{o}}\right) \leq d_{1,\mathfrak{o}}(\xi, \xi') \leq \exp\left(-(\xi \mid \xi')^L_{\mathfrak{o}}\right),$$

where δ_W is from (6.6.6.1).

It is important that we can set the parameter in $d_{*,\mathfrak{o}}$ to 1 and still get a visual metric. This is because, by Theorem 6.3.11, for $i = 1, 2$, $d_{1,\mathfrak{o}_i}$ is a visual metric on $\partial_\infty \operatorname{Con}(\mathfrak{G}_i)$, and $(\partial_\infty \operatorname{Con}(\mathfrak{G}_i), d_{1,\mathfrak{o}_i})$ is bi-Lipschitz to $\mathfrak{G}_i$. The common parameter of 1 will make it easier to control distance distortion of maps between these spaces, see Proposition 6.8.3.

To prove this proposition we need to be able to control arbitrary chains in $\partial_\infty W$. This involves calculating Gromov products of pairs of elements of $\partial_\infty W$. We approximate those Gromov products of boundary elements using Lemma 6.3.6. We then use the tree-of-spaces structure of W via the concept of evaluation, given in Lemma 6.6.2, and its consequences, lemmas 6.6.5 and 6.6.6, to simplify the chain to just its endpoints. We then return to the boundary with a final use of Lemma 6.3.6.

First, we present the following simple observation about sums of maximums, to make the proof of Proposition 6.6.8 more concise.

Lemma 6.6.9. *For any* $h_i \in [0, \infty)$, $0 \le i \le k$, $k \ge 1$,

$$\sum_{i=1}^{k} h_{i-1} \vee h_i \ge h_0 \vee h_k.$$

Proof. Observe that $h_{i-1} \vee h_i \ge 0$ for $1 \le i \le k$ as $h_i \ge 0$ for every i. Thus

$$\begin{aligned}\sum_{i=1}^{k} h_{i-1} \vee h_i &\ge (h_0 \vee h_1) \vee (h_{k-1} \vee h_k), \\ &\ge h_0 \vee h_k.\end{aligned}$$

□

Proof of Proposition 6.6.8. For any $\xi, \xi' \in \partial_\infty W$, by definition of $d_{1,\mathfrak{o}}$ in Definition 6.3.7,

$$d_{1,\mathfrak{o}}(\xi, \xi') \le \exp(-(\xi \mid \xi')^L_{\mathfrak{o}}). \tag{6.6.9.1}$$

The other inequality requires much more work: Let $\xi = \xi_0, \xi_1 \dots, \xi_n = \xi'$ be an arbitrary chain in $\partial_\infty W$. We will prove that

$$\exp\left(-2(2\delta_W + 1 + 2\log 2)\right) \exp\left(-(\xi \mid \xi')^L_{\mathfrak{o}}\right) \le \sum_{i=1}^{n} \exp\left(-(\xi_{i-1} \mid \xi_i)^L_{\mathfrak{o}}\right).$$

For each i, choose a representative sequence for ξ_i, say $\{{}^i w_j\} \in \xi_i$. For consecutive ξ_{i-1}, ξ_i there exists, by Lemma 6.3.6 and the definition of Gromov product on the boundary, N_{i-1} such that, for all $N \ge N_{i-1}$,

$$\begin{aligned}(\xi_{i-1} \mid \xi_i)^L_{\mathfrak{o}} - (2\delta_W + 1) &\le \inf\{({}^{i-1}w_j \mid {}^i w_k)^L_{\mathfrak{o}} \mid j, k \ge N_{i-1}\} \\ &\le ({}^{i-1}w_N \mid {}^i w_N)^L_{\mathfrak{o}} \\ &\le \sup\{({}^{i-1}w_j \mid {}^i w_k)^L_{\mathfrak{o}} \mid j, k \ge N_{i-1}\} \\ &\le (\xi_{i-1} \mid \xi_i)^L_{\mathfrak{o}} + (2\delta_W + 1).\end{aligned}$$

Let N_n be obtained similarly for the pair ξ_0, ξ_n. Let $N = \max\{N_i \mid 0 \le i \le n\}$. Now, we can approximate this chain by the representatives ${}^i w_N$ without significantly effecting the sum. Indeed,

$$\exp\left(-(2\delta_W + 1)\right) \sum_{i=1}^{n} \exp\left(-({}^{i-1}w_N \mid {}^i w_N)^L_{\mathfrak{o}}\right) \le \sum_{i=1}^{n} \exp\left(-(\xi_{i-1} \mid \xi_i)^L_{\mathfrak{o}}\right).$$

Suppose, there exists a consecutive triple ${}^{i-1}w_N, {}^i w_N, {}^{i+1}w_N$ such that ${}^{i-1}w_N, {}^i w_N$ and ${}^i w_N, {}^{i+1}w_N$ evaluate in different W_*. Then, by Lemma 6.6.5,

$$({}^{i-1}w_N \mid {}^{i+1}w_N)^L_{\mathfrak{o}} = ({}^{i-1}w_N \mid {}^i w_N)^L_{\mathfrak{o}} \text{ or } ({}^{i-1}w_N \mid {}^{i+1}w_N)^L_{\mathfrak{o}} = ({}^i w_N \mid {}^{i+1}w_N)^L_{\mathfrak{o}}.$$

Therefore,

$$\exp(-({}^{i-1}w_N \mid {}^{i+1}w_N)^L_{\mathfrak{o}}) \leq \exp(-({}^{i-1}w_N \mid {}^{i}w_N)^L_{\mathfrak{o}}) + \exp(-({}^{i}w_N \mid {}^{i+1}w_N)^L_{\mathfrak{o}}),$$

as both $\exp(-({}^{i-1}w_N \mid {}^{i}w_N)^L_{\mathfrak{o}})$ and $\exp(-({}^{i}w_N \mid {}^{i+1}w_N)^L_{\mathfrak{o}})$ are non-negative. Thus, we can cut ${}^{i}w_N$ out of the chain in the following way:

Let $w_j = {}^{j}w_N$ for $j \leq i-1$ and $w_j = {}^{j+1}w_N$ for $j \geq i$. Then

$$\sum_{i=1}^{n-1} \exp\left(-(w_{i-1} \mid w_i)^L_{\mathfrak{o}}\right) \leq \sum_{i=1}^{n} \exp\left(-({}^{i-1}w_N \mid {}^{i}w_N)^L_{\mathfrak{o}}\right).$$

This new chain is shorter and still has the same endpoints so, inductively, we can find a chain ${}^{0}w_N = w_0, w_1, \ldots, w_k = {}^{n}w_N$ in W such that

$$\sum_{i=1}^{k} \exp\left(-(w_{i-1} \mid w_i)^L_{\mathfrak{o}}\right) \leq \sum_{i=1}^{n} \exp\left(-({}^{i-1}w_N \mid {}^{i}w_N)^L_{\mathfrak{o}}\right),$$

and, for every $1 \leq i < k$, w_{i-1}, w_i and w_i, w_{i+1} evaluate in the same W_*. Say, for $1 \leq i \leq k$, that w_{i-1} and w_i evaluate in W_{v_i}. Then, as neighbouring pairs evaluate in the same W_*, we observe that $v_i = v_{i+1}$ for each $1 \leq i \leq k$. Therefore, inductively, we see that these v_i are all equal and, hence, there exists a single vertex v of T such that w_{i-1}, w_i evaluate in W_v for each $1 \leq i \leq k$. Let $\hat{w}_i = (z_i, h_i) \in W_v$ be the closest-point-projection of w_i to W_v, for each $0 \leq i \leq k$. Then, by Lemma 6.6.2,

$$(w_{i-1} \mid w_i)^L_{\mathfrak{o}} = d(\mathfrak{o}, \mathfrak{o}_v) + (\hat{w}_{i-1} \mid \hat{w}_i)^L_{\mathfrak{o}_v}$$

for each i. Suppose W_v is a copy of $\mathrm{Con}(\mathfrak{G}_m)$ for some $m = 1, 2$. Apply Lemma 6.3.12 to both $\mathrm{Con}(\mathfrak{G}_1)$ and $\mathrm{Con}(\mathfrak{G}_2)$, recalling that, from (6.4.6.5), we assumed $D(\mathfrak{G}_1) = D(\mathfrak{G}_2) = 1$, and that both spaces have base-points of the form $(*, 1)$, to see, independent of whether m equals 1 or 2,

$$-(\hat{w}_{i-1} \mid \hat{w}_i)^L_{\mathfrak{o}_v} \geq \log(d_{\mathfrak{G}_m}(z_{i-1}, z_i) + h_{i-1} \vee h_i) - 2\log 2,$$

and,

$$\log(d_{\mathfrak{G}_m}(z_0, z_k) + h_0 \vee h_k) \geq -(\hat{w}_0 \mid \hat{w}_k)^L_{\mathfrak{o}_v} - 2\log 2.$$

Hence,

$$\begin{aligned}\sum_{i=1}^{k} \exp(-(w_{i-1} \mid w_i)^L_{\mathfrak{o}}) &= \exp(-d(\mathfrak{o}, \mathfrak{o}_v)) \sum_{i=1}^{k} \exp(-(\hat{w}_{i-1} \mid \hat{w}_i)^L_{\mathfrak{o}_v}) \\ &\geq \exp(-(d(\mathfrak{o}, \mathfrak{o}_v) + 2\log 2)) \sum_{i=1}^{k} (d_{\mathfrak{G}_m}(z_{i-1}, z_i) + h_{i-1} \vee h_i)\end{aligned}$$

by repeated use of the triangle inequality,

$$\geq \exp(-(d(\mathfrak{o}, \mathfrak{o}_v) + 2\log 2)) \left(d_{\mathfrak{G}_m}(z_0, z_k) + \sum_{i=1}^{k} h_{i-1} \vee h_i \right)$$

by Lemma 6.6.9,

$$\begin{aligned}&\geq \exp(-(d(\mathfrak{o}, \mathfrak{o}_v) + 2\log 2))\,(d_{\mathfrak{G}_m}(z_0, z_k) + h_0 \vee h_k)\\ &\geq \exp(-(d(\mathfrak{o}, \mathfrak{o}_v) + 2\log 2))\exp(-(\hat{w}_0 \mid \hat{w}_k)^L_{\mathfrak{o}_v} - 2\log 2)\end{aligned}$$

because $\hat{w}_0$ and $\hat{w}_k$ are both contained in W_v, they evaluate in W_v, so

$$= \exp(-4\log 2)\exp(-(\hat{w}_0 \mid \hat{w}_k)^L_{\mathfrak{o}})$$

by Lemma 6.6.6,

$$\geq \exp(-4\log 2)\exp(-(w_0 \mid w_k)^L_{\mathfrak{o}}).$$

Combining all of the above,

$$\begin{aligned}\sum_{i=1}^{n} \exp\left(-(\xi_{i-1} \mid \xi_i)^L_{\mathfrak{o}}\right) &\geq \exp\left(-(2\delta_W + 1)\right) \sum_{i=1}^{n} \exp\left(-({}^{i-1}w_N \mid {}^{i}w_N)^L_{\mathfrak{o}}\right)\\ &\geq \exp\left(-(2\delta_W + 1)\right) \sum_{i=1}^{k} \exp\left(-(w_{i-1} \mid w_i)^L_{\mathfrak{o}}\right)\\ &\geq \exp\left(-(2\delta_W + 1 + 4\log 2)\right)\exp(-(w_0 \mid w_k)^L_{\mathfrak{o}})\\ &= \exp\left(-(2\delta_W + 1 + 4\log 2)\right)\exp(-({}^{0}w_N \mid {}^{n}w_N)^L_{\mathfrak{o}})\end{aligned}$$

recalling that $N \geq N_n$,

$$\begin{aligned}&\geq \exp\left(-(2\delta_W + 1 + 4\log 2)\right)\exp(-(\xi_0 \mid \xi_n)^L_{\mathfrak{o}} - (2\delta_W + 1))\\ &= \exp\left(-2(2\delta_W + 1 + 2\log 2)\right)\exp(-(\xi_0 \mid \xi_n)^L_{\mathfrak{o}}).\end{aligned}$$

□

6.7 Hölder equivalent boundary metrics

In this section, we restrict the class of length functions L that we allow in (6.5.0.4) so that we can eventually conclude, in Proposition 6.7.10, that $\partial_\infty \tilde{X}^{(1)}_\Gamma$ is Hölder equivalent to a subset of $\partial_\infty(W, d_{W,L})$.

Until now, the only property we needed to assume about L was that it takes strictly positive real values. For the purposes of this chapter we will look at the following heavily restricted family of functions:

Fix $l \geq 1$. Let e be an edge in T with endpoints v and v', and, without loss of generality, suppose $d_T(\pi(o), v) < d_T(\pi(o), v')$. Let $x \in \tilde{X}^{(1)}_\Gamma$ be the endpoint of $\pi^{-1}(e)$ contained in X_v, then define

$$L_l(e) := (l - 1)d_{W_v}(\mathfrak{o}_v, F(x)) + l. \tag{6.7.0.1}$$

Note that $L_l(e) \geq l \geq 1 > 0$ so d_{W,L_l} is a metric on W by Lemma 6.5.1. To make notation slightly nicer, denote by d^l_W the metric d_{W,L_l} on W.

In this section we will prove that, for any $l \geq 1$, F induces a bi-Hölder embedding of $(\partial_\infty \tilde{X}_\Gamma^{(1)}, d_{\epsilon_X,o})$ into $(\partial_\infty W, d_{1,\mathfrak{o}})$ when W is equipped with the metric d_W^l. We will do this by using that $d_{\epsilon_X,o}$ is visual on $\partial_\infty \tilde{X}_\Gamma^{(1)}$ and $d_{1,\mathfrak{o}}$ is visual on $\partial_\infty W$, and by controlling the distortion of Gromov products under the boundary map induced by F. We control this distortion by approximating Gromov products of boundary elements by Gromov products of elements of W, and by controlling the distortion of Gromov products of W under F.

Firstly, in Subsection 6.7.1 we use the simplification of Gromov products given in Subsection 6.6.1 to control the distortion of Gromov products under F. We then extend this control to the distortion of Gromov products under an extension of F to the boundary, in Subsection 6.7.2. Finally, in Subsection 6.7.3, we prove that this extension of F to the boundary is a bi-Hölder embedding.

6.7.1 Distortion of Gromov products under F

This subsection illustrates how we can control how F distorts Gromov products and gives linear bounds on the distortion.

Subsection 6.6.1 tells us that to understand arbitrary Gromov products based at the global base-point, we need to understand Gromov products contained within vertex spaces based at their local base-point, and distances from the global base-point to local base-points.

Firstly, we control how F distorts distances between a point and the local base-point of the vertex space containing that point. This control is independent of our choice of L because the distance calculations are within a single vertex space.

Lemma 6.7.1. *There exists $B = B(\lambda, \alpha, \beta, k') \geq 1$ such that for any vertex $v \in V(T)$ and $x \in X_v$, the following holds*

$$\epsilon_X d_{X_v}(o_v, x) - B \leq d_{W_v}(F(o_v), F(x)) \leq \epsilon_X d_{X_v}(o_v, x) + B,$$

where ϵ_X comes from Φ_i being a (ϵ_X, k')-rough similarity, see (6.4.6.3).

Proof. Let v be a vertex in T and $x \in X_v$, recall, from Section 6.5, $F(x) \in W_v$ and W_v is a copy of $\mathrm{Con}(\mathfrak{G}_i)$ for some $i = 1, 2$, so $F(x) = (f_i(\xi_x), h_x)$ for some $\xi_x \in \mathfrak{G}_i$. Recall that f_i, defined in (6.4.6.4), is a (λ, α, β)-bi-Hölder homeomorphism $\partial_\infty \tilde{X}_i^{(1)} \to \mathfrak{G}_i$.

$$\begin{aligned}
d_{W_v}(F(o_v), F(x)) &= d_{\mathrm{Con}(\mathfrak{G}_i)}((f_i(\xi_{o_i}), 1), (f_i(\xi_x), h_x)) \\
&= 2 \log \left(\frac{d_{\mathfrak{G}_i}(f_i(\xi_{o_i}), f_i(\xi_x)) + 1 \vee h_x}{\sqrt{1 h_x}} \right) \\
&\leq 2 \log \left(\frac{\lambda d_{\partial_\infty \tilde{X}_i^{(1)}}(\xi_{o_i}, \xi_x)^\beta + 1}{\sqrt{h_x}} \right) \\
&\leq 2 \log \left(\lambda \left(d_{\partial_\infty \tilde{X}_i^{(1)}}(\xi_{o_i}, \xi_x)^\beta + 1^\beta \right) \right) - 2 \log \left(\sqrt{h_x} \right)
\end{aligned}$$

using (6.3.2.1) with the knowledge that $\beta \le 1$,

$$\le 2\log(\lambda) + 2\log\left(2^{1-\beta}\left(d_{\partial_\infty \tilde{X}_i^{(1)}}(\xi_{o_i},\xi_x)+1\right)^{\beta}\right) - 2\log\left(\sqrt{h_x}\right)$$
$$\le 2\log\left(\lambda 2^{1-\beta}\right) + 2\beta\log\left(d_{\partial_\infty \tilde{X}_i^{(1)}}(\xi_{o_i},\xi_x)+1\right) - 2\log\left(\sqrt{h_x}\right)$$

noting that $d_{\partial_\infty \tilde{X}_i^{(1)}}(\xi_{o_i},\xi_x)+1 \ge 1$ thus $\log\left(d_{\partial_\infty \tilde{X}_i^{(1)}}(\xi_{o_i},\xi_x)+1\right) \ge 0$

$$\le 2\log\left(\lambda 2^{1-\beta}\right) + 2\log\left(d_{\partial_\infty \tilde{X}_i^{(1)}}(\xi_{o_i},\xi_x)+1\right) - 2\log\left(\sqrt{h_x}\right)$$
$$= d_{\mathrm{Con}\left(\partial_\infty \tilde{X}_i^{(1)}\right)}(\Phi_i \circ \iota_v(o_v), \Phi_i \circ \iota_v(x)) + 2\log\left(\lambda 2^{1-\beta}\right)$$

as Φ is a (ϵ_X, k')-rough similarity from (6.4.6.3), and ι_v is an isometry by Lemma 6.4.6,

$$\le \epsilon_X d_{X_v}(o_v, x) + 2\log\left(\lambda 2^{1-\beta}\right) + k'.$$

Similarly, but using (6.3.2.2) instead of (6.3.2.1),

$$d_{W_v}(F(o_v), F(x)) \ge \epsilon_X d_{X_v}(o_v, x) - 2\log\left(\lambda 2^{\alpha-1}\right) - k',$$

Taking $B = \max\left\{2\log\left(\lambda 2^{1-\beta}\right) + k', 2\log\left(\lambda 2^{\alpha-1}\right) + k'\right\}$ gives the desired result. □

For our special family of L given in (6.7.0.1), we can upgrade this local lemma to a global one by comparison of the distance formulas (6.4.2.1) and (6.5.0.4) on $\tilde{X}_\Gamma^{(1)}$ and W, respectively, to get:

Lemma 6.7.2. *There exists* $\tau = \max\{\epsilon_X, B+1, (2B+\epsilon_X)/\epsilon_X\} \ge 1$*, with* ϵ_X *from* (6.4.6.1)*, and* B *as in Lemma 6.7.1, such that, for any* $l \ge 1$ *and* $v \in V(T)$*,*

$$\frac{l}{\tau} d_{\tilde{X}_\Gamma^{(1)}}(o, o_v) \le d_W^l(\mathfrak{o}, \mathfrak{o}_v) \le l\tau d_{\tilde{X}_\Gamma^{(1)}}(o, o_v) + lB.$$

Proof. Let $\pi(o) = v_0, v_1, \ldots v_n = v$ be the geodesic path from $\pi(o)$ to v in T, and let $e_i = (v_{i-1}, v_i)$ for $1 \le i \le n$. For $0 \le i \le n$, let $z_{1,i}$ and $z_{2,i}$ be the closest-point projections of o and o_v in X_{v_i} respectively. Note that, by Definition 6.4.5, $z_{1,i} = o_{v_i}$ for each i. Also note that $z_{2,n} = o_v$, but o_v is also the closest-point projection of o in X_v, so $z_{1,n} = o_v = z_{2,n}$. Recall from Lemma 6.5.2, $F(z_{1,i})$ is the closest-point projection of $F(o) = \mathfrak{o}$ in W_{v_i} and $F(z_{2,i})$ is the closest-point projection of $F(o_v) = \mathfrak{o}_v$ in W_{v_i}. By the distance formula (6.5.0.4),

$$d_W^l(\mathfrak{o}, \mathfrak{o}_v) = \sum_{i=0}^{n} d_{W_{v_i}}(F(z_{1,i}), F(z_{2,i})) + \sum_{i=1}^{n} L_l(e_i).$$

Note that, for each $i \ge 1$, $d_T(\pi(o), v_{i-1}) < d_T(\pi(o), v_i)$, so by (6.7.0.1)

$$L_l(e_i) = (l-1) d_{W_{v_{i-1}}}(F(o_{v_{i-1}}), F(z_{2,i-1})) + l.$$

Using this, $z_{1,i} = o_{v_i}$, and $z_{1,n} = z_{2,n}$, we collect terms to find

$$d^l_W(\mathfrak{o}, \mathfrak{o}_v) = l \sum_{i=0}^{n} d_{W_{v_i}}(F(o_{v_i}), F(z_{2,i})) + nl, \tag{6.7.2.1}$$

by Lemma 6.7.1,

$$\leq l \sum_{i=0}^{n} (\epsilon_X d_{X_{v_i}}(o_{v_i}, z_{2,i}) + B) + nl,$$
$$\leq l\epsilon_X \sum_{i=0}^{n} (d_{X_{v_i}}(o_{v_i}, z_{2,i})) + lB(n+1) + ln,$$

as $\tau \geq \epsilon_X \vee (B+1)$,

$$\leq l\tau \left(\sum_{i=0}^{n} d_{X_{v_i}}(o_{v_i}, z_{2,i}) + n \right) + lB,$$

by the distance formula (6.4.2.1) for $\tilde{X}^{(1)}_\Gamma$,

$$= l\tau d_{\tilde{X}^{(1)}_\Gamma}(o, o_v) + lB.$$

For the other inequality,

$$d_{\tilde{X}^{(1)}_\Gamma}(o, o_v) = \sum_{i=0}^{n} d_{X_{v_i}}(o_{v_i}, z_{2,i}) + n,$$

by Lemma 6.7.1,

$$\leq \sum_{i=0}^{n} \frac{1}{\epsilon_X}(d_{W_{v_i}}(F(o_{v_i}), F(z_{2,i})) + B) + n,$$
$$\leq \frac{1}{\epsilon_X} \sum_{i=0}^{n} (d_{W_{v_i}}(F(o_{v_i}), F(z_{2,i}))) + \frac{1}{\epsilon_X} B(n+1) + n,$$

as $B \geq 1$,

$$\leq \frac{B + \epsilon_X}{\epsilon_X} \left(\sum_{i=0}^{n} (d_{W_{v_i}}(F(o_{v_i}), F(z_{2,i}))) + n \right) + \frac{B}{\epsilon_X},$$

using (6.7.2.1),

$$= \frac{B + \epsilon_X}{l\epsilon_X} d^l_W(\mathfrak{o}, \mathfrak{o}_v) + \frac{B}{\epsilon_X}. \tag{6.7.2.2}$$

To absorb the additive error into the multiplicative one, we observe that $d^l_W(\mathfrak{o}, \mathfrak{o}_v)$ is either 0 or at least l. First, recall that, from Definition 6.4.5, o_v is the closest-point projection of o in X_v. Further, recall that, from Lemma 6.5.2, $\mathfrak{o}_v$ is the closest-point projection of $\mathfrak{o}$ in W_v. By examination of the distance formulas (6.4.2.1) and (6.5.0.4), we see that $d_{\tilde{X}^{(1)}_\Gamma}(o, o_v) = 0$ if and only if $d^l_W(\mathfrak{o}, \mathfrak{o}_v) = 0$. In this case, the lower bound given in the statement of the lemma holds trivially.

If $d^l_W(\mathfrak{o}, \mathfrak{o}_v) \neq 0$, then there is a non-trivial contribution from L_l in the distance formula (6.5.0.4). As $L_l(e) \geq l$ for any edge e in T, this non-trivial contribution means $d^l_W(\mathfrak{o}, \mathfrak{o}_v) \geq l$. By (6.7.2.2),

$$\begin{aligned} d_{\tilde{X}^{(1)}_\Gamma}(o, o_v) &\leq \frac{B + \epsilon_X}{l\epsilon_X} d^l_W(\mathfrak{o}, \mathfrak{o}_v) + \frac{B}{\epsilon_X}, \\ &\leq \left(\frac{B + \epsilon_X}{l\epsilon_X} + \frac{B}{l\epsilon_X} \right) d^l_W(\mathfrak{o}, \mathfrak{o}_v), \end{aligned}$$

as $\tau \geq (2B + \epsilon_X)/\epsilon_X$,

$$\leq \frac{\tau}{l} d^l_W(\mathfrak{o}, \mathfrak{o}_v).$$

Combining these inequalities, we get the desired result. □

Now we have control on the distortion of the distance from the global base-point to local base-points under F, we move onto controlling the distortion of Gromov products contained within vertex spaces.

We will break down the problem of controlling F's distortion of local Gromov products into controlling distortion of Gromov products by its factors: $\bar{f}_i$, Φ_i, and ι_v. The map ι_v is an isometry, by Lemma 6.4.6, so has no distortion, Lemma 6.7.3 will give us control on Φ_i, and Lemma 6.7.4 will give us control on $\bar{f}_i$.

Lemma 6.7.3. *Suppose X and Y are metric spaces and $g\colon X \to Y$ is a (λ, C)-rough similarity of X into Y. Then for all $x, y, w \in X$*

$$\lambda(x \mid y)_w - \frac{3C}{2} \leq (g(x) \mid g(y))_{f(w)} \leq \lambda(x \mid y)_w + \frac{3C}{2}.$$

Proof. Recall, by definition,

$$(g(x) \mid g(y))_w = \frac{1}{2} \left(d_Y(g(x), g(w)) + d_Y(g(y), g(w)) - d_Y(g(x), g(y)) \right),$$

using that g is a rough similarity,

$$\begin{aligned} &\leq \frac{1}{2} \left(\lambda d_X(x, w) + C + \lambda d_X(y, w) + C - (\lambda d_X(x, y) - C) \right), \\ &\leq \lambda \frac{1}{2} \left(d_X(x, w) + d_X(y, w) - d_X(x, y) \right) + \frac{3C}{2}, \\ &\leq \lambda(x \mid y)_w + \frac{3C}{2}. \end{aligned}$$

Similarly,

$$(g(x) \mid g(y))_w \geq \lambda(x \mid y)_w - \frac{3C}{2}.$$ □

Lemma 6.7.4. *Let X, Y be bounded metric spaces with diameter 0 or 1, and $f\colon X \to Y$ be a (λ, α, β)-bi-Hölder embedding. Let $\bar{f}\colon \mathrm{Con}(X) \to \mathrm{Con}(Y)$ be defined by $(x, h) \mapsto (f(x), h)$ for each $x \in X$. For any $z_1 = (x_1, h_1), z_2 = (x_2, h_2) \in \mathrm{Con}(X)$ and $w = (x_0, 1)$ base-point. There exists $K = K(\lambda, \alpha, \beta) \geq 0$ such that*

$$\beta(z_1 \mid z_2)_w - K \leq (\bar{f}(z_1) \mid \bar{f}(z_2))_{\bar{f}(w)} \leq \alpha(z_1 \mid z_2)_w + K.$$

Proof. From Lemma 6.3.12, for any $z_1, z_2 \in \mathrm{Con}(X)$,

$$\begin{aligned}(\bar{f}(z_1) \mid \bar{f}(z_2))_{\bar{f}(w)} &\leq -\log\left(d(f(x_1), f(x_2)) + h_1 \vee h_2\right) + 2\log 2\\ &\leq -\log\left(\frac{1}{\lambda} d(x_1, x_2)^{\alpha} + h_1 \vee h_2\right) + 2\log 2\end{aligned}$$

because f is (λ, α, β)-bi-Hölder,

$$\leq -\log\left(\frac{1}{\lambda}\left(d(x_1, x_2)^{\alpha} + (h_1 \vee h_2)^{\alpha}\right)\right) + 2\log 2$$

because $h_1 \vee h_2 \leq 1$ and $\alpha \geq 1$,

$$\leq -\log\left(\frac{1}{2^{\alpha-1}\lambda}\left(d(x_1, x_2) + h_1 \vee h_2\right)^{\alpha}\right) + 2\log 2$$

by (6.3.2.2), noting that $-\log(\cdot)$ is a decreasing function,

$$\begin{aligned}&= -\alpha \log\left(d(x_1, x_2) + h_1 \vee h_2\right) + \log(2^{\alpha-1}\lambda) + 2\log 2\\ &\leq \alpha(z_1 \mid z_2)_w + \log(2^{\alpha-1}\lambda) + 2(1+\alpha)\log 2\end{aligned}$$

by Lemma 6.3.12 again. Similarly,

$$(\bar{f}(z_1) \mid \bar{f}(z_2))_{\bar{f}(w)} \geq \beta(z_1 \mid z_2)_w - \log(2^{1-\beta}\lambda) - 2(1+\beta)\log 2.$$

Thus we can set $K = \log(\lambda) + 2(1+\alpha)\log 2 + \max\{1-\beta, \alpha-1\}\log 2$ to get the desired inequalities, as $(1+\alpha) \geq (1+\beta)$. □

We can now upgrade to control on arbitrary Gromov products based at the global base-point.

Now that we have restricted to the family of metrics d_W^l from (6.7.0.1), we still wish to keep it clear which metric a Gromov product is being evaluated on, but also keep our notation from getting cumbersome. Therefore, we introduce the following notation.

Notation 6.7.5. For any $w, w', z \in \left(W, d_W^l\right)$ we shall denote by $(w \mid w')_z^l$ the Gromov product of w and w' with respect to z. Similarly, for any $\xi, \xi' \in \partial_\infty \left(W, d_W^l\right)$ we denote by $(\xi \mid \xi')_{\mathfrak{o}}^l$ the Gromov product of ξ and ξ' with respect to $\mathfrak{o}$.

Proposition 6.7.6. *There exists $a \geq 1$, $b \in (0, 1]$, and $K' \geq 0$ such that, for any $l \geq 1$, if W is equipped with d_W^l, then for any $x, x' \in \tilde{X}_\Gamma^{(1)}$,*

$$b(x \mid x')_o - K' \leq (F(x) \mid F(x'))_{\mathfrak{o}}^l \leq la(x \mid x')_o + lK'.$$

Proof. Suppose $x \in X_v$ and $x' \in X_{v'}$. Let $\hat{v}$ be the central vertex of the tripod with vertices $\pi(o), v, v'$ in T. Let $\hat{x}$ and $\hat{x}'$ be the closest-point projections of x and x' in $\hat{v}$ respectively. Let $w = F(x)$ and $w' = F(x')$, and let $\hat{w}$ and $\hat{w}'$ be the closest-point projections of w and w' in $W_{\hat{v}}$, respectively. Then, by Lemma 6.5.2, $F(\hat{x}) = \hat{w}$ and $F(\hat{x}') = \hat{w}'$. Recall that, for any $v \in V(T)$, F restricts to F_v on X_v, and F_v factors as $\bar{f}_i \circ \Phi_i \circ \iota_v$ for appropriate $i \in \{1, 2\}$, see (6.4.6.7). Now,

$$\begin{aligned}(F(x) \mid F(x'))^l_{\mathfrak{o}} &= d^l_W(\mathfrak{o}, \mathfrak{o}_{\hat{v}}) + (F(\hat{x}) \mid F(\hat{x}'))^l_{\mathfrak{o}_{\hat{v}}} && \text{by Lemma 6.6.2,}\\ &= d^l_W(\mathfrak{o}, \mathfrak{o}_{\hat{v}}) + (F_{\hat{v}}(\hat{x}) \mid F_{\hat{v}}(\hat{x}'))^l_{F_{\hat{v}}(o_{\hat{v}})}\end{aligned}$$

supposing $W_{\hat{v}}$ is a copy of $\mathrm{Con}(\mathfrak{G}_i)$, so by Lemmas 6.7.2 and 6.7.4 we see

$$\begin{aligned}&\leq l\tau d_{\tilde{X}^{(1)}_\Gamma}(o, o_{\hat{v}}) + lB\\ &\quad + \alpha(\Phi_i \circ \iota_{\hat{v}}(\hat{x}) \mid \Phi_i \circ \iota_{\hat{v}}(\hat{x}'))_{(\xi_{o_i}, 1)} + K,\end{aligned}$$

by Lemma 6.7.3 using that $\iota_{\hat{v}}$ is an isometry and Φ_i a (ϵ_X, k')-rough similarity,

$$\begin{aligned}&\leq l\tau d_{\tilde{X}^{(1)}_\Gamma}(o, o_{\hat{v}}) + lB + \alpha\epsilon_X(\hat{x} \mid \hat{x}')_{o_{\hat{v}}} + \alpha\frac{3k'}{2} + K,\\ &\leq la(x \mid x')_o + lB + \frac{3\alpha k'}{2} + K && \text{by Lemma 6.6.4,}\end{aligned}$$

where $a = \max\{\tau, \alpha\epsilon_X\}$. Similarly,

$$\begin{aligned}(F(x) \mid F(x'))^l_{\mathfrak{o}} &= d^l_W(\mathfrak{o}, \mathfrak{o}_{\hat{v}}) + (F(\hat{x}) \mid F(\hat{x}'))^l_{F(o_{\hat{v}})},\\ &\geq \frac{l}{\tau} d_{\tilde{X}^{(1)}_\Gamma}(o, o_{\hat{v}}) + \beta(\Phi_i \circ \iota_{\hat{v}}(\hat{x}) \mid \Phi_i \circ \iota_{\hat{v}}(\hat{x}'))_{(\xi_{o_i}, 1)} - K,\\ &\geq \frac{1}{\tau} d_{\tilde{X}^{(1)}_\Gamma}(o, o_{\hat{v}}) + \beta\epsilon_X(\hat{x} \mid \hat{x}')_{o_{\hat{v}}} - \beta\frac{3k'}{2} - K,\\ &\geq b(x \mid x')_o - \frac{3\beta k'}{2} - K,\end{aligned}$$

where $b = \min\{1/\tau, \beta\epsilon_X\}$. Letting $K' = B + 3\alpha k'/2 + K$ and noting $l \geq 1$ gives us the desired result. □

6.7.2 Extending to the boundary

In this subsection we give an extension of F to the boundary, show that it is well defined and injective, and prove an extension of Proposition 6.7.6 to the boundary.

A natural extension of F is $F_\infty \colon \partial_\infty \tilde{X}^{(1)}_\Gamma \to \partial_\infty W$ defined by

$$F_\infty(\{x_i\}) \coloneqq \{F(x_i)\},$$

for any $\{x_i\} \in \partial_\infty \tilde{X}^{(1)}_\Gamma$. The following lemma proves that F_∞ is well-defined when W is equipped with the metric d^l_W.

Lemma 6.7.7. *If $\{x_i\}, \{y_i\} \subset \tilde{X}_\Gamma^{(1)}$ are equivalent sequences that converge at infinity, then $\{F(x_i)\}$ and $\{F(y_i)\}$ are also equivalent sequences that converge at infinity in (W, d_W^l).*

Proof. For any i, j, by Proposition 6.7.6,

$$(F(x_i) \mid F(x_j))_{\mathfrak{o}}^l \geq b(x_i \mid x_j)_o - K'.$$

Hence,

$$\liminf_{i,j\to\infty}(F(x_i) \mid F(x_j))_{\mathfrak{o}}^l = \infty,$$

as $\liminf_{i,j\to\infty}(x_i \mid x_j)_o = \infty$ and $b > 0$. Therefore, $\{F(x_i)\} \subset (W, d_W^l)$, and similarly $\{F(y_j)\} \subset (W, d_W^l)$ converge at infinity. Likewise, for any i, j

$$(F(x_i) \mid F(y_j))_{\mathfrak{o}}^l \geq b(x_i \mid y_j)_o - K'.$$

Hence,

$$\liminf_{i,j\to\infty}(F(x_i) \mid F(y_j))_{\mathfrak{o}}^l = \infty,$$

as $\liminf_{i,j\to\infty}(x_i \mid y_j)_o = \infty$ and $b > 0$. □

Similarly, our control from Proposition 6.7.6 tells us that F_∞ is injective when W is equipped with the metric d_W^l.

Lemma 6.7.8. *$F_\infty \colon \partial_\infty \tilde{X}_\Gamma^{(1)} \to \partial_\infty(W, d_W^l)$ is injective.*

Proof. Suppose $\{x_i\}, \{y_i\} \in \partial_\infty \tilde{X}_\Gamma^{(1)}$ are such that $\{F(x_i)\}$ and $\{F(y_i)\}$ are equivalent. Thus, by definition

$$\liminf_{i,j\to\infty}(F(x_i) \mid F(y_j))_{\mathfrak{o}}^l = \infty.$$

By Proposition 6.7.6, for any i, j

$$\frac{(F(x_i) \mid F(y_j))_{\mathfrak{o}}^l - lK'}{la} \leq (x_i \mid y_j)_o.$$

Recalling that $a > 0$ and $l \geq 1$, we now see that

$$\liminf_{i,j\to\infty}(x_i \mid y_j)_o = \infty,$$

too, allowing us to conclude that $\{x_i\}$ and $\{y_i\}$ are also equivalent. □

We can extend Proposition 6.7.6 to F_∞ and the boundary when W is equipped with the metric d_W^l.

Lemma 6.7.9. *Let a, b, and K' be as in Proposition 6.7.6, and δ_W from* (6.6.6.1)*. Recall, from Notation 6.4.2, $\tilde{X}_\Gamma^{(1)}$ is δ_X-hyperbolic. Define $K'' := K' + 2\delta_W + 2a\delta_X$. If W is equipped with the metric d_W^l then, for any $\gamma, \gamma' \in \partial_\infty \tilde{X}_\Gamma^{(1)}$,*

$$b(\gamma \mid \gamma')_o - K'' \leq (F_\infty(\gamma) \mid F_\infty(\gamma'))_{\mathfrak{o}}^l \leq la(\gamma \mid \gamma')_o + lK''.$$

Proof. Let $\gamma, \gamma' \in \partial_\infty \tilde{X}_\Gamma^{(1)}$ and let $\{x_i\} \in \gamma$, $\{x_i'\} \in \gamma'$ be representative sequences. Combining Lemma 6.3.6 and Proposition 6.7.6, we see that

$$\begin{aligned}(F_\infty(\gamma) \mid F_\infty(\gamma'))^l_{\mathfrak{o}} &\le \liminf_{i,j\to\infty} (F(x_i) \mid F(x_j'))^l_{\mathfrak{o}} + 2\delta_W \\ &\le la \liminf_{i,j\to\infty} (x_i \mid x_j')_o + (lK' + 2\delta_W) \\ &\le la(\gamma \mid \gamma')_o + (lK' + 2\delta_W + 2la\delta_X)\end{aligned}$$

recalling that $l \ge 1$ and $\delta_W \ge 0$,

$$\le la(\gamma \mid \gamma')_o + lK''.$$

Similarly,

$$\begin{aligned}(F_\infty(\gamma) \mid F_\infty(\gamma'))^l_{\mathfrak{o}} &\ge \liminf_{i,j\to\infty} (F(x_i) \mid F(x_j'))^l_{\mathfrak{o}} - 2\delta_W \\ &\ge b \liminf_{i,j\to\infty} (x_i \mid x_j')_o - (K' + 2\delta_W) \\ &\ge b(\gamma \mid \gamma')_o - (K' + 2\delta_W + 2b\delta_X)\end{aligned}$$

recalling that $b \le 1 \le a$ and $\delta_X \ge 0$,

$$\ge b(\gamma \mid \gamma')_o - K''.$$

□

6.7.3 F_∞ is a bi-Hölder embedding

Finally, in this subsection, we conclude that $\left(\partial_\infty \tilde{X}_\Gamma^{(1)}, d_{\epsilon_X,o}\right)$ is Hölder equivalent via F_∞ to its image in $\left(\partial_\infty (W, d_W^l), d_{1,\mathfrak{o}}\right)$, for any $l \ge 1$.

Proposition 6.7.10. *Let a, b, K' be as in Proposition 6.7.6, and δ_W be from* (6.6.6.1)*. Recall, from Notation 6.4.2, $\tilde{X}_{\Gamma^{(1)}}$ is δ_X-hyperbolic, and, from Subsection 6.4.4, $d_{\epsilon_X,o}$ is a visual metric on $\tilde{X}_\Gamma^{(1)}$. Let $\nu \ge 1$ be such that $d_{\epsilon_X,o}$ is a ν-visual metric on $\tilde{X}_\Gamma^{(1)}$, see* (6.3.7.1)*. Then, for any $l \ge 1$, $F_\infty \colon (\partial_\infty \tilde{X}_\Gamma^{(1)}, d_{\epsilon_X,o}) \to (\partial_\infty (W, d_W^l), d_{1,\mathfrak{o}})$ is a $(\lambda', la/\epsilon_X, b/\epsilon_X)$-bi-Hölder embedding, where $\lambda' = \lambda'(a, b, K', \delta_X, \delta_W, \epsilon_X, \nu, l)$.*

The metric Δ_l described in Section 6.2 is the pull back, through F_∞, of the metric $d_{1,\mathfrak{o}}$ restricted to $F_\infty(\partial_\infty \tilde{X}_\Gamma^{(1)}) \subseteq \partial_\infty (W, d_W^l)$.

Proof. Lemma 6.7.8 gives that F_∞ is injective. We now illustrate how to combine Lemma 6.7.9, Proposition 6.6.8, and the fact that $\left(\partial_\infty \tilde{X}_\Gamma^{(1)}, d_{\epsilon_X,o}\right)$ is visual, see (6.4.6.1), to derive the desired result.

Fix $\gamma_1, \gamma_2 \in \partial_\infty \tilde{X}_\Gamma^{(1)}$. Proposition 6.6.8 tells us

$$\exp\left(-2(\delta_W + 1 + 2\log 2)\right) \exp(-(F_\infty(\gamma_1) \mid F_\infty(\gamma_2))^l_{\mathfrak{o}}) \le d_{1,\mathfrak{o}}(F_\infty(\gamma_1), F_\infty(\gamma_2)),$$

and

$$d_{1,\mathfrak{o}}(F_\infty(\gamma_1), F_\infty(\gamma_2)) \le \exp(-(F_\infty(\gamma_1) \mid F_\infty(\gamma_2))^l_{\mathfrak{o}}).$$

Combine this with Lemma 6.7.9 to see

$$\frac{1}{\lambda_1} \exp(-(\gamma_1 \mid \gamma_2)_o)^{la} \le d_{1,\mathfrak{o}}(F_\infty(\gamma_1), F_\infty(\gamma_2)) \le \lambda_2 \exp(-(\gamma_1 \mid \gamma_2)_o)^b,$$

where

$$\lambda_1 = \exp(2\delta_W + 2 + 4\log 2 + lK'')$$

and

$$\lambda_2 = \exp(K'').$$

Finally, using $(\partial_\infty \tilde{X}^{(1)}_\Gamma, d_{\epsilon_X,o})$ is ν-visual,

$$\frac{1}{\nu^{\frac{la}{\epsilon_X}}\lambda_1} d_{\epsilon_X,o}(\gamma_1,\gamma_2))^{\frac{la}{\epsilon_X}} \le d_{1,\mathfrak{o}}(F_\infty(\gamma_1), F_\infty(\gamma_2)) \le \nu^{\frac{b}{\epsilon_X}} \lambda_2 d_{\epsilon_X,o}(\gamma_1,\gamma_2))^{\frac{b}{\epsilon_X}}.$$

Let

$$\lambda' = \max\left\{\nu^{\frac{la}{\epsilon_X}}\lambda_1, \nu^{\frac{b}{\epsilon_X}}\lambda_2\right\},$$

then F_∞ is a $(\lambda', la/\epsilon_X, b/\epsilon_X)$-bi-Hölder embedding. □

6.8 Covering the boundary

In this section we describe a family of covers for the boundary of $\partial_\infty \tilde{X}^{(1)}_\Gamma$, using the tree-of-spaces structure, that will allow us to bound the Hausdorff dimension of the image of F. Each cover will be comprised of two types of subsets; the first will be discussed in subsection 6.8.1, and the second in subsection 6.8.2

First, we introduce ‘stabilization’ as a way of finding subsets of the boundary of a hyperbolic space via subsets of the hyperbolic space itself.

Definition 6.8.1. Let X be a Gromov hyperbolic space, and let $S \subseteq X$. A sequence $\{x_n\} \subset X$ is said to *stabilize* in S if there exists $N \in \mathbb{N}$ such that $x_n \in S$ for all $n \ge N$. Similarly, a geodesic ray $\gamma\colon [0,\infty) \to X$ is said to *stabilize* in S if there exists $T \ge 0$ such that $\gamma(t) \in S$ for all $t \ge T$.

Stabilization gives us a way of finding subsets of the boundary of X: Define $\partial_\infty S \subseteq \partial_\infty X$ as the collection of $\xi \in \partial_\infty X$ such that ξ has a representative sequence $\{x_n\}$ that stabilizes in S.

Thinking about geodesic rays will be useful later in Subsection 6.8.3 when we cover $F_\infty\left(\partial_\infty \tilde{X}^{(1)}_\Gamma\right)$. For now, we shall consider the boundaries of vertex spaces, $\partial_\infty X_v$ and $\partial_\infty W_v$.

6.8.1 Boundaries of vertex spaces

In this subsection we shall examine, for $v \in V(T)$, $\partial_\infty X_v \subset \partial_\infty \tilde{X}^{(1)}_\Gamma$, $\partial_\infty W_v \subset \partial_\infty W$, and the relationship between them. In particular, we shall formalise two pieces of intuition: Firstly, as F maps X_v into W_v, so should F_∞ map $\partial_\infty X_v$ into $\partial_\infty W_v$. Secondly, as each W_v is a copy of $\mathrm{Con}(\mathfrak{G}_i)$ for some $i = 1, 2$, $\partial_\infty W_v$ should be a (bi-Lipschitz) copy of $\partial_\infty \mathrm{Con}(\mathfrak{G}_i)$.

The first is direct from the definition of stabilization.

Lemma 6.8.2. $F_\infty(\partial_\infty X_v) \subseteq \partial_\infty W_v$

Proof. Pick any $\xi \in \partial_\infty X_v$ and let $\{x_i\}$ be a representative sequence that stabilizes in X_v. Thus, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, $x_n \in X_v$. The function F maps X_v to W_v, so $F(x_n) \in W_v$. Thus $F_\infty(\xi) = \{F(x_i)\}$ stabilizes in W_v and, thus, is an element of $\partial_\infty W_v$. ☐

Proving $\partial_\infty W_v$ is bi-Lipschitz to $\partial_\infty \mathrm{Con}(\mathfrak{G}_i)$ is not much harder, but requires a little set-up:

We defined W_v to be a copy of $\mathrm{Con}(\mathfrak{G}_i)$ so to make it clear whether an object is an element of W_v or of $\mathrm{Con}(\mathfrak{G}_i)$, let $\mathfrak{i}_v \colon \mathrm{Con}(\mathfrak{G}_i) \to W_v$ be the identity map. We define a map $\tilde{\mathfrak{i}}_v \colon \partial_\infty \mathrm{Con}(\mathfrak{G}_i) \to \partial_\infty W_v$ as follows: for any $\xi \in \partial_\infty \mathrm{Con}(\mathfrak{G}_i)$, take a representative sequence $\{w_n\} \subset \mathrm{Con}(\mathfrak{G}_i)$ and define $\mathfrak{i}_v(\xi)$ to be the equivalence class of $\partial_\infty W_v$ containing $\{\mathfrak{i}_v(w_n)\}$.

Proposition 6.8.3. *$\tilde{\mathfrak{i}}_v$ is a well-defined bi-Lipschitz homeomorphism. Further, $\partial_\infty W_v \subset \partial_\infty(W, d_{W,L})$ is bi-Lipschitz equivalent to $\mathfrak{G}_i$ for some $i \in \{1, 2\}$.*

Proof. For any $w, w' \in \mathrm{Con}(\mathfrak{G}_i)$, observe that $\mathfrak{i}_v(w), \mathfrak{i}_v(w') \in W_v \subset W$ evaluate in W_v. Thus, by Lemma 6.6.2,

$$(\mathfrak{i}_v(w) \mid \mathfrak{i}_v(w'))^L_{\mathfrak{o}} = d_{W,L}(\mathfrak{o}, \mathfrak{o}_v) + (\mathfrak{i}_v(w) \mid \mathfrak{i}_v(w'))^L_{\mathfrak{o}_v}. \tag{6.8.3.1}$$

Further, by examination of the distance formula (6.5.0.4) restricted to W_v, we see $\mathfrak{i}_v$ is an isometry, and therefore,

$$(\mathfrak{i}_v(w) \mid \mathfrak{i}_v(w'))^L_{\mathfrak{o}_v} = (w \mid w')_{\mathfrak{o}_i}. \tag{6.8.3.2}$$

Thus, $(\mathfrak{i}_v(w) \mid \mathfrak{i}_v(w'))^L_{\mathfrak{o}}$ differs from $(w \mid w')_{\mathfrak{o}_i}$ by a constant, namely $d_{W,L}(\mathfrak{o}, \mathfrak{o}_v)$. Hence, $\{w_n\} \subset \mathrm{Con}(\mathfrak{G}_i)$ converges to infinity if and only if $\{\mathfrak{i}_v(w_n)\} \subset W_v$ does. Similarly, $\{w_n\}, \{w'_n\} \subset \mathrm{Con}(\mathfrak{G}_i)$ are equivalent if and only if $\{\mathfrak{i}_v(w_n)\}, \{\mathfrak{i}_v(w'_n)\}$ are equivalent. Hence, $\tilde{\mathfrak{i}}_v$ is well-defined and injective.

For any $\xi \in \partial_\infty W_v$, by Definition 6.8.1, there exists a representative sequence $\{w_n\} \subset W$ that stabilizes in W_v. Cutting finitely many terms off the start of this sequence gives an equivalent sequence that also converges to infinity, thus, without loss of generality, we may assume that $\{w_n\} \subset W_v$. Further, as mentioned above, $\{w_n\} \subset W_v$ converges to infinity implies $\{\mathfrak{i}_v^{-1}(w_n)\} \subset \mathrm{Con}(\mathfrak{G}_i)$ converges to infinity. Therefore, $\tilde{\mathfrak{i}}_v$ is surjective.

Now, $\tilde{\mathfrak{i}}_v$ is well-defined and a bijection. The metric control comes from a combination of equations (6.8.3.1) and (6.8.3.2), Lemma 6.3.6, and that $\left(\partial_\infty \mathrm{Con}(\mathfrak{G}_i), d_{1,(\xi_{o_i},1)}\right)$ (as defined in Definition 6.3.7) is visual by Theorem 6.3.11 and $(\partial_\infty(W, d_{W,L}), d_{1,\mathfrak{o}})$ is visual by Proposition 6.6.8. The bi-Lipschitz equivalence to $\mathfrak{G}_i$ also comes from Theorem 6.3.11, which says $\left(\partial_\infty \operatorname{Con}(\mathfrak{G}_i), d_{1,(\xi_{o_i},1)}\right)$ is bi-Lipschitz to $\mathfrak{G}_i$ regardless of base-point. □

Essentially the same proof works for extending $\iota_v \colon X_v \to \tilde{X}_i^{(1)}$, from Lemma 6.4.6, to the boundary:

Proposition 6.8.4. *For any $v \in V(T)$, there exists $i \in \{1,2\}$ such that, $(\partial_\infty X_v, d_{\epsilon_X,o})$ is bi-Lipschitz equivalent to $\left(\partial_\infty \tilde{X}_i^{(1)}, d_{\epsilon_X,o_i}\right)$, where $\epsilon_X > 0$ is chosen in Subsection 6.4.4 so that $\left(\partial_\infty \tilde{X}_\Gamma^{(1)}, d_{\epsilon_X,o}\right)$, $\left(\partial_\infty \tilde{X}_1^{(1)}, d_{\epsilon_X,o_1}\right)$, and $\left(\partial_\infty \tilde{X}_2^{(1)}, d_{\epsilon_X,o_2}\right)$ are visual.*

The key points are that ι_v is an isometry, and that both $\partial_\infty X_v \subset \left(\partial_\infty \tilde{X}_\Gamma^{(1)}, d_{\epsilon_X,o}\right)$ and $\left(\partial_\infty \tilde{X}_i^{(1)}, d_{\epsilon_X,o_i}\right)$ are visual metrics for the same parameter, ϵ_X.

6.8.2 Shadows and half-spaces

In this subsection, we present the second type of subset that we shall use to cover the boundary of $\tilde{X}_\Gamma^{(1)}$.

For any $v \in V(T)$, define the *half-space* of $\tilde{X}_\Gamma^{(1)}$ associated to v, denoted H_v, as the set of all $x \in \tilde{X}_\Gamma^{(1)}$ such that the geodesic path in T between $\pi(o)$ and $\pi(x)$ passes through v. Similarly, define the half-space of W associated to v, denoted $\mathfrak{H}_v$, as the set of all $w \in W$ such that the geodesic path in T between $\pi_W(\mathfrak{o})$ and $\pi_W(w)$ passes through v.

Noting that $\pi = \pi_W \circ F$, we find the following relationship between half-spaces of $\tilde{X}_\Gamma^{(1)}$ and W.

Lemma 6.8.5. *$F(H_v) \subseteq \mathfrak{H}_v$.*

This extends to the boundary: If a sequence $\{x_n\}$ stabilizes in H_v, then $\{F(x_n)\}$ stabilizes in $F(H_v) \subseteq \mathfrak{H}_v$. Thus, we get:

Lemma 6.8.6. *$F_\infty(\partial_\infty H_v) \subseteq \partial_\infty \mathfrak{H}_v$.*

Half-spaces will be crucial to obtaining desirable coverings of $\partial_\infty \tilde{X}_\Gamma^{(1)}$ and ultimately coverings of $F_\infty\left(\partial_\infty \tilde{X}_\Gamma^{(1)}\right)$. To see how they combine with $\partial_\infty X_v$ to cover $\tilde{X}_\Gamma^{(1)}$ it will be helpful to give an alternate description of $\partial_\infty H_v$ utilizing the geodesic structure of $\tilde{X}_\Gamma^{(1)}$:

For any $\xi \in \partial_\infty \tilde{X}_\Gamma^{(1)}$ and any γ geodesic ray in $\tilde{X}_\Gamma^{(1)}$, we say that γ *represents* ξ if $\{\gamma(n)\} \in \xi$. For any $x \in \tilde{X}_\Gamma^{(1)}$, the *shadow* of x, denoted U_x, is defined to be the set of points in $\partial_\infty \tilde{X}_\Gamma^{(1)}$ that can be represented by a geodesic emanating from o that passes through x.

Lemma 6.8.7. *$\partial_\infty H_v = U_{o_v}$.*

Proof. As $\tilde{X}_\Gamma^{(1)}$ is a proper geodesic metric space, we can apply the Arzela-Ascoli theorem to find geodesics representing points in the boundary of half-spaces, see [GdLH90, Proposition 7.4]. Essentially, the Arzela-Ascoli theorem means that if one has a sequence $\{x_n\} \subset \tilde{X}_\Gamma^{(1)}$ converging to infinity, and one considers a family of geodesic segments $\{\gamma_n\}$ with γ_n a geodesic between o and x_n, then one can find a geodesic ray γ based at o, and a subsequence $\{\gamma_{n_k}\}$ such that γ_{n_k} limit to γ point-wise. Lemma 6.3.6 means that $\{x_{n_k}\}$ also converges to infinity and is equivalent to $\{x_n\}$. The point-wise convergence of γ_{n_k} to γ means $\{x_{n_k}\}$ is equivalent to $\{\gamma(n)\}$ and so γ represents the equivalence class containing $\{x_n\}$.

Let $\xi \in \partial_\infty H_v$, and suppose $\{x_n\}$ is a representative of ξ that stabilizes in H_v. The above process allows us to find a subsequence $\{x_{n_k}\}$ whose members are the endpoints of geodesic segments γ_{n_k} that limit point-wise to a geodesic ray γ based at o representing ξ. Thus, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $x_{n_k} \in H_v$. By definition, the unique geodesic path in T between $\pi(o)$ and $\pi(x_{n_k})$ passes through v. As $\pi \circ \gamma_{n_k}$ is a non-backtracking path in T between $\pi(o) = \pi(\gamma(0))$ and $\pi(x_{n_k})$, the image of $\pi \circ \gamma_{n_k}$ is this unique geodesic path between $\pi(o)$ and $\pi(x_{n_k})$ and therefore passes through v. Therefore, γ_{n_k} passes through o_v. As γ_{n_k} are all geodesics, they must all pass through o_v at time $t := d_{\tilde{X}_\Gamma^{(1)}}(o, o_v)$. Further, as γ_{n_k} converge point-wise to γ, $\gamma(t) = o_v$ also. Therefore, $\xi \in U_{o_v}$.

For any $\xi \in U_{o_v}$ there exists a geodesic γ based at o that represents ξ and passes through o_v. Hence, there exists $N \in \mathbb{N}$ such that $\gamma(N) = o_v$. Similar to the above, projecting via π to T we see, for any $n \geq N$, the unique geodesic path in T between $\pi(o) = \pi(\gamma(0))$ and $\pi(\gamma(n))$ passes through $\pi(\gamma(N)) = v$. Therefore $\gamma(n) \in H_v$ for each $n \geq N$, so γ stabilizes in H_v and $\xi \in \partial_\infty H_v$. □

Now we can easily find a cover for $\partial_\infty \tilde{X}_\Gamma^{(1)}$.

6.8.3 A cover for $F_\infty\left(\partial_\infty \tilde{X}_\Gamma^{(1)}\right)$

In this subsection, we use the geodesic structure of $\tilde{X}_\Gamma^{(1)}$ to explain how the boundaries of vertex spaces combine with the boundaries of half-spaces to cover $\partial_\infty \tilde{X}_\Gamma^{(1)}$.

Recall, we assigned each edge in $\tilde{X}_\Gamma^{(1)}$ to have length 1 so distances in $\tilde{X}_\Gamma^{(1)}$ take integer values.

Fix $r \in \mathbb{N}$ and consider a geodesic ray γ based at $o \in \tilde{X}_\Gamma^{(1)}$. Observe, either there exists $v \in V(T)$ such that $\pi(\gamma(n)) = v$ for all $n \geq r$, or not. If such a v exists, then γ represents an element of $\partial_\infty X_v$ and $\gamma(k) = o_v$ for some $k \leq r$, so $d_{\tilde{X}_\Gamma^{(1)}}(o, o_v) = k \leq r$. Otherwise, there exists $n \geq r+1$ such that $\pi(\gamma(n-1)) \neq \pi(\gamma(n))$, in which case $\gamma(n) = o_{v'}$ for some v' and γ represents an element of $U_{o_{v'}}$ with $d_{\tilde{X}_\Gamma^{(1)}}(o, o_{v'}) = n \geq r+1$. Let $\mathcal{V}_k$ be the collection of $v \in V(T)$ with $d_{\tilde{X}_\Gamma^{(1)}}(o, o_v) = k$, and let $\mathcal{V}_{\leq r}$ be the collection of $v \in V(T)$ such that $d_{\tilde{X}_\Gamma^{(1)}}(o, o_v) \leq r$. Then, we can cover $\partial_\infty \tilde{X}_\Gamma^{(1)}$ by

$$\{\partial_\infty X_v \mid v \in \mathcal{V}_{\leq r}\} \cup \bigcup_{k \geq r+1} \{U_{o_v} \mid v \in \mathcal{V}_k\}.$$

Hence, using lemmas 6.8.2, 6.8.6, and 6.8.7, we get the following proposition:

Proposition 6.8.8. *For any $r \in \mathbb{N}$,*

$$\{\partial_\infty W_v \mid v \in \mathcal{V}_{\leq r}\} \cup \bigcup_{k \geq r+1} \{\partial_\infty \mathfrak{H}_v \mid v \in \mathcal{V}_k\}$$

is a cover of $F_\infty\left(\partial_\infty \tilde{X}_\Gamma^{(1)}\right)$.

6.8.4 Diameter bound on half-spaces

In this subsection, we explain how we bound the diameter of half-spaces for the purpose of Proposition 6.2.6.

Lemma 6.8.9. *For any $w, w' \in \mathfrak{H}_v \subseteq (W, d_{W,L})$,*

$$(w \mid w')^L_{\mathfrak{o}} \geq d_{W,L}(\mathfrak{o}, \mathfrak{o}_v).$$

Proof. As both w and w' lie in $\mathfrak{H}_v$, v lies on the unique geodesic path from $\pi_W(\mathfrak{o})$ to $\pi_W(w)$ and the unique geodesic path from $\pi_W(\mathfrak{o})$ to $\pi_W(w')$. Therefore, by uniqueness, these paths must agree between $\pi_W(\mathfrak{o})$ and v. In particular, if w and w' evaluate in $W_{\hat{v}}$, then the unique geodesic path in T between $\pi_W(\mathfrak{o})$ and $\hat{v}$ must also pass through v, as $\hat{v}$ is the centre of the tripod $\pi_W(\mathfrak{o}), \pi_W(w), \pi_W(w')$. Considering the distance formula (6.5.0.4), we find $d_{W,L}(\mathfrak{o}, \mathfrak{o}_v) \leq d_{W,L}(\mathfrak{o}, \mathfrak{o}_{\hat{v}})$. Finally, using Lemma 6.6.2 we obtain the desired result. ☐

Proposition 6.8.10. *Let τ be as in Lemma 6.7.2, δ_W be from* (6.6.6.1)*, and $l \geq 1$, then, for any $v \in V(T)$, the subset $\partial_\infty \mathfrak{H}_v \subseteq \left(\partial_\infty(W, d^l_W), d_{1,\mathfrak{o}}\right)$ is such that*

$$\operatorname{diam}(\partial_\infty \mathfrak{H}_v) \leq \exp(2\delta_W + 1)\exp\left(-\frac{l}{\tau} d_{\tilde{X}_\Gamma^{(1)}}(o, o_v)\right).$$

Proof. For any $\xi, \xi' \in \partial_\infty \mathfrak{H}_v$, there exist representative sequences $\{w_n\} \in \xi$ and $\{w'_n\} \in \xi'$ that stabilize in $\mathfrak{H}_v$. Thus, combined with Lemma 6.3.6, for sufficiently large $i, j \in \mathbb{N}$,

$$|(\xi \mid \xi')^l_{\mathfrak{o}} - (w_i \mid w'_j)^l_{\mathfrak{o}}| \leq 2\delta_W + 1, \tag{6.8.10.1}$$

and $w_i, w'_j \in \mathfrak{H}_v$. Hence, by Lemma 6.8.9,

$$(w_i \mid w'_j)^l_{\mathfrak{o}} \geq d^l_W(\mathfrak{o}, \mathfrak{o}_v).$$

Using Lemma 6.7.2, we see

$$(w_i \mid w'_j)^l_{\mathfrak{o}} \geq \frac{l}{\tau} d_{\tilde{X}_\Gamma^{(1)}}(o, o_v),$$

which combines with (6.8.10.1) to give

$$(\xi \mid \xi')^l_{\mathfrak{o}} \geq \frac{l}{\tau} d_{\tilde{X}_\Gamma^{(1)}}(o, o_v) - (2\delta_W + 1).$$

Proposition 6.6.8 allows us to conclude:

$$\begin{aligned} d_{1,\mathfrak{o}}(\xi, \xi') &\leq \exp(-(\xi \mid \xi')^l_{\mathfrak{o}}) \\ &\leq \exp(2\delta_W + 1) \exp\left(-\frac{l}{\tau} d_{\tilde{X}^{(1)}_\Gamma}(o, o_v)\right). \end{aligned}$$ □

6.9 Proof of Theorem 6.1.1

This section is dedicated to proving that Theorem 6.1.1 is a corollary of Theorem 6.2.1. We start by showing that Theorem 6.1.2 is a corollary of Theorem 6.2.1 by induction. Then, in subsection 6.9.2 we show that Theorem 6.1.1 follows from Theorem 6.1.2.

6.9.1 From Theorem 6.2.1 to Theorem 6.1.2

Recall Theorem 6.2.1 was proved in Section 6.2 assuming the contents of Sections 6.4 through 6.8. From it, we aim to prove the following:

Theorem 6.9.1 (Theorem 6.1.2)**.** *Let* $\Gamma = G_1 * G_2 * \cdots * G_n$*, for* $n \geq 2$*, where* G_i *is infinite hyperbolic, for all* $1 \leq i \leq n$*, and not every* G_i *is virtually cyclic. Then*

$$\text{Höldim}\, \partial_\infty \Gamma = \max\{\text{Höldim}\, \partial_\infty G_i \mid 1 \leq i \leq n\} \tag{6.9.1.1}$$

and $\partial_\infty \Gamma$ *attains its Hölder dimension if and only if* $\partial_\infty G_i$ *attains its Hölder dimension for each* i *such that* $\text{Höldim}\, \partial_\infty G_i = \text{Höldim}\, \partial_\infty \Gamma$.

Proof. We proceed by induction on n:

The $n = 2$ case holds by Theorem 6.2.1. Now, assume that Theorem 6.9.1 holds for any free product of length $n - 1$. The equality (6.9.1.1) can be seen as follows.

Consider $\Gamma = G_1 * G_2 * \cdots * G_n$, a free product of infinite hyperbolic groups of length n where not every G_i is virtually cyclic for $1 \leq i \leq n$. Without loss of generality, we may assume, by rearranging the factors, that G_1 is not virtually cyclic. Thus, $\Gamma' = G_1 * G_2 * \cdots * G_{n-1}$ is a free product of infinite hyperbolic groups of length $n - 1$ where not every G_i is virtually cyclic for $1 \leq i \leq n - 1$, so $\text{Höldim}\, \partial_\infty \Gamma' = \max\{\text{Höldim}\, \partial_\infty G_i \mid 1 \leq i \leq n - 1\}$ by the induction assumption. Further, note that Γ' is not virtually cyclic, and both Γ' and G_n are infinite hyperbolic groups. Applying Theorem 6.2.1 to the decomposition $\Gamma = \Gamma' \vee G_n$ gives

$$\begin{aligned} \text{Höldim}\, \partial_\infty \Gamma &= \text{Höldim}\, \partial_\infty \Gamma' \vee \text{Höldim}\, \partial_\infty G_n, \\ &= \max\{\text{Höldim}\, \partial_\infty G_i \mid 1 \leq i \leq n\}, \end{aligned}$$

as desired.

The ‘attainment’ aspect of Theorem 6.9.1 can be seen as follows:

To see the ‘if’ direction, suppose that, for any i such that $\text{Höldim}\, \partial_\infty G_i = \text{Höldim}\, \partial_\infty \Gamma$, $\partial_\infty G_i$ attains its Hölder dimension. For each i, let $\mathfrak{G}_i$ be a space with $\dim_H \mathfrak{G}_i \leq \text{Höldim}\, \partial_\infty \Gamma$.

Let $1 \leq j \leq n$ be such that $\text{Höldim}\,\partial_\infty G_j = \min\{\text{Höldim}\,\partial_\infty G_i \mid 1 \leq i \leq n\}$ and G_j is not the only factor that is not virtually cyclic. Such a choice is possible as the Hölder dimension of the boundary of any infinite virtually cyclic group is 0. Therefore, if the only factor that is not virtually cyclic has boundary with minimal Hölder dimension among all factors, of which the rest are virtually cyclic, then it too must have Hölder dimension 0. This means we can change our choice to one of the virtually cyclic factors and still have chosen a factor whose boundary has minimal Hölder dimension amongst all factors. As $n \geq 2$, $\max\{\text{Höldim}\,\partial_\infty G_i \mid 1 \leq i \leq n\} = \max\{\text{Höldim}\,\partial_\infty G_i \mid 1 \leq i \leq n, i \neq j\}$. Thus $\Gamma_j = G_1 * G_2 * \cdots * G_{j-1} * G_{j+1} * \cdots * G_n$, the same free product as Γ but with G_j omitted, still has Hölder dimension equal to $\text{Höldim}\,\partial_\infty\Gamma$, and $\text{Höldim}\,\partial_\infty G_i$ is attained for each $1 \leq i \leq n, i \neq j$ such that $\text{Höldim}\,\partial_\infty G_i = \text{Höldim}\,\partial_\infty\Gamma = \text{Höldim}\,\partial_\infty\Gamma_j$. As we have expressed Γ_j as a free product of infinite hyperbolic groups of length $n-1$, not all of which are virtually cyclic, our induction assumption tells us that $\partial_\infty\Gamma_j$ attains its Hölder dimension. Note Γ is isomorphic to $\Gamma_j * G_j$. We have shown that Γ_j is not virtually cyclic, as it can be expressed as a free product with a factor that is not virtually cyclic, and $\partial_\infty\Gamma_j$ has Hölder dimension equal to that of $\partial_\infty\Gamma$ and is attained. We do not know if $\partial_\infty G_j$ has the same Hölder dimension as $\partial_\infty\Gamma$, but, by assumption, we know that if it does, then it is attained. We can now apply Theorem 6.2.1 to the decomposition $\Gamma \simeq \Gamma_j * G_j$ to conclude that $\partial_\infty\Gamma$ does indeed attain its Hölder dimension.

For the 'only if' direction, assume, for $\Gamma = G_1 * G_2 * \cdots * G_n$, that $\partial_\infty\Gamma$ attains its Hölder dimension and let $f\colon \partial_\infty\Gamma \to Z$ be a bi-Hölder homeomorphism realising this attainment, for some space Z. Fix j such that $\text{Höldim}(\partial_\infty G_j) = \text{Höldim}\,\partial_\infty\Gamma$. The inclusion $G_j \hookrightarrow \Gamma$ is a quasi-isometric embedding, so induces a quasi-symmetric, and therefore bi-Hölder, embedding $\phi\colon \partial_\infty G_j \hookrightarrow \partial_\infty\Gamma$. Composing ϕ with the restriction of f to $\phi(\partial_\infty G_j)$ gives a bi-Hölder homeomorphism $f_j\colon \partial_\infty G_j \to Z_j$ for some $Z_j \subset Z$. Trivially, $\dim_H Z_j \leq \dim_H Z = \text{Höldim}\,\partial_\infty\Gamma$. However, Z_j is a space Hölder equivalent to $\partial_\infty G_j$, which has Hölder dimension equal to $\text{Höldim}\,\partial_\infty\Gamma$ by assumption. Thus, $\dim_H Z_j \geq \text{Höldim}\,\partial_\infty\Gamma$ too. Therefore, Z_j is a space realising the Hölder dimension of $\partial_\infty G_j$, and we can conclude that $\partial_\infty G_j$ does indeed attain its Hölder dimension. □

6.9.2 From Theorem 6.1.2 to Theorem 6.1.1

In this subsection, we show how Theorem 6.1.1 follows from Theorem 6.1.2. We aim to prove the following:

Theorem 6.9.2 (Theorem 6.1.1)**.** *Suppose Γ is an infinite hyperbolic group that splits as a finite graph of groups $\mathcal{G}$ with finite edge groups. Let $G_1, G_2, \ldots, G_n$ represent all quasi-isometry types of the vertex groups of $\mathcal{G}$ that are infinite and not virtually cyclic. If $n \geq 1$, then* $\text{Höldim}\,\partial_\infty\Gamma = \max\{\text{Höldim}\,\partial_\infty G_i \mid 1 \leq i \leq n\}$ *and $\partial_\infty\Gamma$ attains its Hölder dimension if and only if $\partial_\infty G_i$ attains its Hölder dimension for each i such that* $\text{Höldim}\,\partial_\infty G_i = \text{Höldim}\,\partial_\infty\Gamma$.

If $n = 0$, then $\text{Höldim}(\partial_\infty \Gamma) = 0$ *and the Hölder dimension of* $\partial_\infty \Gamma$ *is attained is and only if* Γ *is virtually cyclic.*

The main tool that will allow us to reduce Theorem 6.1.1 to Theorem 6.1.2 is the following theorem presented by Martin and Świątkowski as a corollary of [PW02, Theorem 0.3].

Theorem 6.9.3 ([MS15, Corollary 5.2])**.** *Let* Γ *be a finitely generated group with infinitely many ends and let* $\mathcal{G}$ *be a graph of group decomposition of* Γ *with all edge groups finite. If* $G_1, \ldots, G_n$ *represent all quasi-isometry types of those infinite vertex groups of* $\mathcal{G}$ *that are not virtually cyclic, then:*

- *if $n = 0$, then Γ is quasi-isometric to the free group F_2,*
- *if $n = 1$, then Γ is quasi-isometric to $G_1 * G_1$,*
- *if $n > 1$, then Γ is quasi-isometric to $G_1 * \cdots * G_n$.*

Proof of Theorem 6.9.2. To use Theorem 6.9.3, we first have to deal with the cases when Γ does not have infinitely many ends.

A group can have 0, 1, 2, or infinitely many ends, see [Sta71, 1.B.6]. If Γ has 0 ends, then Γ is finite, but we are assuming Γ is infinite so this case cannot occur. If Γ has 1 end, then Γ does not split non-trivially as a graph of groups over finite edge groups, so there is at most one quasi-isometry class of vertex groups, that is, $n \leq 1$. Γ is not virtually cyclic, else it would have 2 ends, so $n = 1$ and Γ is quasi-isometric to G_1, and Theorem 6.1.1 trivially holds. If Γ has 2 ends, then Γ is virtually cyclic and its boundary is a pair of points. Thus $n = 0$ and we can verify that, as the Hausdorff dimension of a pair of points is 0 regardless of metric, $\partial_\infty \Gamma$ has Hölder dimension 0 and attains its Hölder dimension.

Thus, we are left with the case that Γ has infinitely many ends. Applying Theorem 6.9.3, if $n = 0$, then Γ is quasi-isometric to F_2, which has boundary equal to a Cantor set with strictly positive Hausdorff dimension. One can easily show that any space Hölder equivalent to one with strictly positive Hausdorff dimension also has strictly positive Hausdorff dimension, so $\partial_\infty \Gamma$ has Hölder dimension 0, by Corollary 4.1.2, and does not attain its Hölder dimension. Finally, if $n \geq 1$, then we are left with Γ quasi-isometric to a free product of groups that are infinite and not virtually cyclic. We then apply Theorem 6.1.2 to conclude the result. □

CHAPTER 7

Open Questions and Concluding Remarks

We conclude this thesis with this chapter discussing a few possible directions for further research. Note that this chapter does not contain every question posed in this thesis, see questions 3.3.6, 6.1.7, 6.1.8, and 6.1.9.

7.1 Integer Hölder dimension

The reader might have noted that every example we have given has had integer Hölder dimension. For our compact, locally self-similar examples, this follows from Corollary 4.1.2. However, even in our non-self-similar examples, Theorem 4.2.10 and Theorem 5.1.2, we still have an integer Hölder dimension. Moreover, Theorem 3.3.5 gives that Hölder dimension cannot take values strictly between 0 and 1. While this gap also appears for conformal dimension, which can take any real value greater than 1, it still leads us to pose the question:

Question 7.1.1. Can Hölder dimension take a non-integer value?

Towards finding examples with non-integer values, I believe one could potentially utilise the rigidity of euclidean spaces under Hölder maps. By this I mean that, in some quantifiable sense, n-dimensional Euclidean spaces not only refuse to have their dimension lowered by Hölder equivalences, but also refuse to have their n-measure lowered too much either, see Lemma 5.3.2, which forces bounds on Hausdorff measure reminiscent of those caused by Lipschitz distortions, as opposed to the weaker trivial bounds from the bi-Hölder metric control. Interlacing Euclidean structures into something with non-integer dimension, might provide enough rigid structure that the overall dimension cannot be lowered by bi-Hölder maps.

However, if Hölder dimension really does always take integer values, this might be because Hölder dimension is equivalent to another notion of dimension that always takes integer values.

Theorems 4.2.10 and 5.1.2 show that Hölder dimension is not equivalent to topological or capacity dimension. We used these notions of dimension for comparison due to their established nature, however, neither of these dimensions are particularly customised to bi-Hölder maps. Perhaps a similar notion of covering dimension with metric control more strongly related to exponents would provide a candidate integer dimension that is equivalent to Hölder dimension.

7.2 Characterising the Hölder classes of nice spaces

For some sufficiently nice spaces, both the class of topologically equivalent and quasi-symmetrically equivalent spaces are well understood. The purpose of this section is to build a rough picture of this understanding to justify our questions about the Hölder structure of such spaces; we omit explanations of some of the terminology mentioned as this would make the section cumbersome, which would be counterproductive to its purpose.

For arcs and circles, there are clean characterisations for both topological and quasi-symmetric equivalence.

Theorem 7.2.1 ([HY12, Theorem 2-27])**.** *A metric space X is topologically equivalent to the closed unit interval $[0,1]$ if and only if X is a non-empty connected, compact metric space with exactly two non-cut points.*

Theorem 7.2.2 ([HY12, Theorem 2-28])**.** *A metric space X is topologically equivalent to the unit circle $\mathbb{S}^1$ if and only if X is a non-empty connected, compact metric space such that, for any distinct $x, y \in X$, $X \setminus \{x, y\}$ is not connected.*

Theorem 7.2.3 ([TV80, Theorem 4.9])**.** *A topological arc or circle is quasi-symmetrically equivalent to the closed unit interval $[0,1]$ or the unit circle $\mathbb{S}^1$, respectively, if and only if it is doubling and is of bounded turning.*

This leads us to ask the following question.

Question 7.2.4. Is there a similar characterisation for topological arcs or circles that are Hölder equivalent to the unit interval or unit circle?

In particular, being able to detect subspaces that are Hölder equivalent to euclidean structures could be potentially useful for showing non-attainment of Hölder dimension as in Theorem 5.1.1.

Cantor sets have their own clean characterisations for both topological and quasi-symmetric equivalence. Their topological characterisation is due to Brouwer in [Bro]. The following form can be found in [Wil04, Corollary 30.4].

Theorem 7.2.5. *The Cantor set is the only totally disconnected, perfect compact metric space up to homeomorphism.*

Theorem 7.2.6 ([DS97, Proposition 15.11])**.** *A metric space is quasi-symmetrically equivalent to the* $1/3$*-Cantor set if and only if it is compact, doubling, uniformly perfect, and uniformly disconnected.*

Via the trivial bounds (1.0.3.1), one can see that two Cantor sets cannot be Hölder equivalent unless they both have either 0, positive and finite, or infinite Hausdorff dimensions. Further, Theorem 5.1.2 provides an example of a Cantor set with positive and finite Hausdorff dimension that is not Hölder equivalent to the 1/3-Cantor set. Therefore, it seems that if a Hölder classification of a type of Cantor set exists, then it would behave more like the quasi-symmetric classification of the 1/3-Cantor set than the topological classification of all Cantor sets. This leads us to ask the following question.

Question 7.2.7. Is there a characterisation of spaces Hölder equivalent to the 1/3-Cantor set?

Our final example for this section is the Sierpiński carpet. This space is obtained by an iterative process in a similar way to the 1/3-Cantor set: Take a square with unit side-length and divide it into 9 equal squares of side-length 1/3, then remove the middle square leaving you with a ring of 8 squares. One can then repeat this process by removing the middle square of side-length 1/9 from each of the remaining 8 squares. If one continues ad infinitum to remove the middle square of side-length a third the length of its parent square, then one is left with the standard 1/3-Sierpiński carpet, see Figure 4.1. Whyburn provided a topological characterisation for the Sierpiński carpet in [Why58, Theorem 4].

Theorem 7.2.8. *A planar, connected, locally connected compact metric space* X *of topological dimension* 1 *is topologically equivalent to the Sierpiński carpet if and only if it has no local separating point.*

There has also been work towards understanding the quasi-symmetric structure of Sierpiński carpets, however this understanding is not so clean. See [BKM09], [Bon11], and [BM13] for some examples of progress. I would be interested in seeing whether the added flexibility of Hölder maps simplifies the task of characterising the Hölder class of the 1/3-Sierpiński carpet to something clean, similar to the arc, circle, and Cantor set cases.

7.3 Hölder profile

When one is studying a new object that has a well-studied parallel it can be tempting to find inspiration only in what one can easily translate from the old object to the new one. However, finding where the new object is fundamentally different from the old can yield powerful results. In light of this, observe that both quasi-symmetric equivalence and Hölder equivalence have quantifiers in their definitions that control the wildness of the metric distortion. However, the quantifier in Hölder equivalence has a clear parametrisation that the quantifier in quasi-symmetric equivalence does not. That is, we can easily restrict the wildness of our Hölder maps by restricting the values of the constants in the definition of Hölder equivalence. Intuitively, for the sake of Hölder dimension, the exponent of the lower bound has the greatest impact, as it is the main constant controlling decreases in dimension. This leads us to make the following definition.

Definition 7.3.1. Let X be a metric space. The *Hölder profile* of X is the function $P_X : \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 0}$ defined as, for any $\alpha \geq 1$,

$$P_X(\alpha) = \inf\{\dim_H(Y) \mid \exists X \to Y \ (\lambda, \alpha, \beta)\text{-bi-Hölder, for some } \lambda > 0, \beta \leq 1\}.$$

Profiles should now allow one to meaningfully study "how fast does a space approach its Hölder dimension?". Such a question does not make much sense for conformal dimension because there is no clear parametrisation of their wildness. If one could derive useful information from studying Hölder profiles, one would have a fundamental benefit for studying Hölder dimension over conformal dimension.

I have not made much progress on understanding Hölder profiles as of yet, but would be greatly interested in studying them because of their aforementioned potential. The following are some preliminary simple results that indicate that profiles at least have some sensible properties.

Observe that, directly from the definition, P_X is monotone decreasing. Further, we can show P_X is uniformly continuous.

Proposition 7.3.2. *For any metric space X, P_X is uniformly continuous.*

First, we present a helpful lemma.

Lemma 7.3.3. *For any metric space X, $\alpha \geq 1$, and $\epsilon > 0$,*

$$P_X(\alpha) \leq (1+\epsilon) P_X(\alpha(1+\epsilon)).$$

Proof. Let Z be any metric space such that there exists a $(\lambda, \alpha(1+\epsilon), \beta)$-bi-Hölder homeomorphism $f : X \to Z$, for some $\lambda > 0, \beta \leq 1$. As $1+\epsilon \geq 1$, we can 'snowflake' Z by $1/(1+\epsilon)$ to get the metric space $Y := \left(Z, {d_Z}^{1/(1+\epsilon)}\right)$. To see that Y is indeed a metric space, one simply needs to observe that the function that sends any non-negative real number t to $t^{1/(1+\epsilon)}$ is a concave function that evaluates to 0 if and only if $t = 0$. Note that, there is a natural map $\mathrm{id} : Z \to Y$, which is the identity on the underlying set Z, that is a $(1, 1/(1+\epsilon), 1/(1+\epsilon))$-bi-Hölder homeomorphism. Composing with f we find that $\mathrm{id} \circ f : X \to Y$ is a $(\lambda', \alpha, \beta/(1+\epsilon))$-bi-Hölder homeomorphism, for some $\lambda' \geq 1$. Further, it is easy to show from the definition of Hausdorff dimension that $\dim_H(Y) = (1+\epsilon)\dim_H(Z)$. Hence,

$$P_X(\alpha) \leq \dim_H(Y) = (1+\epsilon)\dim_H(Z).$$

Thus, as $P_X(\alpha(1+\epsilon))$ is the infimum over all such Z, we obtain

$$P_X(\alpha) \leq (1+\epsilon)P_X(\alpha(1+\epsilon)),$$

as desired. □

Proof of Proposition 7.3.2. For any $\epsilon > 0$, let $\delta = \epsilon/\dim_H(X)$, and take any $\alpha, \alpha' \geq 1$ such that $|\alpha - \alpha'| < \delta$. Without loss of generality, we may assume that $\alpha' < \alpha$. Hence,

$$|P_X(\alpha) - P_X(\alpha')| = P_X(\alpha') - P_X(\alpha)$$

as P_X is monotone decreasing,

$$\leq \left(1 + \frac{\alpha - \alpha'}{\alpha'}\right) P_X\left(\alpha'\left(1 + \frac{\alpha - \alpha'}{\alpha'}\right)\right) - P_X(\alpha)$$

by Lemma 7.3.3 as $(\alpha - \alpha')/\alpha' > 0$,

$$= \frac{\alpha - \alpha'}{\alpha'} P_X(\alpha)$$
$$< \frac{\delta}{\alpha'} P_X(\alpha)$$

by the assumption on α, α',

$$\leq \delta P_X(\alpha)$$

as $\alpha' \geq 1$,

$$\leq \delta P_X(1)$$

as P_X is monotone decreasing,

$$\leq \delta \dim_H(X) = \epsilon$$

as the identity on X is a $(1,1,1)$-bi-Hölder homeomorphism. □

The Hölder profile of a space also comes with a trivial lower bound from the trivial lower bound on Hausdorff dimension distorted under a Hölder map given in (1.0.3.1).

Lemma 7.3.4. *For any metric space X and $\alpha \geq 1$, $P_X(\alpha) \geq (\dim_H(X)/\alpha) \vee \mathrm{H\ddot{o}ldim}(X)$.*

This trivial lower bound is attained by ultrametric spaces. An ultrametric space is a standard variant of a metric space. It is a metric space (X, d) with the following strengthened version of the triangle inequality: For any $x, y, z \in X$, $d(x,z) \leq d(x,y) \vee d(y,z)$.

Proposition 7.3.5. *Let (X, d) be an ultrametric space, then $P_X(\alpha) = \dim_H(X)/\alpha$.*

One can interpret this as the profile decaying as fast as possible when the metric is sufficiently flexible. As a consequence of this proposition, we see that ultrametric spaces have Hölder dimension 0, and attain it if and only if they have Hausdorff dimension 0.

Proof. For any $\alpha \geq 1$, note that (X, d^α) is also an ultrametric space as the function $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that maps, for any $t \in \mathbb{R}_{\geq 0}$, $t \mapsto t^\alpha$ is a monotone increasing bijection. Further, it is easy to show from the definition of Hausdorff dimension that $\dim_H(X, d^\alpha) = \dim_H(X, d)/\alpha$. Combined with Lemma 7.3.4 we see that $P_X(\alpha) = \dim_H(X, d)/\alpha$ as desired. □

We actually computed a lower bound for the profile of $C \times I^n$ in the proof of Theorem 5.3.1 that is better than the trivial bound from Lemma 7.3.4. We proved that the image of $C \times I^n$ under any (λ, α, β)-bi-Hölder map has Hausdorff dimension at least $n + \log(2)/\alpha \log(3)$. In particular, this means there is a space with a 'non-trival' Hölder profile. However, it is far from clear what kinds of Hölder profiles can appear in general. This leads us to ask the following question.

Question 7.3.6. What kinds of functions appear as Hölder profiles?

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