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Intrinsic Ultracontractivity for Domains in Negatively Curved Manifolds

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Abstract

Let M be a complete, non-compact, connected Riemannian manifold with Ricci curvature bounded from below by a negative constant. A sufficient condition is obtained for open and connected sets D in M for which the corresponding Dirichlet heat semigroup is intrinsically ultracontractive. That condition is formulated in terms of capacitary width. It is shown that both the reciprocal of the bottom of the spectrum of the Dirichlet Laplacian acting in $L^2(D)$, and the supremum of the torsion function for D are comparable with the square of the capacitary width for D if the latter is sufficiently small. The technical key ingredients are the volume doubling property, the Poincaré inequality and the Li-Yau Gaussian estimate for the Dirichlet heat kernel at finite scale.

Keywords Intrinsic ultracontractivity · Ricci curvature · First eigenvalue · Heat kernel · Torsion function · Capacitary width

Dedicated to the memory of Professor Walter K. Hayman.

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1 Main Results

Let M be a complete, non-compact, n -dimensional connected Riemannian manifold, without boundary, and with Ricci curvature bounded below by a negative constant, i.e., $\text{Ric} \geq -K$ with non-negative constant K . Throughout the paper, K is reserved for this constant. In this article, we investigate domains (open, and connected sets) in M for which the heat semigroup is intrinsically ultracontractive.

For a domain $D \subset M$ we denote by $p_D(t, x, y)$, $t > 0$, $x, y \in D$, the Dirichlet heat kernel for $\partial/\partial t - \Delta$ in D , i.e., the fundamental solution to $(\partial/\partial t - \Delta)u = 0$ subject to the Dirichlet boundary condition $u(t, x) = 0$ for $x \in \partial D$ and $t > 0$. Davies and Simon [12] introduced the notion of intrinsic ultracontractivity. There are several equivalent definitions for intrinsic ultracontractivity ([12, p. 345]). The following is in terms of the heat kernel estimate.

Definition 1.1 Let $D \subset M$. We say that the semigroup associated with $p_D(t, x, y)$ is *intrinsically ultracontractive* (abbreviated to IU) if the following two conditions are satisfied:

- (i) The Dirichlet Laplacian $-\Delta$ has no essential spectrum and has the first eigenvalue $\lambda_D > 0$ with corresponding positive eigenfunction φ_D normalized by $\|\varphi_D\|_2 = 1$.
- (ii) For every $t > 0$, there exist constants $0 < c_t < C_t$ depending on t such that

$$c_t \varphi_D(x) \varphi_D(y) \leq p_D(t, x, y) \leq C_t \varphi_D(x) \varphi_D(y) \quad \text{for all } x, y \in D. \quad (1.1)$$

For simplicity, we say that D itself is IU if the semigroup associated with $p_D(t, x, y)$ is IU.

Both the analytic and probabilistic aspects of IU have been investigated in detail. For example it turns out that IU implies the Cranston-McConnell inequality, while IU is derived from very weak regularity of the domain. Davis [13] showed that a bounded Euclidean domain above the graph of an *upper semi-continuous* function is IU; no regularity of the boundary function is needed. There are many results on IU for Euclidean domains. Bañuelos and Davis [5, Thm. 1, Thm. 2] gave conditions characterizing IU and the Cranston-McConnell inequality when restricting to a certain class of plane domains, which illustrate subtle differences between IU and the Cranston-McConnell inequality. Méndez-Hernández [16] gave further extensions. See also [1,4,7,8,13], and references therein.

There are relatively few results for domains in a Riemannian manifold. Lierl and Saloff-Coste [15] studied a general framework including Riemannian manifolds. In that paper, they gave a precise heat kernel estimate for a bounded inner uniform domain, which implies IU ([15, Thm. 7.9]). In view of [13], however, the requirement of inner uniformity for IU to hold can be relaxed. See Sect. 7.

Our main result is a sufficient condition for IU for domains in a manifold, which is a generalization of the Euclidean case [1]. Our condition is given in terms of capacity.

It is applicable not only to bounded domains but also to unbounded domains. Let $\Omega \subset M$ be an open set. For $E \subset \Omega$ we define relative capacity by

$$\text{Cap}_\Omega(E) = \inf \left\{ \int_\Omega |\nabla\varphi|^2 d\mu : \varphi \geq 1 \text{ on } E, \varphi \in C_0^\infty(\Omega) \right\},$$

where μ is the Riemannian measure in M and $C_0^\infty(\Omega)$ is the space of all smooth functions compactly supported in Ω . Let $d(x, y)$ be the distance between x and y in M . The open geodesic ball with center x and radius $r > 0$ is denoted by $B(x, r) = \{y \in M : d(x, y) < r\}$. The closure of a set E is denoted by \bar{E} , and so $\bar{B}(x, r)$ stands for the closed geodesic ball of center x and radius r .

Definition 1.2 Let $0 < \eta < 1$. For an open set D we define the *capacitary width* $w_\eta(D)$ by

$$w_\eta(D) = \inf \left\{ r > 0 : \frac{\text{Cap}_{B(x, 2r)}(\bar{B}(x, r) \setminus D)}{\text{Cap}_{B(x, 2r)}(\bar{B}(x, r))} \geq \eta \text{ for all } x \in D \right\}.$$

The next theorem asserts that the parameter η has no significance.

Theorem 1.3 *Let $0 < R_0 < \infty$. If $0 < \eta' < \eta < 1$, then*

$$w_{\eta'}(D) \leq w_\eta(D) \leq C w_{\eta'}(D) \text{ for all open sets } D \text{ with } w_\eta(D) < R_0$$

with $C > 1$ depending only on $\eta, \eta', \sqrt{K} R_0$ and n .

The first condition for IU has a characterization in terms of capacitary width. This is straightforward from Persson’s argument [17], and Theorem 1.6 below. Hereafter we fix $o \in M$.

Theorem 1.4 *Let D be a domain in M . Then D has no essential spectrum if and only if $\lim_{R \rightarrow \infty} w_\eta(D \setminus \bar{B}(o, R)) = 0$.*

We shall prove the following sufficient condition for IU, which looks the same as in the Euclidean case [1]. Nevertheless, the proof is significantly different for negatively curved manifolds. See the remark after Theorem A.

Theorem 1.5 *Suppose M has positive injectivity radius. Then a domain $D \subset M$ is IU if the following two conditions are satisfied:*

- (i) $\lim_{R \rightarrow \infty} w_\eta(D \setminus \bar{B}(o, R)) = 0$.
- (ii) For some $\tau > 0$

$$\int_0^\tau w_\eta(\{x \in D : G_D(x, o) < t\})^2 \frac{dt}{t} < \infty, \tag{1.2}$$

where G_D is the Green function for D .

Our results are based on the relationship between the torsion function

$$v_D(x) = \int_D G_D(x, y)d\mu(y)$$

and the bottom of the spectrum

$$\lambda_{\min}(D) = \inf \left\{ \frac{\|\nabla f\|_2^2}{\|f\|_2^2} : f \in C_0^\infty(D) \text{ with } \|f\|_2 \neq 0 \right\}. \tag{1.3}$$

We note that $\lambda_{\min}(D)$ is the first eigenvalue λ_D if D has no essential spectrum. This is always the case for a bounded domain D . Theorem 1.4 asserts that the same holds even for an unbounded domain D whenever $\lim_{R \rightarrow \infty} w_\eta(D \setminus \overline{B}(o, R)) = 0$. We also observe that the torsion function is the solution to the de Saint-Venant problem:

$$\begin{aligned} -\Delta v_D &= 1 \quad \text{in } D, \\ v_D &= 0 \quad \text{on } \partial D, \end{aligned}$$

where the boundary condition is taken in the Sobolev sense. The second named author [19] proved the following theorem.

Theorem A *Let $K = 0$. If $D \subset M$ satisfies $\lambda_{\min}(D) > 0$, then*

$$\lambda_{\min}(D)^{-1} \leq \|v_D\|_\infty \leq C\lambda_{\min}(D)^{-1}, \tag{1.4}$$

where C depends only on M .

The second inequality of (1.4) does not necessarily hold for negatively curved manifolds. Let \mathbb{H}^n be the n -dimensional hyperbolic space of constant curvature -1 . It is known that

$$\lambda_{\min}(\mathbb{H}^n) = \frac{(n - 1)^2}{4},$$

whereas $v_{\mathbb{H}^n} \equiv \infty$ as \mathbb{H}^n is stochastically complete. Hence the second inequality of (1.4) fails to hold if D is the whole space \mathbb{H}^n .

The point of this paper is that (1.4) still holds if D is limited to a certain class. We make use of (1.4) with this limitation to derive Theorems 1.4 and 1.5. We have the following theorem, which is a key ingredient in their proofs.

Theorem 1.6 *Let $K \geq 0$ and let $0 < \eta < 1$. Then there exist $R_0 > 0$ and $C > 1$ depending only on K, η and n such that if $D \subset M$ satisfies $w_\eta(D) < R_0$, then*

$$\frac{C^{-1}}{w_\eta(D)^2} \leq \frac{1}{\|v_D\|_\infty} \leq \lambda_{\min}(D) \leq \frac{C}{\|v_D\|_\infty} \leq \frac{C^2}{w_\eta(D)^2}. \tag{1.5}$$

Remark 1.7 We actually find $\Lambda_0 > 0$ depending only on K and n such that (1.4) holds for D with $\lambda_{\min}(D) > \Lambda_0$ (Lemma 3.2 below). This is a generalization of Theorem A as $\Lambda_0 = 0$ for $K = 0$. In practice, however, the condition $w_\eta(D) < R_0$ in Theorem 1.6 is more convenient since the capacity width $w_\eta(D)$ can be more easily estimated than the bottom of the spectrum $\lambda_{\min}(D)$.

In Sect. 2 we summarize the key technical ingredients of the proofs: the volume doubling property, the Poincaré inequality and the Li-Yau Gaussian estimate for the Dirichlet heat kernel at finite scale. Observe that these fundamental tools are available not only for manifolds with Ricci curvature bounded below by a negative constant but also for unimodular Lie groups and homogeneous spaces. See [15, Ex. 2.11] and [18, Sect. 5.6]. This observation suggests that our approach is also extendable to those spaces.

We use the following notation. By the symbol C we denote an absolute positive constant whose value is unimportant and may change from one occurrence to the next. If necessary, we use C_0, C_1, \dots , to specify them. We say that f and g are comparable and write $f \approx g$ if two positive quantities f and g satisfy $C^{-1} \leq f/g \leq C$ with some constant $C \geq 1$. The constant C is referred to as the constant of comparison.

2 Preliminaries

We recall that M is a manifold of dimension $n \geq 2$ with $\text{Ric} \geq -K$ with $K \geq 0$. Let us recall the volume doubling property of the Riemannian measure μ , the Poincaré inequality and the Gaussian estimate for the Dirichlet heat kernel $p_M(t, x, y)$ for M . For $B = B(x, r)$ and $\tau > 0$ we write $\tau B = B(x, \tau r)$.

Theorem 2.1 (Volume doubling at finite scale. [18, Thm. 5.6.4]) *Let $0 < R_0 < \infty$. Then for all $B = B(x, r)$ with $0 < r < R_0$*

$$\mu(2B) \leq 2^n \exp(\sqrt{(n-1)K} R_0) \mu(B).$$

Theorem 2.2 (Poincaré inequality [18, Thm. 5.6.6]) *For each $1 \leq p < \infty$ there exist positive constants $C_{n,p}$ and C_n such that*

$$\int_B |f - f_B|^p d\mu \leq C_{n,p} r^p \exp(C_n \sqrt{K} r) \int_{2B} |\nabla f|^p d\mu$$

for all $B = B(x, r)$. Here f_B stands for the average of f on B .

Corollary 2.3 (Poincaré inequality at finite scale) *Let $0 < R_0 < \infty$. Then for all $B = B(x, r)$ with $0 < r < R_0$*

$$\int_B |f - f_B|^2 d\mu \leq C_{n,2} r^2 \exp(C_n \sqrt{K} R_0) \int_{2B} |\nabla f|^2 d\mu.$$

Remark 2.4 If the Ricci curvature of M is non-negative, i.e., $K = 0$, then the estimates in Theorems 2.1, 2.2 and Corollary 2.3 hold with constants independent of $0 < r < \infty$.

The Poincaré inequality yields the Sobolev inequality. We see that if $B = B(x, r)$ with $0 < r < R_0$, then

$$\left(\frac{1}{\mu(B)} \int_B |f|^2 d\mu \right)^{1/2} \leq C_{n,2} r \left(\frac{1}{\mu(B)} \int_B |\nabla f|^2 d\mu \right)^{1/2} \quad \text{for all } f \in C_0^\infty(B)$$

with different $C_{n,2}$. See [18, Thm. 5.3.3] for a more general Sobolev inequality. Hence the characterization of the bottom of the spectrum in terms of Rayleigh quotients (1.3) gives the following:

Corollary 2.5 *Let $0 < R_0 < \infty$. Then there exists a constant $C > 0$ depending only on $\sqrt{K} R_0$ and n such that*

$$\lambda_{\min}(B(x, r)) \geq Cr^{-2} \quad \text{for } 0 < r < R_0.$$

The celebrated theorem by Grigor’yan and Saloff-Coste gives the relationship between the Poincaré inequality, the volume doubling property of the Riemannian measure, the Li-Yau Gaussian estimate for the heat kernel, and the parabolic Harnack inequality. Let $V(x, r) = \mu(B(x, r))$.

Theorem B ([18, Thm. 5.5.1, Thm. 5.5.3]) *Let $0 < R_0 \leq \infty$. Consider the following conditions:*

(i) (PI) *There exists a constant $P_0 > 0$ such that for all $B = B(x, r)$ with $0 < r < R_0$ and all $f \in C^\infty(B)$,*

$$\int_B |f - f_B|^2 d\mu \leq P_0 r^2 \int_{2B} |\nabla f|^2 d\mu.$$

(ii) (VD) *There exists a constant $D_0 > 0$ such that for all $B = B(x, r)$ with $0 < r < R_0$*

$$\mu(2B) \leq D_0 \mu(B).$$

(iii) (PHI) *There exists a constant $A > 0$ such that for all $B = B(x, r)$ with $0 < r < R_0$ and all $u > 0$ with $(\partial_t - \Delta)u = 0$ in $(s - r^2, s) \times B$*

$$\sup_{Q_-} u \leq A \inf_{Q_+} u,$$

where $Q_- = (s - 3r^2/4, s - r^2/2) \times B(x, r/2)$ and $Q_+ = (s - r^2/4, s) \times B(x, r/2)$.

(iv) (GE) *There exists a finite constant $C > 1$ such that for $0 < t < R_0^2$ and $x, y \in M$,*

$$\frac{1}{CV(x, \sqrt{t})} \exp\left(-\frac{Cd(x, y)^2}{t}\right) \leq p_M(t, x, y) \leq \frac{C}{V(x, \sqrt{t})} \exp\left(-\frac{d(x, y)^2}{Ct}\right). \tag{2.1}$$

Then

$$(i) + (ii) \iff (iii) \iff (iv).$$

Theorem 2.1 and Corollary 2.3 assert that (i) and (ii) of Theorem B hold true for $0 < R_0 < \infty$ with constants depending only on K, R_0 and n . Hence, the Li-Yau Gaussian estimate of the heat kernel for the whole manifold M and the parabolic Harnack inequality up to scale R_0 are available in our setting. Observe that the volume doubling inequality $\mu(B(x, 2r)) \leq D_0\mu(B(x, r))$ implies

$$\mu(B(x, r)) \geq C \left(\frac{r}{R}\right)^\alpha \mu(B(x, R)) \quad \text{for } 0 < r < R < R_0 \tag{2.2}$$

with $\alpha = \log D_0 / \log 2$. We also have the following elliptic Harnack inequality since positive harmonic functions are time-independent positive solutions to the heat equation.

Corollary 2.6 (Elliptic Harnack inequality) *Let $0 < r_1 < r_2 < R_0 < \infty$. If h is a positive harmonic function in $B(x, r_2)$, then*

$$C^{-1} \leq \frac{h(y)}{h(x)} \leq C \quad \text{for } y \in B(x, r_1)$$

where $C > 1$ depends only on $\sqrt{K} R_0, r_1/r_2$ and n .

3 Torsion Function and the Bottom of Spectrum

In this section we obtain estimates between the bottom of the spectrum and the torsion function v_D . We shall prove the second and the third inequalities of (1.5).

Since the Green function $G_D(x, y)$ is the integral of the heat kernel $p_D(t, x, y)$ with respect to $t \in (0, \infty)$, we have

$$v_D(x) = \int_0^\infty P_D(t, x) dt,$$

where

$$P_D(t, x) = \int_D p_D(t, x, y) d\mu(y).$$

We note that $P_D(t, x) = \mathbb{P}_x[\tau_D > t]$, i.e., the survival probability that the Brownian motion $(B_t)_{t \geq 0}$ started at x stays in D up to time t , where τ_D is the first exit time from

D . We also observe that $P_D(t, x)$ is considered to be the (weak) solution to

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) u(t, x) &= 0 \quad \text{in } (0, \infty) \times D, \\ u(t, x) &= 0 \quad \text{on } (0, \infty) \times \partial D, \\ u(0, x) &= 1 \quad \text{on } \{0\} \times D. \end{aligned}$$

Let $\pi_D(t) = \sup_{x \in D} P_D(t, x)$. Let us begin with the proof of the second inequality of (1.5).

Lemma 3.1 *If $\lambda_{\min}(D) > 0$, then $\lambda_{\min}(D) \|v_D\|_\infty \geq 1$.*

Proof We follow [1, Lem. 3.2, Lem. 3.3]. Without loss of generality we may assume that $\|v_D\|_\infty < \infty$. It suffices to show the following two estimates:

$$\exp(-\lambda_{\min}(D) t) \leq \pi_D(t) \quad \text{for all } t > 0. \tag{3.1}$$

$$\text{If } C > 1, \text{ then } \pi_D(t) \leq \frac{C}{C-1} \exp\left(-\frac{t}{C\|v_D\|_\infty}\right) \quad \text{for all } t > 0. \tag{3.2}$$

In fact, we obtain from (3.1) and (3.2) that

$$\exp\left(-\lambda_{\min}(D) t + \frac{t}{C\|v_D\|_\infty}\right) \leq \frac{C}{C-1},$$

which holds for all $t > 0$ only if

$$\lambda_{\min}(D) \geq \frac{1}{C\|v_D\|_\infty}.$$

Since $C > 1$ is arbitrary, we have $\lambda_{\min}(D) \|v_D\|_\infty \geq 1$.

Let us prove (3.1). Take $\alpha > \lambda_{\min}(D)$. Then we find $\varphi \in C_0^\infty(D)$ such that $\|\nabla\varphi\|_2^2 / \|\varphi\|_2^2 \leq \alpha$. Take a bounded domain Ω such that $\text{supp } \varphi \subset \Omega \subset D$. Then Ω has no essential spectrum. Let λ_Ω and φ_Ω be the first eigenvalue and its positive eigenfunction with $\|\varphi_\Omega\|_2 = 1$ for Ω , respectively. By definition

$$\lambda_\Omega = \inf \left\{ \frac{\|\nabla\psi\|_2^2}{\|\psi\|_2^2} : \psi \in C_0^\infty(\Omega) \right\} \leq \frac{\|\nabla\varphi\|_2^2}{\|\varphi\|_2^2} \leq \alpha.$$

Since $u(t, x) = \exp(-\lambda_\Omega t) \varphi_\Omega(x)$ is the solution to the heat equation in $(0, \infty) \times \Omega$ such that $u(0, x) = \varphi_\Omega(x)$ and $u(t, x) = 0$ on $(0, \infty) \times \partial\Omega$, it follows from the comparison principle that

$$\exp(-\lambda_\Omega t) \varphi_\Omega(x) \leq \int_\Omega p_D(t, x, y) \varphi_\Omega(y) d\mu(y) \leq \|\varphi_\Omega\|_\infty P_D(t, x) \leq \|\varphi_\Omega\|_\infty \pi_D(t)$$

in $(0, \infty) \times \Omega$. Taking the supremum for $x \in \Omega$, and then dividing by $0 < \|\varphi_\Omega\|_\infty < \infty$, we obtain

$$\exp(-\alpha t) \leq \exp(-\lambda_\Omega t) \leq \pi_D(t).$$

Since $\alpha > \lambda_{\min}(D)$ is arbitrary, we have (3.1).

Let us prove (3.2) to complete the proof of the lemma. Let $C > 1$ and $\beta = 1/(C\|v_D\|_\infty)$. Put

$$w(t, x) = e^{-\beta t}(v_D(x) + (C - 1)\|v_D\|_\infty).$$

Since $-\Delta v_D = 1$ in D , it follows that

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)w &= -\beta e^{-\beta t}(v_D + (C - 1)\|v_D\|_\infty) - e^{-\beta t}\Delta v_D \\ &= e^{-\beta t}\left(-\frac{v_D + (C - 1)\|v_D\|_\infty}{C\|v_D\|_\infty} + 1\right) \\ &\geq e^{-\beta t}\left(-\frac{\|v_D\|_\infty + (C - 1)\|v_D\|_\infty}{C\|v_D\|_\infty} + 1\right) = 0. \end{aligned}$$

Hence w is a super solution to the heat equation. By the comparison principle

$$\begin{aligned} (C - 1)\|v_D\|_\infty P_D(t, x) &\leq w(t, x) \\ &= e^{-\beta t}(v_D(x) + (C - 1)\|v_D\|_\infty) \leq C e^{-\beta t}\|v_D\|_\infty. \end{aligned}$$

Dividing the inequality by $0 < \|v_D\|_\infty < \infty$, and taking the supremum for $x \in D$, we obtain (3.2). □

Next we prove the third inequality of (1.5) under an additional assumption on $\lambda_{\min}(D)$.

Lemma 3.2 *There exist $\Lambda_0 > 0$ and $C_0 > 0$ depending only on K and n such that if either $\lambda_{\min}(D) > \Lambda_0$ or $\|v_D\|_\infty < 1/\Lambda_0$, then*

$$\lambda_{\min}(D) \|v_D\|_\infty \leq C_0. \tag{3.3}$$

Proof In view of Lemma 3.1, we see that $\|v_D\|_\infty < 1/\Lambda_0$ implies $\lambda_{\min}(D) > \Lambda_0$. So, it suffices to show (3.3) under the assumption $\lambda_{\min}(D) > \Lambda_0$ with Λ_0 to be determined later.

For simplicity we write λ_D for $\lambda_{\min}(D)$, albeit $\lambda_{\min}(D)$ need not be an eigenvalue. Let $0 < R_0 < \infty$. By symmetry, the Gaussian estimate (2.1) implies

$$\begin{aligned} \frac{1}{CV(x, \sqrt{t})^{1/2}V(y, \sqrt{t})^{1/2}} \exp\left(-\frac{Cd(x, y)^2}{t}\right) &\leq p_M(t, x, y) \\ &\leq \frac{C}{V(x, \sqrt{t})^{1/2}V(y, \sqrt{t})^{1/2}} \exp\left(-\frac{d(x, y)^2}{Ct}\right) \end{aligned} \tag{3.4}$$

with the same C ; and conversely, (3.4) implies (2.1) with different C depending only on $\sqrt{K} R_0$ and n by volume doubling. Let $0 < t < R_0^2$. By [14, Ex. 10.29] we have

$$\begin{aligned} p_D(t, x, y) &\leq p_D(t, x, y)^{1/2} p_M(t, x, y)^{1/2} \\ &\leq \left(e^{-\lambda_D} \sqrt{p_D(t/2, x, x) p_D(t/2, y, y)} \right)^{1/2} p_M(t, x, y)^{1/2} \\ &\leq e^{-\lambda_D t/4} p_M(t/2, x, x)^{1/4} p_M(t/2, y, y)^{1/4} p_M(t, x, y)^{1/2}, \end{aligned}$$

so that the upper estimates of (2.1) and (3.4), together with volume doubling, show that $p_D(t, x, y)$ is bounded by

$$\begin{aligned} &e^{-\lambda_D t/4} \left\{ \frac{C}{V(x, \sqrt{t/2})} \right\}^{1/4} \cdot \left\{ \frac{C}{V(y, \sqrt{t/2})} \right\}^{1/4} \cdot \\ &\left\{ \frac{C}{V(x, \sqrt{t})^{1/2} V(y, \sqrt{t})^{1/2}} \exp\left(-\frac{d(x, y)^2}{Ct}\right) \right\}^{1/2} \\ &\leq e^{-\lambda_D t/4} \frac{CC'}{V(x, \sqrt{t})^{1/2} V(y, \sqrt{t})^{1/2}} \exp\left(-\frac{d(x, y)^2}{2Ct}\right), \end{aligned}$$

where C' takes care of the various volume doubling factors. By the lower estimate of (3.4) with $2C^2t$ in place of t and volume doubling, we find $C_1 \geq 1$ depending only on $\sqrt{K} R_0$ and n such that

$$p_D(t, x, y) \leq C_1 e^{-\lambda_D t/4} p_M(2C^2t, x, y).$$

Integrating the inequality with respect to $y \in D$, we obtain

$$\begin{aligned} P_D(t, x) &= \int_D p_D(t, x, y) d\mu(y) \\ &\leq C_1 e^{-\lambda_D t/4} \int_D p_M(2C^2t, x, y) d\mu(y) \leq C_1 e^{-\lambda_D t/4}. \end{aligned}$$

Taking the supremum over $x \in D$, we obtain

$$\pi_D(t) \leq C_1 \exp\left(-\frac{\lambda_D t}{4}\right) \quad \text{for } 0 < t < R_0^2. \tag{3.5}$$

Let $T = R_0^2/2$. We claim that (3.3) holds with $C_0 = 8 \log(2C_1)$, and with $\Lambda_0 = 4T^{-1} \log(2C_1)$ or

$$C_1 \exp\left(-\frac{\Lambda_0 T}{4}\right) = \frac{1}{2}. \tag{3.6}$$

Suppose $\lambda_D > \Lambda_0$. Then (3.5) with $t = T$ yields $\pi_D(T) \leq 1/2$. Solving the initial value problem from time T , we see that

$$P_D(t, x) \leq \pi_D(T) \cdot P_D(t - T, x) \leq \frac{1}{2} \quad \text{for } t \geq T.$$

Take the supremum for $x \in D$. We find

$$\pi_D(t) \leq \frac{1}{2} \text{ for } t \geq T.$$

Repeating the same argument, we obtain

$$\pi_D(t) \leq \frac{1}{2^k} \text{ for } kT \leq t < (k + 1)T \text{ with } k = 0, 1, 2, \dots$$

Hence

$$\begin{aligned} v_D(x) &= \int_0^\infty P_D(t, x) dt = \sum_{k=0}^\infty \int_{kT}^{(k+1)T} P_D(t, x) dt \\ &\leq \sum_{k=0}^\infty \int_{kT}^{(k+1)T} \pi_D(t) dt \leq T \sum_{k=0}^\infty \frac{1}{2^k} = 2T \leq \frac{2\Lambda_0 T}{\lambda_D} = \frac{8 \log(2C_1)}{\lambda_D} \end{aligned}$$

by (3.6). Taking the supremum for $x \in D$, we obtain $\lambda_D \|v_D\|_\infty \leq 8 \log(2C_1)$, as required. □

Remark 3.3 If the Gaussian estimate (2.1) holds uniformly for all $0 < t < \infty$, then there exists $C > 0$ such that $\lambda_{\min}(D) \|v_D\|_\infty \leq C$ for all $D \subset M$. This is the case when $K = 0$. See [19].

4 Capacitary Width and Harmonic Measure

By $\omega^x(E, D)$ we denote the harmonic measure of E in D evaluated at x . In this section we give an estimate for harmonic measure in terms of capacitary width. This will be crucial for the proof of Theorem 1.3.

Theorem 4.1 (cf. [1, Thm. 12.7]) *Let $0 < R_0 < \infty$. Let $D \subset M$ be an open set with $w_\eta(D) < R_0$. If $x \in D$ and $R > 0$, then*

$$\omega^x(D \cap \partial B(x, R), D \cap B(x, R)) \leq \exp\left(2C_2 - \frac{C_2 R}{w_\eta(D)}\right),$$

where C_2 depends only on $\sqrt{K} R_0, \eta$ and n .

Let us begin by estimating the torsion function of a ball.

Lemma 4.2 *Let $0 < R_0 < \infty$. Then there exists a constant $C > 1$ depending only on $\sqrt{K} R_0$ and n such that*

$$C^{-1}r^2 \leq \|v_{B(x,r)}\|_\infty \leq Cr^2 \text{ for } 0 < r < R_0.$$

Proof Let $0 < r < R_0$. Write $B = B(x, r)$ for simplicity. We have $\lambda_{\min}(B) \geq Cr^{-2}$ by Corollary 2.5. Since B is bounded, the bottom of the spectrum is an eigenvalue. So let us write λ_B for $\lambda_{\min}(B)$. Let $z \in B$. In view of [14, Ex. 10.29], the Gaussian estimate (2.1) and the volume doubling property, we have

$$\begin{aligned} v_B(z) &= \int_B G_B(z, y) d\mu(y) = \int_0^\infty dt \int_B p_B(t, z, y) d\mu(y) \\ &= \int_0^{r^2} dt \int_B p_B(t, z, y) d\mu(y) + \int_{r^2}^\infty dt \int_B p_B(t, z, y) d\mu(y) \\ &\leq r^2 + \int_{r^2}^\infty e^{-\lambda_B(t-r^2)} dt \int_B \sqrt{p_B(r^2, z, z) p_B(r^2, y, y)} d\mu(y) \\ &\leq r^2 + \frac{1}{\lambda_B} \int_B \frac{C d\mu(y)}{\sqrt{V(z, r) V(y, r)}} \leq r^2 + Cr^2, \end{aligned}$$

where C depends only on $\sqrt{K} R_0$ and n . Hence $\|v_B\|_\infty \leq Cr^2$.

The opposite inequality is an immediate consequence of the combination of Corollary 2.5 and Lemma 3.1. But for later purpose we give a direct proof based on a lower estimate of the Dirichlet heat kernel of a ball: if $x \in M$, then

$$p_B(t, y, z) \geq \frac{C}{V(x, \sqrt{t})} \text{ for } y, z \in \varepsilon B \text{ and } 0 < t < \varepsilon r^2$$

valid for some $0 < \varepsilon < 1$ and $C > 0$. In fact, this lower estimate is equivalent to the Gaussian estimate (2.1). See e.g. [6, (1.5)]. If $y \in \varepsilon B$, then

$$v_B(y) = \int_B G_B(y, z) d\mu(z) \geq \int_0^{\varepsilon r^2} dt \int_{\varepsilon B} p_B(t, y, z) d\mu(z) \geq \frac{\varepsilon r^2 C \mu(\varepsilon B)}{V(x, \sqrt{\varepsilon r^2})} \geq Cr^2$$

by volume doubling. Thus $\|v_B\|_\infty \geq Cr^2$. □

For later use we record the above estimate: if $0 < r < R_0$, then

$$v_{B(x,r)} \geq C_3 r^2 \text{ on } B(x, \varepsilon r), \tag{4.1}$$

where ε and C_3 depends only on $\sqrt{K} R_0$ and n .

Remark 4.3 In case $K > 0$, the inequality (4.1) does not necessarily hold for all $0 < r < \infty$ uniformly. Let \mathbb{H}^n be the n -dimensional hyperbolic space of constant curvature -1 . Then the torsion function for $B(a, r)$ is a radial function $f(\rho)$ of $\rho = d(x, a)$ satisfying

$$-1 = \Delta f(\rho) = \frac{1}{(\sinh \rho)^{n-1}} \frac{d}{d\rho} \left\{ (\sinh \rho)^{n-1} \frac{df}{d\rho} \right\} \text{ for } 0 < \rho < r,$$

$f(r) = 0, f'(0) = 0$ and $f(0) = \|v_{B(a,r)}\|_\infty$. See [11, pp. 176-177] or [14, (3.85)]. Hence

$$\|v_{B(a,r)}\|_\infty = \int_0^r \int_0^\rho \left(\frac{\sinh t}{\sinh \rho}\right)^{n-1} dt d\rho.$$

Since the integrand is less than 1, we have $\|v_{B(a,r)}\|_\infty \leq \frac{1}{2}r^2$ for all $r > 0$. Observe that $t \leq \sinh t$ for $t > 0$ and $\sinh \rho \leq \rho \cosh R_0$ for $0 < \rho < R_0$. Hence, if $0 < r < R_0$, then

$$\|v_{B(a,r)}\|_\infty \geq \int_0^r \int_0^\rho \left(\frac{t}{\rho \cosh R_0}\right)^{n-1} dt d\rho = \frac{r^2}{2n(\cosh R_0)^{n-1}},$$

so that $\|v_{B(a,r)}\|_\infty \approx r^2$. This gives the estimate in Lemma 4.2 with explicit bounds.

On the other hand, if $r > 1$, then $\sinh \rho \geq \frac{1}{2}(1 - e^{-2})e^\rho$ for $1 < \rho < r$, so that

$$\begin{aligned} \|v_{B(a,r)}\|_\infty &\leq \int_0^1 \int_0^\rho dt d\rho + \int_1^r \int_0^\rho \left(\frac{\sinh t}{\sinh \rho}\right)^{n-1} dt d\rho \\ &\leq \frac{1}{2} + \int_1^r \int_0^\rho \left(\frac{e^t}{(1 - e^{-2})e^\rho}\right)^{n-1} dt d\rho \\ &= \frac{1}{2} + \frac{1}{n-1} \int_1^r \frac{e^{(n-1)\rho} - 1}{((1 - e^{-2})e^\rho)^{n-1}} d\rho \leq \frac{1}{2} + \frac{r-1}{(n-1)(1 - e^{-2})^{n-1}}. \end{aligned}$$

Thus $\|v_{B(a,r)}\|_\infty = O(r)$ as $r \rightarrow \infty$, so (4.1) fails to hold uniformly for $0 < r < \infty$. This example illustrates that the assumption $0 < r < R_0$ cannot be dropped in Lemma 4.2.

Next we compare capacity and volume. Observe that $\text{Cap}_D(E)$ coincides with the Green capacity of E with respect to D , i.e.,

$$\text{Cap}_D(E) = \sup \left\{ \|v\| : \text{supp } v \subset E \text{ and } \int_D G_D(x, y) dv(y) \leq 1 \text{ on } D \right\}, \quad (4.2)$$

where $\|v\|$ stands for the total mass of the measure v .

Lemma 4.4 *Let $0 < R_0 < \infty$. There exists a constant $C_4 > 0$ depending only on $\sqrt{K} R_0$ and n such that if $0 < r < R_0$, then*

$$\frac{\mu(E)}{\mu(\overline{B}(x, r))} \leq C_4 \frac{\text{Cap}_{B(x,2r)}(E)}{\text{Cap}_{B(x,2r)}(\overline{B}(x, r))}$$

for every Borel set $E \subset \overline{B}(x, r)$.

Proof Let $0 < r < R_0$. Lemma 4.2 yields

$$\int_E G_{B(x,2r)}(y, z)d\mu(z) \leq \int_{B(x,2r)} G_{B(x,2r)}(y, z)d\mu(z) \leq \|v_{B(x,2r)}\|_\infty \leq Cr^2 \text{ for all } y \in M,$$

where C depends only on $\sqrt{K} R_0$ and n . Hence the characterization (4.2) of capacity gives

$$\text{Cap}_{B(x,2r)}(E) \geq \frac{\mu(E)}{Cr^2}. \tag{4.3}$$

Let $\varphi(y) = \min\{2 - d(y, x)/r, 1\}$. Observe that $\varphi \in W_0^1(B(x, 2r))$, $|\nabla\varphi| \leq 1/r$ and $\varphi = 1$ on $\overline{B}(x, r)$. The definition of capacity and the volume doubling property yield

$$\text{Cap}_{B(x,2r)}(\overline{B}(x, r)) \leq \int_{B(x,2r)} |\nabla\varphi|^2 d\mu \leq \frac{\mu(\overline{B}(x, 2r))}{r^2} \leq \frac{C\mu(\overline{B}(x, r))}{r^2}.$$

This, together with (4.3) for $E = \overline{B}(x, r)$, shows that $\text{Cap}_{B(x,2r)}(\overline{B}(x, r)) \approx r^{-2}\mu(\overline{B}(x, r))$ with the constant of comparison depending only on $\sqrt{K} R_0$ and n . Dividing (4.3) by $\text{Cap}_{B(x,2r)}(\overline{B}(x, r))$, we obtain the lemma. \square

Let us introduce regularized reduced functions, which are closely related to capacity and harmonic measure. See [3, Sect. 5.3-7] for the Euclidean case. Let D be an open set. For $E \subset D$ and a non-negative function u in E , we define the reduced function ${}^D\mathbf{R}_u^E$ by

$${}^D\mathbf{R}_u^E(x) = \inf\{v(x) : v \geq 0 \text{ is superharmonic in } D \text{ and } v \geq u \text{ on } E\} \text{ for } x \in D.$$

The lower semicontinuous regularization of ${}^D\mathbf{R}_u^E$ is called the *regularized reduced function* or *balayage* and is denoted by ${}^D\widehat{\mathbf{R}}_u^E$. It is known that ${}^D\widehat{\mathbf{R}}_u^E$ is a non-negative superharmonic function, ${}^D\widehat{\mathbf{R}}_u^E \leq {}^D\mathbf{R}_u^E$ in D with equality outside a polar set. If u is a non-negative superharmonic function in D , then ${}^D\widehat{\mathbf{R}}_u^E \leq u$ in D . By the maximum principle ${}^D\widehat{\mathbf{R}}_u^E$ is non-decreasing with respect to D and E . If u is the constant function 1, then ${}^D\widehat{\mathbf{R}}_1^E(x)$ is the probability of Brownian motion hitting E before leaving D when it starts at x . In an almost verbatim way we can extend [1, Lem. F] to the present setting. But, for completeness, we shall provide a proof.

Lemma 4.5 *Let $0 < r < R < R_0 < \infty$.*

- (i) $\inf_{\overline{B}(x,r)} {}^{B(x,R)}\widehat{\mathbf{R}}_1^E \leq \frac{\text{Cap}_{B(x,R)}(E)}{\text{Cap}_{B(x,R)}(\overline{B}(x, r))}$ for $E \subset B(x, R)$.
- (ii) $\frac{\text{Cap}_{B(x,R)}(\overline{B}(x, r))}{{}^{B(x,R)}\widehat{\mathbf{R}}_1^E} \leq C \inf_{\overline{B}(x,r)} {}^{B(x,R)}\widehat{\mathbf{R}}_1^E$ for $E \subset \overline{B}(x, r)$ with $C > 1$ depending only on $\sqrt{K} R_0, r/R$ and n .

Proof Let ν_E and ν_B be the capacity measures of E and $\overline{B}(x, r)$, respectively. Then ν_E is supported on \overline{E} , $G_{B(x,R)}\nu_E = {}^{B(x,R)}\widehat{\mathbf{R}}_1^E$ and $\|\nu_E\| = \text{Cap}_{B(x,R)}(E)$; ν_B is supported on $\overline{B}(x, r)$, $G_{B(x,R)}\nu_B = {}^{B(x,R)}\widehat{\mathbf{R}}_1^{\overline{B}(x,r)}$ and $\|\nu_B\| = \text{Cap}_{B(x,R)}(\overline{B}(x, r))$. In particular, $G_{B(x,R)}\nu_B \leq 1$ in $B(x, R)$ and hence

$$\begin{aligned} \text{Cap}_{B(x,R)}(E) &\geq \int G_{B(x,R)}\nu_B d\nu_E = \int G_{B(x,R)}\nu_E d\nu_B \\ &= \int {}^{B(x,R)}\widehat{\mathbf{R}}_1^E d\nu_B \geq \int \left(\inf_{\overline{B}(x,r)} {}^{B(x,R)}\widehat{\mathbf{R}}_1^E \right) d\nu_B \\ &= \left(\inf_{\overline{B}(x,r)} {}^{B(x,R)}\widehat{\mathbf{R}}_1^E \right) \text{Cap}_{B(x,R)}(\overline{B}(x, r)). \end{aligned}$$

Thus (i) follows.

Let $\rho = (r + R)/2$. The elliptic Harnack inequality (Corollary 2.6) implies

$$\begin{aligned} G_{B(x,R)}(z, y) &\approx G_{B(x,R)}(z, x) \quad \text{for } z \in \partial B(x, \rho) \text{ and } y \in \overline{B}(x, r), \\ {}^{B(x,R)}\widehat{\mathbf{R}}_1^{\overline{B}(x,r)} &\approx 1 \quad \text{on } \partial B(x, \rho), \end{aligned}$$

where, and hereafter, the constants of comparison depend only on $\sqrt{K} R_0, r/R$ and n . Let $E \subset \overline{B}(x, r)$. Since $\text{supp } \nu_E \subset \overline{B}(x, r)$, we have for $z \in \partial B(x, \rho)$,

$$\begin{aligned} {}^{B(x,R)}\widehat{\mathbf{R}}_1^E(z) &= \int G_{B(x,R)}(z, y) d\nu_E(y) \approx G_{B(x,R)}(z, x) \text{Cap}_{B(x,R)}(E), \\ {}^{B(x,R)}\widehat{\mathbf{R}}_1^{\overline{B}(x,r)}(z) &= \int G_{B(x,R)}(z, y) d\nu_B(y) \approx G_{B(x,R)}(z, x) \text{Cap}_{B(x,R)}(\overline{B}(x, r)), \end{aligned}$$

so that

$$\frac{\text{Cap}_{B(x,R)}(E)}{\text{Cap}_{B(x,R)}(\overline{B}(x, r))} \approx {}^{B(x,R)}\widehat{\mathbf{R}}_1^E(z).$$

Since $z \in \partial B(x, \rho)$ is arbitrary, the superharmonicity of ${}^{B(x,R)}\widehat{\mathbf{R}}_1^E$ and the maximum principle yield (ii). □

We restate the above lemma in terms of harmonic measure. We recall $\omega^x(E, D)$ stands for the harmonic measure of E in D evaluated at x . We see that if E is a compact subset of $B(x, R)$, then

$$\omega(\partial B(x, R), B(x, R) \setminus E) = 1 - {}^{B(x,R)}\widehat{\mathbf{R}}_1^E \quad \text{on } B(x, R). \tag{4.4}$$

Strictly speaking, the harmonic measure is extended by the right-hand side. Lemma 4.5 reads as follows.

Lemma 4.6 *Let $0 < r < R < R_0 < \infty$.*

- (i) $1 - \frac{\text{Cap}_{B(x,R)}(E)}{\text{Cap}_{B(x,R)}(\overline{B}(x,r))} \leq \sup_{\overline{B}(x,r)} \omega(\partial B(x,R), B(x,R) \setminus E)$ for $E \subset B(x,R)$.
- (ii) $\sup_{\overline{B}(x,r)} \omega(\partial B(x,R), B(x,R) \setminus E) \leq 1 - C^{-1} \frac{\text{Cap}_{B(x,R)}(E)}{\text{Cap}_{B(x,R)}(\overline{B}(x,r))}$ for $E \subset \overline{B}(x,r)$
with $C > 1$ depending only on $\sqrt{K} R_0, r/R$ and n . In particular, if $0 < r < R_0/2$, then

$$\sup_{\overline{B}(x,r)} \omega(\partial B(x,2r), B(x,2r) \setminus E) \leq 1 - C_5^{-1} \frac{\text{Cap}_{B(x,2r)}(E)}{\text{Cap}_{B(x,2r)}(\overline{B}(x,r))},$$

where $C_5 > 1$ depends only on $\sqrt{K} R_0$ and n .

Applying Lemma 4.6 repeatedly, we obtain the following estimate of harmonic measure, which is a preliminary version of Theorem 4.1.

Lemma 4.7 *Let $0 < R_0 < \infty$. Let $D \subset M$ be an open set with $w_\eta(D) < R_0$. Suppose $x \in D$ and $R > 0$. If k is a non-negative integer such that $R - 2kw_\eta(D) > 0$, then*

$$\sup_{D \cap \overline{B}(x, R - 2kw_\eta(D))} \omega(D \cap \partial B(x,R), D \cap B(x,R)) \leq (1 - C_5^{-1}\eta)^k.$$

Proof For simplicity let $\omega_0 = \omega(D \cap \partial B(x,R), D \cap B(x,R))$. By definition we find $r > w_\eta(D)$ arbitrarily close to $w_\eta(D)$ such that

$$\frac{\text{Cap}_{B(y,2r)}(\overline{B}(y,r) \setminus D)}{\text{Cap}_{B(y,2r)}(\overline{B}(y,r))} \geq \eta \text{ for all } y \in D.$$

Hence it suffices to show that $\omega_0 \leq (1 - C_5^{-1}\eta)^k$ in $D \cap \overline{B}(x, R - 2kr)$. Let us prove this inequality by induction on k . The case $k = 0$ holds trivially. Let $k \geq 1$ and suppose $\omega_0 \leq (1 - C_5^{-1}\eta)^{k-1}$ in $D \cap \overline{B}(x, R - 2(k-1)r)$. Take $y \in D \cap \partial B(x, R - 2kr)$ and let $E = \overline{B}(y,r) \setminus D$. Since $D \cap B(y,2r) \subset D \cap \overline{B}(x, R - 2(k-1)r)$, we have

$$\begin{aligned} \omega_0 &\leq (1 - C_5^{-1}\eta)^{k-1} \omega(D \cap \partial B(y,2r), D \cap B(y,2r)) \\ &\leq (1 - C_5^{-1}\eta)^{k-1} \omega(\partial B(y,2r), D \setminus E) \leq (1 - C_5^{-1}\eta)^k \end{aligned}$$

in $D \cap B(y,2r)$. Since $y \in D \cap \partial B(x, R - 2kr)$ is arbitrary, we have $\omega_0 \leq (1 - C_5^{-1}\eta)^k$ on $D \cap \partial B(x, R - 2kr)$, and hence in $D \cap \overline{B}(x, R - 2kr)$ by the maximum principle, as required. □

This lemma and the definition of capacity width yield

Proof of Theorem 4.1 Let k be the integer such that $2kw_\eta(D) < R \leq 2(k + 1)w_\eta(D)$. Lemma 4.7 gives

$$\begin{aligned} \omega^x(D \cap \partial B(x, R), D \cap B(x, R)) &\leq (1 - C_5^{-1}\eta)^k = \exp\left(-k \log \frac{1}{1 - C_5^{-1}\eta}\right) \\ &\leq \exp\left(-\left(\frac{R}{2w_\eta(D)} - 1\right) \log \frac{1}{1 - C_5^{-1}\eta}\right), \end{aligned}$$

which implies the required inequality with

$$C_2 = \frac{1}{2} \log \frac{1}{1 - C_5^{-1}\eta}.$$

5 Proofs of Theorems 1.3 and 1.6

In this section we prove Theorem 1.3 and complete the proof of Theorem 1.6 by showing

Theorem 5.1 *Let $0 < R_0 < \infty$. If $w_\eta(D) < R_0$, then*

$$C^{-1}w_\eta(D)^2 \leq \|v_D\|_\infty \leq Cw_\eta(D)^2 \tag{5.1}$$

where C depends only on $\sqrt{K} R_0, \eta$ and n .

This theorem, together with (3.2) in Lemma 3.1, immediately yields the following estimate of the survival probability, which plays a crucial role in the proof of Theorem 1.5.

Theorem 5.2 *Let $0 < R_0 < \infty$. There exist positive constants C_6 and C_7 depending only on $\sqrt{K} R_0, \eta$ and n such that*

$$P_D(t, x) \leq C_6 \exp\left(-\frac{C_7 t}{w_\eta(D)^2}\right) \text{ for all } t > 0 \text{ and } x \in D, \tag{5.2}$$

whenever $w_\eta(D) < R_0$.

Let us begin with a uniform estimate of the capacity of balls.

Lemma 5.3 *Let $0 < R_0 < \infty$. For $0 < t \leq 1$, define*

$$\kappa(t) = \inf \left\{ \frac{\text{Cap}_{B(x, 2R)}(\overline{B}(x, tR))}{\text{Cap}_{B(x, 2R)}(\overline{B}(x, R))} : x \in M, 0 < R < R_0 \right\}.$$

Then $\lim_{t \rightarrow 1} \kappa(t) = 1$.

Proof Without loss of generality we may assume that $1/2 < t \leq 1$. Let $\Omega = B(x, 2R) \setminus \overline{B}(x, tR)$ and let $E_t = \partial B(x, tR)$. We find $a > 0$ such that for each $y \in E_t$ and $0 < r < \frac{1}{4}R$ there exists a ball of radius ar lying in $B(y, r) \setminus \Omega$. This means that

$$\frac{\mu(B(y, r) \setminus \Omega)}{\mu(B(y, r))} \geq \varepsilon$$

with some $\varepsilon > 0$ depending only on a and the doubling constant. By Lemmas 4.4 and 4.6 we have

$$\sup_{\overline{B}(y, r)} \omega(\partial B(y, 2r), B(y, 2r) \cap \Omega) \leq 1 - \varepsilon' \tag{5.3}$$

with $\varepsilon' > 0$ independent of x, R, t, y and r .

The technique in the proof of [2, Thm. 1] yields a positive superharmonic function s in Ω such that

$$s \approx \text{dist}(\cdot, E_t)^\alpha, \tag{5.4}$$

where $\alpha > 0$ and the constants of comparison are independent of x, R and t . In fact, let $r_k = 4^k, k \in \mathbb{Z}$. For each $k \in \mathbb{Z}$ choose a locally finite covering of E_t by open balls $B(x_{kj}, r_k/4), j \in J_k$; let $B_{kj} = B(x_{kj}, r_k)$. By (5.3) we find a positive continuous function u_{kj} in $\Omega \cap \overline{B}_{kj}$, superharmonic in $\Omega \cap B_{kj}$, such that $\varepsilon'' \leq u_{kj} \leq 2$ in $\Omega \cap B_{kj}$, $u_{kj} \geq 1$ in $\Omega \cap \partial B_{kj}$, $u_{kj} \leq 1 - \varepsilon''$ in $\Omega \cap \frac{1}{2}B_{kj}$, where ε'' is a small positive constant depending only on ε' . Let $A = 1 - \varepsilon''/2$ and extend u_{kj} on $\Omega \setminus \overline{B}_{kj}$ by $u_{kj} = \infty$. Then

$$s(x) = \inf\{A^{-k}u_{kj}(x) : k \in \mathbb{Z}, j \in J_k\}, \quad x \in \Omega$$

is a superharmonic function in Ω satisfying (5.4) with $\alpha = |\log A|/\log 4$. Actually, we can make s a strong barrier. In the present context, however, superharmonicity is enough.

From (5.4), we find a positive constant C independent of x, R and t such that

$$\frac{s}{CR^\alpha} \geq 1 \quad \text{on } \partial B(x, 3R/2).$$

Let u be the capacity potential for $\overline{B}(x, tR)$ in $B(x, 2R)$, i.e.,

$$\begin{aligned} \Delta u &= 0 \quad \text{in } B(x, 2R) \setminus \overline{B}(x, tR), \\ u &= 1 \quad \text{on } \overline{B}(x, tR), \\ u &= 0 \quad \text{on } \partial B(x, 2R), \\ \text{Cap}_{B(x, 2R)}(\overline{B}(x, tR)) &= \int_{B(x, 2R)} |\nabla u|^2 d\mu. \end{aligned}$$

Since $1 - u \leq s/(CR^\alpha)$ on $\partial B(x, 3R/2)$, it follows from the maximum principle

$$1 - u \leq \frac{s}{CR^\alpha} \approx \frac{\text{dist}(\cdot, E_t)^\alpha}{R^\alpha} \quad \text{in } B(x, 3R/2) \setminus \bar{B}(x, tR).$$

Hence

$$u \geq 1 - C \frac{((1-t)R)^\alpha}{R^\alpha} = 1 - C(1-t)^\alpha \quad \text{in } B(x, R) \setminus \bar{B}(x, tR)$$

with another positive constant C . If $1 - C(1-t)^\alpha > 0$, then by definition,

$$\begin{aligned} \text{Cap}_{B(x,2R)}(\bar{B}(x, R)) &\leq \frac{1}{(1 - C(1-t)^\alpha)^2} \int_{B(x,2R)} |\nabla u|^2 d\mu \\ &= \frac{\text{Cap}_{B(x,2R)}(\bar{B}(x, tR))}{(1 - C(1-t)^\alpha)^2}. \end{aligned}$$

Hence

$$\frac{\text{Cap}_{B(x,2R)}(\bar{B}(x, tR))}{\text{Cap}_{B(x,2R)}(\bar{B}(x, R))} \geq (1 - C(1-t)^\alpha)^2,$$

so that the lemma follows as $\lim_{t \rightarrow 1} (1 - C(1-t)^\alpha)^2 = 1$. □

Proof of Theorem 1.3 By definition the first inequality holds for arbitrary open sets D . Let us prove the second inequality. In view of Lemma 5.3, we find an integer $N \geq 2$ depending only on $\sqrt{K} R_0$ and n such that

$$\frac{\text{Cap}_{B(x,2R)}(\bar{B}(x, (1 - N^{-1})R))}{\text{Cap}_{B(x,2R)}(\bar{B}(x, R))} \geq \sqrt{\eta} \tag{5.5}$$

uniformly for $x \in M$ and $0 < R < R_0$. Let C_5 be as in Lemma 4.6 and take an integer $k > 2$ so large that $(1 - C_5^{-1}\eta')^k \leq 1 - \sqrt{\eta}$.

Let $w_\eta(D) < R_0$. We prove the theorem by showing

$$w_\eta(D) \leq 2Nkw_{\eta'}(D). \tag{5.6}$$

If $w_{\eta'}(D) \geq R_0/(2Nk)$, then $w_\eta(D) < R_0 \leq 2Nkw_{\eta'}(D)$, so (5.6) follows. Suppose

$$w_{\eta'}(D) < \frac{R_0}{2Nk}.$$

For simplicity we write $\rho = w_{\eta'}(D)$. Apply Lemma 4.7, with η' in place of η , to $x \in D$ and $R = 2Nk\rho$. We obtain

$$\sup_{D \cap \bar{B}(x, R-2k\rho)} \omega(D \cap \partial B(x, R), D \cap B(x, R)) \leq (1 - C_5^{-1}\eta')^k \leq 1 - \sqrt{\eta}.$$

Let $E = \overline{B}(x, R) \setminus D$. Then the maximum principle yields

$$\omega(\partial B(x, 2R), B(x, 2R) \setminus E) \leq \omega(D \cap \partial B(x, R), D \cap B(x, R)) \text{ in } D \cap B(x, R),$$

so that

$$\omega(\partial B(x, 2R), B(x, 2R) \setminus E) \leq 1 - \sqrt{\eta} \text{ in } \overline{B}(x, R - 2k\rho),$$

where we use the convention $\omega(\partial B(x, 2R), B(x, 2R) \setminus E) = 0$ in E . Hence, Lemma 4.6 (i) with $R - 2k\rho$ and $2R$ in place of r and R gives

$$1 - \frac{\text{Cap}_{B(x, 2R)}(E)}{\text{Cap}_{B(x, 2R)}(\overline{B}(x, R - 2k\rho))} \leq 1 - \sqrt{\eta},$$

so that

$$\frac{\text{Cap}_{B(x, 2R)}(E)}{\text{Cap}_{B(x, 2R)}(\overline{B}(x, R - 2k\rho))} \geq \sqrt{\eta}.$$

Multiplying the inequality and (5.5), we obtain

$$\frac{\text{Cap}_{B(x, 2R)}(E)}{\text{Cap}_{B(x, 2R)}(\overline{B}(x, R))} \geq \eta,$$

as $R - 2k\rho = (1 - N^{-1})R$. Since $x \in D$ is arbitrary, we have $w_\eta(D) < R = 2Nk\rho = 2Nkw_{\eta'}(D)$. Thus we have (5.6). □

Proof of Theorem 5.1 First, let us prove the second inequality of (5.1), i.e., $\|v_D\|_\infty \leq Cw_\eta(D)^2$. In view of the monotonicity of the torsion function, we may assume that D is bounded and hence $\|v_D\|_\infty < \infty$. By definition we find $r, w_\eta(D) \leq r < 2w_\eta(D) < 2R_0$, such that

$$\frac{\text{Cap}_{B(x, 2r)}(\overline{B}(x, r) \setminus D)}{\text{Cap}_{B(x, 2r)}(\overline{B}(x, r))} \geq \eta \text{ for every } x \in D.$$

For a moment we fix $x \in D$ and let $B = B(x, r)$, $B^* = B(x, 2r)$, and $E = \overline{B} \setminus D$ for simplicity. Then $\text{Cap}_{B^*}(E) / \text{Cap}_{B^*}(\overline{B}) \geq \eta$. We compare v_D with

$$v_{B^*} = \int_{B^*} G_{B^*}(\cdot, y) d\mu(y).$$

It is easy to see that $v_D - v_{B^*}$ is harmonic in $D \cap B^*$ and $v_D = 0$ on ∂D outside a polar set. Hence the maximum principle yields

$$v_D - v_{B^*} \leq \|v_D\|_\infty \omega(D \cap \partial B^*, D \cap B^*) \text{ in } D \cap B^*.$$

Since Lemma 4.6 implies that

$$\omega^x(D \cap \partial B^*, D \cap B^*) \leq \omega^x(\partial B^*, B^* \setminus E) \leq 1 - C_5^{-1}\eta,$$

it follows from Lemma 4.2 that

$$v_D(x) \leq v_{B^*}(x) + \|v_D\|_\infty \omega^x(D \cap \partial B^*, D \cap B^*) \leq Cr^2 + \|v_D\|_\infty(1 - C_5^{-1}\eta).$$

Taking the supremum with respect to $x \in D$, we obtain

$$\|v_D\|_\infty \leq CC_5\eta^{-1}r^2 \leq 4CC_5\eta^{-1}w_\eta(D)^2.$$

Second, let us prove the first inequality of (5.1), i.e. $w_\eta(D)^2 \leq C\|v_D\|_\infty$. We distinguish two cases. Suppose first $\|v_D\|_\infty \geq C_3R_0^2/2$ with C_3 as in (4.1). Then

$$\|v_D\|_\infty \geq \frac{1}{2}C_3R_0^2 > \frac{1}{2}C_3w_\eta(D)^2,$$

as required. Suppose next $\|v_D\|_\infty < C_3R_0^2/2$. Take R such that

$$\|v_D\|_\infty = \frac{C_3R^2}{2}. \tag{5.7}$$

Then $0 < R < R_0$. Let $x \in D$. This time, we let $B = B(x, R)$, $B^* = B(x, 2R)$ and $E = \overline{B} \setminus D$ with R as in (5.7). We shall compare v_D with the torsion function

$$v_B = \int_B G_B(\cdot, y)d\mu(y).$$

Observe that $v_B - v_D$ is harmonic in $D \cap B$. By the maximum principle and Lemma 4.2

$$\begin{aligned} v_B - v_D &\leq \sup_E v_B \cdot \omega(\partial E, B \setminus E) = \sup_E v_B \cdot (1 - \omega(D \cap \partial B, B \setminus E)) \\ &\leq CR^2(1 - \omega(\partial B^*, B^* \setminus E)) \quad \text{in } D \cap B, \end{aligned}$$

since $\partial(D \cap B) \subset (B \cap \partial D) \cup (D \cap \partial B) \subset E \cup \partial B$, and since $v_B = 0$ on ∂B . Let $0 < \varepsilon < 1$ be as in (4.1). Taking the infimum over $\overline{B}(x, \varepsilon R)$, we obtain from Lemma 4.6 that

$$\begin{aligned} \inf_{\overline{B}(x, \varepsilon R)} v_B - \|v_D\|_\infty &\leq CR^2 \left(1 - \sup_{\overline{B}(x, \varepsilon R)} \omega(\partial B^*, B^* \setminus E) \right) \\ &\leq CR^2 \frac{\text{Cap}_{B^*}(E)}{\text{Cap}_{B^*}(\overline{B}(x, \varepsilon R))}. \end{aligned}$$

Hence, (4.1) and (5.7) yield

$$C_3 R^2 - \frac{C_3 R^2}{2} \leq C R^2 \frac{\text{Cap}_{B^*}(E)}{\text{Cap}_{B^*}(\overline{B}(x, \varepsilon R))}.$$

Dividing by $C R^2$, we obtain

$$\frac{\text{Cap}_{B^*}(E)}{\text{Cap}_{B^*}(\overline{B}(x, \varepsilon R))} \geq \frac{C_3}{2C},$$

so that, by Lemma 4.4 and volume doubling

$$\begin{aligned} \frac{\text{Cap}_{B^*}(E)}{\text{Cap}_{B^*}(\overline{B}(x, R))} &= \frac{\text{Cap}_{B^*}(E)}{\text{Cap}_{B^*}(\overline{B}(x, \varepsilon R))} \cdot \frac{\text{Cap}_{B^*}(\overline{B}(x, \varepsilon R))}{\text{Cap}_{B^*}(\overline{B}(x, R))} \\ &\geq \frac{C_3}{2C} \cdot \frac{C\mu(\overline{B}(x, \varepsilon R))}{\mu(\overline{B}(x, R))} \geq \eta' \end{aligned}$$

with $0 < \eta' < 1$ depending only on $\sqrt{K} R_0$ and n . Thus

$$\frac{\text{Cap}_{B^*}(\overline{B}(x, R) \setminus D)}{\text{Cap}_{B^*}(\overline{B}(x, R))} \geq \eta'.$$

Since $x \in D$ is arbitrary, we have $w_{\eta'}(D) < R$ and so $w_{\eta}(D) \leq CR$ by Theorem 1.3. Hence $w_{\eta}(D)^2 \leq C\|v_D\|_{\infty}$ by (5.7). The proof is complete. \square

6 Proof of Theorem 1.5

The crucial step of the proof of Theorem 1.5 is the following *parabolic box argument* (cf. [1, Lem. 4.1]),

Lemma 6.1 *Suppose (1.2) holds. If $t > 0$, then*

$$P_D(t, x) \leq C_t G_D(x, o) \quad \text{for } x \in D \quad (6.1)$$

with C_t depending on t .

Proof Without loss of generality we may assume that $\tau = 1$ in (1.2). For notational convenience we shall prove (6.1) with T in place of t . For simplicity we write $w_{\eta}(G_D^o < s) = w_{\eta}(\{x \in D : G_D(x, o) < s\})$. Let $\alpha_j = \exp(-2^j)$. Since

$$\begin{aligned} \int_{\alpha_j}^{\alpha_{j-1}} w_{\eta}(G_D^o < s)^2 \frac{ds}{s} &\geq w_{\eta}(G_D^o < \alpha_j)^2 \int_{\alpha_j}^{\alpha_{j-1}} \frac{ds}{s} \\ &= w_{\eta}(G_D^o < \alpha_j)^2 (2^j - 2^{j-1}) = 2^{j-1} w_{\eta}(G_D^o < \alpha_j)^2, \end{aligned}$$

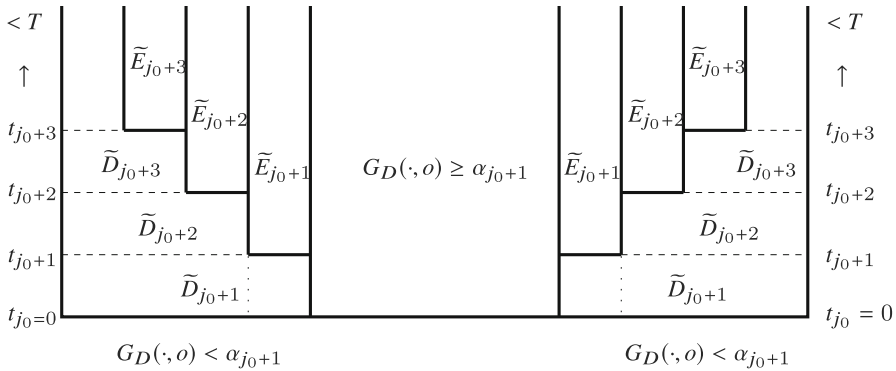


Fig. 1 Parabolic box argument

it follows from (1.2) that $\sum_{j=0}^{\infty} 2^j w_{\eta}(G_D^o < \alpha_j)^2 < \infty$.

Let $w_{\eta}(G_D^o < 1) < R_0 < \infty$ and choose C_6 and C_7 as in Theorem 5.2. We find $j_0 \geq 0$ such that

$$\frac{3}{C_7} \sum_{j=j_0+1}^{\infty} 2^j w_{\eta}(G_D^o < \alpha_j)^2 < T. \tag{6.2}$$

Define

$$t_k = \frac{3}{C_7} \sum_{j=j_0+1}^k 2^j w_{\eta}(G_D^o < \alpha_j)^2 \text{ for } k \geq j_0 + 1,$$

and $t_{j_0} = 0$. Then t_k increases and $\lim_{k \rightarrow \infty} t_k < T$ by (6.2). Observe that

$$\frac{1}{\alpha_{k+1}} \exp\left(-\frac{C_7(t_k - t_{k-1})}{w_{\eta}(G_D^o < \alpha_k)^2}\right) = \exp(2^{k+1} - 3 \cdot 2^k) = \exp(-2^k) \tag{6.3}$$

for $k \geq j_0 + 1$.

Let $D_k = \{x \in D : G_D(x, o) < \alpha_k\}$, $E_k = \{x \in D : \alpha_{k+1} \leq G_D(x, o) < \alpha_k\}$, $\tilde{D}_k = (t_{k-1}, \infty) \times D_k$ and $\tilde{E}_k = (t_k, \infty) \times E_k$. Put

$$q_k = \sup_{(t,x) \in \tilde{E}_k} \frac{P_D(t, x)}{G_D(x, o)}.$$

We claim that $\sup_{k \geq j_0+1} q_k \leq C$, which implies (6.1) with T in place of t , and $C_T = \max\{C, 1/\alpha_{j_0+1}\}$ since $(T, \infty) \times \{x \in D : G_D(x, o) < \alpha_{j_0+1}\} \subset \cup_{k \geq j_0+1} \tilde{E}_k$ by (6.2). See Fig. 1.

By the parabolic comparison principle over \tilde{D}_{j_0+1} we have

$$P_D(t, x) \leq \frac{G_D(x, o)}{\alpha_{j_0+1}} + P_{D_{j_0+1}}(t, x) \text{ for } (t, x) \in \tilde{D}_{j_0+1} = (0, \infty) \times D_{j_0+1}.$$

Divide both sides by $G_D(x, o)$ and take the supremum over \tilde{E}_{j_0+1} . Then (5.2) and (6.3) yield

$$\begin{aligned} q_{j_0+1} &\leq \frac{1}{\alpha_{j_0+1}} + \sup_{(t,x) \in \tilde{E}_{j_0+1}} \frac{P_{D_{j_0+1}}(t, x)}{G_D(x, o)} \\ &\leq \frac{1}{\alpha_{j_0+1}} + \frac{C_6}{\alpha_{j_0+2}} \sup_{t \geq t_{j_0+1}} \exp\left(-\frac{C_7 t}{w_\eta(D_{j_0+1})^2}\right) \\ &\leq \frac{1}{\alpha_{j_0+1}} + \frac{C_6}{\alpha_{j_0+2}} \exp\left(-\frac{C_7(t_{j_0+1} - t_{j_0+1})}{w_\eta(D_{j_0+1})^2}\right) = \exp(2^{j_0+1}) + C_6 \exp(-2^{j_0+1}). \end{aligned}$$

Let $k \geq j_0 + 2$. By the parabolic comparison principle over \tilde{D}_k we have

$$P_D(t, x) \leq q_{k-1} G_D(x, o) + P_{D_k}(t - t_{k-1}, x) \quad \text{for } (t, x) \in \tilde{D}_k = (t_{k-1}, \infty) \times D_k.$$

Divide both sides by $G_D(x, o)$ and take the supremum over \tilde{E}_k . In the same way as above, we obtain from (5.2) and (6.3) that

$$q_k \leq q_{k-1} + \frac{C_6}{\alpha_{k+1}} \exp\left(-\frac{C_7(t_k - t_{k-1})}{w_\eta(D_k)^2}\right) \leq q_{k-1} + C_6 \exp(-2^k).$$

Hence we have the claim as

$$\sup_{k \geq j_0+1} q_k \leq \exp(2^{j_0+1}) + C_6 \sum_{k=j_0+1}^\infty \exp(-2^k) < \infty.$$

The lemma is proved. □

Proof of Theorem 1.5 By Theorem 1.4 we have the first condition for IU. Let us show (1.1) for every $t > 0$. It is known that the lower estimate of (1.1) follows from the upper estimate. Moreover, if $p_D(t_0, x, y) \leq C_{t_0} \varphi_D(x) \varphi_D(y)$ for all $x, y \in D$ with some $t_0 > 0$, then $p_D(t, x, y) \leq C_t \varphi_D(x) \varphi_D(y)$ holds with $C_t \leq C_{t_0} e^{-\lambda_D(t-t_0)}$ for $t \geq t_0$ (See e.g. [1, Prop. 2.1]). Hence, it suffices to show the upper estimate of (1.1) for small $t > 0$.

Since φ_D is superharmonic, and since $G_D(\cdot, o)$ is harmonic outside $\{o\}$, we have $G_D(\cdot, o) \leq C\varphi_D$ apart from a neighborhood of o . So, it is sufficient to show that if $t > 0$ small, then there exists $C_t > 0$ such that

$$p_D(t, x, y) \leq C_t G_D(x, o) G_D(y, o) \quad \text{for } x, y \in D. \tag{6.4}$$

Let i_0 be the injectivity radius of M . It is known that

$$\mu(B(x, r)) \geq Cr^n \quad \text{for } 0 < r < i_0/2 \text{ and } x \in M.$$

where $C > 0$ depends only on M (Croke [9, Prop. 14]). Hence, the Gaussian estimate (2.1) yields

$$p_M(t, x, y) \leq \frac{C}{V(x, \sqrt{t})} \leq Ct^{-n/2} \tag{6.5}$$

for $0 < t < \min\{R_0^2, (i_0/2)^2\}$ and $x, y \in M$. Let $0 < t < \min\{R_0^2, (i_0/2)^2\}$ and $x, y, z \in D$. By (6.5) we have

$$\begin{aligned} p_D(2t, z, y) &= \int_D p_D(t, z, w)p_D(t, w, y)d\mu(w) \\ &\leq \int_D p_M(t, z, w)p_D(t, w, y)d\mu(w) \\ &\leq Ct^{-n/2} \int_D p_D(t, w, y)d\mu(w) = Ct^{-n/2} P_D(t, y), \end{aligned}$$

since the heat kernel is symmetric. Moreover,

$$\begin{aligned} p_D(3t, x, y) &\leq \int_D p_D(t, x, z)p_D(2t, z, y)d\mu(z) \\ &\leq \int_D p_D(t, x, z)Ct^{-n/2} P_D(t, y)d\mu(z) \\ &= Ct^{-n/2} P_D(t, x)P_D(t, y). \end{aligned}$$

Hence Lemma 6.1 yields

$$p_D(3t, x, y) \leq Ct^{-n/2} P_D(t, x)P_D(t, y) \leq C_t G_D(x, o)G_D(y, o).$$

Replacing $3t$ by t , we obtain (6.4) for small $t > 0$. Thus the theorem is proved. \square

Remark 6.2 The assumption on the injectivity radius can be replaced by

$$\inf_{x \in M} \mu(B(x, R_0)) > 0. \tag{6.6}$$

In fact, (2.2) yields

$$\mu(B(x, r)) \geq C \left(\frac{r}{R_0}\right)^\alpha \inf_{x \in M} \mu(B(x, R_0)) \quad \text{for all } x \in M \text{ and } 0 < r < R_0,$$

and hence for small $t > 0$,

$$p_M(t, x, y) \leq \frac{C}{V(x, \sqrt{t})} \leq Ct^{-\alpha/2}.$$

Replacing (6.5) by this inequality, we obtain

$$p_D(3t, x, y) \leq Ct^{-\alpha/2} P_D(t, x)P_D(t, y) \leq C_t G_D(x, o)G_D(y, o),$$

which proves Theorem 1.5. See [10] for further discussion on (6.6).

7 Remarks

Once we obtain the theorems in Sect. 1, we can extend many Euclidean results to the setting of manifolds. Proofs are almost the same as in the Euclidean case. For instance, we relax the requirement of inner uniformity for IU assumed in [15, Thm. 7.9]. For a curve γ in M we denote the length of γ and the subarc of γ between x and y by $\ell(\gamma)$ and $\gamma(x, y)$, respectively. For a domain D in M we define the inner metric in D as

$$d_D(x, y) = \inf\{\ell(\gamma) : \gamma \text{ is a curve connecting } x \text{ and } y \text{ in } D\}.$$

Definition 7.1 Let D be a domain in M and let $\delta_D(x) = \text{dist}(x, M \setminus D)$.

(i) We say that D is a John domain if there exist $o \in D$ and $C \geq 1$ such that every $x \in D$ is connected to o by a rectifiable curve $\gamma \subset D$ with the property

$$\ell(\gamma(x, z)) \leq C\delta_D(z) \quad \text{for all } z \in \gamma.$$

(ii) We say that D is an inner uniform domain if there exists $C \geq 1$ such that every pair of points $x, y \in D$ can be connected by a rectifiable curve $\gamma \subset D$ with the properties $\ell(\gamma) \leq Cd_D(x, y)$ and

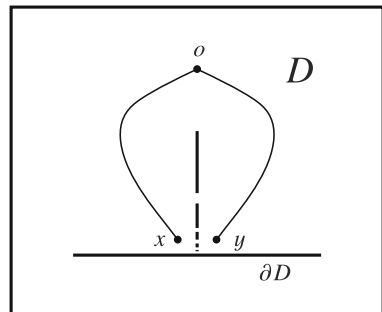
$$\min\{\ell(\gamma(x, z)), \ell(\gamma(z, y))\} \leq C\delta_D(z) \quad \text{for all } z \in \gamma.$$

If we replace $d_D(x, y)$ by the ordinary metric $d(x, y)$ in (ii), then we obtain a uniform domain. By definition a John domain is necessarily bounded. We have the following inclusions for these classes of bounded domains:

$$\text{uniform} \subsetneq \text{inner uniform} \subsetneq \text{John}.$$

Figure 2 depicts a John domain that is not inner uniform. We find a curve connecting x and o with the property of Definition 7.1 (i); yet there is no curve connecting x and

Fig. 2 A John domain that is not inner uniform



y with the properties of Definition 7.1 (ii) if the gaps on the vertical segment shrink sufficiently fast.

Theorem 7.2 *A John domain is IU.*

Proof Let D be a John domain. Observe that $w_\eta(\{x \in D : \delta_D(x) < r\}) \leq Cr$ for small $r > 0$ by definition and $G_D(x, o) \geq C\delta_D(x)^\alpha$ with some $\alpha > 0$ by the Harnack inequality. Hence

$$w_\eta(\{x \in D : G_D(x, o) < t\}) \leq w_\eta(\{x \in D : \delta_D(x) < (t/C)^{1/\alpha}\}) \leq Ct^{1/\alpha},$$

so that (1.2) holds. Therefore Theorem 1.5 asserts that D is IU. \square

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