BOUNDED HOMOMORPHISMS AND FINITELY GENERATED FIBER PRODUCTS OF LATTICES

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Abstract. We investigate when fiber products of lattices are finitely generated and obtain a new characterization of bounded lattice homomorphisms onto lattices satisfying a property we call Dean's condition (D) which arises from Dean's solution to the word problem for finitely presented lattices. In particular, all finitely presented lattices and those satisfying Whitman's condition satisfy (D). For lattice epimorphisms $g\colon A \to D$, $h\colon B \to D$, where A, B are finitely generated and D satisfies (D), we show the following: If g and h are bounded, then their fiber product (pullback) $C = \{(a, b) \in A\times B \mid g(a) = h(b)\}$ is finitely generated. While the converse is not true in general, it does hold when A and B are free. As a consequence we obtain an (exponential time) algorithm to decide boundedness for finitely presented lattices and their finitely generated sublattices satisfying (D). This generalizes an unpublished result of Freese and Nation.

1. Introduction

A *subdirect product* of algebraic structures A and B is a subalgebra C of the direct product $A \times B$ that projects onto both factors. In [7, 6] the second and third authors studied conditions under which direct and subdirect products of various algebras are finitely generated. *Direct* products of finitely generated lattices are finitely generated. On the other hand, note that a congruence α of an algebra A is a subdirect product of two copies of A. If α is finitely generated as subalgebra of A^2, then it is clearly finitely generated as congruence of A as well. For every non-finitely presented quotient $F(X)/\rho$ of a finitely generated free lattice $F(X)$, the congruence ρ is a *subdirect* product of $F(X)$ with itself that is not finitely generated as a lattice. In [6, Example 7.5] an explicit example is given of a congruence ρ such that $F(X)/\rho$ is finite, hence finitely presented, but ρ is not finitely generated as a sublattice of $F(X) \times F(X)$. The present paper is a continuation of that work.

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We start by recalling a standard method for constructing subdirect products. Let A, B be algebras with epimorphisms $g\colon A \to D$ and $h\colon B \to D$ onto the same homomorphic image D. Then the subalgebra

$$C := \{(a, b) \in A \times B \mid g(a) = h(b)\}$$

of $A \times B$ is called a *fiber product* (or *pullback*) of g and h. Clearly C is a subdirect product of A and B. Note that when $B = A$ and $h = g$ the resulting fiber product is precisely the kernel of g as a subdirect product in $A \times A$.

Whether a fiber product of lattices is finitely generated turns out to be connected to the following properties of homomorphisms that originally appeared in the work of McKenzie on lattice varieties [8] and of Jónsson on free lattices [5]. Let A, D be lattices. A homomorphism $g\colon A \to D$ is *lower bounded* if for each $d \in D$ the set $\{x \in A \mid g(x) \geq d\}$ is either empty or has a least element (dually, $\{x \in A \mid g(x) \leq d\}$ is empty or has a greatest element for *upper bounded*). If g is surjective, this condition is equivalent to the preimage $g^{-1}(d)$ having a least element (dually, greatest for upper bounded) for each $d \in D$. Further g is *bounded* if it is both lower and upper bounded.

The existence of a lower bounded epimorphism from a free lattice has a strong universal consequence. By [4, Theorem 2.13] the following are equivalent for any finitely generated lattice D:

(1) There exists a finite set X and a lower bounded epimorphism $f\colon F(X) \to D$ from the free lattice $F(X)$ onto D.
(2) For every finitely generated lattice A, every homomorphism $h\colon A \to D$ is lower bounded.

If D satisfies one, and hence both, conditions (1),(2) above we say that D is *lower bounded*. Of course, the duals of these statements also hold and define *upper bounded lattice*. A lattice that is both upper and lower bounded is said to be *bounded*.

We say a lattice D with finite generating set P satisfies *Dean's condition* (D) if for all finite subsets $S, T \subseteq D$,

$$\bigwedge S \leq \bigvee T \;\Rightarrow\; \begin{cases} \exists s \in S\colon\ s \leq \bigvee T \text{ or} \\ \exists t \in T\colon\ \bigwedge S \leq t \text{ or} \\ \exists p \in P\colon\ \bigwedge S \leq p \leq \bigvee T. \end{cases} \tag{D}$$

Every finitely presented lattice satisfies Dean's condition (D) for an appropriate generating set. More precisely every lattice $F(P)$ that is freely generated by a finite partial lattice P satisfies (D) for P [3, Theorem 2-3.4], and [3, Section 2-3.1] gives a translation from finite presentations into finite partial lattices and conversely. Further every finitely generated lattice satisfying *Whitman's condition* (W),

$$\bigwedge S \leq \bigvee T \;\Rightarrow\; \exists s \in S\colon s \leq \bigvee T \text{ or } \exists t \in T\colon \bigwedge S \leq t, \tag{W}$$

also satisfies Dean's condition (D) for any finite generating set.

Our first result is that boundedness is a sufficient condition for finite generation of fiber products:

Theorem 1.1. *Let A, B, D be finitely generated lattices, and assume D satisfies Dean's condition* (D)*. If $g\colon A \to D$ and $h\colon B \to D$ are bounded epimorphisms, then their fiber product is a finitely generated sublattice of $A \times B$.*

Theorem 1.1 will be proved in Section 2. Its specialization for D a finitely presented lattice resembles a result in congruence permutable varieties: there, for all finitely generated algebras A, B and finitely presented D, every fiber product of epimorphisms $g\colon A \to D$ and $h\colon B \to D$ is finitely generated [6, Proposition 3.3]; for groups see also [1].

Finitely generated lattices satisfying Whitman's condition (W) are not necessarily finitely presented. See McKenzie's example [3, Example 2-9.1] for a finitely generated but not finitely presented sublattice L_1 of a finitely presented lattice. This lattice L_1 satisfies Whitman's condition as was pointed out to us by Freese and Nation. However the bounded finitely generated lattices satisfying Whitman's condition (W) are exactly the projective finitely generated lattices by Kostinsky's Theorem [4, Corollary 5.9]. Hence Theorem 1.1 specialized to D with (W) yields a direct proof, and in fact a strengthening, of the known result that finitely generated projective lattices are finitely presented. Indeed, if $F(X)/\rho$ with X finite is bounded and satisfies Whitman's condition (W), then ρ is finitely generated as sublattice of $F(X) \times F(X)$, hence also as a congruence of $F(X)$.

The following example shows that Dean's condition (D) for D cannot be omitted in Theorem 1.1:

Example 1.2. Let $h\colon F(x_1, x_2, x_3, y_1, y_2, y_3) \to F(x_1, x_2, x_3) \times F(y_1, y_2, y_3)$ with $h(x_i) := (x_i, 0)$ and $h(y_i) := (0, y_i)$ for $i \leq 3$ be the natural epimorphism from the free lattice over 6 generators to the direct square of the free lattice over 3 generators. Note that the latter is bounded since it is a direct product of bounded (free) lattices. However $F(x_1, x_2, x_3) \times F(y_1, y_2, y_3)$ is not finitely presented by [7, Theorem 3.10]. Hence bounded and finitely generated does not imply finitely presented. Moreover $\ker h$ is not finitely generated as a congruence of $F(x_1, x_2, x_3, y_1, y_2, y_3)$ and in particular not finitely generated as a lattice.

The converse of Theorem 1.1 is not true in general as will be shown in Section 3:

Theorem 1.3. *There exists a finitely generated lattice M and an unbounded epimorphism $h\colon M \to L$ onto a finite lattice L such that the kernel of h is finitely generated as a sublattice of $M \times M$.*

However, the converse does hold for fiber products of free lattices and, more generally, of lattices that are generated by join prime and by meet prime elements and satisfy Whitman's condition (W). The latter include in particular all lattices that are freely generated by some ordered set [4, Theorem 5.19].

Theorem 1.4. *Let A, B be lattices that satisfy Whitman's condition* (W) *and are generated by join prime elements as well as by meet prime elements, let D be a lattice. If the fiber product of epimorphisms $g\colon A \to D$ and $h\colon B \to D$ is a finitely generated sublattice of $A \times B$, then g and h are bounded.*

Theorem 1.4 will be proved in Section 4. Together with Theorem 1.1 it yields the following characterization of finitely generated fiber products of free lattices (solving Problem 7.6 of [6]) as well as a new characterization of finitely presented bounded lattices.

Corollary 1.5. *For any finitely generated lattice D that satisfies Dean's condition* (D) *the following are equivalent:*

(1) *D is bounded.*
(2) *There exists a finite set X and an epimorphism $h\colon F(X) \to D$ from the free lattice $F(X)$ onto D such that $\ker h$ is a finitely generated sublattice of $F(X) \times F(X)$.*
(3) *For every finitely generated lattice A and epimorphism $g\colon A \to D$, the kernel of g is a finitely generated sublattice of $A \times A$.*
(4) *For all finitely generated lattices A, B, epimorphisms $g\colon A \to D$ and $h\colon B \to D$, the fiber product of g and h is a finitely generated sublattice of $A \times B$.*

Note that (1)⇒(4) follows from Theorem 1.1; (4)⇒(3)⇒(2) are immediate; (2)⇒(1) follows from Theorem 1.4.

The analogous question of characterizing finite generation of fiber products of free semigroups and monoids was considered by Clayton [2] in the case where the common quotient D is finite or free.

Comparing lattices and congruence permutable varieties again, recall that every finitely generated congruence ρ of a finitely generated Mal'cev algebra A is finitely generated as a subalgebra of $A \times A$. By Corollary 1.5(1)⇒(3) this also holds for congruences ρ of lattices A whenever A/ρ is bounded. In particular for every bounded finitely presented lattice $F(X)/\rho$ with X finite, ρ is not only a finitely generated congruence of the free lattice $F(X)$ but also finitely generated as a sublattice of $F(X) \times F(X)$.

In the course of proving Theorem 1.1 we obtain that an epimorphism $g\colon A \to F(P)$ is bounded if and only if the pre-images of P under g are bounded; see Corollary 2.4.

Combining this with [3, Section 2-3.1] yields (1)⇔(3) in the following reformulation of Corollary 1.5 for finitely presented lattices $F(X)/\rho$.

Corollary 1.6. *Let X be finite. Then the following are equivalent for a congruence ρ of $F(X)$ generated as a congruence by a finite set $R \subseteq F(X) \times F(X)$:*

(1) *$F(X)/\rho$ is bounded.*
(2) *ρ is finitely generated as a sublattice of $F(X) \times F(X)$.*
(3) *If u is an element of X or a subterm of a term occurring in R, the class u/ρ is bounded in $F(X)$.*

By Corollary 2.4 boundedness for D with finite generating set P satisfying Dean's condition (D) is determined by pre-images of P. This yields an algorithm for deciding whether certain types of such lattices D are bounded which we will describe in Section 5. The assertion of the following theorem for the class of finitely presented lattices was already known to Freese and Nation; see [4, page 251].

Theorem 1.7. *Lower boundedness is decidable for finitely presented lattices and their finitely generated sublattices satisfying Dean's condition* (D)*.*

Finally let us add a small observation about subdirect products of lattices that are not fiber products but closely related:

Remark 1.8. If the fiber product of epimorphisms $g\colon A \to D$ and $h\colon B \to D$ is finitely generated, then also

$$C = \{(a, b) \in A \times B \mid g(a) \leq h(b)\}$$

is finitely generated. Indeed if the fiber product of g and h is generated by G, then C is generated by $G \cup \{(0_A, 1_B)\}$.

2. Bounded homomorphisms imply finite generation

To facilitate inductive proofs on the complexity of lattice elements over some generating set, we adapt the notation from [4, Section II.1].

Let A be a lattice with finite generating set X. For a subset W of A we define

$$W^{\wedge} := \Big\{\bigwedge U \;\Big|\; U \text{ is a finite subset of } W\Big\},$$
$$W^{\vee} := \Big\{\bigvee U \;\Big|\; U \text{ is a finite subset of } W\Big\},$$

with the convention that $1 := \bigvee X = \bigwedge \emptyset$ and $0 := \bigwedge X = \bigvee \emptyset$ in A. Next define an ascending chain

$$X = G_{X,0} \subseteq H_{X,0} \subseteq G_{X,1} \subseteq H_{X,1} \subseteq \dots$$

of subsets of A inductively as follows:

$$G_{X,0} := X, \quad H_{X,k} := G_{X,k}^{\wedge}, \quad G_{X,k+1} := H_{X,k}^{\vee} \quad \text{for } k \in \mathbb{N}. \tag{2.1}$$

If the generating set X is clear from the context, we write simply G_k, H_k. Note that G_k for $k \geq 1$ is join-closed, H_k for $k \geq 0$ is meet-closed and

$$A = \bigcup_{k \in \mathbb{N}} G_k = \bigcup_{k \in \mathbb{N}} H_k.$$

For $g\colon A \to D$ an epimorphism, $k \in \mathbb{N}$ and $d \in D$, we define

$$\alpha_{g,k}(d) := \bigvee \{w \in G_k \mid g(w) \leq d\}, \quad \beta_{g,k}(d) := \bigwedge \{w \in H_k \mid g(w) \geq d\}. \tag{2.2}$$

Note that $\alpha_{g,0}(d)$ is not necessarily contained in $G_{X,0} = X$, but for $k \geq 1$ the element $\alpha_{g,k}(d)$ is the greatest $w \in G_{X,k}$ with $g(w) \leq d$. Dually $\beta_{g,k}(d)$ is the least $w \in H_k$ with $g(w) \geq d$ for all $k \in \mathbb{N}$. Furthermore (2.1) yields for all $k \geq 1$

$$\begin{aligned} \alpha_{g,k}(d) &= \bigvee \{w \in H_{k-1} \mid g(w) \leq d\}, \\ \beta_{g,k}(d) &= \bigwedge \{w \in G_k \mid g(w) \geq d\}. \end{aligned} \tag{2.3}$$

If the epimorphism g is clear from the context, we write α_k instead of $\alpha_{g,k}$, etc. Note that α_k, β_k depend on the choice of the generating set X of A.

In [4, Section II.1] H_k and β_k are defined exactly as above. We have introduced the non-standard notions of G_k and α_k for our proof of Theorem 1.1.

Remark 2.1. There is a duality between H_k and β_k on one hand and G_k and α_k on the other. However when referring to this duality one needs to bear in mind the following:

- $G_0 = X$ and $H_0 = X^{\wedge}$ are not dual to each other with respect to X. Still we will obtain completely dual formulas for α_k and β_k for all $k \in \mathbb{N}$ in Lemma 2.2(4),(5) below.
- By (2.3), if a statement about G_k and α_k also refers to H_{k-1} and β_{k-1}, then its dual for H_k and β_k will need to refer to G_k and α_k (and *not* to G_{k-1} and α_{k-1}).

In the following lemma we record some basic properties of our functions that extend those given for β_k in [4, Theorems 2.2, 2.4].

Lemma 2.2. *Let $g\colon A \to D$ be a lattice epimorphism, $a \in A, d, e \in D$, and $k, \ell \in \mathbb{N}$. The following hold:*

(1) *If $d \leq e$, then $\alpha_k(d) \leq \alpha_k(e)$ and $\beta_k(d) \leq \beta_k(e)$.*

(2) *If $k \leq \ell$, then $\alpha_k(d) \leq \alpha_\ell(d)$ and $\beta_k(d) \geq \beta_\ell(d)$.*
(3) *If $d \leq g(a) \leq e$, then $\beta_m(d) \leq a \leq \alpha_{m+1}(e)$ for some $m \in \mathbb{N}$.*
(4) $\alpha_0(d) = \bigvee\{x \in X \mid g(x) \leq d\}$ *and for $k > \ell$*

$$\alpha_k(d) = \bigvee\Big\{\bigwedge U \mid U \subseteq G_{k-1}, g(\bigwedge U) \leq d \text{ but } g(u) \nleq d \text{ for every } u \in U\Big\} \vee \alpha_\ell(d).$$

(5) $\beta_0(d) = \bigwedge\{x \in X \mid g(x) \geq d\}$ *and for $k > \ell$*

$$\beta_k(d) = \bigwedge\Big\{\bigvee U \mid U \subseteq H_{k-1}, g(\bigvee U) \geq d \text{ but } g(u) \ngeq d \text{ for every } u \in U\Big\} \wedge \beta_\ell(d).$$

(6) *For each non-empty finite $E \subseteq D$*

$$\bigwedge \alpha_{k-1}(E) \leq \alpha_k(\bigwedge E), \quad \bigvee \beta_{k-1}(E) \geq \beta_k(\bigvee E).$$

Proof. Parts (1), (2) and (3) are immediate from the definitions.

Item (4) is proved by induction on k. The base case $k = 0$ is just the definition of $\alpha_0(d)$. Let $k \geq 1$. From (2.3), $H_{k-1} = G_{k-1}^{\wedge}$ and $\alpha_k(d) \geq \alpha_{k-1}(d)$ we have

$$\alpha_k(d) = \bigvee\Big\{\bigwedge U \mid U \subseteq G_{k-1}, g(\bigwedge U) \leq d\Big\} \vee \alpha_{k-1}(d). \tag{2.4}$$

If a meetand $u \in U \subseteq G_{k-1}$ satisfies $g(u) \leq d$, then $\bigwedge U \leq u \leq \alpha_{k-1}(d)$, in which case $(\bigwedge U) \vee \alpha_{k-1}(d) = \alpha_{k-1}(d)$ and $\bigwedge U$ can be removed from the join in (2.4). So we are left with

$$\alpha_k(d) = \underbrace{\bigvee\Big\{\bigwedge U \mid U \subseteq G_{k-1}, g(\bigwedge U) \leq d \text{ but } g(u) \nleq d \text{ for every } u \in U\Big\}}_{=:b} \vee \alpha_{k-1}(d).$$

For $k - 1 = \ell$ this is the assertion already. Else for $k - 1 > \ell$ the induction assumption yields

$$\alpha_{k-1}(d) = \underbrace{\bigvee\Big\{\bigwedge U \mid U \subseteq G_{k-2}, g(\bigwedge U) \leq d \text{ but } g(u) \nleq d \text{ for every } u \in U\Big\}}_{=:c} \vee \alpha_\ell(d).$$

Since $G_{k-2} \subseteq G_{k-1}$, we have $c \leq b$ and $\alpha_k(d) = b \vee \alpha_\ell(d)$ as required.

Item (5) is dual to (4). The only place in which their proofs differ is that the base case for $k = 0$ follows from (2.3) instead of the definition.

For (6), note that $\beta_{k-1}(d) \in H_{k-1}$ and $g\beta_{k-1}(d) \geq d$ for all $d \in D$. Then $w := \bigvee\{\beta_{k-1}(d) \mid d \in E\}$ is in $H_{k-1}^{\vee}$ and hence in H_k. Since

$$g(w) = \bigvee\{g\beta_{k-1}(d) \mid d \in E\} \geq \bigvee E,$$

we have $w \geq \beta_k(\bigvee E)$ and the claim follows. The proof of the other assertion is dual. $\square$

Assume that $g\colon A \to D$ is a lower bounded epimorphism. We denote the least element in the preimage of $d \in D$ by

$$\beta_g(d) := \bigwedge g^{-1}(d).$$

Note that for every $d \in D$ there exists $k \in \mathbb{N}$ such that $\beta_g(d) = \beta_{g,k}(d)$.

Dually for an upper bounded epimorphism $g\colon A \to D$ the greatest element in the preimage of $d \in D$ is denoted by

$$\alpha_g(d) := \bigvee g^{-1}(d).$$

It is not hard to see that β_g preserves joins and α_g preserves meets [4, page 27]. In general $\alpha_{g,k}, \beta_{g,k}$ do not preserve any lattice operations. Still we can obtain some useful identities when D satisfies Dean's condition (D).

Lemma 2.3. *Let A be a lattice with finite generating set X, let D be a lattice with finite generating set P satisfying Dean's condition* (D)*, and let $g\colon A \to D$ be an epimorphism. Assume every $p \in P$ has a least pre-image $\beta(p) := \bigwedge g^{-1}(p)$ in A and $\beta(p) \in H_{X,0}$. Let $k \in \mathbb{N}$. Then*

(1) *$\beta_k(\bigwedge E) = \bigwedge \beta_k(E) \wedge \beta_0(\bigwedge E)$ for each finite $E \subseteq D$;*
(2) *$\beta_k(d) = \beta(d)$ for all $d \in H_{P,k}$.*

Proof. (1) For $k = 0$ the statement is immediate from the monotonicity of β_0 by Lemma 2.2(1). Let $k \geq 1$. For $d := \bigwedge E$ let $w \in H^{\vee}_{X,k-1}$ be one of the meetands in the formula for $\beta_k(d)$ in Lemma 2.2(5); specifically, $w = \bigvee U$ for some $U \subseteq H_{X,k-1}$, where $g(\bigvee U) \geq d$ and $g(u) \not\geq d$ for all $u \in U$. We claim that

$$w \geq \bigwedge \beta_k(E) \wedge \beta_0(d). \tag{2.5}$$

By Dean's condition (D), the assumption $g(w) = g(\bigvee U) \geq \bigwedge E = d$ yields $g(u) \geq d$ for some $u \in U$ or $g(w) \geq e$ for some $e \in E$ or $g(w) \geq p \geq d$ for some $p \in P$. We consider each case.

Case 1: $g(u) \geq d$ for some $u \in U$ contradicts our assumption on w.

Case 2: $g(w) \geq e$ for some $e \in E$ yields $w \geq \beta_k(e) \geq \bigwedge \beta_k(E)$.

Case 3: Assume $g(w) \geq p \geq d$ for some $p \in P$. Then

$$\begin{aligned} w &\geq \beta(p) \\ &= \beta_0(p) && \text{by the assumption } \beta(p) \in H_{X,0} \\ &\geq \beta_0(d) && \text{by Lemma 2.2(1).} \end{aligned}$$

In any case we obtain (2.5). Thus by Lemma 2.2(5) we have

$$\beta_k(d) \geq \bigwedge \beta_k(E) \wedge \beta_0(d).$$

The converse inequality follows from Lemma 2.2(1),(2).

(2) We use induction on the complexity of $d \in D$ over the generating set P. The base case is the assumption that $\beta_0(d) = \beta(d)$ for all $d \in G_{P,0} = P$.

For $k \in \mathbb{N}$ we will use two induction steps alternatingly: from $G_{P,k}$ to $H_{P,k}$ and from $H_{P,k}$ to $G_{P,k+1}$. First assume that $\beta_k(e) = \beta(e)$ for all $e \in G_{P,k}$. Let $d \in H_{P,k} = G_{P,k}^{\wedge}$ and write $d = \bigwedge E$ for $E \subseteq G_{P,k}$. For $\ell \in \mathbb{N}$,

$$\begin{aligned}
\beta_{k+\ell}(d) &= \bigwedge \beta_{k+\ell}(E) \wedge \beta_0(d) && \text{by item (1)} \\
&= \bigwedge \beta_k(E) \wedge \beta_0(d) && \text{by induction assumption} \\
&= \beta_k(d) && \text{by item (1).}
\end{aligned}$$

Hence $\beta_k(d)$ is the least element in A that g maps to d and $\beta_k(d) = \beta(d)$ for all $d \in H_{P,k}$.

Next assume that $\beta_k(e) = \beta(e)$ for all $e \in H_{P,k}$. Let $d \in G_{P,k+1} = H_{P,k}^{\vee}$ and write $d = \bigvee E$ for $E \subseteq H_{P,k}$. Then

$$\begin{aligned}
\beta(d) &= \bigvee \beta(E) && \text{since } \beta \text{ preserves joins} \\
&= \bigvee \beta_k(E) && \text{by induction assumption} \\
&\geq \beta_{k+1}(d) && \text{by Lemma 2.2(6).}
\end{aligned}$$

Since the converse inequality holds trivially, $\beta_{k+1}(d) = \beta(d)$ for all $d \in G_{P,k+1} \subseteq H_{P,k+1}$. This concludes both induction steps and the proof of (2). $\square$

Lemma 2.3(2) yields in particular:

Corollary 2.4. *Let $g\colon A \to D$ be an epimorphism from a finitely generated lattice A onto a lattice D with finite generating set P satisfying Dean's condition* (D)*. Then g is lower bounded if and only if the pre-images under g of P are lower bounded.*

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let A, B, D be lattices with finite generating sets X, Y, P, respectively, let D and P satisfy Dean's condition (D), let $g\colon A \to D$ and $h\colon B \to D$ be bounded epimorphisms, and let $E := g(X) \cup h(Y) \cup P$.

We will show that the fiber product

$$C := \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in A \times B \mid g(a) = h(b) \right\}$$

is generated by the finite set

$$Z := \left\{ \begin{pmatrix} x \\ \alpha_h g(x) \end{pmatrix}, \begin{pmatrix} \beta_g h(y) \\ y \end{pmatrix}, \begin{pmatrix} \alpha_g(d) \\ \beta_h(d) \end{pmatrix}, \begin{pmatrix} \beta_g(d) \\ \alpha_h(d) \end{pmatrix} \mid x \in X, y \in Y, d \in E \right\}.$$

Enlarging the original generating sets X, Y by finitely many elements if necessary, we may assume that X and Y actually are the projections of Z onto its first and second components, respectively.

We proceed via a series of technical claims. We begin by observing that the following hold from the definition of Z:

(Z1) For all $x \in X$ we have $(x, \alpha_h g(x))$ and $(x, \beta_h g(x)) \in Z \cap (X \times Y)$.
(Z2) Let $k \in \mathbb{N}$. For all $a_1 \in G_{X,k}$, $a_2 \in H_{X,k}$ there exist $b_1 \in G_{Y,k}$, $b_2 \in H_{Y,k}$, such that $(a_1, b_1), (a_2, b_2) \in \langle Z \rangle$.
(Z3) For every $k \in \mathbb{N}$ we have $g(G_{X,k}) = h(G_{Y,k})$ and $g(H_{X,k}) = h(H_{Y,k})$.

Of course the symmetric versions of statements (Z1), (Z2) with components (as well as X and Y) swapped hold as well.

Claim 1. *The following hold:*

(1) $\forall b \in B\ \exists b' \leq b$: $\begin{pmatrix} \alpha_{g,0} h(b) \\ b' \end{pmatrix} \in \langle Z \rangle$,

(2) $\forall a \in A\ \exists a' \geq a$: $\begin{pmatrix} a' \\ \beta_{h,0} g(a) \end{pmatrix} \in \langle Z \rangle$.

Proof. (1) Let $b \in B$. Recall from Lemma 2.2(4) that

$$\alpha_{g,0} h(b) = \bigvee \{x \in X \mid g(x) \leq h(b)\}.$$

If $\{x \in X \mid g(x) \leq h(b)\} = \emptyset$, then $\alpha_{g,0} h(b) = 0$. Since $(0,0) = \bigwedge Z$, we can take $b' = 0$ in that case. Otherwise consider an arbitrary joinand $x \in X$ from above. Note that $h(\alpha_h g(x) \wedge b) = g(x) \wedge h(b) = g(x)$ and hence $\alpha_h g(x) \wedge b \geq \beta_h g(x)$. Picking any $a \in A$ with $(a, b) \in \langle Z \rangle$, we have

$$\langle Z \rangle \ni \left[\begin{pmatrix} x \\ \alpha_h g(x) \end{pmatrix} \wedge \begin{pmatrix} a \\ b \end{pmatrix} \right] \vee \begin{pmatrix} x \\ \beta_h g(x) \end{pmatrix} = \begin{pmatrix} x \\ \alpha_h g(x) \wedge b \end{pmatrix}.$$

Taking the join of these elements over all x with $g(x) \leq h(b)$ we obtain $(\alpha_{g,0} h(b), b') \in \langle Z \rangle$ for some $b' \leq b$.

(2) By Lemma 2.2(4) and (5) the formulas for α_0 and β_0 are dual to each other. Hence the proof of (2) is just the dual of (1) after swapping first and second components. $\square$

The key technical step in our proof of Theorem 1.1 is to establish the next claim.

Claim 2. *The following hold for every $k \in \mathbb{N}$:*

$$\forall b \in G_{Y,k} \colon \begin{pmatrix} \alpha_{g,k}h(b) \\ b \end{pmatrix} \in \langle Z \rangle, \tag{2k}$$

$$\forall a \in H_{X,k} \colon \begin{pmatrix} a \\ \beta_{h,k}g(a) \end{pmatrix} \in \langle Z \rangle. \tag{2k+1}$$

Proof. We use induction on the index of the statements.

Base case: To prove statement (0) let $k = 0$ and $b \in G_{Y,0} = Y$. Pick $a \in G_{X,0} = X$ such that $(a, b) \in Z$. (The existence of such a, b is guaranteed by the remarks immediately following the definition of Z above.) Then $g(a) = h(b)$ implies $a \leq \alpha_{g,0}h(b)$ by Lemma 2.2(4). By Claim 1(1) we also have $b' \leq b$ such that $(\alpha_{g,0}h(b), b') \in \langle Z \rangle$. It follows that

$$\langle Z \rangle \ni \begin{pmatrix} a \\ b \end{pmatrix} \vee \begin{pmatrix} \alpha_{g,0}h(b) \\ b' \end{pmatrix} = \begin{pmatrix} \alpha_{g,0}h(b) \\ b \end{pmatrix}$$

as required.

Induction step (2k)$\Rightarrow$(2k+1): Let $a \in H_{X,k} = G_{X,k}^{\wedge}$ and write $a = \bigwedge T$ for $T \subseteq G_{X,k}$. Let $d \in g(T)$. By (2k) we have

$$\langle Z \rangle \ni \bigwedge \left\{ \begin{pmatrix} \alpha_{g,k}h(w) \\ w \end{pmatrix} \mid w \in G_{Y,k}, h(w) \geq d \right\} = \begin{pmatrix} \alpha_{g,k}(d) \\ \beta_{h,k}(d) \end{pmatrix}. \tag{2.6}$$

To see that the above equality holds, first note that in the second component we simply have (2.3). For the first component of (2.6) note that $h(w) \geq d$ implies $\alpha_{g,k}h(w) \geq \alpha_{g,k}(d)$ by Lemma 2.2(1). Also, since $d \in g(T) \subseteq g(G_{X,k}) = h(G_{Y,k})$ (see (Z3)), there exists $w \in G_{Y,k}$ with $h(w) = d$ and the equality in the first component of (2.6) follows.

We take the meet over all elements in (2.6) for d in $g(T)$ and use Claim 1(2) to obtain $a' \geq a$ such that

$$\langle Z \rangle \ni \bigwedge \left\{ \begin{pmatrix} \alpha_{g,k}(d) \\ \beta_{h,k}(d) \end{pmatrix} \mid d \in g(T) \right\} \wedge \begin{pmatrix} a' \\ \beta_{h,0}g(a) \end{pmatrix}. \tag{2.7}$$

The second component in the above meet is

$$\bigwedge \beta_{h,k}g(T) \wedge \beta_{h,0}g(a) = \beta_{h,k}\big(\bigwedge g(T)\big) = \beta_{h,k}g(a)$$

by Lemma 2.3(1) (note that $\beta(P) \subseteq H_{Y,0}$ by (Z1)).

The first component of the element in (2.7) is $a'' := \bigwedge_{t \in T} \alpha_{g,k}g(t) \wedge a'$. Let $t \in T$. Then Lemma 2.2(1) and $g(t) \geq g(\bigwedge T)$ imply

$$\alpha_{g,k}g(t) \geq \alpha_{g,k}g(\bigwedge T) = \alpha_{g,k}g(a) \geq a.$$

Thus $\alpha_{g,k}g(t) \geq a$ for all $t \in T$ and $a' \geq a$, which yield $\bigwedge_{t\in T} \alpha_{g,k}g(t) \wedge a' \geq a$. We conclude

$$\begin{pmatrix} a'' \\ \beta_{h,k}g(a) \end{pmatrix} \in \langle Z \rangle \quad \text{and} \quad a'' \geq a.$$

Now let $b \in G_{Y,k}$ with $(a,b) \in \langle Z \rangle$. Then $b \geq \beta_{h,k}g(a)$ and so

$$\langle Z \rangle \ni \begin{pmatrix} a \\ b \end{pmatrix} \wedge \begin{pmatrix} a'' \\ \beta_{h,k}g(a) \end{pmatrix} = \begin{pmatrix} a \\ \beta_{h,k}g(a) \end{pmatrix}.$$

Induction step (2k-1)⇒(2k)**:** This is dual to the proof of (2k) ⇒ (2k+1). For completeness here is the whole argument verbatim except for switching first and second components, meets and joins, as well as the replacements

$$\left.\begin{matrix} H_{X,k}, \alpha_{g,k} \\ G_{X,k}, \beta_{g,k} \end{matrix}\right\} \to \left\{\begin{matrix} G_{Y,k}, \beta_{h,k} \\ H_{Y,k-1}, \alpha_{h,k-1}. \end{matrix}\right.$$

The shift of indices in the last part occurs since $H_{X,k} = G^{\wedge}_{X,k}$ but $G_{Y,k} = H^{\vee}_{Y,k-1}$ in line with Remark 2.1.

Let $b \in G_{Y,k} = H^{\vee}_{Y,k-1}$ and write $b = \bigvee T$ for $T \subseteq H_{Y,k-1}$. Let $d \in h(T)$. By statement (2k-1) we have

$$\langle Z \rangle \ni \bigvee \left\{ \begin{pmatrix} w \\ \beta_{h,k-1}g(w) \end{pmatrix} \mid w \in H_{X,k-1}, g(w) \leq d \right\} = \begin{pmatrix} \alpha_{g,k}(d) \\ \beta_{h,k-1}(d) \end{pmatrix}. \tag{2.8}$$

To see that the above equality holds, first note that in the first component we simply have (2.3). For the second component of (2.8) note that $g(w) \leq d$ implies $\beta_{h,k-1}g(w) \leq \beta_{h,k-1}(d)$ by Lemma 2.2(1). Also, since $d \in h(T) \subseteq h(H_{Y,k-1}) = g(H_{X,k-1})$ (see (Z3)), there exists $w \in H_{X,k-1}$ with $g(w) = d$ and the equality in the second component of (2.8) follows.

We take the join over all elements in (2.8) for d in $h(T)$ and use Claim 1(1) to obtain $b' \leq b$ such that

$$\langle Z \rangle \ni \bigvee \left\{ \begin{pmatrix} \alpha_{g,k}(d) \\ \beta_{h,k-1}(d) \end{pmatrix} \mid d \in h(T) \right\} \vee \begin{pmatrix} \alpha_{g,0}h(b) \\ b' \end{pmatrix}. \tag{2.9}$$

The first component in the above join is

$$\bigvee \alpha_{g,k}h(T) \vee \alpha_{g,0}h(b) = \alpha_{g,k}\Big(\bigvee h(T)\Big) = \alpha_{g,k}h(b)$$

by (the dual of) Lemma 2.3(1). Denote the second component of the element in (2.9) by b''. For $t \in T \subseteq H_{Y,k-1}$ let $d := h(t)$. Then $b \geq t \geq \beta_{h,k-1}(d)$ by the definition of $\beta_{h,k-1}$ in (2.2). So we conclude

$$\begin{pmatrix} \alpha_{g,k}h(b) \\ b'' \end{pmatrix} \in \langle Z \rangle \quad \text{and} \quad b'' \leq b.$$

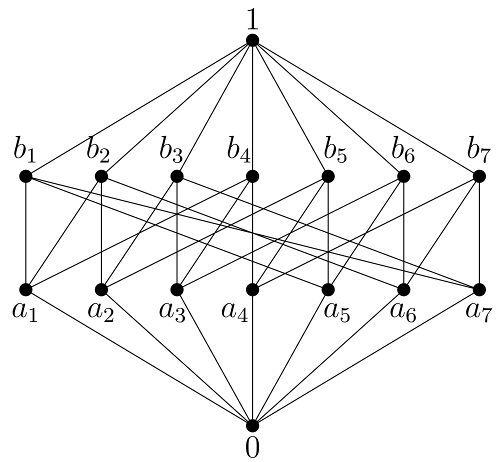

FIGURE 1. The lattice L.

Now let $a \in G_{X,k}$ with $(a, b) \in \langle Z \rangle$. Then $a \leq \alpha_{g,k}h(b)$ and so

$$\langle Z \rangle \ni \begin{pmatrix} a \\ b \end{pmatrix} \vee \begin{pmatrix} \alpha_{g,k}h(b) \\ b'' \end{pmatrix} = \begin{pmatrix} \alpha_{g,k}h(b) \\ b \end{pmatrix}.$$

This completes the induction for Claim (2). □

We can now return to the proof of Theorem 1.1. Let $(a, b) \in C$ be arbitrary. Then there exists $k \in \mathbb{N}$ such that $a \in H_{X,k}$, $b \in H_{Y,k}$. Using statement (2k+1) of Claim 2 and its symmetric version with swapped components

$$\langle Z \rangle \ni \begin{pmatrix} a \\ \beta_{h,k}g(a) \end{pmatrix} \vee \begin{pmatrix} \beta_{g,k}h(b) \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Thus $C = \langle Z \rangle$ as required. □

3. A FINITELY GENERATED FIBER PRODUCT WITH UNBOUNDED HOMOMORPHISMS

Proof of Theorem 1.3. We start with the lattice L of subspaces of the 3-dimensional vector space over the field with 2 elements. Labelling its elements

$$\{0, 1\} \cup \{a_i, b_i \mid i = 1, \ldots, 7\},$$

we obtain the non-trivial comparisons

$$a_i \leq b_k \quad \Leftrightarrow \quad k = i, i+1, i+3 \pmod 7.$$

See Figure 1 for a graphical representation.

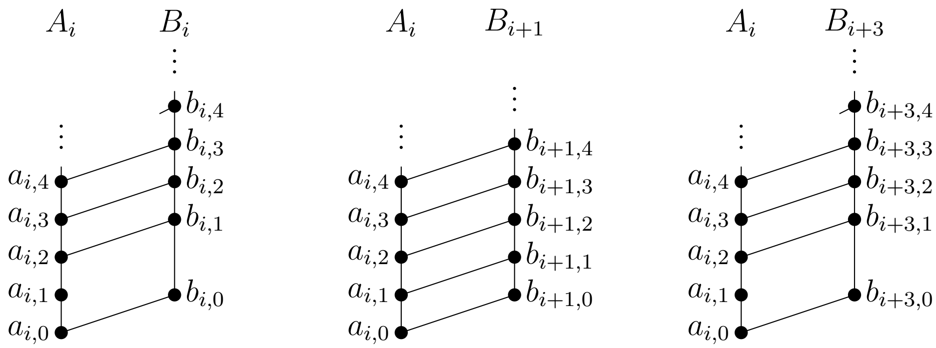

FIGURE 2. Comparisons between A_i and B_i, B_{i+1}, B_{i+3}.

Next, we 'expand' L to an infinite lattice M by 'inflating' each a_i, b_i to an infinite chain isomorphic to $\omega = \{0 < 1 < 2 < \dots\}$. Specifically, the elements are

$$M := \{0,1\} \cup \bigcup_{i=1}^{7} A_i \cup \bigcup_{i=1}^{7} B_i,$$

where

$$A_i := \{a_{i,j} \mid j = 0,1,\dots\} \quad \text{and} \quad B_i := \{b_{i,j} \mid j = 0,1,\dots\}$$

and the comparisons are as follows. First, each A_i, B_i is an increasing chain, i.e. $a_{i,j} \le a_{i,\ell}$ and $b_{i,j} \le b_{i,\ell}$ for all $j \le \ell$. The elements from different A_i are incomparable, as are the elements from different B_i. Comparisons exist between elements of A_i and B_k if and only if $a_i \le b_k$ in L, i.e. if and only if $k = i, i+1, i+3 \pmod 7$, and they are given by

$$a_{i,j} \le b_{k,\ell} \Leftrightarrow \begin{cases} k = i, i+1, i+3 \pmod 7 \text{ and } j \le \ell; \text{or} \\ k = i, i+3 \pmod 7 \text{ and } j = \ell + 1. \end{cases}$$

These comparisons are illustrated in Figure 2. Again, it is easy to verify that M is a lattice. Moreover we claim that

$$M \text{ is generated by the finite set } \{a_{i,0}, a_{i,1} \mid i = 1,\dots,7\}. \tag{3.1}$$

This follows by a straightforward induction on $j = 0,1,\dots$ using that

$$b_{i,j} = a_{i-1,j} \vee a_{i,j} \text{ and } a_{i,j+2} = b_{i,j+1} \wedge b_{i+3,j+1}.$$

Now consider the mapping

$$h : M \to L,\ 0 \mapsto 0,\ 1 \mapsto 1,\ a_{i,j} \mapsto a_i,\ b_{i,j} \mapsto b_i \text{ for } i = 1,\dots,7, j = 0,1,\dots$$

Clearly h is a surjective lattice homomorphism with

$$\ker h = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \cup \bigcup_{i=1}^{7} (A_i \times A_i) \cup \bigcup_{i=1}^{7} (B_i \times B_i).$$

Furthermore, h is not bounded, since none of its kernel classes A_i, B_i have maximal elements. We claim that

$$\ker h \text{ is finitely generated as a sublattice of } M \times M. \tag{3.2}$$

To prove this, consider

$$C_1 := \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \cup \bigcup_{i=1}^{7} \big(\{a_{i,0}\} \times A_i\big) \cup \bigcup_{i=1}^{7} \big(\{b_{i,0}\} \times B_i\big) \subseteq \ker h.$$

Since $\{0,1\} \cup \{a_{i,0}, b_{i,0} \mid i = 1, \ldots, 7\}$ is isomorphic to L, it follows that C_1 is a lattice isomorphic to M. In particular, C_1 is finitely generated by (3.1). By symmetry,

$$C_2 := \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \cup \bigcup_{i=1}^{7} (A_i \times \{a_{i,0}\}) \cup \bigcup_{i=1}^{7} (B_i \times \{b_{i,0}\})$$

is a lattice isomorphic to M and is finitely generated. Any element from $\ker h$ different from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ has the form $\begin{pmatrix} a_{i,j} \\ a_{i,k} \end{pmatrix}$ or $\begin{pmatrix} b_{i,j} \\ b_{i,k} \end{pmatrix}$. Furthermore

$$\begin{pmatrix} a_{i,j} \\ a_{i,k} \end{pmatrix} = \begin{pmatrix} a_{i,0} \\ a_{i,k} \end{pmatrix} \vee \begin{pmatrix} a_{i,j} \\ a_{i,0} \end{pmatrix} \in C_1 \vee C_2,$$

and a dual statement holds for $\begin{pmatrix} b_{i,j} \\ b_{i,k} \end{pmatrix}$. Thus $\ker h$ is generated by its finitely generated sublattices C_1, C_2, which implies (3.2), and completes the proof of Theorem 1.3. $\square$

4. Fiber products of free lattices

The following is our main tool for showing that fiber products are not finitely generated.

Lemma 4.1. *Let A, B be lattices. Assume A is generated by a finite set of join prime elements X and satisfies Whitman's condition (W). Let $g\colon A \to D$, $h\colon B \to D$ be epimorphisms onto a lattice D. Then for each finite subset Z of the fiber product*

$$C := \{(a,b) \in A \times B \mid g(a) = h(b)\}$$

there exists $N \in \mathbb{N}$ such that

$$\forall (a,b) \in \langle Z \rangle,\ \forall k \in \mathbb{N},\ \forall w \in H_{X,k}\colon a \geq w \Rightarrow b \geq \beta_{h,k+N}\, g(w). \tag{4.1}$$

Proof. Since Z is finite, Lemma 2.2(3) implies that there exists $N \in \mathbb{N}$ such that for all $(a, b) \in Z$ we have $b \geq \beta_{h,N} g(a)$. We will show that (4.1) holds for this N by induction on the complexity of (a, b) over the generating set Z. For the base case let $(a, b) \in Z$ and $w \in H_{X,k}$ such that $a \geq w$. Lemma 2.2(1),(2) yield

$$b \geq \beta_{h,N} g(a) \geq \beta_{h,N} g(w) \geq \beta_{h,k+N} g(w).$$

The inductive step splits into two cases:

Case 1: $(a, b) = (a_1, b_1) \wedge (a_2, b_2)$*, where* $(a_1, b_1), (a_2, b_2) \in \langle Z \rangle$. If $a = a_1 \wedge a_2 \geq w$ for $w \in H_{X,k}$, then $a_i \geq w$ for each $i \in \{1, 2\}$. So the induction hypothesis for (4.1) yields $b_i \geq \beta_{h,k+N} g(w)$ for each $i \in \{1, 2\}$. Therefore, $b = b_1 \wedge b_2 \geq \beta_{h,k+N} g(w)$, as desired.

Case 2: $(a, b) = (a_1, b_1) \vee (a_2, b_2)$*, where* $(a_1, b_1), (a_2, b_2) \in \langle Z \rangle$. We use a second induction on $k \in \mathbb{N}$. For the base case $k = 0$, assume $a \geq w \in H_{X,0}$. Then $w = \bigwedge W$ for some $\emptyset \neq W \subseteq X$. By Whitman's condition (W)

$$a_1 \vee a_2 \geq \bigwedge W \;\Rightarrow\; a_1 \geq w \quad \text{or} \quad a_2 \geq w \quad \text{or} \quad a \geq x \text{ for some } x \in W.$$

Since generators X in A are join prime by assumption, the latter case yields $a_1 \geq x$ or $a_2 \geq x$ which implies $a_1 \geq w$ or $a_2 \geq w$ again. Applying the first induction assumption (from the induction on term complexity), we find $b_1 \geq \beta_{h,N} g(w)$ or $b_2 \geq \beta_{h,N} g(w)$. Therefore, $b = b_1 \vee b_2 \geq \beta_{h,N} g(w)$ and the base case is proved.

Next assume $k \geq 1$ and $a \geq w \in H_{X,k}$. By definition $w = \bigwedge W$ for some non-empty $W \subseteq H^{\vee}_{k-1}$. By Whitman's condition (W)

$$a_1 \vee a_2 \geq \bigwedge W \;\Rightarrow\; a_1 \geq w \;\text{ or }\; a_2 \geq w \;\text{ or }\; a \geq u \text{ for some } u \in W. \tag{4.2}$$

The first two alternatives are again straightforward using the first induction assumption on term complexity which implies $b_1 \geq \beta_{h,k+N} g(w)$ or $b_2 \geq \beta_{h,k+N} g(w)$; in either case, $b = b_1 \vee b_2 \geq \beta_{h,k+N} g(w)$. For the third alternative in (4.2) recall that $u = \bigvee U$ for some non-empty $U \subseteq H_{k-1}$. For each $v \in U$, we have $a \geq v$ and hence $b \geq \beta_{h,k-1+N} g(v)$ by the second induction hypothesis (induction on k). Thus

$$\begin{aligned} b &\geq \bigvee \{\beta_{h,k-1+N} g(v) \mid v \in U\} \\ &\geq \beta_{h,k+N}\big(\bigvee \{g(v) \mid v \in U\}\big) && \text{by Lemma 2.2(6)} \\ &= \beta_{h,k+N} g(u) \\ &\geq \beta_{h,k+N} g(w) && \text{by } u \geq w \text{ and Lemma 2.2(1).} \end{aligned}$$

This concludes the induction on k and the proof of (4.1). □

Lemma 4.2. *Let A, B be lattices. Assume A is generated by a set of join prime elements X and satisfies Whitman's condition* (W)*. Let $g\colon A \to D$, $h\colon B \to D$ be epimorphisms onto a lattice D.*

If the fiber product of g and h is a finitely generated sublattice of $A \times B$, then h is lower bounded.

Proof. Using contraposition we assume that h is not lower bounded. Then we have $d \in D$ such that $h^{-1}(d)$ does not have a least element.

Fix a finite subset $Z \subseteq C$ and let N be as in Lemma 4.1 such that (4.1) holds. Let $k \in \mathbb{N}$ such that $g^{-1}(d) \cap H_{X,k} \neq \emptyset$; such k exists since g is surjective and $A = \bigcup_{k \in \mathbb{N}} H_{X,k}$. Let $a \in g^{-1}(d) \cap H_{X,k}$. Since $h^{-1}(d)$ has no least element, there exists $b \in h^{-1}(d)$ such that $b < \beta_{h,k+N}(d)$. Then $(a, b) \in C$ but $(a, b) \notin \langle Z \rangle$ by Lemma 4.1. Since Z was an arbitrary finite subset of C, this proves that C is not finitely generated. □

We are now in a position to prove Theorem 1.4.

Proof of Theorem 1.4. Assume the fiber product of g and h is a finitely generated sublattice of $A \times B$. Since A and B are generated by join prime elements by assumption, h and g are lower bounded by Lemma 4.2. Moreover, since A and B are also generated by meet prime elements, the dual of Lemma 4.2 yields that h and g are upper bounded as well. □

5. Deciding bounded lattices

It is known to be decidable whether a finitely presented lattice is bounded by an unpublished result of Freese and Nation; see [4, page 251]. We give a proof for this and that it is decidable whether a finitely generated sublattice satisfying Dean's condition (D) of a finitely presented lattice is bounded.

Let P be a finite partial lattice, and let $n \in \mathbb{N}$. Then $S := P^{(\vee\wedge)^n\vee}$ is a finite join-subsemilattice of $F(P)$ with the join of the empty set, i.e. $\bigwedge P$, as its least element. Because S is join closed, has a least element and is finite, any $a, b \in S$ have an infimum $\inf(a, b) \in S$. Note that $\inf(a, b) \leq a \wedge b$ where the latter denotes the meet in $F(P)$; equality may hold e.g. if that meet happens to be defined in the partial lattice P. Hence $(S, \inf, \vee)$ is a finite lattice but not necessarily a sublattice of $F(P)$. Instead $(S, \inf, \vee)$ turns out to be a homomorphic image of $F(P)$.

By [3, Lemma 2-6.11] and the subsequent discussion in the extended version of that paper, the *standard homomorphism*

$$f \colon F(P) \to S,\ d \mapsto \bigvee \{w \in S \mid w \leq d\},$$

exists and is a lower bounded epimorphism. For any $d \in S \subseteq F(P)$ we have $f(d) = d$ and consequently $d = \beta_f(d)$.

Lemma 5.1. *Let A be a lattice with finite generating set X, let P be a finite partial lattice, and let $g\colon A \to F(P)$ be a homomorphism. Assume that $g(A)$ satisfies Dean's condition (D) for the generating set $g(X)$ and $g(X) \subseteq P^{(\vee\wedge)^n\vee}$ for $n \in \mathbb{N}$. Then g is lower bounded if and only if its composition $fg\colon A \to P^{(\vee\wedge)^n\vee}$ with the standard homomorphism f is lower bounded.*

Proof. The forward direction follows since the composition of bounded homomorphisms is bounded.

For the backward direction, assume that fg is lower bounded. Let $d \in P^{(\vee\wedge)^n\vee} \cap g(A)$. Then $f(d) = d$ yields $\beta_{fg}(d) = \beta_g(d)$. Hence $g^{-1}(d)$ has a least element for any $d \in P^{(\vee\wedge)^n\vee} \cap g(A)$. In particular $\beta_g g(x)$ exists for any generator $g(x)$ of $g(A)$. Thus $g\colon A \to F(P)$ is lower bounded by Corollary 2.4. □

We can now give the algorithm for deciding boundedness that proves Theorem 1.7.

Proof of Theorem 1.7. For $D = F(P)$ finitely presented, D is lower bounded if and only if the lattice $S := P^{\vee}$ is lower bounded by Lemma 5.1 with A the free lattice over the set P and $g\colon A \to D$ the natural epimorphism.

In case D is generated by some finite subset X of $F(P)$ and satisfies Dean's condition (D), assume $X \subseteq P^{(\vee\wedge)^n\vee}$ for some $n \in \mathbb{N}$. Then D is lower bounded if and only if the sublattice S of $P^{(\vee\wedge)^n\vee}$ that is generated by X is lower bounded by Lemma 5.1with A the free lattice over X and the natural epimorphism $g\colon A \to D$.

In either case it suffices to decide whether the finite lattice S is lower bounded. This can be done in time $O(|S|^2)$ by [4, Theorem 11.20]. Note that $|S|$ is at most exponential in the size of the input P, X, respectively. Hence we can decide whether D is bounded in exponential time. □

For the second case in Theorem 1.7 we note that a sublattice of $F(P)$ trivially satisfies Dean's condition (D) if $F(P)$ satisfies Whitman's condition (W). By Dean's solution to the word problem for $F(P)$ [3, Theorem 2-3.4] this is equivalent to P satisfying Whitman's condition (W) whenever meets and joins are defined in P. In other words, $F(P)$ fails (W) if and only if there is a failure in P using the defined joins and meets.

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