1-7-2021

Cancellable Elements of the Lattice of Monoid Varieties

Sergey V. Gusev
Ural Federal University - Ekaterinburg, Russia

Edmond W. H. Lee
Nova Southeastern University, edmond.lee@nova.edu

CANCELLABLE ELEMENTS OF THE LATTICE OF MONOID VARIETIES

SERGEY V. GUSEV AND EDMOND W. H. LEE

ABSTRACT. The set of all cancellable elements of the lattice of semigroup varieties has recently been shown to be countably infinite. But the description of all cancellable elements of the lattice $\mathbb{MON}$ of monoid varieties remains unknown. This problem is addressed in the present article. The first example of a monoid variety with modular but non-distributive subvariety lattice is first exhibited. Then a necessary condition of the modularity of an element in $\mathbb{MON}$ is established. These results play a crucial role in the complete description of all cancellable elements of the lattice $\mathbb{MON}$. It turns out that there are precisely five such elements.

1. INTRODUCTION AND SUMMARY

The present article is concerned with the lattice $\mathbb{MON}$ of all monoid varieties, where monoids are considered as semigroups with an identity element that is fixed by a 0-ary operation. For many years, results on the lattice $\mathbb{MON}$ were scarce. But recently, interest in this lattice has grown significantly; in particular, the study of its special elements was initiated in the articles [5, 6]. In the present work, we continue these investigations.

Special elements play an important role in general lattice theory; see [3, Section III.2], for instance. We recall definitions of those types of special elements that are relevant here. An element x of a lattice L is

cancellable if $\quad \forall\, y, z \in L\colon\ \ x \vee y = x \vee z \ \&\ x \wedge y = x \wedge z \longrightarrow y = z;$

modular if $\quad \forall\, y, z \in L\colon\ \ y \le z \longrightarrow (x \wedge z) \vee y = (x \vee y) \wedge z.$

It is easy to see that every cancellable element is modular.

Our main goal is to describe all cancellable elements of the lattice $\mathbb{MON}$. To formulate our main result, we need some definitions and notation. Let $\mathfrak{X}^+$ [respectively, $\mathfrak{X}^*$] denote the free semigroup [respectively, monoid] over a countably infinite alphabet $\mathfrak{X}$. Elements of $\mathfrak{X}$ are called *letters* and elements of $\mathfrak{X}^*$ are called *words*. Words unlike letters are written in bold. An identity is written as $\mathbf{u} \approx \mathbf{v}$, where $\mathbf{u}, \mathbf{v} \in \mathfrak{X}^*$.

Let $\mathbf{T}$, $\mathbf{SL}$, and $\mathbf{MON}$ denote the variety of trivial monoids, the variety of semilattice monoids, and the variety of all monoids, respectively. For any identity system Σ, let $\operatorname{var}\Sigma$ denote the variety of monoids given by Σ. Put

$$\mathbf{C}_2 = \operatorname{var}\{x^2 \approx x^3,\, xy \approx yx\} \ \text{ and } \ \mathbf{D} = \operatorname{var}\{x^2 \approx x^3,\, x^2y \approx xyx \approx yx^2\}.$$

2010 *Mathematics Subject Classification.* 20M07, 08B15.
Key words and phrases. Monoid, variety, lattice of varieties, cancellable element of a lattice, modular element of a lattice.
The first author is supported by the Ministry of Science and Higher Education of the Russian Federation (project FEUZ-2020-0016).

Then the following is our main result.

Theorem 1.1. *A monoid variety is a cancellable element of the lattice* $\mathbb{MON}$ *if and only if it coincides with one of the varieties* $\mathbf{T}$, $\mathbf{SL}$, $\mathbf{C}_2$, $\mathbf{D}$ *or* $\mathbf{MON}$.

Many articles were devoted to special elements of different types in the lattice $\mathbb{SEM}$ of all semigroup varieties; an overview of results published before 2015 can be found in the survey [16].[1] It is natural to compare Theorem 1.1 with the description of cancellable elements of the lattice $\mathbb{SEM}$ that was found in 2019 [15]. Theorem 1.1 shows that there are only five cancellable elements in the lattice $\mathbb{MON}$. In contrast, the set of all cancellable elements of the lattice $\mathbb{SEM}$ is countably infinite.

In general, the set of cancellable elements in a lattice need not form a sublattice. For example, the elements x and y of the lattice in Fig. 1 are cancellable but their join $x \vee y$ is not. However, the class of all cancellable elements of $\mathbb{SEM}$ forms a distributive sublattice of $\mathbb{SEM}$; see Corollary 3.14 in the extended version of the survey [16]. Theorem 1.1 shows that the same is true for monoid varieties too; in fact, the five cancellable elements in $\mathbb{MON}$ constitute a chain.

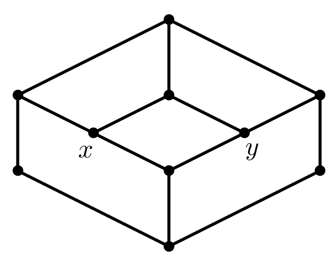

FIGURE 1.

Now since the chain $\mathbf{T} \subset \mathbf{SL} \subset \mathbf{C}_2 \subset \mathbf{D}$ coincides with the lattice $\mathfrak{L}(\mathbf{D})$ of subvarieties of $\mathbf{D}$ (see Fig. 2), a monoid variety $\mathbf{V}$ is a cancellable element of the lattice $\mathbb{MON}$ if and only if either $\mathbf{V} \subseteq \mathbf{D}$ or $\mathbf{V} = \mathbf{MON}$. It is routinely verified that the variety $\mathbf{D}$ can be given by the single identity $x^3yz \approx yxzx$. Therefore it is easy to check the cancellability of proper elements of the lattice $\mathbb{MON}$; a monoid variety is *proper* if it is different from $\mathbf{MON}$.

Corollary 1.2. *Suppose that* M *is any monoid that generates a proper subvariety* $\mathbf{V}$ *of* $\mathbf{MON}$. *Then* $\mathbf{V}$ *is a cancellable element of the lattice* $\mathbb{MON}$ *if and only if* M *satisfies the identity* $x^3yz \approx yxzx$.

The article consists of five sections. Section 2 contains definitions, notation, certain known results and their simple corollaries. In Section 3, the first example of a monoid variety with modular but non-distributive subvariety lattice is given. In Section 4, a necessary condition of the modularity of an element in $\mathbb{MON}$ is established in Proposition 4.3. Results from Sections 3 and 4 will then be used in Section 5 to prove Theorem 1.1.

[1] An extended version of this survey, which is periodically updated as new results are found and/or new articles are published, is available at http://arxiv.org/abs/1309.0228v20.

2. Preliminaries

Acquaintance with rudiments of universal algebra is assumed of the reader. Refer to the monograph [2] for more information.

Recall that a variety is *periodic* if it consists of periodic monoids. Equivalently, a variety is periodic if and only if it satisfies the identity $x^n \approx x^{n+m}$ for some $n, m \geq 1$. For any word $\mathbf{w}$ and any set X of letters, the word obtained from $\mathbf{w}$ by deleting all the letters of X is denoted by $\mathbf{w}_X$. The *content* of a word $\mathbf{w}$, denoted by $\mathrm{con}(\mathbf{w})$, is the set of letters occurring in $\mathbf{w}$. The partition lattice over a set X is denoted by $\mathrm{Part}(X)$. Let $\mathcal{L}_{\mathrm{FIC}(\mathfrak{X}^*)}$ denote the lattice of all fully invariant congruences on the monoid $\mathfrak{X}^*$, and for any variety $\mathbf{V}$ of monoids, let $\mathrm{FIC}(\mathbf{V})$ denote the fully invariant congruence on $\mathfrak{X}^*$ corresponding to $\mathbf{V}$. It is well known that the mapping $\mathrm{FIC}\colon \mathbb{MON} \longrightarrow \mathcal{L}_{\mathrm{FIC}}(\mathfrak{X}^*)$ is an anti-isomorphism of lattices; see [2, Theorem 11.9 and Corollary 14.10], for instance. For any $\mathbf{u}, \mathbf{v} \in \mathfrak{X}^+$, we put $\mathbf{u} \preceq \mathbf{v}$ if $\mathbf{v} = \mathbf{a}\xi(\mathbf{u})\mathbf{b}$ for some words $\mathbf{a}, \mathbf{b} \in \mathfrak{X}^*$ and some endomorphism ξ of $\mathfrak{X}^+$. It is easily seen that the relation $\preceq$ on $\mathfrak{X}^+$ is a quasi-order. For an arbitrary anti-chain $A \subseteq \mathfrak{X}^+$ under the relation $\preceq$, let L_A denote the set of all monoid varieties $\mathbf{V}$ for which A is a union of $\mathrm{FIC}(\mathbf{V})$-classes. Define the map $\varphi_A\colon \mathrm{L}_A \longrightarrow \mathrm{Part}(A)$ by the rule $\varphi_A(\mathbf{V}) = \mathrm{FIC}(\mathbf{V})|_A$ for any $\mathbf{V} \in \mathrm{L}_A$.

Lemma 2.1 ([4, Lemma 3])**.** *Let A be any anti-chain under the quasi-order $\preceq$. Suppose that for any words $\mathbf{u}, \mathbf{v} \in A$ and any nonempty set $X \subseteq \mathrm{con}(\mathbf{u})$, the equalities $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v})$ and $\mathbf{u}_X = \mathbf{v}_X$ hold. Then*

(i) *the set L_A is a sublattice of the lattice $\mathbb{MON}$;*
(ii) *the map φ_A is a surjective anti-homomorpism of the lattice L_A onto the lattice $\mathrm{Part}(A)$;*
(iii) *for any partition $\beta \in \mathrm{Part}(A)$, there exists a non-periodic monoid variety $\mathbf{V} \in \mathrm{L}_A$ such that $\varphi_A(\mathbf{V}) = \beta$.*

Recall that a band is *left regular* if it is a semilattice of left zero bands. It is well known that the class of left regular band monoids coincides with the variety

$$\mathbf{LRB} = \mathrm{var}\{xy \approx xyx\}.$$

The *initial part* of a word $\mathbf{w}$, denoted by $\mathrm{ini}(\mathbf{w})$, is the word obtained from $\mathbf{w}$ by retaining the first occurrence of each letter. The following assertion is well known and easily verified.

Lemma 2.2. *An identity $\mathbf{u} \approx \mathbf{v}$ holds in $\mathbf{LRB}$ if and only if $\mathrm{ini}(\mathbf{u}) = \mathrm{ini}(\mathbf{v})$.*

For any $n \geq 2$, the variety

$$\mathbf{C}_n = \mathrm{var}\{x^n \approx x^{n+1},\, xy \approx yx\}$$

is generated by the monoid $\langle a, 1 \mid a^n = 0\rangle$ [1, Corollary 6.1.5]. Note that the variety $\mathbf{C}_2$ has already been introduced in Section 1. A word $\mathbf{w}$ is an *isoterm* for a variety $\mathbf{V}$ if the $\mathrm{FIC}(\mathbf{V})$-class of $\mathbf{w}$ is singleton. The following result is easily deduced from [10, Lemma 3.3].

Lemma 2.3. *Let $n \geq 1$. For any monoid variety $\mathbf{V}$, the following are equivalent:*

a) *x^n is not an isoterm for $\mathbf{V}$;*
b) *$\mathbf{V}$ satisfies the identity $x^n \approx x^{n+m}$ for some $m \geq 1$;*
c) *$\mathbf{C}_{n+1} \nsubseteq \mathbf{V}$.*

A monoid is *completely regular* if it is a union of its maximal subgroups. A variety is *completely regular* if it consists of completely regular monoids. It is well known that a monoid variety is completely regular if and only if it satisfies the identity $x \approx x^{n+1}$ for some $n \geq 1$.

Lemma 2.4 ([7, Lemma 2.14])**.** *If a monoid variety* $\mathbf{V}$ *is non-completely regular and noncommutative, then* $\mathbf{D} \subseteq \mathbf{V}$*.*

Lemma 2.5. *Let* $\mathbf{V}$ *be any monoid variety such that* $\mathbf{C}_2 \subseteq \mathbf{V}$*. Suppose that* $\mathbf{V}$ *does not contain the variety*

$$\mathbf{E} = \mathrm{var}\{x^2 \approx x^3,\, x^2y \approx xyx,\, x^2y^2 \approx y^2x^2\}.$$

Then $\mathbf{V}$ *satisfies the identity* $x^pyx^q \approx yx^r$ *for some* $p, q \geq 1$ *and* $r \geq 2$*.*

Proof. If $\mathbf{D} \subseteq \mathbf{V}$, then the result follows from [7, Lemma 4.1 and Proposition 4.2]. Therefore suppose that $\mathbf{D} \nsubseteq \mathbf{V}$, so that by Lemma 2.4, the variety $\mathbf{V}$ is either completely regular or commutative. But $\mathbf{V}$ cannot be completely regular because $\mathbf{C}_2 \subseteq \mathbf{V}$. Hence $\mathbf{V}$ is commutative and satisfies the identity $xyx \approx yx^2$. □

3. Monoid variety with modular but non-distributive subvariety lattice

There are many examples of monoid varieties with non-distributive subvariety lattice; see [5, 6, 13], for instance. However, all these varieties have non-modular subvariety lattice as well. In this section, we present the first example of a monoid variety whose subvariety lattice is modular but non-distributive. To this end, the following varieties are required: the variety $\mathbf{D}_2$ generated by the monoid

$$\langle a, b, 1 \mid a^2 = b^2 = bab = 0\rangle = \{a, b, ab, ba, aba, 1, 0\},$$

the variety $\mathbf{R}$ generated by the monoid

$$\langle a, b, 1 \mid a^3 = b^2 = ba = 0\rangle = \{a, b, a^2, ab, a^2b, 1, 0\}$$

and the variety $\mathbf{R}^\delta$ dual to $\mathbf{R}$. It is proved in [9, Lemmas 2.2.8 and 2.2.9] that

$$\begin{aligned}\mathbf{D}_2 = \mathrm{var}\{&x^3 \approx x^2,\, x^3yzt \approx yxzxtx,\\ &xyzxty \approx yxzxty,\, xzxyty \approx xzyxty,\, xtyzxy \approx xtyzyx\},\\ \mathbf{R} \vee \mathbf{R}^\delta = \mathrm{var}\{&x^4 \approx x^3,\, x^3yzt \approx yxzxtx,\\ &xyzxty \approx yxzxty,\, xzxyty \approx xzyxty,\, xtyzxy \approx xtyzyx\}.\end{aligned}$$

It is easily seen that $\mathbf{D}_2 = (\mathbf{R} \vee \mathbf{R}^\delta) \wedge \mathrm{var}\{x^3 \approx x^2\}$.

Proposition 3.1. *The lattice* $\mathfrak{L}(\mathbf{R} \vee \mathbf{R}^\delta)$ *of subvarieties of* $\mathbf{R} \vee \mathbf{R}^\delta$ *is given in Fig. 2. In particular, this lattice is modular but not distributive.*

Proof. It is easily shown that $\mathbf{C}_3 \subseteq \mathbf{R}\vee\mathbf{R}^\delta$. According to Lemma 2.3, any subvariety $\mathbf{V}$ of $\mathbf{R} \vee \mathbf{R}^\delta$ such that $\mathbf{C}_3 \nsubseteq \mathbf{V}$ satisfies the identity $x^3 \approx x^2$, whence $\mathbf{V} \subseteq \mathbf{D}_2$. Therefore, the lattice $\mathfrak{L}(\mathbf{R} \vee \mathbf{R}^\delta)$ is the disjoint union of the lattice $\mathfrak{L}(\mathbf{D}_2)$ and the interval $[\mathbf{C}_3, \mathbf{R}\vee\mathbf{R}^\delta]$. It is proved in [10, Lemmas 4.4 and 4.5] that the lattice $\mathfrak{L}(\mathbf{D}_2)$ coincides with the 5-element chain in Fig. 2. Thus it remains to describe the interval $[\mathbf{C}_3, \mathbf{R} \vee \mathbf{R}^\delta]$. It follows from [12, Proposition 4.1] that every noncommutative variety in this interval is defined within $\mathbf{R}\vee\mathbf{R}^\delta$ by some of the identities $xyx \approx x^2y$,

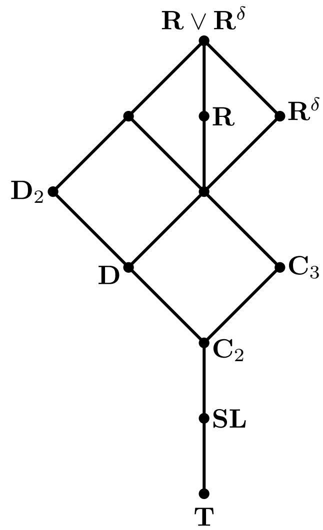

FIGURE 2. The subvariety lattice $\mathfrak{L}(\mathbf{R} \vee \mathbf{R}^\delta)$

$xyx \approx yx^2$ or $x^2y \approx yx^2$. It is then routinely shown that the interval $[\mathbf{C}_3, \mathbf{R} \vee \mathbf{R}^\delta]$ is as described in Fig. 2, where

$$\mathbf{R} = (\mathbf{R} \vee \mathbf{R}^\delta) \wedge \mathrm{var}\{xyx \approx yx^2\},$$
$$\mathbf{D}_2 \vee \mathbf{C}_3 = (\mathbf{R} \vee \mathbf{R}^\delta) \wedge \mathrm{var}\{x^2y \approx yx^2\},$$

and $\mathbf{D} \vee \mathbf{C}_3 = (\mathbf{D}_2 \vee \mathbf{C}_3) \wedge \mathbf{R} = (\mathbf{D}_2 \vee \mathbf{C}_3) \wedge \mathbf{R}^\delta = \mathbf{R} \vee \mathbf{R}^\delta$. The proof of this proposition is thus complete. □

4. NECESSARY CONDITION OF THE MODULARITY OF AN ELEMENT IN $\mathbb{MON}$

Given any word $\mathbf{w}$ and letter x, let $\mathrm{occ}_x(\mathbf{w})$ denote the number of occurrences of x in $\mathbf{w}$. Let λ denote the empty word. Let $W = W_1 \cup W_2$, where

$$W_1 = \{y^{r_1}xt^{r_2}z^{r_3}y^{r_4}t^{r_5}xz^{r_6} \mid r_1, r_2, r_3, r_4, r_5, r_6 \geq 2\},$$
$$W_2 = \{y^{r_1}xt^{r_2}z^{r_3}xy^{r_4}t^{r_5}xz^{r_6} \mid r_1, r_2, r_3, r_4, r_5, r_6 \geq 2\}.$$

Let us fix the following two words:

$$\mathbf{p} = y^2xt^2z^2y^2t^2xz^2 \quad \text{and} \quad \mathbf{q} = y^2xt^2z^2xy^2t^2xz^2.$$

Put $\mathbf{K} = \mathrm{var}\{\mathbf{p} \approx \mathbf{q}\}$.

Lemma 4.1. *The set W is a* FIC($\mathbf{K}$)*-class.*

Proof. Let $\mathbf{u} \approx \mathbf{v}$ be any identity of $\mathbf{K}$ with $\mathbf{u} \in W$. We need to verify that $\mathbf{v} \in W$. By assumption, there is a deduction of the identity $\mathbf{u} \approx \mathbf{v}$ from the identity $\mathbf{p} \approx \mathbf{q}$, that is, a sequence $\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_m$ of words such that $\mathbf{w}_0 = \mathbf{u}$, $\mathbf{w}_m = \mathbf{v}$ and, for each $i = 0, 1, \ldots, m-1$, there are words $\mathbf{a}_i, \mathbf{b}_i \in \mathfrak{X}^*$ and an endomorphism ξ_i of $\mathfrak{X}^*$ such that $\mathbf{w}_i = \mathbf{a}_i\xi_i(\mathbf{s}_i)\mathbf{b}_i$ and $\mathbf{w}_{i+1} = \mathbf{a}_i\xi_i(\mathbf{t}_i)\mathbf{b}_i$, where $\{\mathbf{s}_i, \mathbf{t}_i\} = \{\mathbf{p}, \mathbf{q}\}$. By trivial induction on m, it suffices to only consider the case when $\mathbf{u} = \mathbf{a}\xi(\mathbf{s})\mathbf{b}$ and $\mathbf{v} = \mathbf{a}\xi(\mathbf{t})\mathbf{b}$ for some words $\mathbf{a}, \mathbf{b} \in \mathfrak{X}^*$, an endomorphism ξ of $\mathfrak{X}^*$ and words $\mathbf{s}$ and $\mathbf{t}$ such that $\{\mathbf{s}, \mathbf{t}\} = \{\mathbf{p}, \mathbf{q}\}$.

Since any subword of $\mathbf{u}$ of the form ab, where a an b are distinct letters, occurs only once in $\mathbf{u}$ and all letters occurring in $\mathbf{s}$ are multiple, the following holds:

(I) For any $a \in \mathrm{con}(\mathbf{s})$, either $\xi(a) = \lambda$ or $\xi(a)$ is a power of some letter.

Further, since $\mathrm{occ}_x(\mathbf{u}) \leq 3$ and $\mathrm{occ}_y(\mathbf{s}) = \mathrm{occ}_z(\mathbf{s}) = \mathrm{occ}_t(\mathbf{s}) = 4$, we have

(II) $x \notin \mathrm{con}(\xi(yzt))$.

We note that if $\xi(\mathbf{s}) = \lambda$ or $\xi(\mathbf{s})$ is a power of some letter, then the required statement is evident. So, we may assume that

(III) $|\,\mathrm{con}(\xi(\mathbf{s}))| \geq 2$.

Let $\mathbf{u} = y^{\ell_1} x t^{\ell_2} z^{\ell_3} x^c y^{\ell_4} t^{\ell_5} x z^{\ell_6}$, where $c \in \{0,1\}$ and $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6 \geq 2$, and let

$$d = \begin{cases} 0 & \text{if } \mathbf{s} = \mathbf{p}, \\ 1 & \text{if } \mathbf{s} = \mathbf{q}. \end{cases}$$

If $\xi(x) = \lambda$, then $\xi(\mathbf{s}) = \xi(\mathbf{t})$, whence $\mathbf{v} = \mathbf{u} \in W$. So, it remains to consider the case when $\xi(x) \neq \lambda$. Then (I) implies that $\xi(x)$ is a power of some letter.

Suppose that $\xi(x)$ is a power of y. Then (III) implies that $\mathrm{con}(\xi(t^2z^2x^dy^2t^2))$ contains one of the letters x, z and t. This is only possible when $\xi(t^2z^2x^dy^2t^2) = y^p x t^{\ell_2} z^{\ell_3} x^c y^q$ for some $0 \leq p \leq \ell_1$ and $0 \leq q \leq \ell_4$. But since $x \notin \{y\} = \mathrm{con}(\xi(x))$ by assumption and $x \notin \mathrm{con}(\xi(yzt))$ by (II), the contradiction $x \notin \mathrm{con}(\xi(t^2z^2x^dy^2t^2))$ is deduced. Therefore, $\xi(x)$ cannot be a power of y. Similarly, $\xi(x)$ cannot be a power of z as well.

Suppose now that $\xi(x)$ is a power of t. Then (III) implies that $\mathrm{con}(\xi(t^2z^2x^dy^2t^2))$ contains one of the letters x, y and z. This is only possible when $\xi(t^2z^2x^dy^2t^2) = t^p z^{\ell_3} x^c y^{\ell_4} t^q$ for some $0 \leq p \leq \ell_2$ and $0 \leq q \leq \ell_5$. Then by (I), either $\xi(t) = \lambda$ or $\xi(t)$ is a power of t. This implies that $\xi(z^2x^dy^2) = z^{\ell_3} x^c y^{\ell_4}$. Taking into account that $\xi(x)$ is a power of t, we apply (I) again and obtain that $\xi(z^2) = z^{\ell_3}$, $\xi(y^2) = y^{\ell_4}$ and $c = d = 0$. This is only possible when $\xi(xt^2z^2x^d) = y^r x t^{\ell_2} z^{\ell_3} x^c$ for some $0 \leq r \leq \ell_1$. But since $x \notin \{t\} = \mathrm{con}(\xi(x))$ by assumption and $x \notin \mathrm{con}(\xi(yzt))$ by (II), the contradiction $x \notin \mathrm{con}(\xi(xt^2z^2x^d))$ is deduced. Therefore, $\xi(x)$ cannot be a power of t.

Finally, suppose that $\xi(x)$ is a power of x. Then since x^2 is not a subword of $\mathbf{u}$, we have $\xi(x) = x$.

Suppose that $c = 0$. Then $d = 0$ because otherwise, $\mathrm{occ}_x(\mathbf{u}) < \mathrm{occ}_x(\xi(\mathbf{s}))$. Then $\xi(t^2z^2y^2t^2) = t^{\ell_2} z^{\ell_3} y^{\ell_4} t^{\ell_5}$. It follow from (I) that $\xi(z^2) = z^{\ell_3}$, $\xi(y^2) = y^{\ell_4}$ and $\xi(t^2) = t^{\ell_2} = t^{\ell_5}$. Then $\xi(\mathbf{s}) = y^{\ell_4} x t^{\ell_2} z^{\ell_3} y^{\ell_4} t^{\ell_5} x z^{\ell_3}$, $\mathbf{a} = y^{\ell_1 - \ell_4}$ and $\mathbf{b} = z^{\ell_6 - \ell_3}$. Therefore, $\xi(\mathbf{t}) = y^{\ell_4} x t^{\ell_2} z^{\ell_3} x y^{\ell_4} t^{\ell_5} x z^{\ell_3}$, whence $\mathbf{v} = y^{\ell_1} x t^{\ell_2} z^{\ell_3} x y^{\ell_4} t^{\ell_5} x z^{\ell_6} \in W$, and we are done.

Suppose now that $c = 1$. If $x \in \mathrm{con}(\mathbf{b})$, then $d = 0$ because otherwise, $\mathrm{occ}_x(\mathbf{u}) < \mathrm{occ}_x(\xi(\mathbf{s})\mathbf{b})$. This is only possible when

$$\mathbf{a}\xi(y^2) = y^{\ell_1},\ x\xi(t^2z^2y^2t^2)x = x t^{\ell_2} z^{\ell_3} x \ \text{ and } \ \xi(z^2)\mathbf{b} = y^{\ell_4} t^{\ell_5} x z^{\ell_6}.$$

The second equality implies that $\xi(t^2z^2y^2t^2) = t^{\ell_2} z^{\ell_3}$. Clearly, $\xi(t^2) = \lambda$, whence $\xi(z^2y^2) = t^{\ell_2} z^{\ell_3}$. In view of (I), we have $\xi(z^2) = t^{\ell_2}$ and $\xi(y^2) = z^{\ell_3}$. But this contradicts the fact that $\xi(z^2)\mathbf{b} = z^{\ell_4} t^{\ell_5} x z^{\ell_6}$. Therefore, $x \notin \mathrm{con}(\mathbf{b})$. Analogously, one can verify that $x \notin \mathrm{con}(\mathbf{a})$. It follows that $d = 1$. Then

$$\mathbf{a}\xi(y^2) = y^{\ell_1},\ x\xi(t^2z^2)x\xi(y^2t^2)x = x t^{\ell_2} z^{\ell_3} x y^{\ell_4} t^{\ell_5} x \ \text{ and } \ \xi(z^2)\mathbf{b} = z^{\ell_6}.$$

It follows from (I) that $\xi(z^2) = z^{\ell_3}$, $\xi(y^2) = y^{\ell_4}$ and $\xi(t^2) = t^{\ell_2} = t^{\ell_5}$. Then $\xi(\mathbf{s}) = y^{\ell_3}xt^{\ell_2}z^{\ell_3}xy^{\ell_4}t^{\ell_5}xz^{\ell_4}$, $\mathbf{a} = y^{\ell_1-\ell_3}$ and $\mathbf{b} = z^{\ell_6-\ell_4}$. Therefore, $\xi(\mathbf{t}) = y^{\ell_3}xt^{\ell_2}z^{\ell_3}xy^{\ell_4}t^{\ell_5}xz^{\ell_4}$, whence $\mathbf{v} = y^{\ell_1}xt^{\ell_2}z^{\ell_3}y^{\ell_4}t^{\ell_5}xz^{\ell_6} \in W$, and we are done. □

For any $n \geq 1$, put
$$\mathbf{B}_n = \mathrm{var}\{x^n \approx x^{n+1}\}.$$

Lemma 4.2. *Suppose that* $\mathbf{V}$ *is any proper monoid variety that is a modular element of the lattice* $\mathbb{MON}$*. Then* $\mathbf{V}$ *is periodic.*

Proof. Seeking a contradiction, suppose that $\mathbf{V}$ is not periodic, so that $\mathbf{V}$ contains the variety $\mathbf{COM}$ of all commutative monoids. Since $\mathbf{V}$ is proper and non-periodic, it satisfies some nontrivial identity $\mathbf{u} \approx \mathbf{v}$ such that every letter from $\mathrm{con}(\mathbf{uv})$ occurs n times on both sides for some $n \geq 1$, that is, $n = \mathrm{occ}_a(\mathbf{u}) = \mathrm{occ}_a(\mathbf{v})$ for all $a \in \mathrm{con}(\mathbf{uv})$. Then by [14, Lemma 3.2], there exist two distinct letters x and y such that the identity obtained from $\mathbf{u} \approx \mathbf{v}$ by retaining x and y is nontrivial. Therefore we may assume that $\mathrm{con}(\mathbf{u}) = \mathrm{con}(\mathbf{v}) = \{x, y\}$ with $n = \mathrm{occ}_x(\mathbf{u}) = \mathrm{occ}_x(\mathbf{v}) = \mathrm{occ}_y(\mathbf{u}) = \mathrm{occ}_y(\mathbf{v})$.

Suppose that $\mathbf{LRB} \subseteq \mathbf{V}$. In view of Lemma 2.2, we may assume without loss of generality that $\mathrm{ini}(\mathbf{u}) = \mathrm{ini}(\mathbf{v}) = xy$. Let $\mathbf{u}'$ and $\mathbf{v}'$ be words that obtain from $\mathbf{u}$ and $\mathbf{v}$, respectively, by making the substitution $(x, y) \mapsto (y, x)$. Then $\mathrm{ini}(\mathbf{u}') = \mathrm{ini}(\mathbf{v}') = yx$. Put
$$A = \{\mathbf{w} \in \{x, y\}^+ \mid \mathrm{occ}_x(\mathbf{w}) = n + 1,\, \mathrm{occ}_y(\mathbf{w}) = n\}.$$
Let $\mathbf{w}, \mathbf{w}' \in A$ and $\mathbf{w} \preceq \mathbf{w}'$. This means that $\mathbf{w}' = \mathbf{a}\xi(\mathbf{w})\mathbf{b}$ for some words $\mathbf{a}, \mathbf{b} \in \mathfrak{X}^*$ and some endomorphism ξ of $\mathfrak{X}^+$. Since the length of $\mathbf{w}$ equals to the length of $\mathbf{w}'$, we have $\mathbf{a} = \mathbf{b} = \lambda$. Then $\mathbf{w}' = \xi(\mathbf{w})$. But this is only possible when $\xi(x) = x$ and $\xi(y) = y$ because $\mathrm{occ}_y(\mathbf{w}) < \mathrm{occ}_x(\mathbf{w})$. Hence $\mathbf{w} = \mathbf{w}'$. So, A is an anti-chain under the quasi-order $\preceq$. Then L_A is a sublattice of $\mathbb{MON}$ by Lemma 2.1(i) and $\mathbf{V} \in \mathrm{L}_A$.

Clearly, $\mathbf{u}x, \mathbf{u}'x, \mathbf{v}x, \mathbf{v}'x \in A$. Evidently, $\mathrm{ini}(\mathbf{u}x) = \mathrm{ini}(\mathbf{v}x) = xy$ and $\mathrm{ini}(\mathbf{u}'x) = \mathrm{ini}(\mathbf{v}'x) = yx$. In view of Lemma 2.2, the words $\mathbf{u}x$ and $\mathbf{u}'x$ lie in distinct $\mathrm{FIC}(\mathbf{V})$-classes. Then, since $\mathbf{V}$ satisfies the nontrivial identities $\mathbf{u}x \approx \mathbf{v}x$ and $\mathbf{u}'x \approx \mathbf{v}'x$, the equivalence $\gamma = \varphi(\mathbf{V})$ contains at least two non-singleton classes. It is verified in [11, Proposition 2.2] that a partition $\rho \in \mathrm{Part}(X)$ is a modular element in $\mathrm{Part}(X)$ if and only if ρ has at most one non-singleton class. This result implies that γ is not a modular element of the lattice $\mathrm{Part}(A)$. Then there are $\alpha, \beta \in \mathrm{Part}(A)$ such that $\alpha \subset \beta$ and
$$(\gamma \wedge \beta) \vee \alpha \subset (\gamma \vee \alpha) \wedge \beta. \tag{4.1}$$
According to Lemma 2.1, we can find a non-periodic variety $\mathbf{X} \in \mathrm{L}_A$ such that $\varphi(\mathbf{X}) = \alpha$. Put
$$\mathbf{Y} = \mathbf{X} \wedge \mathrm{var}\{\mathbf{w} \approx \mathbf{w}' \mid (\mathbf{w}, \mathbf{w}') \in \beta\}.$$
Clearly, $\mathbf{Y} \in \mathrm{L}_A$ and $\varphi(\mathbf{Y}) = \beta$. Then
$$(\mathbf{V} \wedge \mathbf{X}) \vee \mathbf{Y} \subset (\mathbf{V} \vee \mathbf{Y}) \wedge \mathbf{X}$$
because otherwise, the inclusion (4.1) does not hold. We see that $\mathbf{V}$ is not a modular element of the lattice $\mathbb{MON}$, which is a contradiction.

Suppose now that $\mathbf{LRB} \nsubseteq \mathbf{V}$. Then Lemma 2.2 allows us to assume that $\mathbf{u}$ starts with the letter x but $\mathbf{v}$ starts with the letter y. Let
$$\mathbf{Z} = \mathrm{var}\{x^{n+1} \approx x^{n+2},\, x^n\mathbf{v} \approx x^{n+1}\mathbf{v}\}.$$

We note that $\mathbf{V} \wedge \mathbf{B}_{n+1} \subseteq \mathbf{Z}$. Indeed, $\mathbf{V} \wedge \mathbf{B}_{n+1}$ satisfies the identities

$$x^n\mathbf{v} \approx x^n\mathbf{u} \approx x^{n+1}\mathbf{u} \approx x^{n+1}\mathbf{v}$$

and so the identity $x^n\mathbf{v} \approx x^{n+1}\mathbf{v}$. Clearly, the word $x^n\mathbf{v}$ is an isoterm for both $\mathbf{V} \vee \mathbf{Z}$ and $\mathbf{B}_{n+1}$. It follows that $x^n\mathbf{v}$ is an isoterm for $(\mathbf{V} \vee \mathbf{Z}) \wedge \mathbf{B}_{n+1}$ as well. However, $x^n\mathbf{v}$ is not an isoterm for $\mathbf{Z}$ because $\mathbf{Z}$ satisfies $x^n\mathbf{v} \approx x^{n+1}\mathbf{v}$. Therefore,

$$(\mathbf{V} \wedge \mathbf{B}_{n+1}) \vee \mathbf{Z} = \mathbf{Z} \subset (\mathbf{V} \vee \mathbf{Z}) \wedge \mathbf{B}_{n+1}.$$

This means that $\mathbf{V}$ is not a modular element of the lattice $\mathbb{MON}$, which again is a contradiction. □

The following is the main result of this section.

Proposition 4.3. *Suppose that* $\mathbf{V}$ *is any proper monoid variety that is a modular element of the lattice* $\mathbb{MON}$. *Then* $\mathbf{V}$ *satisfies the identities*

$$x^2 \approx x^3, \tag{4.2}$$

$$x^2y \approx yx^2. \tag{4.3}$$

Proof. By Lemma 4.2, the variety $\mathbf{V}$ is periodic and so it satisfies the identity $x^n \approx x^{n+m}$ for some $n, m \geq 1$; we may assume n and m to be the least possible.

First, suppose that $n = 1$, so that $\mathbf{V}$ is completely regular. If $\mathbf{X}$ is a noncommutative completely regular variety, then it is verified in [5, Lemma 3.1] that

$$(\mathbf{X} \wedge \mathbf{D}) \vee \mathbf{C}_2 \subset (\mathbf{X} \vee \mathbf{C}_2) \wedge \mathbf{D},$$

whence $\mathbf{X}$ is not a modular element of the lattice $\mathbb{MON}$. If $\mathbf{X}$ is a commutative variety containing a nontrivial group, then it is proved in [5, Lemma 3.2] that

$$(\mathbf{X} \wedge \mathbf{B}_2) \vee \mathbf{Q} \subset (\mathbf{X} \vee \mathbf{Q}) \wedge \mathbf{B}_2,$$

where $\mathbf{Q} = \mathrm{var}\{yxyzxy \approx yxzxyxz\}$, whence $\mathbf{X}$ is again not a modular element of the lattice $\mathbb{MON}$. In view of these two facts, the variety $\mathbf{V}$ is commutative and does not contain any nontrivial group. Since $\mathbf{V}$ is also completely regular, it is idempotent and so is contained in $\mathbf{SL}$. Obviously, $\mathbf{SL}$ satisfies (4.2) and (4.3).

So, it remains to consider the case when $n > 1$. Then $\mathbf{C}_n \subseteq \mathbf{V}$ and $\mathbf{C}_{n+1} \nsubseteq \mathbf{V}$ by Lemma 2.3. It follows from [6, Lemma 2] that $\mathbf{E} \nsubseteq \mathbf{V}$. Then by Lemma 2.5, $\mathbf{V}$ satisfies the identity $x^{p_1}yx^{q_1} \approx yx^{r_1}$ for some $p_1, q_1 \geq 1$ and $r_1 \geq 2$. The dual arguments imply that $\mathbf{V}$ also satisfies the identity $x^{p_2}yx^{q_2} \approx x^{r_2}y$ for some $p_2, q_2 \geq 1$ and $r_2 \geq 2$. Since one can substitute x^n for x in these identities and $\mathbf{V}$ satisfies $x^n \approx x^{n+m}$, we may assume without loss of generality that

$$p_1, p_2, q_1, q_2, r_1, r_2 \in \{n, n+1, \ldots, n+m-1\}.$$

Evidently, there exist ℓ_1 and ℓ_2 such that the identities

$$x^{p_1}yx^{q_1+\ell_1} \approx yx^{r_1+\ell_1} \quad \text{and} \quad x^{p_2+\ell_2}yx^{q_2} \approx x^{r_2+\ell_2}y$$

are equivalent modulo $x^n \approx x^{n+m}$ to the identities

$$x^{p_1}yx^{q_2} \approx yx^{r_1+\ell_1} \quad \text{and} \quad x^{p_1}yx^{q_2} \approx x^{r_2+\ell_2}y,$$

respectively. Therefore $\mathbf{V}$ satisfies $x^{r_2+\ell_2}y \approx yx^{r_1+\ell_1}$, whence it satisfies

$$x^ky \approx yx^k \tag{4.4}$$

for some $k \geq n$. It follows that the meet $\mathbf{V} \wedge \mathbf{B}_2$ satisfies the identities (4.2) and (4.4); it also satisfies the identity $\mathbf{p} \approx \mathbf{q}$ because

$$\begin{aligned}\mathbf{p} = y^2xt^2z^2y^2t^2xz^2 &\stackrel{(4.2)}{\approx} y^kxt^kz^ky^kt^kxz^k \\ &\stackrel{(4.4)}{\approx} x^2y^{2k}z^{2k}t^{2k} \\ &\stackrel{(4.2)}{\approx} x^3y^{2k}z^{2k}t^{2k} \\ &\stackrel{(4.4)}{\approx} y^kxt^kz^kxy^kt^kxz^k \\ &\stackrel{(4.2)}{\approx} y^2xt^2z^2xy^2t^2xz^2 = \mathbf{q}.\end{aligned}$$

Therefore $\mathbf{V} \wedge \mathbf{B}_2 \subseteq \mathbf{K}$, so that $(\mathbf{V} \wedge \mathbf{B}_2) \vee \mathbf{K} = \mathbf{K}$.

Suppose that $n > 2$ or $m > 1$. Recall from the beginning of the section that

$$W_1 = \{y^{r_1}xt^{r_2}z^{r_3}y^{r_4}t^{r_5}xz^{r_6} \mid r_1, r_2, r_3, r_4, r_5, r_6 \geq 2\}.$$

Let $\mathbf{a} \approx \mathbf{b}$ be any identity of $\mathbf{V} \vee \mathbf{K}$ with $\mathbf{a} \in W_1$. If $n > 2$, then $\mathbf{b} \in W_1$ by Lemmas 2.3 and 4.1. Clearly, $\mathbf{V}$ contains the variety $\mathbf{A}_m$ of all Abelian groups of exponent m. It is well known and easily verified that an identity $\mathbf{w} \approx \mathbf{w}'$ holds in $\mathbf{A}_m$ if and only if $\mathrm{occ}_a(\mathbf{w}) \equiv \mathrm{occ}_a(\mathbf{w}') \pmod m$ for all $a \in \mathfrak{X}$. This fact and Lemma 4.1 imply that if $m > 1$, then $\mathbf{b} \in W_1$. We see that if $n > 2$ or $m > 1$, then $\mathbf{b} \in W_1$ in either case. Evidently, if $\mathbf{B}_2$ satisfies an identity $\mathbf{c} \approx \mathbf{d}$ with $\mathbf{c} \in W_1$, then $\mathbf{d} \in W_1$. This implies that if an identity of the form $\mathbf{p} \approx \mathbf{w}$ holds in $(\mathbf{V} \vee \mathbf{K}) \wedge \mathbf{B}_2$, then $\mathbf{w} \in W_1$. In particular, $(\mathbf{V} \vee \mathbf{K}) \wedge \mathbf{B}_2$ violates $\mathbf{p} \approx \mathbf{q}$. Therefore,

$$(\mathbf{V} \wedge \mathbf{B}_2) \vee \mathbf{K} = \mathbf{K} \subset (\mathbf{V} \vee \mathbf{K}) \wedge \mathbf{B}_2.$$

This means that $\mathbf{V}$ is not a modular element in $\mathbb{MON}$. It follows that $n = 2$ and $m = 1$. Then $\mathbf{V}$ satisfies (4.2). Besides that, since (4.4) holds in the variety $\mathbf{V}$, this variety satisfies (4.3).

Proposition 4.3 is thus proved. □

5. Proof of Theorem 1.1

Necessity. Let $\mathbf{V}$ be any proper monoid variety that is a cancellable element of the lattice $\mathbb{MON}$. Since any cancellable element is modular, Proposition 4.3 implies that $\mathbf{V}$ satisfies the identities (4.2) and (4.3). If $\mathbf{V}$ does not coincide with any of the varieties $\mathbf{T}$, $\mathbf{SL}$, $\mathbf{C}_2$ and $\mathbf{D}$, then $\mathbf{V}$ contains the variety $\mathbf{D}_2$ by [8, Lemma 3.3(i)]. Proposition 3.1 and the fact that $\mathbf{C}_3 \nsubseteq \mathbf{V}$ imply that $\mathbf{V} \vee \mathbf{R} = \mathbf{V} \vee \mathbf{R}^\delta$ and $\mathbf{V} \wedge \mathbf{R} = \mathbf{V} \wedge \mathbf{R}^\delta = \mathbf{D}$, contradicting the assumption that $\mathbf{V}$ is a cancellable element of $\mathbb{MON}$. Hence $\mathbf{V}$ coincides with one of the varieties $\mathbf{T}$, $\mathbf{SL}$, $\mathbf{C}_2$ and $\mathbf{D}$.

Sufficiency. Obviously, $\mathbf{T}$ and $\mathbf{MON}$ are cancellable elements of $\mathbb{MON}$. An element x of a lattice L is *costandard* if

$$\forall\, y, z \in L\colon \quad (x \wedge z) \vee y = (x \vee y) \wedge (z \vee y).$$

It is easily seen that any costandard element is cancellable. It is shown in [5, Theorem 1.2] that the varieties $\mathbf{SL}$ and $\mathbf{C}_2$ are costandard elements of the lattice $\mathbb{MON}$. Therefore, these varieties are cancellable elements of this lattice.

So, it remains to establish that $\mathbf{D}$ is a cancellable element in $\mathbb{MON}$. Let $\mathbf{X}$ and $\mathbf{Y}$ be monoid varieties such that $\mathbf{D} \vee \mathbf{X} = \mathbf{D} \vee \mathbf{Y}$ and $\mathbf{D} \wedge \mathbf{X} = \mathbf{D} \wedge \mathbf{Y}$. If $\mathbf{D} \subseteq \mathbf{X}$, then $\mathbf{D} = \mathbf{D} \wedge \mathbf{X} = \mathbf{D} \wedge \mathbf{Y}$, so that $\mathbf{D} \subseteq \mathbf{Y}$, whence $\mathbf{X} = \mathbf{D} \vee \mathbf{X} = \mathbf{D} \vee \mathbf{Y} = \mathbf{Y}$ and we are done. Therefore by symmetry, we may assume that $\mathbf{D} \nsubseteq \mathbf{X}$ and $\mathbf{D} \nsubseteq \mathbf{Y}$.

Now the subvariety lattice $\mathfrak{L}(\mathbf{D})$ is the chain $\mathbf{T} \subset \mathbf{SL} \subset \mathbf{C}_2 \subset \mathbf{D}$; see Fig. 2. It follows that $\mathbf{D} \wedge \mathbf{X} = \mathbf{D} \wedge \mathbf{Y} \in \{\mathbf{T}, \mathbf{SL}, \mathbf{C}_2\}$. If $\mathbf{D} \wedge \mathbf{X} = \mathbf{D} \wedge \mathbf{Y} = \mathbf{T}$, then $\mathbf{X}$ and $\mathbf{Y}$ are varieties of groups by [7, Lemma 2.1]. Then $\mathbf{X} \vee \mathbf{Y}$ is a variety of groups too and so $\mathbf{SL} \nsubseteq \mathbf{X} \vee \mathbf{Y}$, whence

$$\mathbf{D} \wedge (\mathbf{X} \vee \mathbf{Y}) = \mathbf{D} \wedge \mathbf{X} = \mathbf{D} \wedge \mathbf{Y}. \tag{5.1}$$

If $\mathbf{D} \wedge \mathbf{X} = \mathbf{D} \wedge \mathbf{Y} = \mathbf{SL}$, then $\mathbf{X}$ and $\mathbf{Y}$ are completely regular varieties by [7, Corollary 2.6]. Then $\mathbf{X} \vee \mathbf{Y}$ is completely regular and so $\mathbf{C}_2 \nsubseteq \mathbf{X} \vee \mathbf{Y}$, whence the equality (5.1) is true. Finally, if $\mathbf{D} \wedge \mathbf{X} = \mathbf{D} \wedge \mathbf{Y} = \mathbf{C}_2$, then $\mathbf{X}$ and $\mathbf{Y}$ are commutative by Lemma 2.4. Then $\mathbf{X} \vee \mathbf{Y}$ is commutative and so $\mathbf{D} \nsubseteq \mathbf{X} \vee \mathbf{Y}$, whence the equality (5.1) is true again. We see that the equality (5.1) holds in any case.

Clearly,

$$\mathbf{D} \vee (\mathbf{X} \vee \mathbf{Y}) = \mathbf{D} \vee \mathbf{X} = \mathbf{D} \vee \mathbf{Y}. \tag{5.2}$$

Then

$$\begin{aligned}
\mathbf{X} &= (\mathbf{D} \wedge \mathbf{X}) \vee \mathbf{X} && \text{because } \mathbf{D} \wedge \mathbf{X} \subset \mathbf{X}\\
&= (\mathbf{D} \wedge (\mathbf{X} \vee \mathbf{Y})) \vee \mathbf{X} && \text{by (5.1)}\\
&= (\mathbf{D} \vee \mathbf{X}) \wedge (\mathbf{X} \vee \mathbf{Y}) && \text{by [6, Proposition 7]}\\
&= (\mathbf{D} \vee (\mathbf{X} \vee \mathbf{Y})) \wedge (\mathbf{X} \vee \mathbf{Y}) && \text{by (5.2)}\\
&= (\mathbf{X} \vee \mathbf{Y}) && \text{because } \mathbf{X} \vee \mathbf{Y} \subset \mathbf{D} \vee (\mathbf{X} \vee \mathbf{Y}).
\end{aligned}$$

We see that $\mathbf{X} = \mathbf{X} \vee \mathbf{Y}$. By symmetry, $\mathbf{Y} = \mathbf{X} \vee \mathbf{Y}$, whence $\mathbf{X} = \mathbf{Y}$. Therefore, $\mathbf{D}$ is a cancellable element in $\mathbb{MON}$. □

References

[1] J. Almeida, Finite Semigroups and Universal Algebra, World Scientific, Singapore, 1994. Zbl 0844.20039, MR1331143, DOI 10.1142/2481

[2] S. Burris, H. P. Sankappanavar, A Course in Universal Algebra, Springer Verlag, New York, 1981. Zbl 0478.08001, MR0648287

[3] G. Grätzer, Lattice Theory: Foundation, Springer Basel AG, 2011. Zbl 1233.06001, MR2768581, DOI 10.1007/978-3-0348-0018-1

[4] S. V. Gusev, *On the lattice of overcommutative varieties of monoids*, Izv. VUZ. Matem. No. 5 (2018), 28–32 [Russian; Engl. translation: Russ. Math. Izv. VUZ, **62**, No. 5 (2018), 23–26. Zbl 1398.20070, MR3890382, DOI 10.3103/S1066369X18050043].

[5] S. V. Gusev, *Special elements of the lattice of monoid varieties*, Algebra Universalis, **79** (2018), Article 29, 1–12. Zbl 06904414, MR3788808, DOI 10.1007/s00012-018-0513-0

[6] S. V. Gusev, *Standard elements of the lattice of monoid varieties*, Algebra i Logika, submitted [Russian, Engl. translation available at: http://arxiv.org/abs/1909.13013].

[7] S. V. Gusev, B. M. Vernikov, *Chain varieties of monoids*, Dissert. Math., **534** (2018), 1–73. Zbl 1434.20042, MR3878627, DOI 10.4064/dm772-2-2018

[8] S. V. Gusev, B. M. Vernikov, *Two weaker variants of congruence permutability for monoid varieties*, Semigroup Forum, submitted; available at: http://arxiv.org/abs/2009.11494

[9] M. Jackson, Small Semigroup Related Structures with Infinite Properties, Ph.D. thesis, University of Tasmania, Hobart, 1999.

[10] M. Jackson, *Finiteness properties of varieties and the restriction to finite algebras*, Semigroup Forum, **70** (2005), 154–187. Zbl 1073.20052, MR2129773, DOI 10.1007/s00233-004-0161-x

[11] J. Ježek, *The lattice of equational theories. Part* I*: modular elements*, Czechosl. Math. J., **31** (1981), 127–152. Zbl 0477.08006, MR0604120, DOI 10.21136/CMJ.1981.101731

[12] E. W. H. Lee, *Maximal Specht varieties of monoids*, Mosc. Math. J., **12** (2012), 787–802. Zbl 1295.20063, MR3076856

[13] E. W. H. Lee, *Varieties generated by* 2*-testable monoids*, Studia Sci. Math. Hungar, **49** (2012), 366–389. Zbl 1274.20057, MR3099233, DOI 10.1556/sscmath.49.2012.3.1211

[14] O. Sapir, *Finitely based words*, Int. J. Algebra Comput., **10** (2000), 457–480. Zbl 1010.20043, MR1776051, DOI 10.1142/S0218196700000224

[15] V. Yu. Shaprynskiĭ, D. V. Skokov, B. M. Vernikov, *Cancellable elements of the lattices of varieties of semigroups and epigroups*, Comm. Algebra, **47** (2019), 4697–4712. Zbl 1444.20036, MR3991044, DOI 10.1080/00927872.2019.1590585

[16] B. M. Vernikov, *Special elements in lattices of semigroup varieties*, Acta Sci. Math. (Szeged), **81** (2015), 79–109. Zbl 1363.20050, MR3381875, DOI 10.14232/actasm-013-072-0

Institute of Natural Sciences and Mathematics, Ural Federal University, Lenina str. 51, 620000 Ekaterinburg, Russia

Email address: sergey.gusb@gmail.com

Department of Mathematics, Nova Southeastern University, Fort Lauderdale, FL 33314, USA

Email address: edmond.lee@nova.edu