

Noname manuscript No.  
(will be inserted by the editor)

---

## Antiplane Stoneley waves propagating at the interface between two couple stress elastic materials

Andrea Nobili<sup>1,2,3</sup> · Valentina Volpini<sup>1,2</sup> ·  
Cesare Signorini<sup>3</sup>

the date of receipt and acceptance should be inserted later

**Abstract** We investigate antiplane Stoneley waves, localized at the discontinuity surface between two half-spaces in perfect contact. Both half-spaces are elastic linear isotropic and possess a microstructure that is described within the theory of couple stress materials with micro-inertia. In contrast to classical elasticity, where antiplane Stoneley waves are never supported and in-plane Stoneley waves exist only inasmuch as the ratio of the shear velocity of the half-spaces is small enough, we find that propagation is possible under broad conditions. However, Stoneley waves only propagate beyond a cuton frequency, for which an explicit expression is provided. For a given frequency above cuton, this expression lends the admissible range of material parameters that allows propagation. In particular, significant contrast between the adjoining materials is possible, provided that Stoneley waves propagate at high-enough frequency. Therefore, micro-inertia plays an important role in determining the features of propagation. Considerations concerning existence and uniqueness of antiplane Stoneley waves are given: it is found that evanescent and decaying/exploding modes are also admitted. Results may be especially useful when accounting for microstructure in Non-Destructive Testing (NDT) and near-surface prospection.

**Keywords** Antiplane Stoneley waves · microstructure · couple-stress · existence and uniqueness

---

Andrea Nobili  
Department of Engineering Enzo Ferrari, University of Modena and Reggio Emilia, via Vivarelli 10, 41125 Modena, Italy  
E-mail: [andrea.nobili@unimore.it](mailto:andrea.nobili@unimore.it)

Valentina Volpini  
Research Centre CRICT, via Vivarelli 10, 41125 Modena, Italy

Cesare Signorini  
Centre En&Tech, Tecnopolo, p.le Europa 1, 42124 Reggio Emilia, Italy

## 20 1 Introduction

21 The quest for proving existence of new types of localized waves, similar in nature  
 22 to Rayleigh waves occurring at a free surface, begun shortly after the discovery of  
 23 these in 1885 [30]. In 1911, Love [16, p.165] investigated the possibility of waves  
 24 propagating at the free surface of a layer in perfect contact with a half-space, in an  
 25 attempt to explain the problematic (from the theoretical standpoint) appearance  
 26 of shear horizontal Rayleigh waves in seismograms [19]. Later, Stoneley investi-  
 27 gated the existence of waves localised at the surface of discontinuity between two  
 28 materials [28, 29]. As he points out “Whereas, however, Prof. Love’s problem is  
 29 concerned with a disturbance confined chiefly to the free surface, the present paper  
 30 deals with a wave motion that is greatest at the surface of separation of the two  
 31 media” [29]. Indeed, these waves go under the name of generalized Rayleigh waves  
 32 or, quite fittingly, Stoneley waves. Stoneley concludes that “we can definitely as-  
 33 sert that when the wave-velocities are not too widely different for the two media,  
 34 a wave of the Rayleigh type can exist at the interface”. This finding is somewhat  
 35 surprising, in that it allows localized waves to exist only at “weak” boundaries,  
 36 whereas the exact opposite could be expected. At the time of Stoneley, the main  
 37 motivation behind the investigation was to determine whether seismic energy could  
 38 escape at the Gutenberg-Wiechert boundary between the Earth’s mantle and the  
 39 core. Indeed, existence conditions were supposed to correspond to Wiechert con-  
 40 ditions, expressing equality of mass density and Lamé moduli across the surface  
 41 of discontinuity. Precise quantification of the range of existence of Stoneley waves  
 42 came much later, by Scholte [26]. Scholte conditions are very restrictive for, as  
 43 pointed out in [23, 11], they are satisfied merely by 31 combinations among 900  
 44 isotropic materials.

45 Existence and uniqueness for Stoneley waves have been investigated in the  
 46 monograph [3, Chap.4], by applying the argument principle to the correspond-  
 47 ing Rayleigh function. Extension to anisotropic materials, which admit *generalized*  
 48 Stoneley waves, was given by Lin and Musgrave [15] and by Chadwick and Borejko  
 49 [4] (notably, Chadwick has been Robert Stoneley’s research student).

50 Recently, the role of material microstructure has attracted considerable atten-  
 51 tion in connection with wave propagation and related phenomena [22, 8, 10, 27,  
 52 9, 20, 19]. One way of encompassing material micro-structure in the models is by  
 53 means of polar theories, among which couple stress (CS) theory is perhaps the  
 54 simplest [17, 24]. Couple stress theory builds on top of classical elasticity (CE) by  
 55 adding an extra kinematical field, named the micro-rotation. However, in contrast  
 56 to micro-polar theory, micro-rotation is taken to be entrained by the displacement  
 57 field [13]. As a result, a length scale is introduced in the system and the theory  
 58 is no longer self-similar. Consequently, some unphysical results appearing in CE  
 59 are repaired, such as the non-dispersive nature of bulk and Rayleigh waves and  
 60 the lack of support for anti-plane (or shear horizontally polarized) Rayleigh and  
 61 interfacial (Stoneley) waves. It is precisely the last issue that is addressed in this  
 62 paper.

63 In recent times, a number of contributions have appeared in the literature con-  
 64 cerning propagation of Stoneley waves. In [14], propagation of Stoneley waves is  
 65 considered within the context of the modified couple stress theory, which is a spe-  
 66 cial case wherein symmetry of the couple stress tensor is enforced [33]. The speed  
 67 of Stoneley waves guided by a perfect interface between two elastic half-spaces is

determined analytically in [32] for the case of equal bulk moduli. Consideration of Stoneley waves propagating at a loose interface between two elastic half-spaces is given in [31]. Similarly, guided propagation occurring between two half-spaces in elastically constrained contact is considered in [1]. Since classical elasticity does not support antiplane Stoneley waves, every listed contribution deals with waves polarized in the sagittal plane, to which the term Stoneley waves is traditionally attached. As a notable exception, in [6] antiplane Stoneley waves are shown to occur at the interface between two half-spaces in the presence of surface elasticity.

As illustrated in [19], the presence of the microstructure provides new pathways for energy transport, which take the form of novel wave propagation patterns being supported. In this paper, we show that, in contrast to CE, anti-plane Stoneley waves are supported in CS materials under very general conditions concerning the elastic contrast between the media in contact. However, propagation is only permitted beyond a cuton frequency that is an increasing function of this contrast. Indeed, the role of rotational inertia is especially important for determining the range of admissible parameters for propagation to occur. Consequently, despite its appealing simplicity, the approximation of zero rotation inertia should be avoided. We also discuss existence and uniqueness of Stoneley waves.

Traditionally, Stoneley waves have been exploited in borehole seismics to determine the shear-wave velocities at different depths. Also, thin-film applications are possible, as it is demonstrated in [25]. A pioneering application to Non-Destructive Testing (NDT) is experimentally investigated in [11]. The recent monograph [5] collects 14 real-life case-study applications of surface and near-surface waves, ranging from geophysics to civil engineering, from geotechnics to moonquakes. Today, investigation of Stoneley waves is sustained by modern promising developments in the field of acoustic NDT, as very recently discussed in [12]. This paper aims to add a further tool to the investigator, in the form of anti-plane interfacial waves reflecting the microstructure underneath the continuous media.

## 2 Two couple stress elastic half-spaces in perfect contact

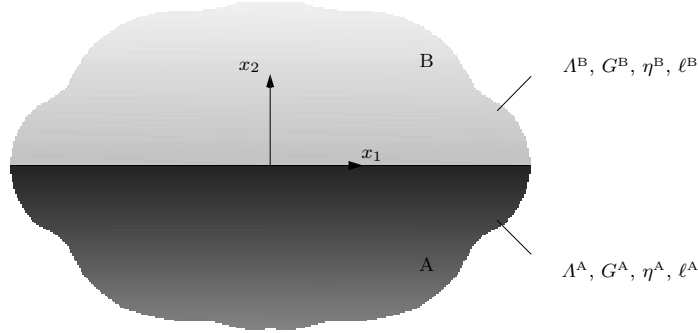
We consider two half-spaces, named A and B, in perfect contact along a plane surface. We introduce a right-handed Cartesian coordinate system  $(O, x_1, x_2, x_3)$ , whose axes are directed along the relevant unit vectors  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ . The co-ordinate system is located in such a way that the plane  $x_2 = 0$  corresponds to the contact surface between A and B, see Fig.1. Both half-spaces possess a microstructure, which is described within the theory of linear couple stress (CS) elasticity. Their relevant properties are henceforth denoted by the superscript  $k \in \{A, B\}$ .

The stress state in each half-space depends not only on the classical Cauchy force stress tensor  $\mathbf{s}^k$ , but also on the couple stress tensor  $\boldsymbol{\mu}^k$ . The latter characterises the polar behaviour of the material such that, for any directed surface of unit normal  $\mathbf{n}^k$ , it determines the internal reduced couple vector  $\mathbf{q}^k$ , acting across that surface

$$\mathbf{q}^k = (\boldsymbol{\mu}^k)^T \mathbf{n}^k, \quad (1)$$

where the superscript T denotes the transposed tensor.  $\mathbf{s}^k$  is conveniently decomposed into its symmetric and skew-symmetric part

$$\mathbf{s}^k = \boldsymbol{\sigma}^k + \boldsymbol{\tau}^k, \quad \boldsymbol{\sigma}^k = \text{Sym } \mathbf{s}^k, \quad \boldsymbol{\tau}^k = \text{Skw } \mathbf{s}^k, \quad (2)$$



**Fig. 1** Schematics, mechanical and microstructural properties of the two half-spaces A and B

111 whereas  $\boldsymbol{\mu}^k$  is decomposed in the deviatoric and spherical part

$$\boldsymbol{\mu}^k = \boldsymbol{\mu}_D^k + \boldsymbol{\mu}_S^k, \quad (3)$$

112 Specifically, the latter reads

$$\boldsymbol{\mu}_S^k = \frac{1}{3} (\boldsymbol{\mu}^k \cdot \mathbf{1}) \mathbf{1}, \quad (4)$$

113 wherein a dot denotes the scalar product and  $\mathbf{1}$  is the rank-2 identity tensor.

114 In each half-space, occupying the volume  $\mathcal{B}^k$ , the internal virtual work may be  
115 expressed as follows (see, e.g., [13, 22]):

$$\int_{\mathcal{B}^k} (\boldsymbol{\sigma}^k \cdot \text{grad } \mathbf{u}^k + \boldsymbol{\mu}^k \cdot \text{grad}^T \boldsymbol{\varphi}^k) dV, \quad (5)$$

116 where grad denotes the gradient operator. Here,  $\mathbf{u}^k$  and  $\boldsymbol{\varphi}^k$  are the displacement  
117 and the micro-rotation vector fields. Within the theory of CS materials (see, for  
118 instance, [13]), the displacement field  $\mathbf{u}^k$  and the micro-rotation field  $\boldsymbol{\varphi}^k$  are related  
119 by

$$\boldsymbol{\varphi}^k = \frac{1}{2} \text{curl } \mathbf{u}^k, \quad \Leftrightarrow \quad \varphi_i^k = \frac{1}{2} \mathbb{E}_{ijs}^k u_{s,j}^k, \quad (6)$$

120 where it is understood that a subscript comma denotes partial differentiation, i.e.  
121  $u_{i,j} = (\text{grad } \mathbf{u})_{ij} = \partial u_i / \partial x_j$ , and  $\mathbb{E}$  is the rank-3 Levi-Civita (permutation) tensor.

122 By standard arguments of linear elasticity, the displacement field is related to  
123 the linear strain tensor  $\boldsymbol{\varepsilon}^k$  through

$$\boldsymbol{\varepsilon}^k = \text{Sym grad } \mathbf{u}^k, \quad (7)$$

124 and this is work-conjugated to  $\boldsymbol{\sigma}$  (see (5)). In similar manner, we introduce the  
 125 *torsion-flexure* (or *wryness*) tensor

$$\boldsymbol{\chi}^k = \text{grad } \boldsymbol{\varphi}^k, \quad (8)$$

126 which is work-conjugated to  $\boldsymbol{\mu}^k$ .

127 By combining Eqs.(5,6) and (8), we infer

$$\boldsymbol{\chi}^k = \boldsymbol{\chi}_D^k \xrightarrow{\text{Eq.(5)}} \boldsymbol{\mu}^k = \boldsymbol{\mu}_D^k, \quad (9)$$

128 that is, the torsion-flexure tensor is purely deviatoric and the couple stress tensor,  
 129 to all intents and purposes, may be replaced by the sole deviatoric part  $\boldsymbol{\mu}_D$ . To  
 130 light notation, hereinafter we write  $\boldsymbol{\mu}^k$  with the understanding that  $\boldsymbol{\mu}_D^k$  is meant.

## 131 2.1 Constitutive prescriptions

132 We assume hyperelastic isotropic material behaviour for both the half-spaces A and  
 133 B. Accordingly, four material parameters are introduced for each space, namely  
 134 the classical Lamé moduli,  $\Lambda^k > 0$  and  $G^k > 0$ , alongside  $\ell^k > 0$  and  $-1 < \eta^k < 1$ ,  
 135 characterising the microstructure. More specifically,  $\eta^k$  plays a role similar to that  
 136 of the Poisson's ratio, whereas  $\ell^k$  is defined as the characteristic length of the  
 137 microstructure. Both parameters may be related to the characteristic length in  
 138 bending and in torsion,  $\ell_b^k$  and  $\ell_t^k$ , respectively, through the following expressions  
 139 [24, 20]

$$\ell_b^k = \ell^k / \sqrt{2}, \quad (10a)$$

140

$$\ell_t^k = \ell^k \sqrt{1 + \eta^k}. \quad (10b)$$

In polycrystalline metals,  $\eta = -1$  and the length in torsion  $\ell_t$  vanishes, while the  
 case

$$\eta = 0 \quad \Rightarrow \quad \ell_t = \ell = \sqrt{2}\ell_b$$

141 relates to the so-called strain gradient effect, taken into account in [34].

Following [13], in each half-space we define a free-energy density  $U^k(\boldsymbol{\varepsilon}^k, \boldsymbol{\chi}^k)$ ,  
 such that the following constitutive relations hold:

$$\boldsymbol{\sigma}^k = \frac{\partial U^k}{\partial \boldsymbol{\varepsilon}^k} \quad \Rightarrow \quad \boldsymbol{\sigma}^k = 2G^k \boldsymbol{\varepsilon}^k + \Lambda^k (\text{tr } \boldsymbol{\varepsilon}^k) \mathbf{1}, \quad (11a)$$

$$\boldsymbol{\mu}^k = \frac{\partial U^k}{\partial \boldsymbol{\chi}^k} \quad \Rightarrow \quad \boldsymbol{\mu}^k = 2G^k (\ell^k)^2 \left[ (\boldsymbol{\chi}^k)^T + \eta^k \boldsymbol{\chi}^k \right]. \quad (11b)$$

142 We observe that, in the special case  $\eta = 1$ , the CS theory at hand specializes to  
 143 the modified CS theory of elasticity introduced in [33], in which  $\ell_b = \ell_t/2 = \ell/\sqrt{2}$   
 144 and the CS tensor is symmetric, owing to a postulated balance law concerning  
 145 equilibrium of the torque of torques.

146 2.2 Equations of motion

In each half-space, the equations of motion, in the absence of body forces, read

$$\operatorname{div} \mathbf{s}^k = \rho^k \ddot{\mathbf{u}}^k, \quad (12a)$$

$$2 \operatorname{axial} \boldsymbol{\tau}^k + \operatorname{div} \boldsymbol{\mu}^k = J^k \ddot{\boldsymbol{\varphi}}^k, \quad (12b)$$

having indicated time differentiation with a superposed dot. Here,  $\rho^k$  and  $J^k \geq 0$  are the mass density and the rotational inertia per unit volume, respectively. Besides,  $(\operatorname{axial} \boldsymbol{\tau})_i^k = \frac{1}{2} \mathbb{E}_{ij}^k \tau_{kj}^k$  denotes the axial vector attached to the skew-symmetric tensor  $\boldsymbol{\tau}$ . Naturally, Eqs.(12) reduce to the equations of classical elasticity (CE), upon setting

$$\ell = 0 \quad \text{and} \quad J = 0 \quad \Rightarrow \quad \boldsymbol{\mu} = \boldsymbol{\tau} = \mathbf{0}.$$

147 2.3 Antiplane shear deformations

148 We assume antiplane shear deformations, such that, in each half-space, the dis-  
149 placement field  $\mathbf{u}^k$  reduces to the out-of-plane component only

$$u_3^k(x_1, x_2, t),$$

and there is no dependence on the  $x_3$  co-ordinate. Within this framework, according to Eqs.(6,7,8), the only non-vanishing components of the strain and of the torsion-flexure tensors are

$$\varepsilon_{13}^k = \frac{1}{2} u_{3,1}^k, \quad \varepsilon_{23}^k = \frac{1}{2} u_{3,2}^k, \quad (13a)$$

$$\varphi_1^k = \frac{1}{2} u_{3,2}^k, \quad \varphi_2^k = -\frac{1}{2} u_{3,1}^k, \quad (13b)$$

$$\chi_{11}^k = -\chi_{22}^k = \frac{1}{2} u_{3,12}^k, \quad \chi_{21}^k = -\frac{1}{2} u_{3,11}^k, \quad \chi_{12}^k = \frac{1}{2} u_{3,22}^k. \quad (13c)$$

Besides, accounting for the constitutive relations (11) and in light of (13), we get

$$\sigma_{13}^k = G^k u_{3,1}^k, \quad \sigma_{23}^k = G^k u_{3,2}^k, \quad (14a)$$

$$\mu_{11}^k = -\mu_{22}^k = G^k (\ell^k)^2 (1 + \eta^k) u_{3,12}^k, \quad \mu_{21}^k = G^k (\ell^k)^2 (u_{3,22}^k - \eta^k u_{3,11}^k), \quad (14b)$$

$$\mu_{12}^k = -G^k (\ell^k)^2 (u_{3,11}^k - \eta^k u_{3,22}^k), \quad (14c)$$

in which the Lamé constant  $\Lambda^k$  does not play any role. The balance equations (12) reduce to

$$\sigma_{13,1}^k + \sigma_{23,2}^k + \tau_{13,1}^k + \tau_{23,2}^k = \rho^k \ddot{u}_3^k, \quad (15a)$$

$$\mu_{11,1}^k + \mu_{21,2}^k + 2\tau_{23}^k = J^k \ddot{\varphi}_1^k, \quad (15b)$$

$$\mu_{12,1}^k + \mu_{22,2}^k - 2\tau_{13}^k = J^k \ddot{\varphi}_2^k. \quad (15c)$$

150 Now, we solve Eq.(12b) for  $\boldsymbol{\tau}^k$

$$\boldsymbol{\tau}^k = \frac{1}{2} \mathbb{E}^k (\operatorname{div} \boldsymbol{\mu}^k - J^k \ddot{\boldsymbol{\varphi}}^k), \quad (16)$$

and substitute Eqs. (13b) and (14) into (16), to obtain (see [18])

$$\tau_{13}^k = -\frac{1}{2}G^k(\ell^k)^2 \hat{\Delta} u_{3,1}^k + \frac{J^k}{4} \ddot{u}_{3,1}^k, \quad (17a)$$

$$\tau_{23}^k = -\frac{1}{2}G^k(\ell^k)^2 \hat{\Delta} u_{3,2}^k + \frac{J^k}{4} \ddot{u}_{3,2}^k, \quad (17b)$$

151 in which the symbol  $\hat{\Delta}$  indicates the two-dimensional Laplace operator in  $x_1$  and  
 152  $x_2$ . Finally, considering homogeneous media, Eqs.(14a,15a) and (17) lead to the  
 153 *governing equation in terms of displacement*

$$G^k \left( \frac{1}{2}(\ell^k)^2 \hat{\Delta} \hat{\Delta} u_3^k - \hat{\Delta} u_3^k \right) - \frac{J^k}{4} \hat{\Delta} \ddot{u}_3^k + \rho^k \ddot{u}_3^k = 0. \quad (18)$$

154 Eq.(18) correctly reduces to the corresponding expressions developed in [7, 34, 35]  
 155 in the context of statics and in the absence of rotational inertia.

### 156 2.3.1 Reduced force traction vector and tangential part of couple stress traction

157 We introduce the *reduced force traction* vector  $\mathbf{p}^k$  acting across a surface with unit  
 158 normal  $\mathbf{n}$  [13]

$$\mathbf{p}^k = (\mathbf{s}^k)^T \mathbf{n}^k + \frac{1}{2} \text{grad } \mu_{nn}^k \times \mathbf{n}^k, \quad (19)$$

159 where  $\mu_{nn}^k = \mathbf{n}^k \cdot \boldsymbol{\mu}^k \mathbf{n}^k = \mathbf{q}^k \cdot \mathbf{n}^k$  and  $\times$  denotes the cross product between vectors.  
 160 Similarly, for the *couple stress traction* vector  $\bar{\mathbf{q}}^k$ , only the tangential part can be  
 161 really imposed on a boundary

$$\bar{\mathbf{q}}^k = (\boldsymbol{\mu}^k)^T \mathbf{n}^k - \mu_{nn}^k \mathbf{n}^k. \quad (20)$$

162 Considering now the boundary surface  $x_2 = 0$  separating the two half-spaces, we  
 163 have

$$\mathbf{n}^A = \mathbf{e}_2 = -\mathbf{n}^B,$$

164 so that the out-of-plane component of the reduced force traction and the in-plane  
 165 components of the couple stress traction read, respectively,

$$p_3^A = s_{23}^A + \frac{1}{2} \mu_{22,1}^A, \quad \bar{q}_1^A = \mu_{21}^A, \quad \text{with } \bar{q}_2^A = 0, \quad (21a)$$

166 for media A, and

$$p_3^B = -\left( s_{23}^B + \frac{1}{2} \mu_{22,1}^B \right), \quad \bar{q}_1^B = -\mu_{21}^B, \quad \text{with } \bar{q}_2^B = 0, \quad (21b)$$

167 for media B.

### 3 Nondimensional equations and time-harmonic solution

#### 3.1 Nondimensional form of governing equations

We are now in a position to bring in nondimensional form the governing equations of Sec.2, introducing suitable normalizing quantities. Although the problem is symmetric under  $A \leftrightarrow B$  inversion, we prefer, for the sake of simplicity, to normalize against either half-space, say A. Thus,

$$\boldsymbol{\xi} = \frac{\boldsymbol{x}}{\Theta \ell^A}$$

is the new dimensionless set of coordinates. The quantity  $\Theta \ell^A$  is a reference length that is proportional, through the yet-unspecified factor  $\Theta$ , to the microstructure characteristic length for the half-space A. Similarly, we introduce the reference time  $T^A = \ell^A / c_s^A$  and let the dimensionless time

$$\tau = \frac{t}{T^A}.$$

Here,  $c_s^A = \sqrt{G^A / \rho^A}$  is the shear wave speed of CE for material A. In a similar fashion,  $T^B = \ell^B / c_s^B$  is the reference time and  $c_s^B = \sqrt{G^B / \rho^B}$  the shear wave speed of CE for material B. It proves convenient to introduce the ratios

$$\beta = \frac{\ell^B}{\ell^A}, \quad (22a)$$

$$v = \frac{T^A}{T^B}, \quad (22b)$$

whereupon  $v\beta = c_s^B / c_s^A$ . We observe that the limiting case in which the half-space B is constituted by classical elastic isotropic material, i.e. in the absence of microstructure for B, can still be retrieved by taking

$$v \rightarrow +\infty, \quad \beta \rightarrow 0, \quad \text{s.t.} \quad \beta v < \infty. \quad (23)$$

The corresponding classical limit for material A cannot be accessed directly on account of the performed nondimensionalization, although this is no limitation for A and B are interchangeable

Substituting the thus introduced nondimensional variables in Eq.(18) lends the nondimensional governing equations

$$\Delta \Delta u_3^A - 2\Theta^2 \Delta u_3^A - 2\Theta^4 \left[ \frac{(\ell_0^A)^2}{\Theta^2} \Delta u_{3,\tau\tau}^A - u_{3,\tau\tau}^A \right] = 0, \quad (24a)$$

$$\Delta \Delta u_3^B - 2\frac{\Theta^2}{\beta^2} \Delta u_3^B - 2\Theta^4 \left[ \frac{(\ell_0^B)^2}{\Theta^2 v^2 \beta^2} \Delta u_{3,\tau\tau}^B - \frac{1}{v^2 \beta^4} u_{3,\tau\tau}^B \right] = 0, \quad (24b)$$

holding in A and B, respectively. Here, the symbol  $\Delta$  indicates the two-dimensional Laplace operator with respect to  $\xi_1$  and  $\xi_2$ , whereas the dimensionless parameters  $\ell_0^k$  are defined as (see also [18, 20]):

$$\ell_0^k = \frac{\ell_d^k}{\ell^k} \quad \text{with} \quad \ell_d^k = \frac{1}{2} \sqrt{\frac{J^k}{\rho^k}}.$$

We note that  $\ell_d^k$  is proportional to the dynamic characteristic length introduced in [27].



## 192 3.2 Time-harmonic solution

193 We consider time-harmonic and straight-crested antiplane waves moving in the  
194 sagittal plane  $(\xi_1, \xi_2)$ ,

$$u_3^k(\xi_1, \xi_2, \tau) = W^k(\xi_1, \xi_2) \exp(-i\Omega\tau), \quad (25)$$

where  $i$  is the imaginary unit and  $\Omega = \omega T^A > 0$  indicates the nondimensional time frequency. Substituting the solution form (25) into Eqs.(24), we obtain the pair of meta-biharmonic partial differential equations (PDEs)

$$\left[ \Delta\Delta - 2 \left( 1 - (\ell_0^A)^2 \Omega^2 \right) \Theta^2 \Delta - 2\Omega^2 \Theta^4 \right] W^A = 0, \quad (26a)$$

$$\left[ \Delta\Delta - 2 \left( 1 - \frac{(\ell_0^B)^2}{v^2} \Omega^2 \right) \frac{\Theta^2}{\beta^2} \Delta - 2\Omega^2 \frac{\Theta^4}{v^2 \beta^4} \right] W^B = 0. \quad (26b)$$

195 Eq.(26a) may be factored out as [20]

$$\left( \Delta + \delta^2 \right) (\Delta - 1) W^A = 0, \quad (27)$$

provided that we make the proper choice for  $\Theta$ . Indeed, applying Vieta's theorem to Eqs.(26a,27), we get

$$\delta^2 = 2\Omega^2 \Theta^4, \quad \text{and} \quad 1 - \delta^2 = 2\Theta^2 \left( 1 - (\ell_0^A)^2 \Omega^2 \right),$$

whence we need to let the dimensionless factor  $\Theta$  as the positive root of the bi-quadratic equation

$$2\Omega^2 \Theta^4 + 2 \left[ 1 - (\ell_0^A)^2 \Omega^2 \right] \Theta^2 - 1 = 0,$$

196 that is

$$\Theta^2 = \frac{\sqrt{(1 - (\ell_0^A)^2 \Omega^2)^2 + 2\Omega^2} - 1 + (\ell_0^A)^2 \Omega^2}{2\Omega^2}. \quad (28)$$

197 It is easily seen that  $\Theta$  is a bounded function of  $\Omega$ , while  $\delta$  is the wavenumber of  
198 shear horizontal (SH) travelling bulk waves. The latter may be rewritten as

$$\delta = 2\delta_{\text{cr}} \Theta^2, \quad (29)$$

199 having let

$$\ell_{0\text{cr}}^A = \frac{1}{\sqrt{2}}, \quad \text{and} \quad \delta_{\text{cr}} = \ell_{0\text{cr}}^A \Omega = \frac{\Omega}{\sqrt{2}}.$$

200 Combining Eqs.(28) and (29), we obtain the connection

$$\delta(\Omega, \ell_0^A) = \frac{1}{2\delta_{\text{cr}}} \left[ \sqrt{(1 - (\ell_0^A)^2 \Omega^2)^2 + 2\Omega^2} - 1 + (\ell_0^A)^2 \Omega^2 \right]. \quad (30)$$

201 This connection becomes simply  $\ell_0^A = \ell_{0\text{cr}}^A$  in the special case  $\delta = \delta_{\text{cr}}$ . In general,  
202 Eq.(30) may be inverted to yield  $\Omega$  as a function of  $\delta$ , and, in the special case  
203  $\ell_0^A = \ell_{0\text{cr}}^A$ , this relationship is linear. However, *when rotational inertia is zero*, that  
204 is for  $\ell_0^A = 0$ , a positive frequency  $\Omega$  exists only within the interval  $\delta \in (-1, 1)$ .

205 With such definitions, Eq.(26b) may be factored as

$$\left(\Delta + \delta_1^2\right) \left(\Delta - \delta_2^2\right) W^B = 0, \quad (31)$$

where we have let the dimensionless wavenumbers for bulk travelling and bulk evanescent waves in medium B

$$\delta_1^2 = \frac{\delta\psi}{\beta^2 v^2}, \quad \delta_2^2 = \frac{\delta}{\beta^2 \psi}.$$

206 In particular,  $\psi$  may be interpreted as a generalization to material B of the pa-  
207 rameter  $\delta$

$$\psi = v \delta \left( \frac{\Omega}{v}, \ell_0^B \right) = \frac{v^2}{2\delta_{\text{cr}}} \left[ \sqrt{\left(1 - (\ell_0^B)^2 \frac{\Omega^2}{v^2}\right)^2 + 2 \frac{\Omega^2}{v^2} - 1 + (\ell_0^B)^2 \frac{\Omega^2}{v^2}} \right]. \quad (32)$$

Indeed, in the special case of a single homogeneous full-space, that is for  $\beta = v = 1$ , we have  $\psi = \delta$ . The following asymptotics hold

$$\begin{aligned} \psi &\rightarrow \sqrt{2}\Omega(\ell_0^B)^2 + O(v^2), & \text{as } v &\rightarrow 0, \\ \psi &\rightarrow \delta_{\text{cr}} + O(v^{-2}), & \text{as } v &\rightarrow +\infty. \end{aligned}$$

We introduce the shorthands

$$(\kappa_1, \kappa_2) = \beta v (\delta_1, \delta_2),$$

208 whereby we obtain the bulk travelling and bulk evanescent (dimensional) wavespeeds  
209 for material B

$$c_{\text{SH}}^B = \frac{\Omega\Theta}{\kappa_1} c_s^B, \quad c_{\text{iSH}}^B = \frac{\Omega\Theta}{\kappa_2} c_s^B. \quad (33)$$

210 The corresponding wavenumbers  $\delta_{1,2}$  may be used in (33) instead of  $\kappa_{1,2}$  when  
211 expressing the corresponding bulk wavespeeds in terms of  $c_s^A$ .

212 In the absence of microstructure for B, that is in the limit (23), we have  $\kappa_1 \rightarrow$   
213  $\sqrt{\delta\delta_{\text{cr}}}$ ,  $\kappa_2/v \rightarrow \sqrt{\delta/\delta_{\text{cr}}}$  and we retrieve the classical bulk SH wavespeed

$$c_{\text{SH}}^B \rightarrow \frac{\Omega\Theta}{\sqrt{\delta\delta_{\text{cr}}}} c_s^B = c_s^B.$$

214 Conversely, evanescent bulk waves become stationary,

$$c_{\text{iSH}}^B = \frac{\Omega\Theta}{v\sqrt{\delta/\delta_{\text{cr}}}} c_s^B \rightarrow 0,$$

215 and (31) factors out in the product of the Laplacian with the Helmholtz operator

$$\Delta \left( \Delta + \delta\delta_{\text{cr}} \frac{c_s^{A2}}{c_s^{B2}} \right) W^B = 0.$$

216 Consequently, the solution is given by two terms, the first providing the time-  
217 harmonic solution of CE and the second being an harmonic function [20].

For the boundary conditions, Eqs.(21a) take on the form

$$p_3^A = -\frac{G^A}{2\Theta^3} \left[ (\delta^2 - 1) W_{,2}^A + (\eta^A + 2) W_{,112}^A + W_{,222}^A \right], \quad (34a)$$

$$\bar{q}_1^A = \frac{G^A \ell^A}{\Theta^2} (W_{,22}^A - \eta W_{,11}^A), \quad (34b)$$

while Eqs.(21b) become

$$p_3^B = -\frac{G^B}{2\Theta^3} \left\{ \beta^2 [(\eta^B + 2) W_{,112}^B + W_{,222}^B] + \frac{\kappa_2^2}{v^2} \left( \frac{\psi^2}{v^2} - 1 \right) W_{,2}^B \right\}, \quad (35a)$$

$$\bar{q}_1^B = \frac{G^B \ell^B}{\Theta^2} \beta^2 (W_{,22}^B - \eta^B W_{,11}^B). \quad (35b)$$

218 In the absence of microstructure for B, Eqs.(35) reproduce the classical result

$$p_3^B = \frac{G^B}{\Theta} W_{,2}^B, \quad \text{and} \quad \bar{q}_1^B = 0, \quad (36)$$

219 expressing the absence of the reduced couple-stress vector and the force traction  
220 vector being simply the product of the shear modulus with the shear deformation.

#### 221 4 Antiplane Stoneley waves

For guided waves propagating along the interface  $\xi_2 = 0$ , we take

$$W^k(\xi_1, \xi_2) = \ell^A w^k(\xi_2) \exp(\iota \kappa \xi_1),$$

222 wherein we have introduced the shorthand  $\kappa = \Theta K$ , with  $K = k \ell^A$  denoting the  
223 dimensionless (spatial) wavenumber in the propagation direction  $\xi_1$ . Moreover, we  
224 define the dimensional phase speed in the propagation direction

$$c = \frac{\omega}{k} = \frac{\Omega}{\kappa} \Theta c_s^A. \quad (37)$$

225 The general decaying solution of Eq.(27) is

$$w^A(\xi_2) = a_1 \exp(A_1 \xi_2) + a_2 \exp(A_2 \xi_2) \quad (38)$$

226 and for Eq.(31)

$$w^B(\xi_2) = b_1 \exp(-B_1 \xi_2) + b_2 \exp(-B_2 \xi_2), \quad (39)$$

227 where the coefficients  $a_{1,2}$  and  $b_{1,2}$  are four amplitudes to be determined. Here,  
228 we have let the decay indices

$$A_1 = \sqrt{\kappa^2 - \delta^2}, \quad A_2 = \sqrt{\kappa^2 + 1}, \quad (40)$$

229 for material A, and, similarly,

$$B_1 = \sqrt{\kappa^2 - \delta_1^2}, \quad B_2 = \sqrt{\kappa^2 + \delta_2^2}, \quad (41)$$

for material B. Here, the square root is made definite by taking the branch that corresponds to the positive square root of any positive real argument (equivalently,

we may demand that  $A_{1,2}$  and  $B_{1,2} \rightarrow \sqrt{p}$ , as  $\kappa = p \rightarrow +\infty$  real and positive). Branch cuts for the square root are taken parallel to the imaginary axis in anti-symmetric fashion, see the discussion in [21, §1.1]. By this choice, the square roots  $A_2$  and  $B_2$  are positive real along the whole real axis, while  $A_1$  (and  $B_1$ ) is positive real outside the interval  $|\kappa| < \delta$  (respectively  $|\kappa| < \delta_1$ ) and purely imaginary inside (see [20]). Also,  $A_1$  and  $B_1$  are purely imaginary on the imaginary axis, while  $A_2$  (and  $B_2$ ) is positive real inside  $|\kappa| < 1$  (respectively  $|\kappa| < \delta_2$ ) and purely imaginary outside, with opposite sign on either side of the cuts. Consequently, continuation by Schwarz's reflection principle  $f(s^*) = f^*(s)$  is warranted outside the interval  $|\kappa| < \delta$  (respectively  $|\kappa| < \delta_1$ ) and by its opposite,  $f(s^*) = -f^*(s)$ , inside. Here, a superscript asterisk denotes complex conjugation, i.e. given  $s = \Re(s) + i\Im(s)$ , we have  $s^* = \Re(s) - i\Im(s)$ . It is emphasized that this is not the choice for the cuts taken in [3], where a finite cut is considered instead. However, according to this choice, we get an odd real-valued function along the real axis, and this jeopardizes decay. At any rate, Rayleigh and Stoneley waves need to be slower than the slowest bulk mode. Besides, it is easily proved that

$$A_1 < A_2, \quad B_1 < B_2, \quad \text{for } \kappa \in \mathbb{R}.$$

Clearly,  $iA_{1,2}$  and  $iB_{1,2}$  are the (dimensionless) wavenumbers in the thickness direction  $\xi_2$ , in the relevant material. Consideration of the dimensional wavenumbers squared

$$\begin{aligned} -\frac{A_1^2}{(\ell^A)^2 \Theta^2} &= -k^2 + \frac{\delta^2}{(\ell^A)^2 \Theta^2} = -k^2 + \frac{1}{(\ell^A)^2} \left( \sqrt{1 + 2(\ell^A)^2 \omega^2 (c_s^A)^{-2}} - 1 \right), \\ -\frac{A_2^2}{(\ell^A)^2 \Theta^2} &= -k^2 - \frac{1}{(\ell^A)^2 \Theta^2} = -k^2 - \frac{1}{(\ell^A)^2} \left( \sqrt{1 + 2(\ell^A)^2 \omega^2 (c_s^A)^{-2}} + 1 \right), \end{aligned}$$

230 matches favourably with the corresponding results given in [7]. As discussed in  
 231 [19], for  $\kappa > \delta$ , the solution (38) corresponds to a localized travelling wave moving  
 232 slower than the corresponding bulk wave in material A. Similarly, for  $\kappa > \delta_1$ , the  
 233 solution (39) corresponds to a localized travelling wave moving slower than the  
 234 corresponding bulk wave in material B.

#### 235 4.1 Rayleigh function: existence of the Rayleigh zeros

236 We define the general form of the Rayleigh function

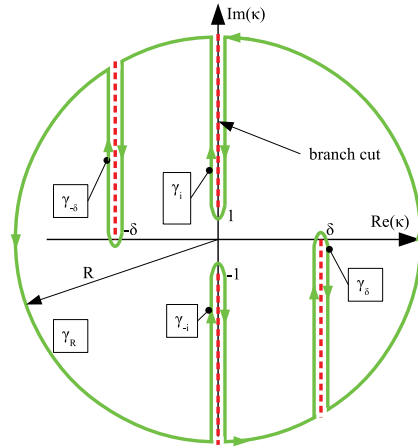
$$R_0(\kappa, \lambda_1, \lambda_2, \eta) = (\eta\kappa^2 - \lambda_1\lambda_2)^2 - \lambda_1\lambda_2(\lambda_1 + \lambda_2)^2, \quad (42)$$

237 that is valid for either half-space, provided that we substitute  $\lambda_{1,2}$  with the relevant  
 238 decay index along  $\xi_2$ . Indeed, for the half-space A, we have  $\lambda_{1,2} = A_{1,2}$ ,  $\eta = \eta^A$ , and  
 239 we get  $R_0^A(\kappa) = R_0(\kappa, A_1, A_2, \eta^A)$ . Multiplication by  $(A_1 - A_2)$  lends the Rayleigh  
 240 function in the form already exposed in [19, 20]

$$R^A(\kappa) = (A_1 - A_2) R_0^A(\kappa) = \left[ (1 + \eta^A)\kappa^2 + 1 \right]^2 A_1 - \left[ (1 + \eta^A)\kappa^2 - \delta^2 \right]^2 A_2. \quad (43)$$

241 In similar fashion, for the half-space B, we have  $R_0^B(\kappa) = R_0(\kappa, B_1, B_2, \eta^B)$  and  
 242 multiplication by  $(B_1 - B_2)$  gives

$$R^B(\kappa) = \left[ (\eta^B + 1)\kappa^2 + \delta_2^2 \right]^2 B_1 - \left[ (\eta^B + 1)\kappa^2 - \delta_1^2 \right]^2 B_2, \quad (44)$$



**Fig. 2** Simple curve  $\gamma$  (green, solid) whose mapping through the Rayleigh function  $R^A(\gamma)$  is used to determine the number of Rayleigh roots. Branch cuts (red, dashed) and branch points are chosen to warrant depthwise decay of the solution (38)

243 that is a generalization of Eq.(43) wherein the role of the material parameters is  
 244 not concealed behind the scaling.

245 On the real axis,

$$R_0(\delta, \lambda_1, \lambda_2, \eta) = \eta^2 \delta^4 \geq 0, \quad (45)$$

246 for the travelling bulk wavenumber  $\kappa = \delta$  (or  $\kappa = \delta_1$ ), while we have the asymp-  
 247 totics

$$R_0(\kappa, \lambda_1, \lambda_2, \eta) = -(3 + 2\eta - \eta^2)\kappa^4 + O(\kappa^2), \quad \text{for } \kappa \rightarrow +\infty, \quad (46)$$

248 whereupon the Rayleigh function eventually becomes real negative. We conclude,  
 249 by continuity, that at least a real root is admitted. In fact, we can prove, by the argu-  
 250 ment principle, that three pairs of central-symmetric roots are present: one real  
 251 pair corresponding to Rayleigh waves, one purely imaginary pair, corresponding  
 252 to Rayleigh-like waves, and a third pair of complex conjugated roots, as discussed  
 253 in [19]. It is observed that in [19] a different choice is made for the cuts, according  
 254 to which the complex-conjugated pair of zeros may fall outside the Riemann sheet.  
 255 In fact, with our choice for the cuts, existence of all roots is always warranted.

256 We determine the number of zeros of (42) in the cut complex plane through the  
 257 argument principle [2]. To fix ideas, we give the proof for  $R^A(\kappa)$ , but the argument  
 258 easily extends to  $R^B(\kappa)$ . We construct the mapping of the simple curve  $\gamma$  through  
 259 the Rayleigh function,  $R^A(\gamma)$ , and count its index (or winding number). Looking at  
 260 Fig.2, we see that  $\gamma$  consists of the circle  $\gamma_R$  of arbitrarily large radius  $R$ , together  
 261 with the loops  $\gamma_{\pm\delta}$  around the centrally symmetric pair of cuts  $[\pm\delta, \pm\delta \mp i\infty)$  and  
 262 the loops  $\gamma_{\pm i}$  around  $[\pm i, \pm i \mp \infty)$ . By the asymptotics (46), we infer that, when  
 263 moving along  $\gamma_R$ , the image point makes four complete turns around the origin.  
 264 We now turn to the loops around the cuts and, in light of the central-symmetry  
 265 property, only loops sitting in either half-plane are considered and the resulting  
 266 winding number is then doubled. On the loop  $\gamma_{-i}$ , we have  $A_1$  and  $A_2$  purely  
 267 imaginary, whence Eq.(42) remains in the same form but now in terms of real  
 268 numbers. In the limit as this loop shrinks down to the cut,  $\gamma_{-i}$  is mapped onto  
 269 a open curve approaching the real line from above, i.e. from positive imaginary

270 numbers. Conversely, the loop  $\gamma_\delta$  is mapped onto an S-shaped open curve as in  
 271 Fig.4, which intersects the real axis three times, named  $d_1 < d_2 < d_3$ . In particular,  
 272  $d_1 < 0$  is located to the left of the origin, while  $d_2 = R^A(\delta) = \delta^4 \eta^2 \geq 0$  is always  
 273 to the right. Together,  $D(\gamma_{-\iota}) \cup D(\gamma_{-\iota}\delta)$  form a non-simple curve winding once  
 274 around the origin, that is closed when including the points at infinity. We conclude  
 275 that six order 1 roots are expected.

Roots should be sought among the zeros of the bi-quartic polynomial

$$\eta^4 \kappa^8 + 2(1 + \eta)\kappa^6 + (\delta_2^2 - \delta_1^2)(2\eta + 1)\kappa^4 - (2\eta\delta_2^2\delta_1^2 + \delta_1^4 + \delta_2^4)\kappa^2 + \delta_1^2\delta_2^2(\delta_2^2 - \delta_1^2) = 0.$$

276 Basically, this is a singularly perturbed polynomial equation inasmuch as  $\eta$  is  
 277 assumed to be small. In this context, we observe that for the case  $\eta = 0$ , corre-  
 278 sponding to the strain-gradient theory, Rayleigh waves collapse into bulk waves as  
 279 Eq.(42) reduces to

$$\lambda_1 \lambda_2 (\lambda_2^2 - \lambda_1^2),$$

280 whose real roots corresponds to bulk waves  $\lambda_{1,2} = 0$ . In fact, Rayleigh roots are  
 281 generally perturbations around either bulk wave speed, see [19].

## 282 5 Frequency equation

For perfect adhesion between the half-spaces at the joining surface  $\xi_2 = 0$ , we enforce the boundary conditions

$$w^A(0) = w^B(0), \quad (47a)$$

$$\frac{dw^A}{d\xi_2}(0) = \frac{dw^B}{d\xi_2}(0), \quad (47b)$$

$$\bar{q}_1^A(0) = \bar{q}_1^B(0), \quad (47c)$$

$$p_3^A(0) = p_3^B(0). \quad (47d)$$

283 Plugging the solutions (38,39) into the boundary conditions (47) lends a homoge-  
 284 neous system of linear algebraic equations in the unknown amplitudes  $a_{1,2}, b_{1,2}$ .  
 285 This system admits non-trivial solutions inasmuch as the following *secular (or fre-*  
 286 *quency) equation* is satisfied:

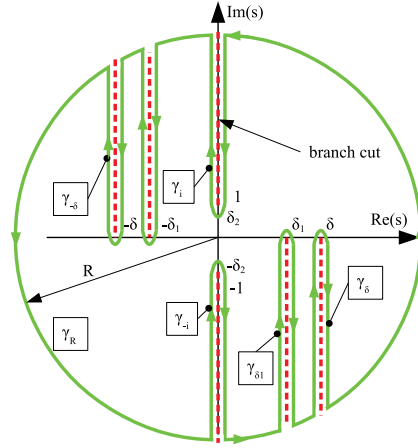
$$\Delta(\kappa) = 0, \quad (48)$$

287 in which  $\Delta$  is the determinant of the linear system. Introducing the ratio  $\Gamma =$   
 288  $G^B/G^A$ , the determinant in equation (48) may be written as

$$\Delta(\kappa) = \Gamma\beta^2 (A_1 - A_2)(B_1 - B_2)D_0(\kappa), \quad (49)$$

289 with

$$D_0(\kappa) = \frac{1}{\Gamma\beta^2}R_0^A(\kappa) - 2D_1(\kappa) + \Gamma\beta^2 R_0^B(\kappa), \quad (50)$$



**Fig. 3** Simple curve  $\gamma$  (green, solid) whose mapping through the Stoneley frequency equation  $D_0(\gamma)$  is used to determine existence and uniqueness of antiplane Stoneley roots. Here, to fix ideas, we have assumed  $\delta_1 < \delta$  and  $\delta_2 < 1$

where  $R_0^A(\kappa)$  and  $R_0^B(\kappa)$  are the Rayleigh functions for the relevant half-space and  $D_1(\kappa)$  is the coupling term. For the latter, we have

$$D_1(\kappa) = \left( \eta^A \kappa^2 - A_1 A_2 \right) \left( \eta^B \kappa^2 - B_1 B_2 \right) + \frac{1}{2} (A_1 A_2 + B_1 B_2) (A_1 + A_2) (B_1 + B_2). \quad (51)$$

290 Here, dependence on  $\Omega$  through  $\delta$ ,  $\delta_1$  and  $\delta_2$  is implicitly assumed. Eq.(50) is the CS  
291 counterpart of the Rayleigh function [3, Eq.(4-26)], valid for isotropic CE media.

The Rayleigh function (50) exhibits symmetry with respect to  $A \leftrightarrow B$  inversion, recalling that we also have  $\Gamma \leftrightarrow \Gamma^{-1}$  and  $\beta \leftrightarrow \beta^{-1}$ . Moreover, when  $A = B$ , that is

$$(\Gamma, \beta, v, \psi, \eta^k) = (1, 1, 1, \delta, \eta^A = \eta^B),$$

292 we obtain

$$D_{0,A=B}(\kappa) = -4A_1 A_2 (A_1 + A_2)^2,$$

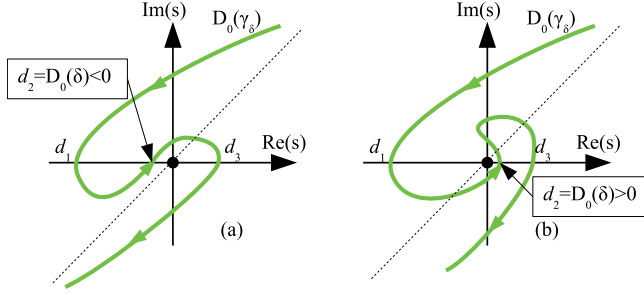
293 whence propagation is possible only for  $A_{1,2} = 0$ , that amounts to finding bulk  
294 waves. Similarly, in the absence of either half-space, that is for  $\Gamma = 0$  or  $\Gamma \rightarrow \infty$ ,  
295 we find the Rayleigh function of the remaining half-space, that is  $R_0^A(\kappa)$  or  $R_0^B(\kappa)$ ,  
296 respectively.

## 297 6 Existence and uniqueness of antiplane Stoneley waves

We have the asymptotics

$$D_0(\kappa) = - \left( 3 - \eta^B + \frac{\eta^A + 1}{\beta^2 \Gamma} \right) \left[ 3 - \eta^A + \beta^2 \Gamma (\eta^B + 1) \right] \kappa^4 + O(\kappa^2),$$

as  $\kappa \rightarrow \infty$ , (52)



**Fig. 4** Image through  $D_0(s)$  of the loop  $\gamma_\delta$  for  $\Omega < \Omega_{cuton}$  (left, index  $-\frac{1}{2}$ ) and  $\Omega > \Omega_{cuton}$  (right, index  $\frac{1}{2}$ ). The dashed line is the common asymptote for the curve at infinity.

whence, on the real axis, the frequency equation eventually becomes negative. For a given triple  $\delta, \delta_1$  and  $\delta_2$ ,  $D_0(\kappa)$  is monotonic decreasing and the possibility of a real root for the frequency equation hinges on the fact that

$$D_0(\delta_M) \geq 0, \quad \delta_M = \max(\delta, \delta_1). \quad (53)$$

This simple analysis may be put into the wider perspective of determining existence and uniqueness of antiplane Stoneley waves. To this aim, we enlarge our viewpoint and think of  $D_0$  as a function of the complex variable  $s$ . Then,  $D_0(s)$  appears centrally symmetric, i.e.  $D_0(s) = D_0(-s)$ . We determine the number of zeros of  $D_0(s)$  in the cut complex plane through the argument principle. Accordingly, we determine the index (winding number) of the curve  $D_0(\gamma)$ , where  $\gamma = \gamma_R \cup \gamma_{\pm\delta} \cup \gamma_{\pm\delta_1} \cup \gamma_{\pm\delta_2}$  is the simple curve shown in Fig.3. Here, to fix ideas, we assume  $\delta_1 < \delta$  and  $\delta_2 < 1$ .

When  $\Gamma$  is small enough, the following analysis resembles that given for the Rayleigh function. By the asymptotics (52), as the point  $\kappa$  moves on the curve  $\gamma_R$ , its mapping  $D_0(\kappa)$  makes four complete turns about the origin, whence the index is 4.

As in Fig.4,  $\gamma_\delta$  is mapped into a open loop having three intersections with the real axis,  $d_1 < 0$ ,  $d_2$  and  $d_3 > 0$ , with  $d_2 = D_0(\delta)$ . The explicit expression for  $d_2$  is given in the Appendix. In contrast,  $d_1$  and  $d_3$  may be found numerically imposing the condition  $\Im[D_0(\delta \mp \varepsilon - iy)] = 0$ , respectively, with  $\varepsilon \rightarrow 0^+$  and  $y > 0$ . When  $\Gamma$  is small enough, this loop looks just like the S-shaped curved encountered in the Rayleigh case, but, unlike there, its intersection  $d_2$  with the real axis is not necessarily positive. Indeed, this loop has index  $-\frac{1}{2}$  inasmuch as  $d_2 < 0$ , that occurs for small values of  $\Omega$ . In this situation,  $D_0(s)$  possesses two pairs of roots: a complex-conjugated pair and a purely imaginary pair. Upon increasing  $\Omega$ , the cuton frequency  $\Omega_{cuton}$  is reached such that  $d_2 = 0$  and the real root  $\kappa_S$  is located precisely at the bulk wavenumber  $\delta$ . In consideration of the fact that  $\delta$  is a monotonic increasing function of  $\Omega$  and so is  $D_0(\Omega)$ , for  $\Omega > \Omega_{cuton}$  we have that  $D_0(\gamma_\delta)$  winds around the origin as in Fig.4(b). Thus, we find three pairs of roots: a complex-conjugated pair, a purely imaginary pair and a real pair.

Similarly,  $\gamma_{\delta_1}$  is mapped into a loop closed at infinity which never encircles the origin and contribute nothing to the index. Finally, the loop  $\gamma_{-\delta}$  is mapped into the real axis from above (i.e. from the side of positive imaginary part) moving



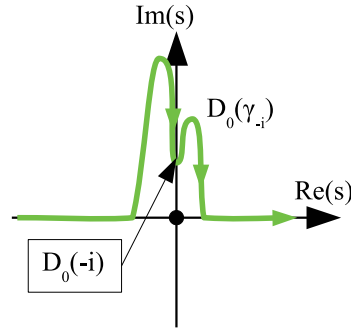


Fig. 5 Image through  $D_0(s)$  of the loop  $\gamma_{-i}$  (index  $\frac{1}{2}$ )

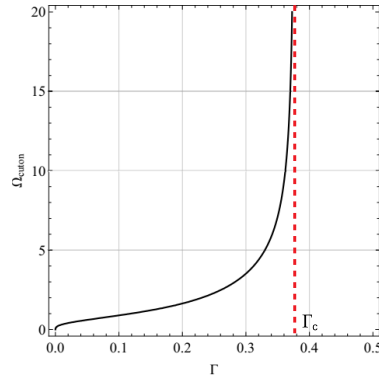
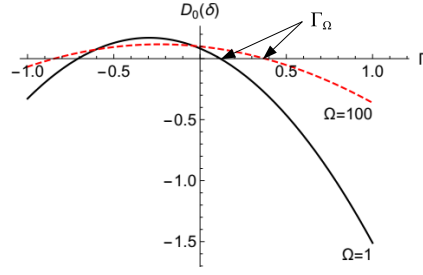


Fig. 6 Cuton frequency as a function of the ratio  $\Gamma$  between the shear moduli of media A and B (solid, black) and vertical asymptote (dashed, red)

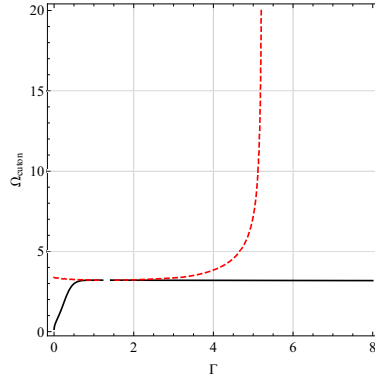
330 from left to right, see Fig.5. This curve brings an index  $\frac{1}{2}$  regardless of  $\Omega$ . We  
 331 conclude that we have the following scenarios:

- 332 – for  $\Omega < \Omega_{cuton}$ , the number of roots is 4, two complex-conjugated, located in  
 333 the second/fourth quadrant, and two opposite purely imaginary. As discussed  
 334 in [19], complex roots represent waves decaying/exploding in every direction  
 335 and have little significance in unbounded media. Conversely, purely imaginary  
 336 roots represent Stoneley-like waves travelling in the interior of the medium  
 337 and decaying/exploding along the interface. Such roots are important in semi-  
 338 infinite situations.
- 339 – For  $\Omega > \Omega_{cuton}$ , the number of roots is 6 and, alongside the previous four  
 340 zeros, a pair of real opposite roots, corresponding to travelling Stoneley waves,  
 341 appears.

342 Fig.6 shows that the cuton frequency is a monotonic increasing function of  $\Gamma$   
 343 exhibiting a vertical asymptote. Consequently, a critical value  $\Gamma_c$  exists for the ratio  
 344  $\Gamma$ , beyond which propagation is blocked. Indeed, for large values of  $\Gamma$ , the root  
 345 landscape, as it appears from the argument principle, becomes more involved and,  
 346 for instance, real (and purely imaginary) roots are eventually lost. A full analysis of  
 347 all possible scenarios rests outside the scope of this paper. We merely observe that,  
 348 for a given  $\Omega$ , the condition (53) demands positivity of a quadratic function of  $\Gamma$ ,



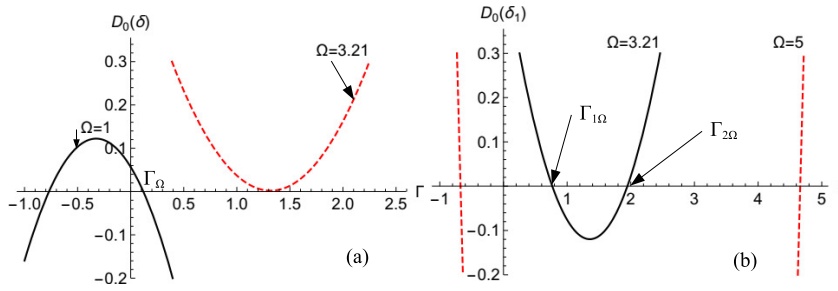
**Fig. 7** The root  $\Gamma_\Omega$  defines the admissible range  $\Gamma < \Gamma_\Omega$  for propagation of antiplane Stoneley waves to occur at a given frequency:  $\Omega = 1$  (solid, black) and  $\Omega = 100$  (dashed, red). We have taken the parameter set:  $\ell_0^B = 0.5$ ,  $\beta = \nu = 1.1$ ,  $\eta^A = 0.8$ ,  $\eta^B = 0.5$  and  $\ell_0^A = 0.5$



**Fig. 8** Cuton frequency as a function of the shear modulus ratio  $\Gamma$  for small values of rotational inertia ( $\ell_0^A = 0.3$ ). We see that cuton, as a function of  $\Gamma$ , possesses a horizontal asymptote and a second branch appears (red, dashed) which is obtained considering  $\kappa = \delta_1 > \delta$  in condition (53)

generally concave upwards, which intersects the  $\Gamma$ -axis to the right of the origin, in light of (45), at  $\Gamma_\Omega < \Gamma_c$ , see Fig.7. As a consequence, a real interval of admissible shear modulus ratios is highlighted,  $0 < \Gamma < \Gamma_\Omega$ , which accommodates propagation at and beyond the specified frequency  $\Omega$ . Fig.7 supports the observation that this admissible interval increases with  $\Omega$  up to the asymptotic value  $\Gamma_c$ . In fact, for large  $\Omega$ , the quadratic positive real root stabilizes very close to  $\Gamma_c$ .

A different situation develops when rotational inertia in medium A is smaller than medium B, for example, in our parameter set, we consider  $\ell_0^A = 0.3$ . Then,  $\delta_1$  grows with  $\Omega$  faster than  $\delta$  and eventually overtakes it. This behaviour reflects itself in that the cuton function exhibits a horizontal asymptote at  $\Omega_{cuton,asym} \approx 3.20663$ , corresponding to this overtaking, see Fig.8. Upon reaching  $\Omega_{cuton,asym}$  from below, the quadratic function of  $\Gamma$  eventually reverse convexity and moves above the  $\Gamma$ -axis: thus, propagation is admitted for any  $\Gamma$ , as in Fig.9a). Beyond  $\Omega_{cuton,asym}$ , the assumption  $\delta > \delta_1$  is violated: a new admissibility interval may be determined considering the roots  $\Gamma_{1\Omega}$  and  $\Gamma_{2\Omega}$  of  $D_0(\delta_1)$ . However, given that convexity has reversed, propagation now occurs *outside* the interval  $[\Gamma_{1\Omega}, \Gamma_{2\Omega}]$ . We thus see, in contrast to CE, that propagation of antiplane Stoneley waves is largely possible, even when material properties are significantly different.



**Fig. 9** When rotational inertia of medium A,  $\ell_0^A = 0.3$ , is less than rotational inertia of medium B,  $\ell_0^B = 0.5$ , we find that Stoneley waves propagate (a) within the finite admissible range  $(0, \Gamma_\Omega]$  for  $\Gamma$  at  $\Omega = 1$  (solid, black); this interval grows and eventually becomes unbounded upon reaching the asymptotic frequency  $\Omega = 3.21$  (dashed, red). Beyond this frequency (b), we have  $\delta_1 > \delta$  and admissibility demands  $D_0(\delta_1) \geq 0$ , which sets the unbounded admissibility interval  $(0, \Gamma_{1\Omega}] \cup [\Gamma_{2\Omega}, \infty)$

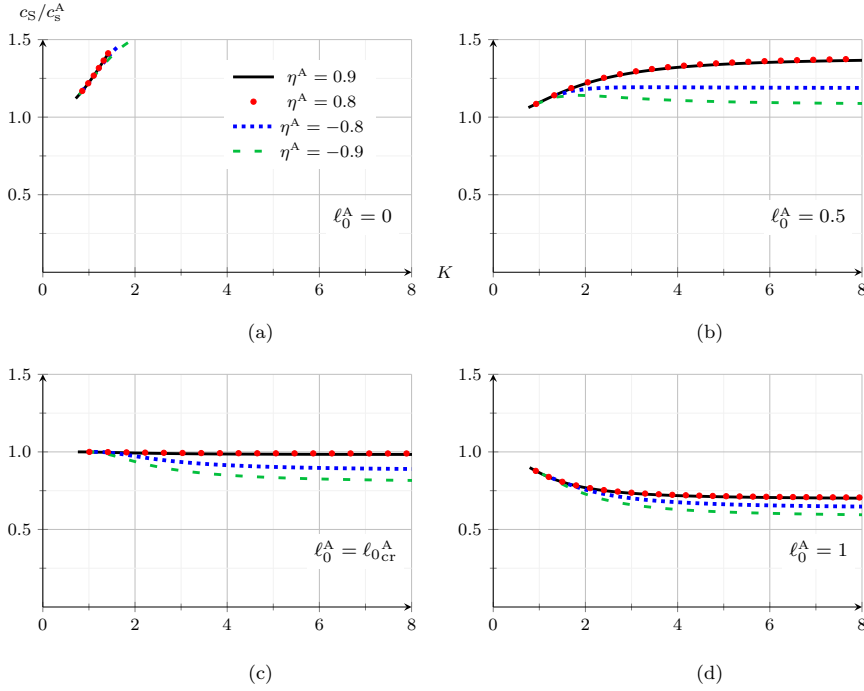
367 Finally, we observe that the case  $\eta^A = 0$  is special, for then intersection with  
 368 the ordinate axis occurs at the origin, that is a double root for  $\Gamma$ , i.e.  $\Gamma_\Omega = 0$ .  
 369 However, we have already proved that, in this situation, Stoneley waves collapse  
 370 into bulk waves.

## 371 7 Dispersion curves

372 Travelling wave solutions are possible inasmuch as a set of real solution pairs  
 373  $(\kappa_S, \Omega_S)$  may be found for the frequency equation (49). This is possible in the open  
 374 interval  $\kappa > \delta_M$ , whence we retrieved the well-known fact, already pointed out in  
 375 [3], that Stoneley waves are slower than the slowest bulk wave. In our example,  
 376  $\delta > \delta_1$ , whence, in light of (33),  $\delta_1$  is the wavenumber of the fastest bulk wave while,  
 377 in its neighbourhood, sits the fastest Rayleigh wave  $\kappa_{1R}$ . Also, we can show that  
 378 Stoneley waves are faster than the the slowest Rayleigh wave, whose wavenumber  
 379 is  $\kappa_R \gtrsim \delta$ . Indeed, looking at (50), we see that, for  $\kappa = \kappa_R$ , the first term drops  
 380 out (by definition of  $\kappa_R$ ) and only negative terms remain, in light of the Rayleigh  
 381 function  $R^B(\kappa)$  being monotonic decreasing (and zero at  $\kappa_{1R} \gtrsim \delta_1$ ). A solution  $\kappa_S$   
 382 for (50), thought of as a function of  $\Gamma$ , is possible only inasmuch as  $R^A(\kappa_S) > 0$ ,  
 383 and this occurs only for  $\kappa_S < \kappa_R$ . We thus prove the result already *observed* in  
 384 [11] and in [15]. Indeed, following the latter, “for all geometries examined [in the  
 385 context of anisotropic CE], the interface wave velocity is found to lie between the  
 386 higher free surface (generalized Rayleigh) wave velocity and the slowest bulk wave  
 387 velocity”.

388 In the following, for the sake of definiteness, when plotting dispersion curves  
 389 we assume the parameters  $\Gamma = 0.1$ ,  $\nu = \beta = 1.1$ ,  $\ell_0^A \leq \ell_0^B = 0.5$  and  $\eta^B = 0.5$ ,  
 390 whereby bulk (and Rayleigh) waves are faster in B, i.e.  $\delta_1 < \delta$ . Fig.10 presents  
 391 dispersion curves for Stoneley waves expressing the wavespeed  $c_S$  as a multiple of  
 392 the shear bulk wave speed of CE,  $c_s^A$ , as in Eq.(37).

393 Similarly to what occurs for Rayleigh waves [20, 19], Stoneley waves are ini-  
 394 tially dispersive, yet they soon develop a horizontal asymptote that is located  
 395 above (below) the shear wave speed of CE, according to  $\ell_0^A \gtrless \ell_{0\text{cr}}^A$ . Therefore,  
 396 the dispersive nature of the wave is restricted to small wavenumbers. However,



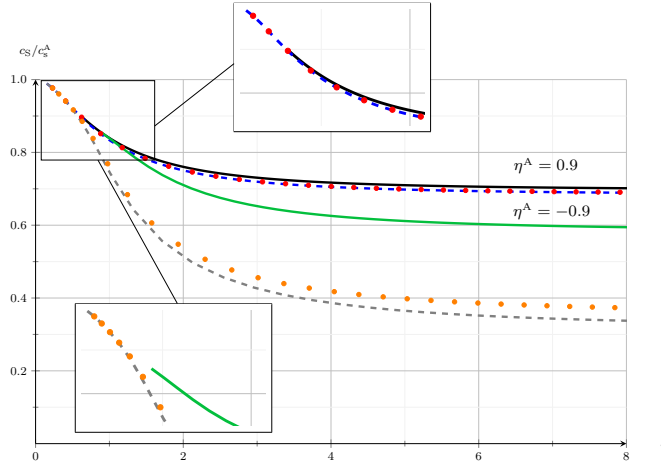
**Fig. 10** Dispersion curves for Stoneley waves where the phase speed  $c_S$  is expressed as a multiple of  $c_s^A$ , that is the SH bulk wave speed of classical elasticity for medium A ( $\Gamma = 0.1$ ,  $\nu = \beta = 1.1$ ,  $\ell_0^B = 0.5$ , and  $\eta^B = 0.5$ )

397 in the absence of rotational inertia in the half-space A (recall  $\Gamma$  is small), that  
 398 is for  $\ell_0^A = 0$  (see Fig.10a)), the curve is monotonic increasing and dispersion is  
 399 always warranted. As already discussed, bulk nondispersive waves, moving with  
 400 the constant speed  $c_s^A$ , are found for either  $\eta^A = 0$  or  $\ell_0^A = \ell_{0cr}^A$ .

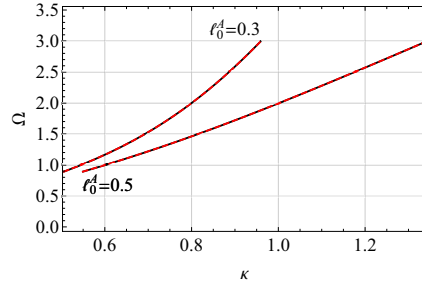
401 So far, the behaviour of Stoneley waves is very similar to that of Rayleigh  
 402 waves, see [20, §5], of which they are perturbations through  $\Gamma$ . However, in con-  
 403 trast to Rayleigh waves, a cuton frequency is met below which propagation is  
 404 prevented. Therefore, Stoneley waves present a zero-frequency block-band and  
 405 their propagation follows an optical branch. Fig.11, illustrates the role of  $\Gamma$  and  $\eta^A$   
 406 in determining the cuton frequency. Also, propagation is possible below a critical  
 407 value  $\Gamma_\Omega$  (case  $\delta > \delta_1$ ) and outside the finite interval  $\Gamma_{1\Omega} < \Gamma < \Gamma_{2\Omega}$  (case  $\delta_1 > \delta$ ),  
 408 as discussed in §6. Especially, in stark contrast to CE, Stoneley wave propagation  
 409 occurs in CS elasticity under pretty general conditions, well beyond Wiechert's  
 410 conditions, and they appear to be the rule rather than the exception.

411 Still, similarly to Rayleigh waves, Stoneley waves are perturbations of the trav-  
 412 elling bulk modes. Indeed, following [19], a convenient approach to the determi-  
 413 nation of Stoneley wavenumbers is obtained expanding Eq.(48) around  $\delta$  (or  $\delta_1$ ,  
 414 depending which is the largest) through setting

$$\kappa_s = \delta(1 + \epsilon_S^2), \quad \epsilon_S \ll 1.$$



**Fig. 11** Dispersion curves for antiplane Stoneley waves at  $\Gamma = 0.1$  (solid),  $\Gamma = 0.01$  (dots), and  $\Gamma = 0.001$  (dashed)



**Fig. 12** Linear approximation (54) (dashed, red) superposed onto the numerically evaluated frequency spectrum (solid, black) for  $l_0^A = 0.3$  and  $l_0^A = 0.5$ . Curves perfectly overlap in the considered domain.

415 Solutions of the resulting linear equation in  $\epsilon_S$

$$d_2 + a_1 \epsilon_S = 0, \quad (54)$$

416 are plotted in Fig.12 against the numerically evaluated frequency spectra. It is  
 417 seen that approximated and "exact" spectra perfectly overlap in the whole range  
 418 considered. The coefficients  $d_2$  and  $a_1$  are given in the Appendix.

## 419 8 Conclusions

420 Propagation of antiplane Stoneley waves is investigated within the context of cou-  
 421 ple stress theory, in an attempt to discuss the role of material microstructure  
 422 in developing new pathways for energy transport. Remarkably, it is found that  
 423 antiplane Stoneley waves are supported under rather general conditions, and this  
 424 outcome stands in marked contrast with the findings of classical elasticity, wherein  
 425 antiplane Stoneley waves are not supported altogether and in-plane Stoneley waves

426 are possible only for a very restricted range for the ratio between the material con-  
427 stants of the media in contact. In fact, according to Wiechert's conditions, material  
428 pairs supporting in-plane Stoneley waves within classical elasticity ought to be very  
429 similar.

430 The question of existence and uniqueness of such waves is also addressed by  
431 the argument principle. It is found that, besides travelling waves, evanescent and  
432 decaying/exploding modes are also admitted, in a complex wave pattern. Interest-  
433 ingly, propagation is possible only beyond a cuton frequency, for which an explicit  
434 expression is given. Indeed, it appears that lack of propagation, typical of clas-  
435 sical elasticity, is relaxed to high-frequency propagation by the presence of the  
436 microstructure. As a result, rotational inertia plays an important role as it affects  
437 the admissibility range for propagation.

438 Stoneley waves, just like Rayleigh waves, are perturbations of the relevant bulk  
439 modes. As a result, an approximated linear (in the wavenumber) expression for  
440 locating Stoneley roots is given that proves extremely accurate when compared  
441 to plain numerics in a wide frequency range. The possibility of Stoneley wave  
442 propagation under general conditions has great importance in seismology and it  
443 is a valuable asset in non-destructive testing of materials.

## 444 **9 Declarations**

### 445 9.1 Funding

446 This research was supported under the grant POR FESR 2014-2020 ASSE 1  
447 AZIONE 1.2.2 awarded to the project "IMPreSA" CUP E81F18000310009

### 448 9.2 Conflicts of interest/Competing interests

449 Authors declare no competing interests.

### 450 9.3 Availability of data and material

451 This paper contains no data.

### 452 9.4 Code availability

453 Not applicable.

454 **Acknowledgements** Enrico Radi's help in simplifying the dispersion relation is gratefully  
455 acknowledged.

## 456 A Linear approximation to cuton

In this short Appendix we gather the analytic expressions of the coefficients in the linear approximation (54). They are

$$\begin{aligned}
 d_2 = \beta^2 \Gamma \left[ \left( \sqrt{(\delta^2 - \delta_1^2)(\delta^2 + \delta_2^2)} - \delta^2 \eta^B \right)^2 \right. \\
 \left. - \sqrt{(\delta^2 - \delta_1^2)(\delta^2 + \delta_2^2)} \left( \sqrt{\delta^2 - \delta_1^2} + \sqrt{\delta^2 + \delta_2^2} \right)^2 \right] \\
 + \frac{\delta^4 (\eta^A)^2}{\beta^2 \Gamma} - 2\delta^4 \eta^A \eta^B + 2\sqrt{(\delta^2 - \delta_1^2)(\delta^2 + \delta_2^2)} \delta^2 \eta^A \\
 - \sqrt{(\delta^2 + 1)(\delta^2 - \delta_1^2)(\delta^2 + \delta_2^2)} \left( \sqrt{\delta^2 - \delta_1^2} + \sqrt{\delta^2 + \delta_2^2} \right),
 \end{aligned}$$

and

$$\begin{aligned}
 a_1 = \frac{\sqrt{2}\delta}{\beta^2 \Gamma} \left\{ \delta^2 \left[ -2\beta^2 \Gamma \left( \sqrt{\delta^2 - \delta_1^2} + \sqrt{\delta^2 + \delta_2^2} - \eta^B \sqrt{\delta^2 + 1} \right) - \sqrt{\delta^2 + 1} (2\eta^A + 1) \right] \right. \\
 \left. - \beta^2 \Gamma \left[ \sqrt{\delta^2 - \delta_1^2} \delta_2^2 + \sqrt{\delta^2 - \delta_1^2} - \delta_1^2 \sqrt{\delta^2 + \delta_2^2} \right. \right. \\
 \left. \left. + \sqrt{\delta^2 + \delta_2^2} + 2\sqrt{(\delta^2 + 1)(\delta^2 - \delta_1^2)(\delta^2 + \delta_2^2)} \right] - \sqrt{\delta^2 + 1} \right\}.
 \end{aligned}$$

457 Naturally, in the special case  $\Gamma \rightarrow 0$ , we retrieve the result already obtained for Rayleigh waves  
 458 in [19].

## 459 References

- 460 1. VTN Anh, LT Thang, PC Vinh, and TT Tuan. Stoneley waves with spring contact and  
 461 evaluation of the quality of imperfect bonds. *Zeitschrift für angewandte Mathematik und*  
 462 *Physik*, 71(1):36, 2020.
- 463 2. AF Beardon. *Complex analysis: The argument principle in analysis and topology*. Courier  
 464 Dover Publications, 2019.
- 465 3. L Cagniard. *Reflection and refraction of progressive seismic waves*. McGraw-Hill, 1962.
- 466 4. P Chadwick and P Borejko. Existence and uniqueness of stoneley waves. *Geophysical*  
 467 *Journal International*, 118(2):279–284, 1994.
- 468 5. G Dal Moro. *Surface wave analysis for near surface applications*. Elsevier, 2014.
- 469 6. V A Eremeyev, G Rosi, and S Naili. Surface/interfacial anti-plane waves in solids with  
 470 surface energy. *Mechanics Research Communications*, 74:8–13, 2016.
- 471 7. H Fan and L Xu. Love wave in a classical linear elastic half-space covered by a surface  
 472 layer described by the couple stress theory. *Acta Mechanica*, 229(12):5121–5132, 2018.
- 473 8. HG Georgiadis and EG Velgaki. High-frequency Rayleigh waves in materials with  
 474 micro-structure and couple-stress effects. *International Journal of Solids and Structures*,  
 475 40(10):2501–2520, 2003.
- 476 9. PA Gourgiotis and HG Georgiadis. Torsional and SH surface waves in an isotropic and  
 477 homogenous elastic half-space characterized by the Toupin–Mindlin gradient theory. *In-*  
 478 *ternational Journal of Solids and Structures*, 62:217–228, 2015.
- 479 10. PA Gourgiotis, HG Georgiadis, and I Neocleous. On the reflection of waves in half-  
 480 spaces of microstructured materials governed by dipolar gradient elasticity. *Wave Motion*,  
 481 50(3):437–455, 2013.
- 482 11. TM Hsieh, EA Lindgren, and M Rosen. Effect of interfacial properties on Stoneley wave  
 483 propagation. *Ultrasonics*, 29(1):38–44, 1991.
- 484 12. AV Ilyashenko. Stoneley waves in a vicinity of the wiechert condition. *International*  
 485 *Journal of Dynamics and Control*, pages 1–3, 2020.

- 486 13. WT Koiter. Couple-stress in the theory of elasticity. In *Proc. K. Ned. Akad. Wet.*, vol-  
487 ume 67, pages 17–44. North Holland Pub, 1964.
- 488 14. R Kumar, S Devi, and SM Abo-Dahab. Stoneley waves at the boundary surface of modified  
489 couple stress generalized thermoelastic with mass diffusion. *Journal of Applied Science*  
490 *and Engineering*, 21(1):1–8, 2018.
- 491 15. TC Lim and MJP Musgrave. Stoneley waves in anisotropic media. *Nature*, 225(5230):372–  
492 372, 1970.
- 493 16. AE Love. Some problems of geodynamics, 1911.
- 494 17. R D Mindlin. Micro-structure in linear elasticity. *Archive for Rational Mechanics and*  
495 *Analysis*, 16(1):51–78, 1964.
- 496 18. G Mishuris, A Piccolroaz, and E Radi. Steady-state propagation of a mode III crack in  
497 couple stress elastic materials. *International Journal of Engineering Science*, 61:112–128,  
498 2012.
- 499 19. A Nobili, E Radi, and C Signorini. A new Rayleigh-like wave in guided propaga-  
500 tion of antiplane waves in couple stress materials. *Proceedings of the Royal Society A*,  
501 476(2235):20190822, 2020.
- 502 20. A Nobili, E Radi, and A Vellender. Diffraction of antiplane shear waves and stress con-  
503 centration in a cracked couple stress elastic material with micro inertia. *Journal of the*  
504 *Mechanics and Physics of Solids*, 124:663–680, 2019.
- 505 21. B Noble. *Methods based on the Wiener-Hopf technique for the solution of partial differ-*  
506 *ential equations, International Series of Monographs on Pure and Applied Mathematics.*  
507 *Vol. 7.* Pergamon Press, New York, 1958.
- 508 22. N S Ottosen, M Ristinmaa, and C Ljung. Rayleigh waves obtained by the indeterminate  
509 couple-stress theory. *European Journal of Mechanics-A/Solids*, 19(6):929–947, 2000.
- 510 23. TE Owen. Surface wave phenomena in ultrasonics. *Progress in Applied Material Research*,  
511 6:71–87, 1964.
- 512 24. E Radi. On the effects of characteristic lengths in bending and torsion on mode III crack in  
513 couple stress elasticity. *International Journal of Solids and Structures*, 45(10):3033–3058,  
514 2008.
- 515 25. SI Rokhlin, M Hefets, and M Rosen. An elastic interface wave guided by a thin film  
516 between two solids. *Journal of Applied Physics*, 51(7):3579–3582, 1980.
- 517 26. JG Scholte. The range of existence of rayleigh and stoneley waves. *Geophysical Supple-*  
518 *ments to the Monthly Notices of the Royal Astronomical Society*, 5(5):120–126, 1947.
- 519 27. HM Shodja, A Goodarzi, MR Delfani, and H Haftbaradaran. Scattering of an anti-plane  
520 shear wave by an embedded cylindrical micro-/nano-fiber within couple stress theory with  
521 micro inertia. *International Journal of Solids and Structures*, 58:73–90, 2015.
- 522 28. R Stoneley. Elastic waves at the surface of separation of two solids. *Proceedings of the*  
523 *Royal Society of London. Series A, Containing Papers of a Mathematical and Physical*  
524 *Character*, 106(738):416–428, 1924.
- 525 29. R Stoneley. Rayleigh waves in a medium with two surface layers.(first paper). *Geophysical*  
526 *Supplements to the Monthly Notices of the Royal Astronomical Society*, 6(9):610–615,  
527 1954.
- 528 30. JW Strutt. On waves propagated along the plane surface of an elastic solid. *Proceedings*  
529 *of the London Mathematical Society*, 17(1):4–11, 1885.
- 530 31. PC Vinh and PTH Giang. On formulas for the velocity of stoneley waves propagating along  
531 the loosely bonded interface of two elastic half-spaces. *Wave Motion*, 48(7):647–657, 2011.
- 532 32. PC Vinh, PG Malischewsky, and PTH Giang. Formulas for the speed and slowness of  
533 Stoneley waves in bonded isotropic elastic half-spaces with the same bulk wave velocities.  
534 *International Journal of Engineering Science*, 60:53–58, 2012.
- 535 33. FACM Yang, ACM Chong, DCC Lam, and P Tong. Couple stress based strain gradient  
536 theory for elasticity. *International Journal of Solids and Structures*, 39(10):2731–2743,  
537 2002.
- 538 34. L Zhang, Y Huang, JY Chen, and KC Hwang. The mode III full-field solution in elastic  
539 materials with strain gradient effects. *International Journal of Fracture*, 92(4):325–348,  
540 1998.
- 541 35. T Zisis. Anti-plane loading of microstructured materials in the context of couple stress  
542 theory of elasticity: half-planes and layers. *Archive of Applied Mechanics*, 88(1-2):97–110,  
543 2018.