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ON TOTAL VERTEX IRREGULARITY STRENGTH OF SOME CLASSES OF TADPOLE CHAIN GRAPHS

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ABSTRACT. A total k-labeling f that assigns $V \cup E$ into $\{1, 2, \dots, k\}$ on graph G is named vertex irregular if $wt_f(u) \neq wt_f(v)$ for dissimilar vertices u, v in G with the weights $wt_f(u) = f(u) + \sum_{ux \in E(G)} f(ux)$. We call the minimum number k utilized in total labeling f as a total vertex irregularity strength of G, symbolized by tvs(G). In this research, we focus on tadpole chain graphs that are chain graphs which contain tadpole graphs in their blocks. We investigate tvs of some classes of tadpole chain graphs,. i.e., $T_r(4, n)$ and $T_r(5, n)$ with length r. Some formulas are derived as follows: $tvs(T_r(4, n)) = \left\lceil \frac{(n+1)r+3}{3} \right\rceil$ and $tvs(T_r(5, n)) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil$.

Keywords: Vertex irregular total k-labeling, tvs, tadpole graph, chain graph.
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1. Introduction

We assume that $G(V, E)$ is a finite, undirected, and simple graph. A mapping f that assigns $V(G) \cup E(G)$ into a set of integers is named a total labeling. Further, the integers used in f are called as labels [14]. In addition, the total k-labeling f is called a total vertex irregular if the vertex weights $wt_f(u) \neq wt_f(v)$ for distinct vertices $u \neq v$ in G with $wt_f(u) = f(u) + \sum_{ux \in E(G)} f(ux)$. Baca et al. [5] initiated the notion of total vertex irregularity strength of graph G, denoted by tvs(G), that is defined as the minimum number k in such a way that G has a vertex irregular total k-labeling.

Bača et al. [5] proposed the lower bound in the following: $\left\lceil \frac{(p+\delta)}{(\Delta+1)} \right\rceil \leq tvs(G) \leq p + \Delta - 2\delta + 1$ for any graph $G(V, E)$ where $p = |V(G)|$, $\delta = \min\{d(v) | \forall v \in V(G)\}$, and $\Delta = \max\{d(v) | \forall v \in V(G)\}$, respectively. Whereas, Anholcer, et al. provided another bounds in [3]. Another way to get the lower bound of tvs for any connected graph G was given by Nurdin et al. [11]. Let G be a connected graph where the number of vertices of

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degrees i is n_i, for $i = \delta, \delta+1, \delta+2, \ldots, \Delta$.

$$tvs(G) \geq \max\left\{ \left\lceil \frac{\delta(G) + n_{\delta(G)}}{\delta(G)+1} \right\rceil, \left\lceil \frac{\delta(G) + n_{\delta(G)} + n_{\delta(G)+1}}{\delta(G)+2} \right\rceil, \ldots, \left\lceil \frac{\delta(G) + \sum_{i=\delta(G)}^{\Delta(G)} n_i}{\Delta(G)+1} \right\rceil \right\}. \quad (1)$$

Nowdays, Many scholars have found total vertex irregularity strength of some graph classes such as in [5],[7], [10], [1], [2], [9], etc. Meanwhile, the results related to cactus chain graphs has been invented in [4] and [12]. Recently, Rosyida et al. published a result of tvs of $T_r(4,1)$ tadpole chain graph [13]. To continue the result in [13], we investigate tvs of some tadpole chain graphs in this paper, i.e. $tvs(T_r(4,n))$ and $tvs(T_r(5,n))$.

2. Main Results

In this part, we present formulas of tvs of $T_r(4,n)$ and $T_r(5,n)$. Let us consider the chain graph in Definition 2.1.

Definition 2.1. *A graph which consists of a cycle graph C_m and a path graph P_n connected with a bridge is called as a tadpole graph, denoted by $T(m,n)$. Given a connected graph G. A bipartite graph (B,C), where B is a set of blocks in graph G and C consists of cut vertices on each block in B, is named a block cut vertex of G. The edges of G join cut vertices with those blocks to which they belong. Further, G is called as a chain graph of length r if it contains r-blocks such that each pair of two blocks B_i and B_{i+1} has one common cut vertex for which the block cut vertex is a path (*[8], [6]*). Furthermore, tadpole chain graphs $T_r(4,n)$ and $T_r(5,n)$ are chain graphs which their blocks are $T(4,n)$ and $T(5,n)$, respectively.*

2.1. Total vertex irregularity strength of $T_r(5,n)$.

The chain graph $T_r(5,n)$ consists of:

- $\{y_1^n, y_2^n, \ldots, y_r^n\}$, i.e. a set of vertices with degrees 1
- $\{y_1^{n-1}, y_2^{n-1}, \ldots, y_r^{n-1}, y_1^{n-2}, y_2^{n-2}, \ldots, y_r^{n-2}, \ldots, y_1^2, y_2^2, \ldots, y_r^2, y_1^1, y_2^1, \ldots, y_r^1\} \cup \{u_1, u_2, \ldots, u_{2r}, u_{2r+1}, u_{2r+2}\}$, i.e. a set of vertices of degrees 2;
- $\{x_1, x_2, \ldots, x_r\}$ that is a set of vertices with degrees 3; and
- $\{v_1, v_2, \ldots, v_{r-1}\}$ that is a set of vertices with degrees 4.

Theorem 2.1. *If $T_r(5,1)$ is a tadpole chain graph of length $r (r \geq 2)$, then*

$$tvs(T_r(5,1)) = r + 1.$$

Proof. Let $y_1, y_2, \ldots, y_r$ be vertices of $T_r(5,1)$ with degrees 1.

Table 1. Labels of vertices and edges of $T_r(5,1)$

Labels of vertices	**Labels of edges**
$f(u_1) = 1,$	$f(u_{2i-1}u_{2i}) = i, 1 \leq i \leq r,$
$f(u_{2i-1}) = i-1, 2 \leq i \leq r,$	$f(u_1u_{2r+1}) = r,$
$f(u_{2i}) = r, 1 \leq i \leq r,$	$f(u_{2r}u_{2r+2}) = r+1,$
$f(u_{2r+1}) = f(u_{2r+2}) = r+1,$	$f(u_{2i}v_i) = r+1, 1 \leq i \leq r-1,$
$f(v_i) = r-1, 1 \leq i \leq r-1,$	$f(u_{2i+1}v_i) = r+1, 1 \leq i \leq r-1,$
$f(x_i) = r+1, 1 \leq i \leq r,$	$f(v_ix_i) = r+1, 1 \leq i \leq r-1,$
$f(y_i) = 1, 1 \leq i \leq r.$	$f(v_ix_{i+1}) = r+1, 1 \leq i \leq r-1,$
	$f(u_{r+1}x_1) = r+1, f(u_{r+2}x_r) = r+1,$
	$f(x_iy_i) = i, 1 \leq i \leq r.$

According to Inequality (1), we obtain

$$tvs(T_r(5,1)) \geq \max\left\{\left\lceil\frac{r+1}{2}\right\rceil, \left\lceil\frac{3r+3}{3}\right\rceil, \left\lceil\frac{4r+3}{4}\right\rceil, \left\lceil\frac{5r+2}{5}\right\rceil\right\} = \left\lceil\frac{3r+3}{3}\right\rceil = r+1. \quad (2)$$

To proof that $tvs(T_r(5,1)) \leq r+1$, we provide a total k-labeling $f : V \cup E \to \{1,2,\ldots,k\}$ with $k = r+1$ and define labels of vertices and edges as in Table 1.

Under labeling f, we obtain that each vertex has the weight below:

$$\begin{aligned} wt_f(y_i) &= i+1, && 1 \leq i \leq r,\\ wt_f(u_1) &= r+2, &&\\ wt_f(u_i) &= i+r+1, && 1 \leq i \leq 2r,\\ wt_f(u_{2r+1}) &= 3r+2, &&\\ wt_f(u_{r+2}) &= 3(r+1), &&\\ wt_f(v_i) &= i+4r+3, && 1 \leq i \leq r-1,\\ wt_f(x_i) &= i+3(r+1), && 1 \leq i \leq r. \end{aligned}$$

The minimum label of vertices and edges is 1 and the maximum label is $r+1$. Also, it is shown that the vertex weights are all diverse. Hence, we get upper bound $tvs(T_r(5,1)) \leq r+1$. Combining with Lower bound (2), it is proved that $tvs(T_r(5,1)) = r+1$. □

Theorem 2.2. *If $T_r(5,n)$ are tadpole chain graphs with length $r(r \geq 2)$, $n = 4 \mod 3$, and $n \geq 4$, then $tvs(T_r(5,n)) = \left\lceil\frac{(n+2)r+3}{3}\right\rceil$.*

Proof. Based on (1), we acquire the lower bound

$$tvs(T_r(5,n)) \geq \max\left\{\left\lceil\frac{r+1}{2}\right\rceil, \left\lceil\frac{(n+2)r+3)}{3}\right\rceil, \left\lceil\frac{(n+3)r+3}{4}\right\rceil, \left\lceil\frac{(n+4)r+2}{5}\right\rceil\right\} = \left\lceil\frac{(n+2)r+3}{3}\right\rceil. \quad (3)$$

We prove the upper bound through 3 cases.
Case 1. For $n = 4$.

TABLE 2. Labels of vertices and edges of $T_r(5,4)$

Labels of vertices	**Labels of edges**
$f(u_{2i-1}) = i+r-2, 1 \leq i \leq r$,	$f(u_{2i-1}u_{2i}) = i+r+1, 1 \leq i \leq r$,
$f(u_{2i}) = (2r+1)-2 = 2r-1, 1 \leq i \leq r$,	$f(u_1u_{2r+1}) = 2r+1$,
$f(u_{2r+1}) = 2r+1-1 = 2r$,	$f(u_{2r}u_{2r+2}) = 2r+1$,
$f(u_{2r+2}) = 2r+1$,	
$f(v_i) = 1, 1 \leq i \leq r-1$,	$f(u_{2i}v_i) = i+r+1, 1 \leq i \leq r-1$,
$f(x_i) = 2r+1, 1 \leq i \leq r$,	$f(u_{2i+1}v_i) = 2r+1, 1 \leq i \leq r-1$,
$f(y_i^1) = 2r-1, 1 \leq i \leq r-1$,	$f(v_ix_i) = 2r-1, 1 \leq i \leq r-1$,
$f(y_r^1) = 2r+1$,	$f(v_ix_{i+1}) = 2r+1, 1 \leq i \leq r-1$,
$f(y_i^2) = r+1, 1 \leq i \leq r$,	$f(u_{2r+1}x_1) = 2r+1, f(u_{2r+2}x_r) = 2r+1$,
$f(y_i^3) = r, 1 \leq i \leq r$,	$f(x_iy_i^1) = i+2, 1 \leq i \leq r-1$,
$f(y_i^4) = i, 1 \leq i \leq r$.	$f(x_ry_r^1) = r$
	$f(y_i^1y_i^2) = r, 1 \leq i \leq r$,
	$f(y_i^2y_i^3) = i; f(y_i^3y_i^4) = 1, 1 \leq i \leq r$,

We assume that $f : V \cup E \to \{1, 2, \ldots, k\}$ is a total k-labeling with $k = \left\lceil \frac{6r+3}{3} \right\rceil = 2r+1$. Vertex and edge labels are defined in Table 2.

Under labeling f, we derive the weights of vertices below:

$$\begin{array}{ll}
wt_f(y_i^4) = i+1, & 1 \le i \le r,\\
wt_f(y_i^3) = i+r+1, & 1 \le i \le r,\\
wt_f(y_i^2) = i+2r+1, & 1 \le i \le r,\\
wt_f(y_i^1) = i+3r+1, & 1 \le i \le r,\\
wt_f(u_i) = i+4r+1, & 1 \le i \le 2r,\\
wt_f(u_{2r+1}) = 3(2r+1)-1 = 6r+2, & \\
wt_f(u_{r+2}) = 3(2r+1), & \\
wt_f(v_i) = i+2(2r+1)+3r+1 = i+7r+3, & 1 \le i \le r-1,\\
wt_f(x_i) = i+2(2r+1)+2r+1 = i+6r+3, & 1 \le i \le r.
\end{array}$$

We observe that each vertex has a distinct weight. Also, vertex and edge labels used are less than or equal to $2r+1$. Hence, $tvs(T_r(5,4)) \le \left\lceil \frac{6r+3}{3} \right\rceil = 2r+1$. Combining with the lower bound, we have $tvs(T_r(5,4)) = 2r+1$.

Case 2. For $n = 7$.

We construct a function $f : V \cup E \to \{1, 2, \ldots, k\}$ which is a total k-labeling with $k = \left\lceil \frac{9r+3}{3} \right\rceil = 3r+1$. We establish labels of vertices and edges as in Table 3.

TABLE 3. Labels of vertices and edges of $T_r(5,7)$

Labels of vertices	**Labels of edges**
$f(u_{2i-1}) = i+2r-2, 1 \le i \le r,$	$f(u_{2i-1}u_{2i}) = i+2r+1, 1 \le i \le r,$
$f(u_{2i}) = (3r+1)-2 = 3r-1, 1 \le i \le r,$	$f(u_1u_{2r+1}) = 3r+1,$
$f(u_{2r+1}) = 3r+1-1 = 3r,$	$f(u_{2r}u_{2r+2}) = 3r+1,$
$f(u_{2r+2}) = 3r+1,$	$f(u_{2i}v_i) = i+2r+1, 1 \le i \le r-1,$
$f(v_i) = 1, 1 \le i \le r-1,$	$f(u_{2i+1}v_i) = 3r+1, 1 \le i \le r-1,$
$f(x_i) = 3r+1, 1 \le i \le r,$	$f(v_ix_i) = 2r-1, 1 \le i \le r-1,$
$f(y_i^1) = 2r-1, 1 \le i \le r-1,$	$f(v_ix_{i+1}) = 3r+1, 1 \le i \le r-1,$
$f(y_r^1) = 3r+1,$	
$f(y_i^2) = 2r+1, 1 \le i \le r,$	$f(u_{2r+1}x_1) = 3r+1,$
$f(y_i^3) = 3r+1, 1 \le i \le r,$	$f(u_{2r+2}x_r) = 3r+1,$
$f(y_i^4) = 2r+1, 1 \le i \le r,$	$f(x_iy_i^1) = i+r+2, 1 \le i \le r-1,$
$f(y_i^5) = 2r, 1 \le i \le r,$	$f(x_ry_r^1) = r; f(y_i^1y_i^2) = 3r, 1 \le i \le r,$
$f(y_i^6) = r, 1 \le i \le r,$	$f(y_i^2y_i^3) = i,$
$f(y_i^7) = 1, 1 \le i \le r.$	$f(y_i^3y_i^4) = r,$
	$f(y_i^4y_i^5) = i,\ f(y_i^5y_i^6) = 1,$
	$f(y_i^6y_i^7) = i, 1 \le i \le r$

By means of labeling f, we have the weights of vertices as follows:

$$
\begin{array}{ll}
wt_f(u_i) = i + 7r + 1, & 1 \le i \le 2r, \\
wt_f(u_{2r+1}) = 3(3r+1) - 1, & \\
wt_f(u_{r+2}) = 3(3r+1), & \\
wt_f(v_i) = i + 2(3r+1) + 4r + 1 = i + 10r + 3, & 1 \le i \le r-1, \\
wt_f(x_i) = i + 2(3r+1) + 3r + 1 = i + 9r + 3, & 1 \le i \le r, \\
wt_f(y_i^l) = i + (7-l)r + 1, & 1 \le i \le r; 1 \le l \le 7.
\end{array}
$$

The vertex weights are all distinct and the labels used are at most $\left\lceil \frac{9r+3}{3} \right\rceil = 3r+1$. Thus, $tvs(T_r(5,7)) = 3r+1$.

Case 3. For $n = 3j + 10$ with $j \ge 0$.

We construct a total k-labeling $f : V \cup E \to \{1, 2, \ldots, k\}$ with $k = \left\lceil \dfrac{(n+2)r+3}{3} \right\rceil$. We create vertex labels in the following:

$$
\begin{array}{ll}
f(u_{2i-1}) = i + (\frac{n-1}{3})r - 2, & 1 \le i \le r, \\
f(u_{2i}) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil - 2, & 1 \le i \le r, \\
f(u_{2r+1}) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil - 1, & \\
f(u_{2r+2}) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil, & \\
f(v_i) = 1, & 1 \le i \le r-1,
\end{array}
$$

$$
\begin{array}{ll}
f(x_i) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil, & 1 \le i \le r-1, \\
f(x_r) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil - \big((\frac{n-7}{3})r - 1\big), & \\
f(y_i^1) = \big(\frac{n-1}{3}\big)r - 2, & 1 \le i \le r-1, \\
f(y_r^1) = \big(\frac{n+2}{3}\big)r + 1, & \\
f(y_i^2) = f(y_i^3) = \ldots = f(y_i^{2+2j-1}) = \big(\frac{n+2}{3}\big)r + 1, & 1 \le i \le r; j \ge 1, \\
f(y_i^{2+2j}) = \big(\frac{n+2}{3}\big)r, & 1 \le i \le r; j \ge 0, \\
f(y_i^{2+2j+1}) = \big(\frac{n+2}{3}\big)r + 1, & 1 \le i \le r; j \ge 0, \\
f(y_i^{2+2j+2}) = \big(\frac{n-1}{3}\big)r + 1, & 1 \le i \le r; j \ge 0, \\
f(y_i^{2+2j+3}) = \big(\frac{n+2}{3}\big)r + 1, & 1 \le i \le r; j \ge 0, \\
f(y_i^{2+2j+4}) = \big(\frac{n-1}{3}\big)r + 1, & 1 \le i \le r; j \ge 0, \\
f(y_i^{2+2j+5}) = \big(\frac{n-1}{3}\big)r, & 1 \le i \le r; j \ge 0, \\
f(y_i^{2+2j+l}) = \big(\frac{n-1}{3} - (l-5)\big)r, & 1 \le i \le r; 6 \le l \le (\frac{n+2}{3} + 2); j \ge 0, \\
f(y_i^{2+2j+l}) = r, & 1 \le i \le r; l = \frac{n+2}{3} + 3; j \ge 0, \\
f(y_i^n) = i, & 1 \le i \le r; \text{n is even}, \\
f(y_i^n) = 1, & 1 \le i \le r; \text{n is odd}.
\end{array}
$$

Meanwhile, each edge label is constructed below:

$f(u_{2i-1}u_{2i}) = i + \big(\frac{n-1}{3}\big)r + 1,$ $\quad 1 \le i \le r,$
$f(u_1u_{2r+1}) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil, f(u_{2r}u_{2r+2}) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil,$
$f(u_{2i}v_i) = i + \big(\frac{n-1}{3}\big)r + 1,$ $\quad 1 \le i \le r-1,$
$f(u_{2i+1}v_i) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil,$ $\quad 1 \le i \le r-1,$
$f(v_ix_i) = 2r - 1,$ $\quad 1 \le i \le r-1,$
$f(v_ix_{i+1}) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil,$ $\quad 1 \le i \le r-1,$
$f(u_{2r+1}x_1) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil;$
$f(u_{2r+2}x_r) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil,$
$f(x_iy_i^1) = i + \big(\frac{n-4}{3}\big)r + 2,$ $\quad 1 \le i \le r-1,$
$f(x_ry_r^1) = \big(\frac{n-4}{3}\big)r - 1,$
$f(y_i^{2l-1}y_i^{2l}) = \big(\frac{n+2}{3} - (l-1)\big)r + 1,$ $\quad 1 \le i \le r; 1 \le l \le j+1; j \ge 0,$
$f(y_i^{2l}y_i^{2l+1}) = i + \big(\frac{n-4}{3} - (l+1)\big)r - 1,$ $\quad 1 \le i \le r; 1 \le l \le j; j \ge 1,$
$f(y_i^{2j+2}y_i^{2j+3}) = i; f(y_i^{2j+3}y_i^{2j+4}) = 3r,$ $\quad 1 \le i \le r; j \ge 0,$
$f(y_i^{2j+4}y_i^{2j+5}) = i; f(y_i^{2j+5}y_i^{2j+6}) = r,$ $\quad 1 \le i \le r; j \ge 0,$
$f(y_i^{2j+6}y_i^{2j+7}) = i; f(y_i^{2j+7}y_i^{2j+8}) = 1,$ $\quad 1 \le i \le r; j \ge 0,$
$f(y_i^{2j+8}y_i^{2j+9}) = i; f(y_i^{2j+9}y_i^{2j+10}) = 1,$ $\quad 1 \le i \le r; j \ge 0,$
$\ldots,$
$f(y_i^{n-1}y_i^n) = 1,$ $\quad 1 \le i \le r;$ n is even,
$f(y_i^{n-1}y_i^n) = i,$ $\quad 1 \le i \le r;$ n is odd.

Under labeling f, we derive the vertex weights in the following:

$$wt_f(u_i) = i + nr + 1, \quad 1 \le i \le 2r,$$
$$wt_f(u_{2r+1}) = 3\left\lceil \frac{(n+2)r+3}{3} \right\rceil - 1,$$
$$wt_f(u_{r+2}) = 3\left\lceil \frac{(n+2)r+3}{3} \right\rceil,$$
$$wt_f(v_i) = i + 2\left\lceil \frac{(n+2)r+3}{3} \right\rceil + (\tfrac{n+5}{3})r + 1, \quad 1 \le i \le r-1,$$
$$wt_f(x_i) = i + 2\left\lceil \frac{(n+2)r+3}{3} \right\rceil + (\tfrac{n+2}{3})r + 1, \quad 1 \le i \le r,$$
$$wt_f(y_i^l) = i + (n-l)r + 1, \quad 1 \le i \le r; 1 \le l \le n.$$

It is obvious that each vertex has a different weight and the labels used are not more than $\left\lceil \frac{(n+2)r+3}{3} \right\rceil$. Therefore, $tvs(T_r(5,n)) \le \left\lceil \frac{(n+2)r+3}{3} \right\rceil$. Combining with Lower bound (3), we get $tvs(T_r(5,n)) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil$.

□

Figure 1 describes the pattern of vertex and edge labels to get $tvs(T_5(5,4)) = 11$.

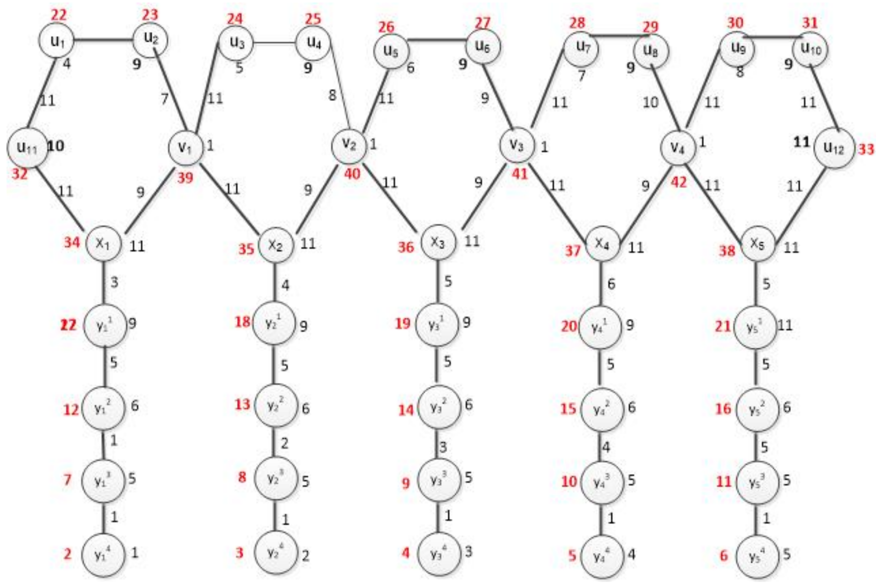

FIGURE 1. The pattern of vertex irregular total 11-labeling of $T_5(5, 4)$

2.2. **Total vertex irregularity strength of $T_r(4, n)$.** The graph $T_r(4, n)$ contains:

- a set of vertices with degrees 1, i.e. $\{y_1^n, y_2^n, \ldots, y_r^n\}$;
- a set of vertices with degrees 2, i.e. $\{y_1^{n-1}, y_2^{n-1}, \ldots, y_r^{n-1}, y_1^{n-2}, y_2^{n-2}, \ldots, y_r^{n-2}, \ldots, y_1^2, y_2^2, \ldots, y_r^2, y_1^1, y_2^1, \ldots, y_r^1\} \cup \{u_1, u_2, \ldots, u_r, u_{r+1}, u_{r+2}\}$;
- a set of vertices with degrees 3, i.e. $\{x_1, x_2, \ldots, x_r\}$; and
- a set of vertices with degrees 14, i.e. $\{v_1, v_2, \ldots, v_{r-1}\}$.

Lemma 2.1. *If $T_r(4, 2)$ is a tadpole chain graph with length r ($r \geq 2$), then*

$$tvs(T_r(4, 2)) = r + 1.$$

Proof. Based on (1), we get the lower bound as follows:

$$tvs(T_r(4,2)) \geq \max\left\{\left\lceil\frac{r+1}{2}\right\rceil, \left\lceil\frac{3r+3}{3}\right\rceil, \left\lceil\frac{4r+3}{4}\right\rceil, \left\lceil\frac{5r+2}{5}\right\rceil\right\} = \left\lceil\frac{3r+3}{3}\right\rceil = r + 1. \quad (4)$$

To establish that $tvs(T_r(4, 2)) \leq r + 1$, we define a function f from $V \cup E$ into a set of integers $\{1, 2, \ldots, k\}$ which is a total k-labeling with $k = \left\lceil\frac{3r+3}{3}\right\rceil = r + 1$.

We construct labels of vertices and edges as in Table 4.

We have the weights of vertices under the labeling f as follows:

$$\begin{aligned}
wt_f(y_i^2) &= i + 1, & 1 \leq i \leq r,\\
wt_f(y_i^1) &= i + r + 1, & 1 \leq i \leq r,\\
wt_f(u_i) &= i + 2r + 1, & 1 \leq i \leq r + 2,\\
wt_f(v_i) &= i + 3(r + 1) + r, & 1 \leq i \leq r - 1,\\
wt_f(x_i) &= i + 3(r + 1), & 1 \leq i \leq r.
\end{aligned}$$

TABLE 4. Labels of vertices and edges on $T_r(4,2)$

Labels of vertices	**Labels of edges**
$f(u_i) = r-1, 1 \le i \le r,$	$f(u_{r+1}u_1) = r+1,$
$f(u_{r+1}) = r,$	$f(u_{r+2}u_r) = r+1,$
$f(u_{r+2}) = r+1,$	$f(u_iv_i) = i+1, 1 \le i \le r-1,$
T $f(v_i) = r-1, 1 \le i \le r-1,$	$f(u_{i+1}v_i) = r+1, 1 \le i \le r-1,$
$f(x_i) = r+1, 1 \le i \le r,$	$f(v_ix_i) = r+1; f(v_ix_{i+1}) = r+1, 1 \le i \le r-1$
$f(y_i^1) = r, 1 \le i \le r,$	$f(u_{r+1}x_1) = r+1; f(u_{r+2}x_r) = r+1,$
$f(y_i^2) = i, 1 \le i \le r,$	$f(x_iy_i^1) = i; f(y_i^1y_i^2) = 1, 1 \le i \le r.$

We observe that each vertex has distinct label. Also, vertex and edge labels are less than or equal to $\left\lceil \frac{3r+3}{3} \right\rceil = r+1$. Hence, we get upper bound $tvs(T_r(4,2)) \le r+1$. Thus, $tvs(T_r(4,2)) = r+1$. □

Theorem 2.3. *If $T_r(4,n)$ are tadpole chain graphs of length r where $r \ge 2$, $n = 5 \mod 3$, and $n \ge 5$, then tvs $(T_r(4,n)) = \left\lceil \dfrac{(n+1)r+3}{3} \right\rceil$.*

Proof. According to (1), we obtain the lower bound as follows:

$$tvs(T_r(4,n)) \ge \max\left\{ \left\lceil \frac{r+1}{2} \right\rceil, \left\lceil \frac{(n+1)r+3)}{3} \right\rceil, \left\lceil \frac{(n+2)r+3}{4} \right\rceil, \left\lceil \frac{(n+3)r+2}{5} \right\rceil \right\} = \left\lceil \frac{(n+1)r+3}{3} \right\rceil. \tag{5}$$

We create a total k-labeling f from $V \cup E$ into $\{1, 2, \ldots, k\}$ with $k = \left\lceil \dfrac{(n+1)r+3}{3} \right\rceil$. Further, we prove the upper bound by means of 2 cases.

Case 1. For $n = 5$ and $n = 8$.

Firstly, we define Labels of vertices as follows:

$$f(u_i) = \begin{cases} 2r-1, & 1 \le i \le r, n = 5, \\ 3r-1, & 1 \le i \le r, n = 8. \end{cases}$$

$$\begin{aligned} f(u_{r+1}) &= \left\lceil \tfrac{(n+1)r+3}{3} \right\rceil - 1, && n = 5, 8 \\ f(u_{r+2}) &= \left\lceil \tfrac{3r+3}{3} \right\rceil, && n = 5, 8 \\ f(v_i) &= 1, && 1 \le i \le r-1, n = 5, 8 \\ f(x_i) &= \left\lceil \tfrac{(n+1)r+3}{3} \right\rceil, && 1 \le i \le r, n = 5, 8, \end{aligned}$$

$$f(y_i^1) = \begin{cases} 2r-1, & 1 \le i \le r-1, n = 5, \\ 2r-2, & 1 \le i \le r-1, n = 8. \end{cases}$$

$$f(y_r^1) = \begin{cases} 2r, & n = 5, \\ 3r, & n = 8. \end{cases}$$

$$f(y_i^2) = \begin{cases} r, & 1 \le i \le r, n = 5, \\ 3r, & 1 \le i \le r, n = 8. \end{cases}$$

$$f(y_i^3) = \begin{cases} 2r, & 1 \le i \le r, n = 5, \\ 3r+1, & 1 \le i \le r, n = 8. \end{cases}$$

$$f(y_i^4) = \begin{cases} r, & 1 \le i \le r, n = 5, \\ 2r+1, & 1 \le i \le r, n = 8. \end{cases}$$

$$f(y_i^5) = \begin{cases} 1, & 1 \le i \le r, n = 5, \\ 3r, & 1 \le i \le r, n = 8. \end{cases}$$

$$f(y_i^6) = 2r; f(y_i^7) = r; f(y_i^8) = i, \quad 1 \le i \le r, n = 8.$$

Secondly, we construct labels of edges as follows:

$$\begin{aligned}
&f(u_{r+1}u_1) = \left\lceil \tfrac{(n+1)r+3}{3} \right\rceil; f(u_{r+2}u_r) = \left\lceil \tfrac{(n+1)r+3}{3} \right\rceil, && n = 5, 8 \\
&f(u_{i+1}v_i) = \left\lceil \tfrac{(n+1)r+3}{3} \right\rceil, 1 \le i \le r-1, && n = 5, 8, \\
&f(v_ix_i) = 2r-1, 1 \le i \le r-1, && n = 5, 8 \\
&f(v_ix_{i+1}) = \left\lceil \tfrac{(n+1)r+3}{3} \right\rceil, 1 \le i \le r-1, && n = 5, 8, \\
&f(u_{r+1}x_1) = \left\lceil \tfrac{(n+1)r+3}{3} \right\rceil; f(u_{r+2}x_r) = \left\lceil \tfrac{(n+1)r+3}{3} \right\rceil,
\end{aligned}$$

$$f(u_iv_i) = \begin{cases} i+r+1, & 1 \le i \le r-1, n = 5, \\ i+2r+1, & 1 \le i \le r-1, n = 8. \end{cases}$$

$$f(x_iy_i^1) = \begin{cases} i+2, & 1 \le i \le r-1, n = 5, \\ i+2r+2, & 1 \le i \le r-1, n = 8. \end{cases}$$

$$f(x_ry_r^1) = \begin{cases} r, & n = 5, \\ 2r, & n = 8. \end{cases}$$

$$f(y_i^1y_i^2) = \begin{cases} 2r+1, & 1 \le i \le r, n = 5, \\ 3r+2, & 1 \le i \le r, n = 8. \end{cases}$$

$$f(y_i^2y_i^3) = i; f(y_i^4y_i^5) = i, \quad 1 \le i \le r, n = 5, 8,$$

$$f(y_i^3y_i^4) = \begin{cases} 1, & 1 \le i \le r, n = 5, \\ 2r, & 1 \le i \le r, n = 8. \end{cases}$$

$$f(y_i^5y_i^6) = 1; f(y_i^6y_i^7) = i; f(y_i^7y_i^8) = 1, \quad 1 \le i \le r, n = 8.$$

We have the vertex weights below:

$$wt_f(u_i) = \begin{cases} i + 2\left\lceil \frac{(n+1)r+3}{3} \right\rceil + (r-1), & 1 \le i \le r+2, n = 5, \\ i + 2\left\lceil \frac{(n+1)r+3}{3} \right\rceil + (2r-1), & 1 \le i \le r+2, n = 8. \end{cases}$$

$$wt_f(v_i) = \begin{cases} i + 2\left\lceil \frac{(n+1)r+3}{3} \right\rceil + (3r+1), & 1 \le i \le r-1, n = 5, \\ i + 2\left\lceil \frac{(n+1)r+3}{3} \right\rceil + (4r+1), & 1 \le i \le r-1, n = 8. \end{cases}$$

$$wt_f(x_i) = \begin{cases} i + 2\left\lceil \frac{(n+1)r+3}{3} \right\rceil + (2r+1), & 1 \le i \le r, n = 5, \\ i + 2\left\lceil \frac{(n+1)r+3}{3} \right\rceil + (3r+1), & 1 \le i \le r, n = 8. \end{cases}$$

$$wt_f(y_i^l) = i + (n-l)r + 1, \quad 1 \le i \le r, 1 \le l \le n, n = 5, 8.$$

It is obvious that each vertex has a different weight. Also, the labels are not more than $\left\lceil \frac{(n+1)r+3}{3} \right\rceil$. Hence, we establish the upper bound $tvs(T_r(4,n)) \le \left\lceil \frac{(n+1)r+3}{3} \right\rceil$. According to Lower bound (5), we have $tvs(T_r(4,n)) = \left\lceil \frac{(n+1)r+3}{3} \right\rceil$.

Case 2. For $n \ge 11$.
Analog to the proof of Theorem 2.2 in Case 3, wet $tvs(T_r(4,n)) = \left\lceil \frac{(n+1)r+3}{3} \right\rceil$. □

Figure 2 shows the pattern of vertex and edge labels to obtain $tvs(T_5(4,5)) = 11$.

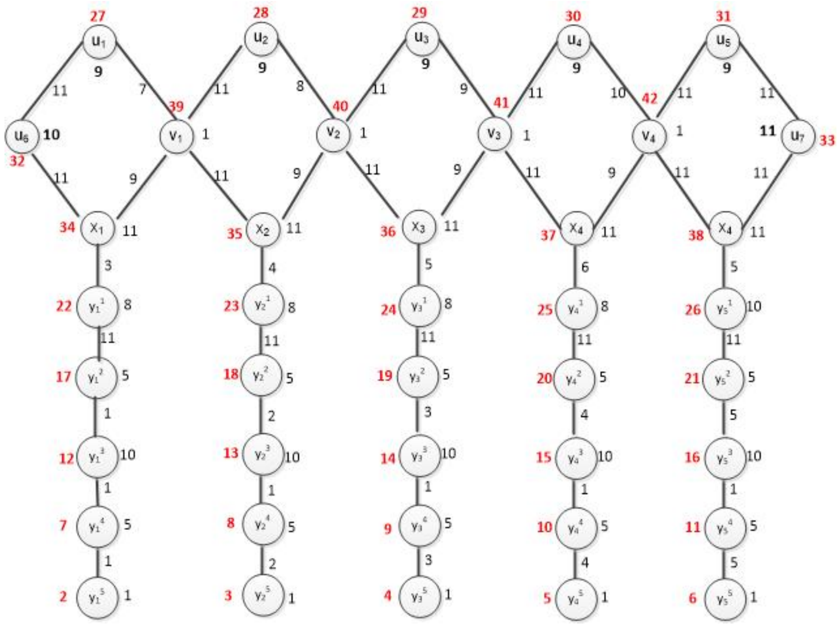

FIGURE 2. The pattern of vertex irregular total 11-labeling of $T_5(4,5)$

3. Conclusions

We have verified tvs of $T_r(4,n)$ and $T_r(5,n)$ in this paper. The patterns to get tvs of the graphs were presented in the theorems. We have proved the upper bounds and got $tvs(T_r(4,n)) = \left\lceil \frac{(n+1)r+3}{3} \right\rceil$ and $tvs(T_r(5,n)) = \left\lceil \frac{(n+2)r+3}{3} \right\rceil$. In further research, we are going to investigate tvs of general tadpole chain graph $T_r(m,n)$ for $m \ge 6$. Also, we will verify the formulas using algorithmic approaches.

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