# GEODETIC DOMINATION INTEGRITY IN GRAPHS

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ABSTRACT. Let $G$ be a simple graph. A subset $S \subseteq V(G)$ is a said to be a geodetic set if every vertex $u \notin S$ lies on a shortest path between two vertices from $S$. The minimum cardinality of such a set $S$ is the geodetic number $g(G)$ of $G$. A subset $D \subseteq V(G)$ is a dominating set of $G$ if every vertex $u \notin D$ has at least one neighbor in $D$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. A subset is said to be a geodetic dominating set of $G$ if it is both a geodetic and a dominating set. The geodetic domination number $\gamma_g(G)$ is the minimum cardinality among all geodetic dominating sets in $G$. The geodetic domination integrity of a graph $G$ is defined by $DI_g(G) = min\{|S| + m(G - S) : S$ is a geodetic dominating set of $G\}$, where $m(G - S)$ denotes the order of the largest component in $G - S$. In this paper, we study the concepts of geodetic dominating integrity of some families of graphs and derive some bounds for the geodetic domination integrity. Also we obtain geodetic domination integrity of some cartesian product of graphs.

Keywords: Geodetic Sets, Geodetic Dominating Sets, Geodetic Domination Integrity Sets

AMS Subject Classification: 05C12, 05C69,05C40, 68M10, 68R10, 05C76

## 1. INTRODUCTION

Let $G = (V, E)$ be a finite graph without loops and multiple edges. The set of vertices and edges of $G$ are denoted by $V(G)$ and $E(G)$, respectively. The order and size of $G$ are $n = |V(G)|$ and $m = |E(G)|$, respectively. The induced subgraph $G[S]$ is a subgraph induced by $S \subseteq V(G)$. The open neighborhood of a vertex $v \in V(G)$ is the set of all vertices adjacent to $v$, denoted by $N(v)$ and the closed neighborhood of $v$ is $N[v] = N(v) \bigcup \{v\}$. The degree $d(v)$ of a vertex $v$ is defined by $d(v) = |N(v)|$. Let $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$. The distance $d(u, v)$ is the length of a shortest $u - v$ path in $G$. The diameter of a connected graph is defined as $diam(G) = \max\limits_{u,v \in V(G))} d(u, v)$. A $u - v$ geodesic is the $u - v$ path of length $d(u, v)$. For a graph $G$, $I_G[u, v]$ denote the set of all vertices lying on a $u - v$ geodesic. Let $I_G[S]$ denote the union of all $I_G[u, v]$ for all $u, v \in S$. Let $P_n, C_n$, and $K_{r,s}$ be the path on $n$ vertices, the cycle on $n$ vertices, and the complete bipartite graph in which one partite set has $r$ vertices and the other partite set has $s$ vertices,

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respectively. A subset $S \subseteq V(G)$ is a geodetic set of $G$ if $I_G[S] = V(G)$. The geodetic number $g(G)$ of a graph $G$ is the minimum cardinality of a set $S$ with $I_G[S] = V(G)$. This parameter was introduced in [4, 14] and further studied in [9]. Geodetic sets are studied in [4, 5, 7–9, 14, 18]. A subset $S$ of vertices of $G$ is a dominating set if each vertex of $G$ is dominated by some vertex of $S$. That is, every vertex in $V - S$ is adjacent with at least one vertex in $S$. The domination number of $G$ is the minimum cardinality of a dominating set of $G$ and is denoted by $\gamma(G)$ in [15].

Since, in general a dominating set is not a geodetic set in a graph $G$ and vice versa, this motivated them to study the new domination conception of geodetic domination [10]. A geodetic dominating set is a subset of vertices of a graph is both a geodetic and a dominating set. The minimum cardinality of a geodetic dominating set of $G$ is called the geodetic domination number $\gamma_g(G)$. Since $V(G)$ is a geodetic dominating set for any graph $G$, the existence of a geodetic domination set of a graph is guaranteed. A geodetic dominating set of size $\gamma_g(G)$ is said to be a $\gamma_g(G)$-set. In $G = K_{1,n-1}$, $n \geq 3$ and the support vertex $v$ in $G$ is a dominating set which is not a geodetic set of $G$. But $S = V(G) \backslash \{v\}$ is a geodetic set of $G$. $S$ is also a minimum geodetic set and so $\gamma_g(K_{1,n-1}) = g(K_{1,n-1}) = n - 1$. Chartrand, Harary and Zhang [5] showed that $g(K_{r,s}) = min\{r, s, 4\}$ for $r, s \geq 2$, and thus we obtain $\gamma_g(K_{r,s}) = g(K_{r,s}) = min\{r, s, 4\}$ for $r, s \geq 2$.

Any communication network can be modeled as a graph whose nodes are the processors (stations) and a communication link as an edge between corresponding nodes. The stability of a communication network is a very important factor for the network designers to reconstruct the it after the failure of certain stations or communication links. Two essential quantities in an analysis of the vulnerability of a communication network are (1) the number of nodes that are not functioning and (2) the size of a maximum order of a remaining sub network within which mutual communications can still occur. C. A. Barefoot, et. al. [2] introduced the concept of integrity. It is an useful measure of vulnerability and it is defined as follows. $I(G) = min\{|S| + m(G - S) : S \subset V(G)\}$, where $m(G - S)$ denotes the order of the largest component in $G - S$. Further this parameter was studied in [11]. Unlike the connectivity measures, integrity shows not only the difficulty to break down the network but also measures the damage caused. Some of the vulnerability parameters like connectivity, toughness, integrity, binding number etc. are available in the literature. A small group of people have effective communication links with other members of the organization and they take important decisions in an administrative set up. Domination in graphs provides a model for such a concept. A minimum dominating set of nodes provides a link with the rest of the nodes in a network, If the removal of such a set, results huge impact in the network. That is, the decision making process is paralyzed but also the communication between the remaining members is minimized. The damage will be more when the dominating set of nodes are under attack. This motivated the study of domination integrity when the sets of nodes disturbed are dominating sets. Sundareswaran et. al introduced the concept of Domination Integrity of a graph and studied in [22–24] as another measure of vulnerability of a graph which is defined as follows. $DI(G) = min\{|S| + m(G - S)\}$, where $S$ is a dominating set of $G$ and $m(G - S)$ denotes the order of the largest component in $G - S$ and is denoted by $DI(G)$.

In section 2, we introduce a new vulnerability parameter geodetic domination integrity and obtain the geodetic domination integrity of certain classes of Graphs. In section 3, we derive some results relating the new parameter on trees. In section 4, we present some bounds for this parameter on the Cartesian product of graphs. In the last section, we studied the application of this parameter in sewage networks.

## 2. Geodetic Domination Integrity

The problem of finding a geodetic path in any network plays an important role for network engineers to optimize time and cost. The removal of a geodetic set increases vulnerability of a network. This motivates to study the concept of geodetic domination integrity in graphs.

**Definition 2.1.** *The geodetic domination integrity of a graph $G$ is defined by $DI_g(G) = min\{|S| + m(G - S) : S$ is a geodetic dominating set of $G\}$, where $m(G - S)$ denotes the order of the largest component in $G - S$. A subset $S$ of $V(G)$ is said to be a $DI_g$-set, if $DI_g(G) = |S| + m(G - S)$.*

**Observation 2.1.**

- The complete graph $K_n$ of $n$ vertices has $DI_g(K_n) = n$.
- The star graph $K_{1,n-1}$ of $n$ vertices has $DI_g(K_{1,n-1}) = n$.
- The complete bipartite graph $K_{r,s}$ on $r + s$ vertices with $r, s \geq 2$ have $DI_g(K_{r,s}) = min\{r, s\} + 1$.
- The wheel graph $W_n$ of $n$ vertices has $DI_g(W_n) = \lceil \frac{(n-1)}{2} \rceil + 2$,for $n \geq 5$.
- The Cycle $C_n$ of $n$ vertices has $DI_g(C_n) = \lceil \frac{n}{3} \rceil + 2, n \geq 6$.
- The Path $P_n$ of $n$ vertices has $DI_g(P_n) = \lceil \frac{(n+2)}{3} \rceil + 2$.
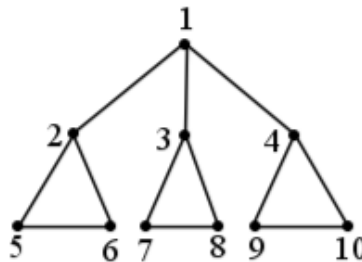- The Petersen graph $G$ has $DI_g(G) = 6$.

**Example 2.1.**



Figure 1. For this graph $G$, $I(G) = 4$, $DI(G) = 5$, $DI_g(G) = 8$.

### 2.1. **Bounds on Geodetic Domination Integrity.**

**Observation 2.2.**

- For any connected Graph $G$, $2 \leq DI_g(G) \leq n$. The bounds are sharp.
- $2 \leq \gamma_g(G) + 1 \leq DI_g(G) \leq n$.
- $max\{\gamma(G), g(G)\} \leq \gamma_g(G) \leq DI_g(G) \leq n$.

**Theorem 2.1.** *Let $G$ be a connected graph. Then $DI_g(G) = 2$ if and only if $G \cong K_2$.*

**Proof.** Let $G = K_2$. Then clearly $DI_g(G) = 2$. Conversely, let $DI_g(G) = 2$ and let $S$ be a $DI_g$-set of $G$. Clearly, $S$ is a singleton set or a set with two vertices.
**Case 1:** Suppose $S = \{u\}$. Then $m(G - S) = 0$ is a $DI_g$-set of $G$. Since $DI_g(G) \geq 2, S \neq \{u\}$. Therefore $|S| \neq 1$. But $DI_g(G) = 2$. Therefore $|S| = 2$.
**Case 2:** Suppose $|S| = 2$. Then $S = \{u, v\}$ is a $DI_g$-set of $G$. Then $m(G - S) = 0$. Therefore $G$ contains only two vertices. Therefore $G \cong K_2$.

**Theorem 2.2.** *Let $G$ be a connected graph then $DI_g(G) = 3$ if and only if $G \cong K_3$ (or) there exists a geodetic set $S = \{u, v\}$ such that $u$ and $v$ have common independent neighbours.*

**Proof.**

Let $DI_g(G) = 3$. Let $S$ be a $DI_g$-Set of $G$. Then $|S| = 1$ (or) 2 (or) 3.

**Case (i):** Suppose $|S| = 3$. Then $m(G - S) = 0$. Then clearly $G \cong K_3$.

**Case (ii):** Suppose $|S| = 2$. Then $m(G - S) = 1$. Let $S = \{u, v\}$. Since $m(G - S) = 1$, $(G - S)$ is totally disconnected and are all adjacent with $u$ and $v$.

**Case (iii):** Suppose $|S| = 1$. $m(G - S) = 2$. Since $DI_g(G) \geq 2$, this cannot happen.

The converse is obvious.

**Theorem 2.3.** *Let $G$ be a connected graph then $DI_g(G) = n$ if and only if $G \cong K_n$ (or) $K_{1,n-1}$.*

**Proof.** Let $G = K_n$. Then $DI_g(G) = n$ and if $G \cong K_{1,n-1}$, then $DI_g(G) = n$. Conversely, By Theorems 2.1 and 2.2, the case when $DI_g(G) = 2, 3$ are proved. Assume that $DI_g(G) = n \geq 4$. If $S$ is a $DI_g$-set of $G$ with $|S| = n$ or $n - 1n$ then clearly $G \cong K_n$ or $G \cong K_{1,n-1}$, respectively. Suppose if $|S| \leq n - 2$, then $m(G - S) \geq 2$. Then $< G - S >$ contains at least a $K_2$, in which no vertex of $< G - S >$ is covered by any geodetic path of $S$. Therefore $|S| \geq n - 1$. Hence the theorem.

**Theorem 2.4.** *For any two positive integers $a$ and $n$ with $2 \leq a \leq n$, there exists a connected graph $G$ such that $DI_g(G) = a$ and $|V(G)| = n$.*

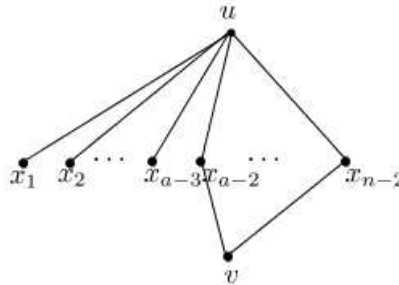**Proof.** The construction is shown in the following figure.



FIGURE 2. Construction of $G$ with $DI_g(G) = a$ for a given $a$, where the $DI_g$-set is $\{x_1, x_2, \ldots, x_{a-3}, u, v\}$

## 3. Geodetic Domination Integrity of $k$-ary trees

In this section, we will consider the geodetic domination integrity of $k$-ary trees $H_n^k$, where $H_n^k$ is the rooted complete $k$-ary tree of height $n - 1$, each vertex except leaves have $k$ children, and each leave is at distance $n - 1$ from the root. Order of the graph $H_n^k$ is $(k^n - 1)/(k - 1)$.

**Theorem 3.1.** *[24] For $k \geq 2$, integrity of the complete $k$-ary tree of height $n - 1$ is given by* $I(H_n^k) = \begin{cases} \frac{k^{n+1}/2 - 1}{k - 1}, & \text{if } n \text{ is odd} \\ \frac{(2k-1)k^{n/2-1} - 1}{k - 1}, & \text{if } n \text{ is even} \end{cases}$

**Theorem 3.2.** *[24] For $k \geq 2$, domination number* $\gamma(H_n^k) = \begin{cases} \frac{k(k^{(n/3)}-1)}{7}, & \text{if } n \equiv 0 (mod\ 3) \\ 1 + \frac{k^2(k^{(\frac{n-1}{3})}-1)}{7} & \text{if } n \equiv 1 (mod\ 3) \\ 1 + \frac{k^3(k^{(\frac{n-2}{3})}-1)}{7} & \text{if } n \equiv 2 (mod\ 3) \end{cases}$

**Theorem 3.3.** *[24] For $k \geq 2$, the domination integrity of the complete $k$-ary tree of height $n - 1$ is given by $DI(H_n^k) = \gamma(H_n^k) + (k + 1)$.*

**Theorem 3.4.** *For $n, k \geq 2$, the geodetic domination integrity of the complete $k$-ary tree of height $n - 1$ is given by $DI_g(H_n^k) = DI(H_{n-2}^k) + k^{n-1}$.*

**Proof.** Consider the graph $H_n^k$, $n, k \geq 2$. The geodetic set of $H_n^k$ is the set of all leaves. Hence $g(H_n^k) = k^{n-1}$. As the geodetic set dominates the vertices in the level $n - 1$ also. Hence it is enough to find the domination integrity of the remaining graph after deleting the last two levels, that is, we consider only the graph $H_{n-2}^k$. Thus the geodetic domination integrity of $H_n^k$ is equal to $DI(H_{n-2}^k) + g(H_n^k) = DI(H_{n-2}^k) + k^{n-1}$. Hence the theorem.

**Corollary 3.1.** *For $n \geq 2$, the geodetic domination integrity of the complete binary tree of height $n - 1$ is given by $DI_g(H_n^2) = DI(H_{n-2}^2) + 2^{n-1}$.*

## 4. Geodetic Domination Integrity of Cartesian Product of Graphs

**Definition 4.1.** *The* cartesian product *of the graphs $G$ and $H$, denoted by $G \square H$, has vertex set $V(G) \times V(H)$ in which $(g_1, h_1)(g_2, h_2)$ is an edge whenever $g_1 = g_2$ and $h_1 h_2 \in E(H)$ or, $h_1 = h_2$ and $g_1 g_2 \in E(G)$, see Figure 3.*
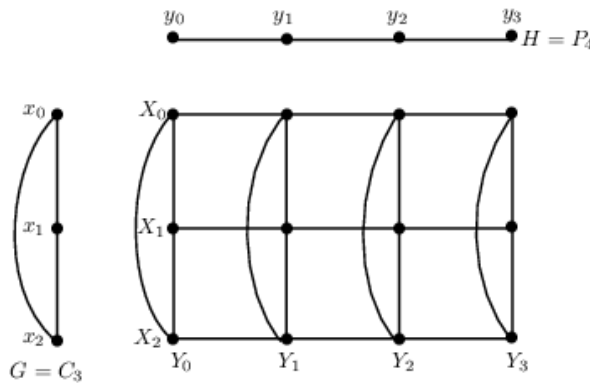


FIGURE 3. $X_i = \{x_i\} \times V(H)$ and $Y_j = V(G) \times \{y_j\}$
The graph $C_3 \square P_4$.

Let $G$ and $H$ be simple graphs with vertex sets $V(G) = \{x_0, x_1, \ldots, x_{m-1}\}$ and $V(H) = \{y_0, y_1, \ldots, y_{n-1}\}$. Then $V(G \times H) = V(G) \times V(H)$. For our convenience, we write $V(G) \times V(H) = \bigcup_{i=0}^{m-1} X_i$, where $X_i$ stands for $\{x_i\} \times V(H)$. Further, we shall denote the vertices of $X_i$, $0 \leq i \leq m - 1$, by $\{x_{i,j} \mid 0 \leq j \leq n - 1\}$, where $x_{i,j}$ stands for the vertex $(x_i, y_j)$. We shall call $X_i$, the $i^{th}$ *layer* of $G \times H$, see Figure 3.

Similarly, for a set $S$ of $V(G \square H)$, we define two sets, namely $S_1$ and $S_2$, the projection of $S$ onto $G$ and $H$, respectively, as follows: Let $S_1 = \{x \mid (x, y) \in S\}$ and $S_2 = \{y \mid (x, y) \in S\}$. Clearly, $S_1 \subseteq V(G)$ and $S_2 \subseteq V(H)$.

**Lemma 4.1.** *[6, 16] Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be two graphs. Then for any set $S \subseteq V(G \square H)$, $I[S] \subseteq I[S_1] \times I[S_2]$.*

**Lemma 4.2.** *[6, 16] Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be two graphs. Then for $a \in V(G \square H)$ and $S \subseteq V(G \square H)$, we have (i). $N[a] \subseteq N[a_1] \times N[a_2]$ for all $a \in V(G \square H)$ and (ii). $N[S] \subseteq N[S_1] \times N[S_2]$.*

In the following theorem, we will present some weak lower bound for $DI_g(G \square H)$.

**Theorem 4.1.** *Let $G$ and $H$ be graphs of order at least three, then*
$DI_g(G \square H) \geq max\{DI_g(G), DI_g(H)\}.$

**Proof.** Let $S$ be the $DI_g$-set of $G \square H$ and let $X$ be the $m((G \square H) \setminus S)$. Let $S_1$ and $S_2$ be the projections of $S$ onto $G$ and $H$, respectively. Similarly, let $X_1$ and $X_2$ be the projections of $X$ onto $G$ and $H$, respectively. Let $X_1^* = X_1 \setminus S_1$ and $X_2^* = X_2 \setminus S_2$.

As observed earlier, $S_1$ and $X_1^*$ are subsets of $V(G)$. As $S$ is a dominating set, of $G \square H$, $S_1$ must be a dominating set of $G$. By Lemmas 4.1 and 4.2, $S_1$ is a geodetic set of $G$. Also after deleting $S_1$ from $V(G)$, $X_1^*$ is the component with maximum order. Hence $DI_g(G) \leq |S_1| + |X_1^*|$. Same arguments hold for the graph $H$. Hence $DI_g(H) \leq |S_2| + |X_2^*|$.

Thus $DI_g(G \square H) = |S| \geq max\{|S_1| + |X_1^*|, |S_2| + |X_2^*|\} \geq max\{DI_g(G), DI_g(H)\}$.

Here is the upper bound for the cartesian product of graphs. Let $S_1$ be the $DI_g$-set of $G$ and $X_1$ be the maximum component of $V(G) \setminus S_1$ and the vertices of remaining components after the deletion of $S_1 \cup X_1$ from $V(G)$ are combined as a set $Y_1$. Therefore, $DI_g(G) = |S_1| + |X_1|$. Similarly, we have sets $S_2, X_2, Y_2$, respectively, of the graph $H$.

**Theorem 4.2.** *For any two graphs $G$ and $H$ with order greater than or equal to three, with $DI_g(G \square H) \leq |S_1||V(H)| + |S_2||V(G)| - |S_1||S_2| + |X_1||X_2|$.*

**Proof.** Let $V(G) = S_1 \cup X_1 \cup Y_1$ and $V(H) = S_2 \cup X_2 \cup Y_2$, where $DI_g(G) = |S_1| + |X_1|$ and $DI_g(H) = |S_2| + |X_2|$. Now consider the set $S = (S_1 \times V(H)) \cup (S_2 \times V(G))$. Since $S_1$ and $S_2$ are $DI_g$-sets of $G$ and $H$, by Lemmas 4.1 and 4.2, $S_1 \times S_2$ is a geodetic set. It is easy to verify that $S$ is a dominating set of $G \square H$. Consider $(G \square H) \setminus S$. Since $X_1$ and $X_2$ are the maximum components of $G$ and $H$ after the deletion of $S_1$ and $S_2$, respectively, we have $x_1 \times X_2$ as the maximum component of $(G \square H) \setminus S$. Hence $S$ is a geodetic dominating set of $G \square H$ and $X_1 \times X_2$ is the maximum component after the deltion of $S$, thus we have $DI_g(G \square H) \leq |S| + |X_1 \times X_2| = |S_1||V(H)| + |S_2||V(G)| - |S_1||S_2| + |X_1||X_2|$.

**Observation 4.1.** *The integrity of a graph $G$ can be found from the problem of finding a maximum size bipartite subgraph of $G$ as follows: Let $H$ be a maximum size bipartite subgraph of $G$ and let $(X_1, X_2)$ be a bipartition of $V(H)$. Now it is an easy observation that $DI(G) = min\{|X_1| + m(G - X_1), |X_2| + m(G - X_2)\}$.*

**Theorem 4.3.** *For $n \geq 2$, $DI_g(K_2 \square P_n) = n + 1$.*

**Proof.** Consider the graph $K_2 \square P_n$ with $V(K_2 \square P_n) = X_1 \cup X_2$, $X_1 = \{x_1, x_2, \ldots, x_n\}$, $X_2 = \{y_1, y_2, \ldots, y_n\}$. It is known that, the graph $K_2 \square P_n$ is a bipartite graph and let the bipartition be $(Y_1, Y_2)$ with $Y_1 = \{x_1, x_3, x_5, \ldots, x_n\} \cup \{y_2, y_4, \ldots, y_{n-1}\}$ and $Y_2 = \{x_2, x_4, \ldots, x_{n-1}\} \cup \{y_1, y_3, \ldots, y_n\}$, if $n$ is odd. Hence by Observation 4.1, we have $DI(K_2 \square P_n) = min\{|Y_1| + m((K_2 \square P_n) - Y_1), |Y_2| + m((K_2 \square P_n) - Y_2)\} = n + 1$. Clearly, both $Y_1$ and $Y_2$ are domminating and geodetic set. Thus $DI_g(K_2 \square P_n) = n + 1$. Similarly, we can prove for $n$ is even.

**Theorem 4.4.** *For $n \geq 3$, $DI_g(K_2 \square C_n) = \begin{cases} n + 2, & n \text{ is odd} \\ n + 1, & n \text{ is even} \end{cases}$ .*

**Proof.** Consider the graph $K_2 \square C_n$ with $V(K_2 \square P_n) = X_1 \cup X_2$, $X_1 = \{x_1, x_2, \ldots, x_n\}$, $X_2 = \{y_1, y_2, \ldots, y_n\}$. We prove this theorem in two cases.

**Case 1.** $n$ is odd.

In this case maximum size bipartite subgraph is obtained with bipartition $(Y_1, Y_2)$, where $Y_1 = \{x_1, x_3, x_5, \ldots, x_n\} \cup \{y_2, y_4, \ldots, y_{n-1}\}$ and $Y_2 = \{x_2, x_4, \ldots, x_{n-1}\} \cup \{y_1, y_3, \ldots, y_n\}$. After deleting $Y_1$ or $Y_2$,) size of the maximum component is two. Hence by Observation 4.1, we have $DI(K_2 \square C_n) = n + 2$. Both $Y_1$ and $Y_2$ are dominating and geodetic sets. Thus $DI_g(K_2 \square C_n) = n + 2$.

**Case 2.** $n$ is even.

As $C_n$ is a bipartite graph, the proof is similar to the proof of Theorem 4.3.

**Theorem 4.5.** *For $n \geq 2$, $DI_g(K_2 \square K_n) = \lceil \frac{3n}{2} \rceil$.*

**Proof.** Consider the graph $K_2 \square P_n$ with $V(K_2 \square K_n) = X_1 \cup X_2$, $X_1 = \{x_1, x_2, \ldots, x_n\}$, $X_2 = \{y_1, y_2, \ldots, y_n\}$. We divide the proof of this theorem into two cases.

**Case 1.** $n$ is odd.

As in the previous Theorem 4.4, the maximum size bipartite subgraph is obtained by taking the partite sets to be $(Y_1, Y_2)$, where $Y_1 = \{x_1, x_3, x_5, \ldots, x_n\} \cup \{y_2, y_4, \ldots, y_{n-1}\}$ and $Y_2 = \{x_2, x_4, \ldots, x_{n-1}\} \cup \{y_1, y_3, \ldots, y_n\}$. After deleting $Y_1$ or $Y_2$, the maximum order component is a $K_{\lceil \frac{n}{2} \rceil}$ and therefore by Observation 4.1, $DI(K_2 \square K_n) = n + \lceil \frac{n}{2} \rceil = \lceil \frac{3n}{2} \rceil$. Clearly, both $Y_1$ and $Y_2$ are dominating and geodetic sets. Hence $DI_g(K_2 \square K_n) = \lceil \frac{3n}{2} \rceil$.

**Case 2.** $n$ is even.

In this case, the maximum size bipartite subgraph is obtained by taking the partite sets to be $(Y_1, Y_2)$, where $Y_1 = \{x_1, x_3, x_5, \ldots, x_{n-1}\} \cup \{y_2, y_4, \ldots, y_n\}$ and $Y_2 = \{x_2, x_4, \ldots, x_n\} \cup \{y_1, y_3, \ldots, y_{n-1}\}$. After deleting $Y_1$ or $Y_2$, the maximum order component is a $K_{\frac{n}{2}}$ and therefore by Observation 4.1, $DI(K_2 \square K_n) = n + \frac{n}{2} = \frac{3n}{2}$. Clearly, both $Y_1$ and $Y_2$ are dominating and geodetic sets. Hence $DI_g(K_2 \square K_n) = \frac{3n}{2}$.

## 5. Applications of Geodetic Domination Integrity in PMU Placement in Electric Power System

A phasor measurement unit (PMU) is considered to be the very important device which is used to investigate and control the power system by calculating the voltage phase in electrical nodes and transmission lines. From this measurement, positive sequence voltage and swing angles can be assessed, which can bring about the estimation of power system stability. These components are linked and synchronized with a universal positioning system and send measurement data to a centralized unit. PMU is a key component in smart grid that needs some attentions, if one elects to implement it in a power system. Since a PMU wants a massive investment, it is important to optimize the number of PMUs without disturbing the depth of observability of the power system. Cost is one of the reasons that want to be put into consideration when deciding PMUs execution. For that purpose, it is the highest priority for electric utilities to plan properly PMUs installation where reliability is sustained while minimizing the cost involved. The examination to find the minimum number of PMUs required while maintaining the whole observability of a power system is mentioning to the optimal PMUs placement problem. This problem mainly motivations on finding a way to determine the minimum number of PMUs required in a power system without compromising power system observability. Several methods are available for finding the optimum number and optimum placement of PMUs in the literature [1, 17, 21].

TABLE 1. Number of PMUs in various methods

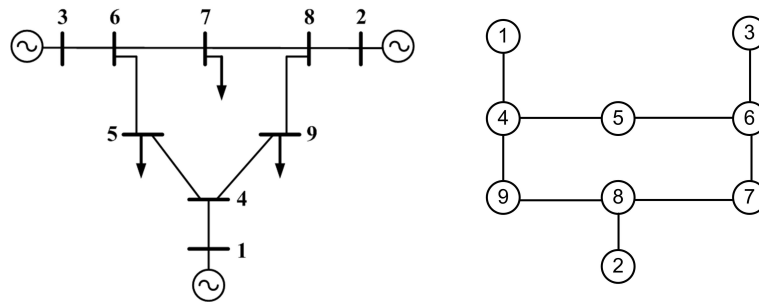| Method | IEEE 14 | IEEE 30 | IEEE 57 |
|---|---|---|---|
| Power domination | 3 | 7 | 11 |
| Topology Transformation | 3 | 7 | 11 |
| Immunity genetic algorithm | 3 | 7 | 11 |
| Genetic Algorithm | 3 | 7 | 12 |
| Binary particle swarm optimization | 3 | 7 | 13 |



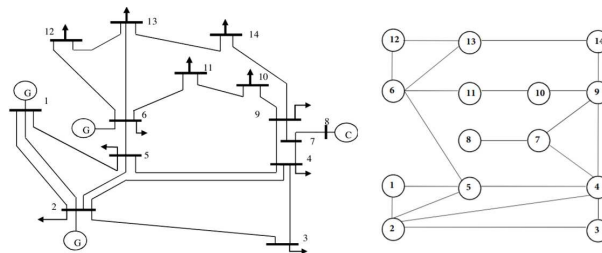FIGURE 4. IEEE 9-bus system and its Graph network model



FIGURE 5. IEEE 14-bus system and its Graph network model

We apply geodetic domination integrity on IEEE 9 and IEEE 14 bus systems. The proposed method of PMU placement using geodetic domination integrity determines the minimum number of strategic bus locations where a PMU must be placed for complete observability of the power system. Table 1 [20] gives the comparative results of minimum number of PMUs required for the complete observability of different systems.

For the IEEE 9-bus test system, the minimal geodetic dominating sets are $S_1 = \{1, 2, 3, 4, 6\}, S_2 = \{1, 2, 3, 4, 7\}, S_3 = \{1, 2, 3, 4, 8\}, S_4 = \{1, 2, 3, 5, 8\}, S_5 = \{1, 2, 3, 6, 8\}$, and $S_6 = \{1, 2, 3, 6, 9\}$. The geodetic domination integrity number is $\gamma_g(G) = min\{5 + 3, 5 + 2, 5 + 3, 5 + 2, 5 + 3, 5 + 2\} = 7$. Thus, the minimum geodetic domination integrity sets are $S_2$, $S_4$ and $S_6$ which have a geodetic domination integrity number 7 and that locations are optimum for PMU placement. For the IEEE 14-bus test system, the geodetic domination integrity number is 9 and corresponding geodetic dominating integrity set is $\{1, 3, 4, 6, 8, 9, 12\}$ which gives the optimum location. Similarly we can apply this method on IEEE 30 , IEEE 57 bus system to find the locations for placing the PMUs.

**Conclusion.** In this paper, a new vulnerability parameter geodetic domination integrity $DI_g(G)$)has been defined and studied for some classes of graphs and obtained few bounds for it. An application of this parameter in IEEE buses has been discussed. Currently

we are deriving a bound for the maximum number of edges in terms geodetic domination integrity of $G$ and its complexity.

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