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BALANCED RANK DISTRIBUTION LABELING OF LADDER GRAPHS, COMPLETE GRAPHS AND COMPLETE BIPARTITE GRAPHS

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Abstract. A balanced rank distribution labeling of a graph G of order n is a new kind of vertex labeling from $\{1, 2, 3, ..., k\}(n \leq k \in Z^+)$ which leads to a balanced edge labeling of G called edge ranks. In this paper, the balanced rank distribution labeling of ladder graphs $L_{n/2}$ for even $n \geq 6$, complete graphs K_n for $n \geq 3$ and complete bipartite graphs $K_{n/2,n/2}$ for even $n \geq 4$ have been investigated and obtained the results on balanced rank distribution number ($\mathbf{brd}(G)$) for the given graphs as follows:

(i) $\mathbf{brd}(L_{n/2}) = 3n - 15$, for even $n \geq 12$

(ii) $\mathbf{brd}(K_n) = n$, for $n \geq 3$

(iii) $\mathbf{brd}(K_{n/2,n/2}) = n$, for even $n \geq 4$

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1. Introduction

All graphs $G(V, E)$ considered here are finite, simple and undirected. Let P_n and K_n denote a path and a complete graph on n vertices respectively. The cartesian product $G \square H$ of graphs G and H is a graph such that (i) the vertex set of $G \square H$ is cartesian product $V(G) \times V(H)$ and (ii) two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G \square H$ if and only if either $u_1 = v_1$ and u_2 is adjacent to v_2 in H, or $u_2 = v_2$ and u_1 is adjacent to v_1 in G. The ladder graph L_p is a planar graph with $2p$ vertices and $3p - 2$ edges. It is the cartesian product of two path graphs, one is P_2 and other one is P_p. For positive integers p and q, $K_{p,q}$ denotes the complete bipartite graph with vertex partitions of cardinality p and q. For a real x, $\lfloor x \rfloor$ and $[x]$ respectively denote the floor function and greatest integer function that gives the greatest integer less than or equal to x as the output and $\lceil x \rceil$ is the ceiling function that gives the least integer greater than or equal to x as the output. A graph labeling is an assignment of values to the vertices or edges subject to specific constraints. The three significant features of most interesting graph labeling problems are

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(i) a set of numbers from which the vertex labels are chosen (ii) a rule that assigns a value to each edge and (iii) a condition that these values must satisfy. Graph labeling has its origin in the middle of 1960. More than 200 types of graph labeling techniques have been introduced and which can be seen in the dynamic survey of graph labeling by J.A. Gallian [1]. Somasundaram and Ponraj [6] have introduced the notion of mean labeling of graphs in 2003. A graph G of order n and size m is said to be a mean graph if there exists an injective function $f : V(G) \to \{0, 1, 2, ..., m\}$ such that when each edge uv has assigned the weight $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$, the resulting weights are distinct. The concept of vertex equitable graph was introduced by Lourdusamy and Seenivasan [4] in 2008. Given a graph with n edges and a labeling $f : V(G) \to \{0, 1, 2, ..., [m/2]\}$, a vertex equitable labeling f^* is defined by $f^*(uv) = f(u) + f(v)$ for all $uv \in E(G)$. If for all i,j and each vertex the number of vertices labeled with i and the number of vertices labeled with j differ by at most one and the edge labels induced by f^* are $\{1, 2, 3, ...\}$. Lee, Liu and Tan [3] considered a new vertex labeling of a graph G as a mapping f from $V(G)$ into the set $\{0, 1\}$ called partial labeling. For each vertex labeling f of G, a partial edge labeling f^* of G is defined in the following way. For each edge uv in G, $f^*(uv) = 0$ if $f(u) = f(v) = 0$, $f^*(uv) = 1$, if $f(u) = f(v) = 1$. Note that if $f(u) \neq f(v)$ then the edge uv is not labeled by f^*. Thus f^* is a partial function from $E(G)$ into the set $\{0, 1\}$. Let $v_f(0)$ and $v_f(1)$ denote the number of vertices of G that are labeled by 0 and 1 under the mapping f respectively. Likewise, let $e_f(0)$ and $e_f(1)$ denote the number of edges of G that are labeled by 0 and 1 under the induced partial function f^* respectively. Pradeep and Devadas [5] considered a labeling $f : V(G) \to A = \{0, 1\}$ that induces a partial edge labeling $f^* : E(G) \to A = \{0, 1\}$ defined by $f^*(xy) = x$, if and only if $f(x) = f(y)$ for each edge $xy \in E(G)$. For $i \in A$, let $v_f(i) = |\{v \in V(G) : f(v) = i\}|$ & $e_{f^*}(i) = |\{e \in E(G) : f^*(e) = i\}|$. A labeling f of a graph G is said to be friendly if $|v_f(0) - v_f(1)| \leq 1$. A friendly labeling is called balanced if $|e_{f^*}(0) - e_{f^*}(1)| \leq 1$. The balanced index set of the graph $G, BI(G)$ is defined as $|e_{f^*}(0) - e_{f^*}(1)|$, if the vertex labeling f is friendly.

We introduced a new labeling called balanced rank distribution labeling which is defined as follows: Let G be a simple undirected connected graph of order $n \geq 2$. If k is a positive integer such that $k \geq n$, then we define an injective function $f : V(G) \to \{1, 2, ..., k\}$ and an onto function $\phi : E(G) \to B \subset N$ by $\phi(uv) = \left\lceil \frac{f(u)}{d(u)} + \frac{f(v)}{d(v)} \right\rceil$ or $\left\lfloor \frac{f(u)}{d(u)} + \frac{f(v)}{d(v)} \right\rfloor$ and for any two edges uv and wx of G, $\phi(uv) = \phi(wx)$ only if $\left| \left(\frac{f(u)}{d(u)} + \frac{f(v)}{d(v)} \right) - \left(\frac{f(w)}{d(w)} + \frac{f(x)}{d(x)} \right) \right| < 1$. Here, $B = \{i | \min \phi(E) \leq i \leq \max \phi(E)\}$, we have many B's for a given graph G and we choose the one with minimum cardinality. Such a set B is denoted by $\mathbf{er}(G)$ and the elements of $\mathbf{er}(G)$ are called the edge ranks of G and f is known as the balanced rank distribution labeling of G. Note that $|\mathbf{er}(G)| \leq \delta(G) \leq \Delta(G)$ and if

(i) $|\mathbf{er}(G)| = 1$, then G is said to be a *Strongly balanced rank distribution* graph

(ii) $\delta(G) < |\mathbf{er}(G)| \leq \Delta(G)$, then G is said to be a *Weakly balanced rank distribution* graph

(iii) $|\mathbf{er}(G)| > \Delta(G)$, then G is said to be a non-balanced rank distribution graph.

The balanced rank distribution number of a graph G denoted by $\mathbf{brd}(G)$ is the minimum k such that $f : V(G) \to \{1, 2, ..., k\}$ is a balanced rank distribution labeling of G where the induced edge label set B has minimum cardinality.

Many electrical and communication networks have the graph structures like paths, cycles, ladder graphs, complete graphs and complete bipartite graphs. There are many problems in interconnection networks that can be addressed through this balanced rank distribution labeling in which balancing load/data transfer through links is required. Thus the study of $\mathbf{brd}(G)$ of such graphs G gain importance in this context. In this paper, the balanced rank distribution labeling of ladder graphs, complete graphs and complete bipartite graphs have been investigated.

2. Balanced Rank Distribution Labeling of Ladder Graphs

Theorem 2.1. *Let L_p be a ladder graph of order $n = 2p$ and $k \geq n$ be the given positive integer. Then*

(a) *L_p is weakly balanced rank graph, if*
 (i) $n = 8$ & $k = 8$
 (ii) $n = 10$ to 14 & $2n\text{-}10 \leq k \leq 2n\text{-}8$
 (iii) $n = 16$ & $22 \leq k \leq 26$
 (iv) $n \geq 18$ & $3n\text{-}27 \leq k \leq 3n\text{-}22$

(b) *L_p is a balanced rank graph, if*
 (i) $n = 6$ & $k = 6$
 (ii) $n = 8$ & $9 \leq k \leq 11$
 (iii) $n = 10$ & $13 \leq k \leq 15$
 (iv) $n{=}12$ & $17 \leq k \leq 20$
 (v) $n \geq 14$ & $3n\text{-}21 \leq k \leq 3n\text{-}16$

(c) *L_p is a strongly balanced rank graph, if*
 (i) $n = 6$ & $k \geq 7$
 (ii) $n = 8$ & $k \geq 12$
 (iii) $n = 10$ & $k \geq 16$
 (iv) $n \geq 12$ & $k \geq 3n\text{-}15$

Proof. For a given k, let $A = \{f(v_i)\}$ for given $i \in \{1, 2, 3, ..., n\}$ be an ordered set of vertex labeling of G and $\mathbf{er}(G) = \{\phi(e) | e \in E(G)\}$ denotes the edge labeling of G which results from the definiton of $\mathbf{brd}(G)$.

Now, we have two cases accordingly $n = 6$ and $n \geq 8$.

Case(i): $n = 6$. Then $k \geq 6$. According to the value of k, the vertex labeling and the corresponding edge ranks of L_3 are given below:

k	A	$\mathbf{er}(G)$
6	$\{1, 6, 2, 3, 5, 4\}$ or $\{1, 6, 2, 4, 3, 5\}$	$\{3, 4\}$
7 to 10	$\{k-3, k-4, k-2, k-6, k, k-5\}$	$\{k-4\}$
11	$\{5, 11, 6, 7, 9, 8\}$	$\{7\}$
≥ 11, $k \equiv 0 \pmod 3$	$\{\frac{2k-9}{3}+1, k, \frac{2k-9}{3}+2, \frac{2k-9}{3}+3, k-2, \frac{2k-9}{3}+4\}$	$\{\frac{2k-3}{3}\}$
≥ 11, $k \equiv 1 \pmod 3$	$\{\frac{2k-8}{3}+1, k, \frac{2k-8}{3}+2, \frac{2k-8}{3}+3, k-2, \frac{2k-8}{3}+4\}$	$\{\frac{2k-2}{3}\}$
≥ 11, $k \equiv 2 \pmod 3$	$\{\frac{2k-10}{3}+1, k, \frac{2k-10}{3}+2, \frac{2k-10}{3}+3, k-2, \frac{2k-10}{3}+4\}$	$\{\frac{2k-1}{3}\}$

Table 2.1.1

Thus, L_3 is a graph with
(i) balanced rank distribution if $k = 6$, since $|\mathbf{er}(G)| = 2$ and
(ii) strongly balanced rank distribution if $k \geq 7$, since $|\mathbf{er}(G)| = 1$
Therefore, $\mathbf{brd}(L_3) = 7$.

Case(ii): Let $n \geq 8$. We define two partial sets of the edge rank set $\mathbf{er}(G)$, namely B_1 and B_2 such that $\mathbf{er}(G) = B_1 \cup B_2$ as
$B_1 = \{\phi(e)\}$ for every $e = v_i v_j$ where both v_i and v_j are of degree 3 and
$B_2 = \{\phi(v_1 v_2), \phi(v_{\frac{n}{2}-1} v_{\frac{n}{2}}), \phi(v_{\frac{n}{2}} v_{\frac{n}{2}+1}), \phi(v_{\frac{n}{2}+1} v_{\frac{n}{2}+2}), \phi(v_{n-1} v_n), \phi(v_n v_1)\}$.

For all values of $k \geq 8$, the vertex labeling to the vertices $v_2, v_3, ..., v_{\frac{n}{2}-1}$ & $v_{\frac{n}{2}+2}, v_{\frac{n}{2}+3}, v_{n-1}$ of degree 3 are as follows :
If $n \equiv 0(mod\ 4)$,

$$f(v_i) = \begin{cases} k-n+3+i, & for\ i = 2,4,6,...,\frac{n}{2}-2 \\ k+2-i, & for\ i = 3,5,7,...,\frac{n}{2}-1 \\ k-n+1+i, & for\ i = \frac{n}{2}+3, \frac{n}{2}+5, ..., n-1 \\ k+4-i, & for\ i = \frac{n}{2}+2, \frac{n}{2}+4, ..., n-2 \end{cases}$$

If $n \equiv 2(mod\ 4)$,

$$f(v_i) = \begin{cases} k-n+3+i, & for\ i = 2,4,6,...,\frac{n}{2}-1 \\ k+2-i, & for\ i = 3,5,7,...,\frac{n}{2}-2 \\ k-n+1+i, & for\ i = \frac{n}{2}+2, \frac{n}{2}+4, ..., n-1 \\ k+4-i, & for\ i = \frac{n}{2}+3, \frac{n}{2}+5, ..., n-2 \end{cases}$$

The respective induced edge labeling will be balanced by taking $\lceil \phi(e) \rceil$ or $\lfloor \phi(e) \rfloor$ for all $e \in E(G)$ as given below:

$\phi(v_i v_{n+1-i}) = \lceil \frac{2k-n+5}{3} \rceil$ or $\lfloor \frac{2k-n+5}{3} \rfloor$, for $i = 2, 3, 4, ..., \frac{n}{2} - 1$ and

$\phi(v_i v_{i+1}) = \lceil \frac{2k-n+4}{3} \rceil$ or $\lfloor \frac{2k-n+4}{3} \rfloor$,
for $i = 2, 4, 6, ..., \frac{n}{2} - 3$ & $i = \frac{n}{2} + 2, \frac{n}{2} + 4, ..., n - 3$ if $n \equiv 2(mod\ 4)$ and
for $i = 2, 4, 6, ..., \frac{n}{2} - 1$ & $i = \frac{n}{2} + 3, \frac{n}{2} + 5, ..., n - 3$ if $n \equiv 0(mod\ 4)$

Further, $\phi(v_i v_{i+1}) = \lceil \frac{2k-n+6}{3} \rceil$ or $\lfloor \frac{2k-n+6}{3} \rfloor$,
for $i = 3, 5, 7, ..., \frac{n}{2} - 2$ & $i = \frac{n}{2} + 3, \frac{n}{2} + 5, ..., n - 2$ if $n \equiv 2(mod\ 4)$ and
for $i = 3, 5, 7, ..., \frac{n}{2} - 3$ & $i = \frac{n}{2} + 2, \frac{n}{2} + 4, ..., n - 2$ if $n \equiv 0(mod\ 4)$
Thus,

$$B_1 = \begin{cases} \{\frac{2k-n+4}{3}\}, \text{ if } 2k-n+4 \equiv 0 \text{ (mod 3)} \\ \{\frac{2k-n+5}{3}\}, \text{ if } 2k-n+5 \equiv 0 \text{ (mod 3)} \\ \{\frac{2k-n+6}{3}\}, \text{ if } 2k-n+6 \equiv 0 \text{ (mod 3)} \end{cases} \quad (1)$$

for all the values of n & $k \geq 8$.
Next we label the vertices of degree 2 namely $v_1, v_{\frac{n}{2}}, v_{\frac{n}{2}+1}$ and v_n in $L_{n/2}$ according to specific values of n & k.
Subcase(i): $n = 8$. Here the vertices of degree 2 are v_1, v_4, v_5 and v_8. For the values of $k \geq 8$, the labelings of these vertices and the resulting partial edge rank set B_2 are given in Table 2.1.2. Thus, from Table 2.1.2 and Equation (1), L_4 is a graph with

k	$f(v_1)$	$f(v_4)$	$f(v_5)$	$f(v_8)$	B_2
8 9,10,11 12 13,14 15	$k-7$	$k-5$	$k-6$	$k-4$	$\{3,4,5\}$ $\{k-4, k-5\}$ $\{7\}$ $\{8\}$ $\{9\}$
≥ 16, $k \equiv 0$ (mod 3)	$\frac{2k-9}{3}+4$	$\frac{2k-9}{3}+2$	$\frac{2k-9}{3}+3$	$\frac{2k-9}{3}+1$	$\{\frac{2k-3}{3}\}$
≥ 16, $k \equiv 1$ (mod 3)	$\frac{2k-11}{3}+4$	$\frac{2k-11}{3}+2$	$\frac{2k-11}{3}+3$	$\frac{2k-11}{3}+1$	$\{\frac{2k-2}{3}\}$
≥ 16, $k \equiv 2$ (mod 3)	$\frac{2k-10}{3}+4$	$\frac{2k-10}{3}+2$	$\frac{2k-10}{3}+3$	$\frac{2k-10}{3}+1$	$\{\frac{2k-4}{3}\}$

Table 2.1.2

(i) weakly balanced rank distribution if $k = 8$, since $|\mathbf{er}(G)| = 3$,

(ii) balanced rank distribution if $k = 9, 10, 11$, since $|\mathbf{er}(G)| = 2$ and
(iii) strongly balanced rank distribution if $k \geq 12$, since $|\mathbf{er}(G)| = 1$.
Therefore, $\mathbf{brd}(L_4) = 12$.

Subcase(ii): $n = 10$. Here the vertices of degree 2 are v_1, v_5, v_6 and v_{10}. For the values of $k \geq 10$, the labelings of these vertices and the resulting partial edge rank set B_2 can be found in Table 2.1.3. The single edge rank x in Table 2.1.3 is given by

k	$f(v_1)$	$f(v_5)$	$f(v_6)$	$f(v_{10})$	B_2
10,11,12 13,14,15	$k-6$	$k-7$	$k-8$	$k-9$	$\{k-7, k-6, k-5\}$ $\{k-7, k-6\}$
≥ 16	$[\frac{2k}{3}]$	$[\frac{2k}{3}]-1$	$[\frac{2k}{3}]-2$	$[\frac{2k}{3}]-3$	$\{x\}$
≥ 19, $k \equiv 0 \pmod 3$	$[\frac{2k}{3}]+1$	$[\frac{2k}{3}]-1$	$[\frac{2k}{3}]-2$	$[\frac{2k}{3}]-4$	$\{x\}$
≥ 19, $k \equiv 1$ or $2 \pmod 3$	$[\frac{2k}{3}]+1$	$[\frac{2k}{3}]$	$[\frac{2k}{3}]-2$	$[\frac{2k}{3}]-3$	$\{x\}$

Table 2.1.3

$$x = \begin{cases} \left[\frac{2k-4}{3}\right], & \text{if } 2k-4 \equiv 0 \pmod 3 \\ \left[\frac{2k-5}{3}\right], & \text{if } 2k-5 \equiv 0 \pmod 3 \\ \left[\frac{2k-6}{3}\right], & \text{if } 2k-6 \equiv 0 \pmod 3 \end{cases}$$

Thus, from Table 2.1.3 and Equation (1), L_5 is a graph with
(i) weakly balanced rank distribution if $k = 10, 11, 12$, since $|\mathbf{er}(G)| = 3$
(ii) balanced rank distribution if $k = 13, 14, 15$, since $|\mathbf{er}(G)| = 2$ and
(iii) strongly balanced rank distribution if $k \geq 16$, since $|\mathbf{er}(G)| = 1$.
Therefore, $\mathbf{brd}(L_5) = 16$.

Subcase(iii): $n = 12$ and $12 \leq k \leq 21$. Here the vertices of degree 2 are v_1, v_6, v_7 and v_{12}. The labelings of these vertices and the resulting partial edge rank set B_2 are tabulated below:

k	$f(v_1)$	$f(v_6)$	$f(v_7)$	$f(v_{12})$	B_2
12,13 14 to 16 17 to 20 21	$k-8$	$k-10$	$k-9$	$k-11$	$\{k-9, k-8, k-7, k-6, k-5\}$ $\{k-9, k-8, k-7\}$ $\{k-9, k-8\}$ $\{12\}$

Table 2.1.4

Thus, from Table 2.1.4 and Equation (1), L_6 is a graph with
(i) weakly balanced rank distribution if $k = 14, 15, 16$, since $|\mathbf{er}(G)| = 3$
(ii) balanced rank distribution if $k = 17, 18, 19, 20$, since $|\mathbf{er}(G)| = 2$ and
(iii) strongly balanced rank distribution if $k = 21$, since $|\mathbf{er}(G)| = 1$.
Therefore, $\mathbf{brd}(L_6) = 21$.

Subcase(iv): $n = 14$ and $14 \leq k \leq 27$. Here the vertices of degree 2 are v_1, v_7, v_8 and v_{14}. The labelings of these vertices and the resulting partial edge rank set B_2 are given in Table 2.1.5.

Thus, from Table 2.1.5 and Equation (1), L_7 is a graph with
(i) weakly balanced rank distribution if $k =$ 18,19,20, since $|\mathbf{er}(G)| = 3$
(ii) balanced rank distribution if $k = 21, 22, 23, 24, 25, 26$, since $|\mathbf{er}(G)| = 2$ and
(iii) strongly balanced rank distribution if $k = 27$, since $|\mathbf{er}(G)| = 1$
Therefore, $\mathbf{brd}(L_7) = 27$.

Subcase(v): $n = 16$ and $22 \leq k \leq 32$. Here the vertices of degree 2 are v_1, v_8, v_9 and v_{16}. The labelings of these vertices and the resulting partial edge rank set B_2 are tabulated in Table 2.1.6.

k	$f(v_1)$	$f(v_7)$	$f(v_8)$	$f(v_{14})$	B_2
14,15					$\{k-12, k-11, k-10, k-9, k-8\}$
16,17	$k-10$	$k-11$	$k-12$	$k-13$	$\{k-11, k-10, k-9, k-8\}$
18,19,20					$\{k-11, k-10, k-9\}$
21					$\left\{\frac{2k-9}{3}, \frac{2k-9}{3}-1\right\}$
					$\left\{\frac{2k-9}{3}, \frac{2k-9}{3}-1\right\}$ for $k \equiv 0 \pmod 3$
22 to 26	$k-10$	$k-11$	$k-12$	$\left\lceil\frac{2(k-9)}{3}\right\rceil+1$	$\left\{\frac{2k-8}{3}, \frac{2k-7}{3}-1\right\}$ for $k \equiv 1 \pmod 3$
					$\left\{\frac{2k-7}{3}, \frac{2k-9}{3}-1\right\}$ for $k \equiv 2 \pmod 3$
27	17	16	15	13	$\{18\}$

Table 2.1.5

k	$f(v_1)$	$f(v_8)$	$f(v_9)$	$f(v_{16})$	B_2
22 to 26					
$\frac{2k-10}{3} \equiv 0(mod\ 3)$					$\left\{\frac{2k-10}{3}, \frac{2k-13}{3}, \frac{2k-16}{3}\right\}$
$\frac{2k-11}{3} \equiv 0(mod\ 3)$	$k-12$	$k-13$	$k-14$	$k-15$	$\left\{\frac{2k-11}{3}, \frac{2k-14}{3}, \frac{2k-17}{3}\right\}$
$\frac{2k-12}{3} \equiv 0(mod\ 3)$					$\left\{\frac{2k-12}{3}, \frac{2k-15}{3}, \frac{2k-18}{3}\right\}$
27 to 32					
$\frac{2k-10}{3} \equiv 0(mod\ 3)$					$\left\{\frac{2k-10}{3}, \frac{2k-13}{3}\right\}$
$\frac{2k-11}{3} \equiv 0(mod\ 3)$	$k-12$	$k-13$	$k-14$	$\left\lceil\frac{2(k-11)}{3}\right\rceil+1$	$\left\{\frac{2k-11}{3}, \frac{2k-14}{3}\right\}$
$\frac{2k-12}{3} \equiv 0(mod\ 3)$					$\left\{\frac{2k-12}{3}, \frac{2k-15}{3}\right\}$

Table 2.1.6

Thus, from Table 2.1.6 and Equation (1), L_8 is a graph with
(i) weakly balanced rank distribution if $k =$ 22,23,24,25,26, since $|\mathbf{er}(G)| = 3$ and
(ii) balanced rank distribution if $k =$ 27,28,29,30,31,32, since $|\mathbf{er}(G)| = 2$.

Subcase(vi): $n \geq 18$ and $k = 3n-26$ to $3n-16$.
The vertices of degree 2 are $v_1, v_{\frac{n}{2}}, v_{\frac{n}{2}+1}$ and v_n. The labeling of these vertices and the resulting partial edge rank set B_2 are given in Table 2.1.7.

k	$f(v_1)$	$f(v_{\frac{n}{2}})$	$f(v_{\frac{n}{2}+1})$	$f(v_n)$	B_2
$3n-26$ to $3n-22$					
$n \equiv 2(mod\ 4)$	$k-n+4$	$k-n+3$	$k-n+2$	$\left\lceil\frac{2k-2n+10}{3}\right\rceil$	X
	$k-n+4$	$\left\lceil\frac{2k-n+6}{3}\right\rceil$	$\left\lceil\frac{2k-n+6}{3}\right\rceil-1$	$\left\lceil\frac{2k-2n+10}{3}\right\rceil+1$	
$3n-26$ to $3n-22$					
$n \equiv 0(mod\ 4)$	$k-n+4$	$k-n+2$	$k-n+3$	$\left\lceil\frac{2k-2n+10}{3}\right\rceil$	X
	$k-n+4$	$\left\lceil\frac{2k-n+6}{3}\right\rceil-1$	$\left\lceil\frac{2k-n+6}{3}\right\rceil$	$\left\lceil\frac{2k-2n+10}{3}\right\rceil+1$	
$3n-21$ to $3n-16$					
$n \equiv 2(mod\ 4)$	$k-n+4$	$\left\lceil\frac{2k-n+6}{3}\right\rceil$	$\left\lceil\frac{2k-n+6}{3}\right\rceil-1$	$\left\lceil\frac{2k-2n+10}{3}\right\rceil$	Y
$n \equiv 0(mod\ 4)$	$k-n+4$	$\left\lceil\frac{2k-n+6}{3}\right\rceil-1$	$\left\lceil\frac{2k-n+6}{3}\right\rceil$	$\left\lceil\frac{2k-2n+10}{3}\right\rceil$	
$3n-21$ to $3n-16$					
$n \geq 20$, $n \equiv 2(mod\ 4)$	$k-n+4$	$\left\lceil\frac{2k-n+6}{3}\right\rceil$	$\left\lceil\frac{2k-n+6}{3}\right\rceil-1$	$\left\lceil\frac{2k-2n+10}{3}\right\rceil$	Y
$n \geq 20$, $n \equiv 0(mod\ 4)$	$k-n+4$	$\left\lceil\frac{2k-n+6}{3}\right\rceil-1$	$\left\lceil\frac{2k-n+6}{3}\right\rceil$	$\left\lceil\frac{2k-2n+10}{3}\right\rceil$	

Table 2.1.7

The sets X and Y in Table 2.1.7 are given by

$$X = \begin{cases} \left\{\frac{2k-n+6}{3}, \frac{2k-n+3}{3}, \frac{2k-n}{3}\right\}, & \text{if } \frac{2k-n+6}{3} \equiv 0 \pmod 3 \\ \left\{\frac{2k-n+5}{3}, \frac{2k-n+2}{3}, \frac{2k-n-1}{3}\right\}, & \text{if } \frac{2k-n+5}{3} \equiv 0 \pmod 3 \\ \left\{\frac{2k-n+4}{3}, \frac{2k-n+1}{3}, \frac{2k-n-2}{3}\right\}, & \text{if } \frac{2k-n+4}{3} \equiv 0 \pmod 3 \end{cases}$$

$$Y = \begin{cases} \left\{\frac{2k-n+6}{3}, \frac{2k-n+3}{3}\right\}, & \text{if } \frac{2k-n+6}{3} \equiv 0 \pmod 3 \\ \left\{\frac{2k-n+5}{3}, \frac{2k-n+2}{3}\right\}, & \text{if } \frac{2k-n+5}{3} \equiv 0 \pmod 3 \\ \left\{\frac{2k-n+4}{3}, \frac{2k-n+1}{3}\right\}, & \text{if } \frac{2k-n+4}{3} \equiv 0 \pmod 3 \end{cases}$$

Thus, from Table 2.1.7 and Equation (1), for $n \geq 18$, $L_{\frac{n}{2}}$ is a graph with
(i) weakly balanced rank distribution if $3n-26 \leq k \leq 3n-22$ and
(ii) balanced rank distribution if $3n-21 \leq k \leq 3n-16$. **Subcase(vii):** $n \geq 12$ and $k \geq 3n-15$.
The labelings of the vertices of degree 2 and the resulting partial edge rank set B_2 are given in Table 2.1.8.

k	$f(v_1)$	$f(v_{\frac{n}{2}})$	$f(v_{\frac{n}{2}+1})$	$f(v_n)$	B_2
$k = 3n-15$ & $n \equiv 0(mod\ 4)$	$2n-11$ $2n-11$	$\left\lceil\frac{2k-n+6}{3}\right\rceil - 1$ $\left\lceil\frac{2k-n+6}{3}\right\rceil - 1$	$\left\lceil\frac{2k-n+6}{3}\right\rceil$ $\left\lceil\frac{2k-n+6}{3}\right\rceil + 1$	$\left\lceil\frac{2k-2n+10}{3}\right\rceil$ $\left\lceil\frac{2k-2n+10}{3}\right\rceil$	B_1
$k = 3n-15$ & $n \equiv 2(mod\ 4)$	$2n-11$ $2n-11$	$\left\lceil\frac{2k-n+6}{3}\right\rceil$ $\left\lceil\frac{2k-n+6}{3}\right\rceil + 1$	$\left\lceil\frac{2k-n+6}{3}\right\rceil - 1$ $\left\lceil\frac{2k-n+6}{3}\right\rceil - 1$	$\left\lceil\frac{2k-2n+10}{3}\right\rceil$ $\left\lceil\frac{2k-2n+10}{3}\right\rceil + 1$	B_1
$k > 3n-15$ & $n \equiv 0(mod\ 4)$	$\left\lfloor\frac{2k}{3}\right\rfloor$ $\left\lceil\frac{2k}{3}\right\rceil$	$\left\lceil\frac{2k-n+6}{3}\right\rceil - 1$ $\left\lceil\frac{2k-n+6}{3}\right\rceil - 1$	$\left\lceil\frac{2k-n+6}{3}\right\rceil$ $\left\lceil\frac{2k-n+6}{3}\right\rceil + 1$	$\left\lceil\frac{2k-2n+10}{3}\right\rceil$ $\left\lceil\frac{2k-2n+10}{3}\right\rceil$	B_1
$k > 3n-15$ & $n \equiv 2(mod\ 4)$	$\left\lfloor\frac{2k}{3}\right\rfloor$ $\left\lceil\frac{2k}{3}\right\rceil$	$\left\lceil\frac{2k-n+6}{3}\right\rceil$ $\left\lceil\frac{2k-n+6}{3}\right\rceil + 1$	$\left\lceil\frac{2k-n+6}{3}\right\rceil - 1$ $\left\lceil\frac{2k-n+6}{3}\right\rceil - 1$	$\left\lceil\frac{2k-2n+10}{3}\right\rceil$ $\left\lceil\frac{2k-2n+10}{3}\right\rceil + 1$	B_1

Table 2.1.8

Thus, from Table 2.1.8 and Equation (1), $L_{\frac{n}{2}}$ has a strongly balanced rank distribution for $n \geq 12$ and $k \geq 3n-15$. Therefore, $\mathbf{brd}(L_{\frac{n}{2}}) = 3n-15$ for $n \geq 12$.

□

Example 2.1. Take $p = 8, n = 16 \equiv 0(mod\ 4)$. Then the balanced rank distribution labeling of the vertices with degree 3 are as follows:

$$f(v_i) = \begin{cases} k-13+i, & for\ \text{ i = 2, 4, 6} \\ k+2-i, & for\ \text{i = 3, 5, 7} \\ k-15+i, & for\ \text{i = 11, 13, 15} \\ k+4-i, & for\ \text{i = 10, 12, 13} \end{cases} \tag{2}$$

Case(i): Let $k = 25$
Label the even degree vertices by $f(v_1) = 13, f(v_8) = 11, f(v_9) = 12, f(v_{16}) = 10$ and labeling of vertices of degree 3 follows from Equation (2).
Then $\mathbf{er}(G) = \{11, 12, 13\}$ which yields a weakly balanced rank distribution labeling of L_8.

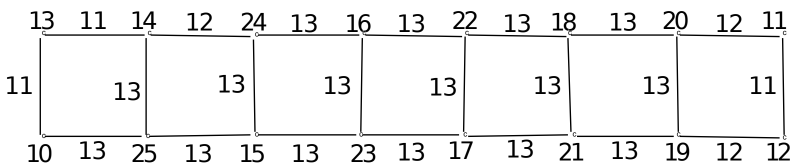

Figure 2.1 : Weakly balanced rank distribution labeling of L_8

Case(ii): Let $k = 30$
Label the even degree vertices by $f(v_1) = 18, f(v_8) = 17, f(v_9) = 16, f(v_{16}) = 13$ and labeling of vertices of degree 3 follows from Equation (2).
Then $\mathbf{er}(G) = \{15, 16\}$ which yields a balanced rank distribution labeling of L_8.

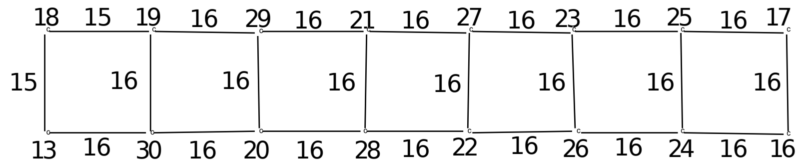

Figure 2.2 : Balanced rank distribution labeling of L_8

Case(iii): Let $k = 33$

Label the even degree vertices by $f(v_1) = 21, f(v_8) = 18, f(v_9) = 19, f(v_{16}) = 15$ and labeling of vertices of degree 3 follows from Equation (2).

Then $\mathbf{er}(G) = \{18\}$ which yields a strongly balanced rank distribution labeling of L_8.

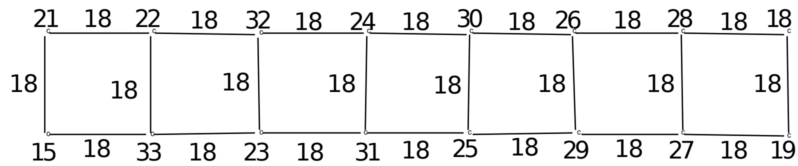

Figure 2.3 : Strongly balanced rank distribution labeling of L_8

From Figure 2.1, 2.2 and 2.3 it is clear that, $\mathbf{brd}(L_8) = 33$.

3. Balanced Rank Distribution Labeling of Complete Graphs

Theorem 3.1. *Every complete graph K_n of order $n \geq 4$ has a balanced rank distribution labeling with balanced rank dristribution number $\boldsymbol{brd}(K_n) = n$.*

Proof. Let K_n be a complete graph of order n. Name the vertices as $v_1, v_2, ..., v_n$ where each vertex is of degree $n-1$. For any given $k \geq n$, define the vertex labeling $f(v_i) = k - i + 1$, $i = 1, 2, ..., n$. For any $i, j \in \{1, 2, ..., n\}$, $\phi(v_i v_j) = \left\lceil \frac{2k+2-i-j}{n-1} \right\rceil$ or $\left\lfloor \frac{2k+2-i-j}{n-1} \right\rfloor$. Here v_1, v_2 are the vertices with highest ranks and v_{n-1}, v_n are the vertices with lowest ranks.

$\phi(v_1 v_2) = \left\lceil \frac{2k-1}{n-1} \right\rceil$ or $\left\lfloor \frac{2k-1}{n-1} \right\rfloor$, $\phi(v_{n-1} v_n) = \left\lceil \frac{2k-2n+3}{n-1} \right\rceil$ or $\left\lfloor \frac{2k-2n+3}{n-1} \right\rfloor$.

$|\phi(v_1 v_2) - \phi(v_{n-1} v_n)| = \left\lceil \frac{2n-4}{n-1} \right\rceil$ or $\left\lfloor \frac{2n-4}{n-1} \right\rfloor$.

For any $n \geq 4$, $\frac{2(n-2)}{n-1}$ is a value lies between 1 and 2.

Therefore, for $i, j, s, t \in \{1, 2, ...n\}, 1 < |\phi(v_i v_j) - \phi(v_s v_t)| < 2$.

Hence, $|\mathbf{er}(K_n)| = 2 < \Delta(K_n)$ for any $k \geq 4$. Therefore, any K_n with $n \geq 4$ has a balanced rank distribution labeling with $\mathbf{er}(K_n) = \left\{ \left\lceil \frac{2k-1}{n-1} \right\rceil, \left\lceil \frac{2k-2n+3}{n-1} \right\rceil \right\}$ or $\left\{ \left\lfloor \frac{2k-1}{n-1} \right\rfloor, \left\lceil \frac{2k-2n+3}{n-1} \right\rceil \right\}$ and the least value of k for which the balanced rank distribution labeling exists is n.

i.e., $\mathbf{brd}(K_n) = n$. □

Example 3.1 When $n = 5$, the balanced rank distribution labeling of K_5 is as follows:

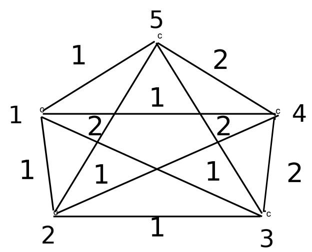

Figure 3.1 : Balanced rank distribution labeling of K_5

From Figure 3.1 it is clear that, $\mathbf{brd}(K_5) = 5$.

4. Balanced Rank Distribution Labeling of Complete Bipartite graphs

Theorem 4.1. *Any complete bipartite graph $K_{p,p}(p \geq 3)$ has a balanced rank distribution labeling with balanced rank distribution number $brd(K_{p,p}) = 2p$.*

Proof. Let $K_{p,p}$ be a complete bipartite graph with vertex partitions U and V of same cardinality p. Assign the vertex labeling f in ascending order to the vertices $u_1, u_2, u_3, ..., u_p$ of U and to the vertices $v_1, v_2, v_3, ..., v_p$ of V by taking $f(u_i) = k - i + 1, i = 1, 2, ..., p$ and $f(v_j) = k - p - j + 1, j = 1, 2, ..., p$.
Now $|\phi(u_1v_1) - \phi(u_pv_p)| = |\left(\frac{k}{p} + \frac{k-p}{p}\right) - \left(\frac{k-p+1}{p} + \frac{k-2p+1}{p}\right)| = |2 - \frac{2}{p}| < 2$ for any finite positive integer p.

Therefore, $1 < |\phi(u_1v_1) - \phi(u_pv_p)| < 2$ and $\mathbf{er}(K_{p,p}) = \left\{\left\lceil\frac{2k-p}{n-1}\right\rceil, \left\lceil\frac{2k-3p+2}{n-1}\right\rceil\right\}$ or $\left\{\left\lfloor\frac{2k-p}{n-1}\right\rfloor, \left\lceil\frac{2k-3p+2}{n-1}\right\rceil\right\}$. Thus, $|\mathbf{er}(K_{p,p})| = 2$. Therefore, for $p \geq 3$, $K_{p,p}$ has balanced rank distribution labeling and $\mathbf{brd}(K_{p,p}) = 2p$. □

Example 4.1

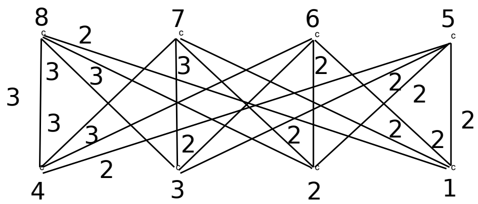

Figure 4.1 : Balanced rank distribution labeling $K_{4,4}$

From Figure 4.1 it is clear that, $\mathbf{brd}(K_{4,4}) = 8$.

5. Conclusion

In this paper the balanced rank distribution number has been computed for the ladder graphs $L_{n/2}$ for even $n \geq 6$, complete graphs K_n for $n \geq 3$, complete bipartite graphs $K_{n/2,n/2}$ for even $n \geq 4$ and obtained the following results:

(i) $\mathbf{brd}(L_{n/2}) = 3n - 15$, for even $n \geq 12$
(ii) $\mathbf{brd}(K_n) = n$, for $n \geq 3$
(iii) $\mathbf{brd}(K_{n/2,n/2}) = n$, for even $n \geq 4$

Existence of balanced rank distribution labeling of ladder graphs and complete graphs are completely settled. Further, the balanced rank distribution labeling of $K_{p,q}$ where $p \neq q$ and $p + q = n$ is yet to be investigated.

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