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# THE MINIMUM MEAN MONOPOLY ENERGY OF A GRAPH

M. V. CHAKRADHARA RAO[1], K. A. VENKATESH[2], D. V. LAKSHMI[3], §

Abstract. The motivation for the study of the graph energy comes from chemistry, where the research on the so-called total $\pi$ - electron energy can be traced back until the 1930s. This graph invariant is very closely connected to a chemical quantity known as the total $\pi$ - electron energy of conjugated hydro carbon molecules. In recent times analogous energies are being considered, based on Eigen values of a variety of other graph matrices. In 1978, I.Gutman [1] defined energy mathematically for all graphs. Energy of graphs has many mathematical properties which are being investigated. The ordinary energy of an undirected simple finite graph G is defined as the sum of the absolute values of the Eigen values of its associated matrix. i.e. if $\mu_1, \mu_2, ..., \mu_n$ are the Eigen values of adjacency matrix A(G), then energy of graph is E(G) $= \sum_{i=1}^{n} |\mu_i|$. Laura Buggy, Amalia Culiuc, Katelyn Mccall and Duyguyen [9] introduced the more general M-energy or Mean Energy of G is then defined as $E^M$ (G) $= \sum_{i=1}^{n} |\mu_i - \bar{\mu}|$, where $\bar{\mu}$ is the average of $\mu_1, \mu_2, ..., \mu_n$.
A subset M $\subseteq$ V (G), in a graph G (V, E), is called a monopoly set of G if every vertex v $\in$ (V - M) has at least $\frac{d(v)}{2}$ neighbors in M. The minimum cardinality of a monopoly set among all monopoly sets in G is called the monopoly size of G, denoted by mo(G).Ahmed Mohammed Naji and N.D.Soner [7] introduced minimum monopoly energy $E_{MM}$ [G] of a graph G. In this paper we are introducing the minimum mean monopoly energy, denoted by $E_{MM}^M$ (G), of a graph G and computed minimum monopoly energies of some standard graphs. Upper and lower bounds for $E_{MM}^M$ (G)are also established.

Keywords: Monopoly Set, Monopoly Size, Minimum Monopoly Matrix, Minimum Monopoly Eigenvalues, Minimum Monopoly Energy and Minimum Mean Monopoly Energy of a graph
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## 1. Introduction

In this paper, a graph G(V, E) mean a simple non-empty undirected graph, having no loops and no multiple edges. Let n and m be the number of vertices and edges, respectively, of G. The degree of a vertex v in a graph G, denoted by d(v), is the number of edges incident on v. For any vertex v of a graph G, the open neighborhood of v is the set

[1] Department of Mathematics, School of Engineering, Presidency University, Bangalore, India.
e-mail: chakrimv@yahoo.com; ORCID: https://orcid.org/0000-0002-2340-8100.
[2] Department of Mathematics and Comp. Science, Myanmar Institute of Information Technology, Myanmar.
e-mail:prof.kavenkatesh@gmail.com; ORCID: https://orcid.org/0000-0002-1341-2659.
[3] Department of Mathematics, Bapatla Engineering College, Bapatla, India.
e-mail:himaja96@gmail.com; ORCID: https://orcid.org/0000-0003-2020-314X.
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N(v) = $\{u \in V :< u, v >\in E(G)\}$. For a subset S ⊆ V (G) the degree of a vertex v ∈ V (G) with respect to a subset S is $d_s$ (V) = | N(v) ∩ S |. For graph theoretic terminology we refer to Harary book [5]

A subset M of V in G = (V, E) is called a monopoly set if for every vertex v ∈ (V - M) has at least $\frac{d(v)}{2}$ neighbors in M, the monopoly size of a graph G, denoted by mo(G), is a minimum cardinality of a monopoly set in G.

In particular, monopolies are dynamic monopoly (dynamos) that, when colored black at a certain time step, will cause the entire graph to be colored black in the next time step under an irreversible majority conversion process.

The energy of a graph was introduced by I. Gutman [1], [6] in the year 1978. Let G be a graph with n vertices { $v_1, v_2, ..., v_n$ } and m edges. Let A = $(a_{ij})$ be the adjacency matrix of the graph. The Eigen values $m_1, m_2, ..., m_n$ of A, assumed in non-increasing order, are the Eigen values of the graph G.
The energy of graph is defined as E(G) = $\sum_{i=1}^{n} |m_i|$. The minimum mean energy of G is then defined, by Laura Buggy, Amaliailiuc, Katelyn Mccall, Duyguyen [9], as $E^M$(G) =$\sum_{i=1}^{n} |m_i - \bar{m}|$, where $\bar{m}$ is the mean of the Eigen values. To know further basic concepts about energy of graphs refer [2], [3], [4].

## 2. The Minimum Mean Monopoly Energy of Graphs

Let G be a graph of order n with vertex set V (G) = { $v_1, v_2, ..., v_n$ } and edge set E. Let M be a minimum monopoly set of G. The minimum monopoly matrix of G is the n × n matrix, denoted by $A_{MM}(G) = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 \quad if \ < v_i, v_j >\in E \\ 1 \quad if \ i = j, v_i \in M \\ 0 \qquad otherwise \end{cases}$$

Since $A_{MM}$ (G) is real and symmetric, its Eigen values $m_1, m_2, ..., m_n$ are real and positive. The Minimum Monopoly energy of graph G is defined as $E_{MM}$ (G) = $\sum_{i=1}^{n} |m_i|$, where $m_1, m_2, ..., m_n$ are Eigen values of minimum monopoly matrix $A_{MM}$ (G).The Minimum Mean Monopoly Energy of G is then defined as $E_{MM}^{M}$ (G) = $\sum_{i=1}^{n} |m_i - \bar{m}|$, where $\bar{m}$ is the mean of $m_1, m_2, ..., m_n$. For further basic concepts about mean energy refer [8], [9].

**Remark 2.1.** *The minimum monopoly energy of a graph G depends [7] on the choice of the minimum monopoly set. i.e. the minimum monopoly energy is not a graph invariant.*

## 3. Minimum Mean Monopoly Energy of Some Standard Graphs

**Theorem 3.1.** *For n ≥ 2, the minimum mean monopoly energy of complete graph $K_n$ is*

$$\begin{cases} (n-2) + \sqrt{n^2+1}, \quad if \ n \ is \ even \\ \\ \frac{(n-1)^2}{n} + \sqrt{n^2-1}, \quad if \ n \ is \ odd \end{cases}$$

*Proof.* $K_n$ is complete graph with vertex set V = $\{v_1, v_2, ..., v_n\}$. Then the minimum monopoly size is mo($K_n$ ) = $\lfloor \frac{n}{2} \rfloor = \begin{cases} \frac{n}{2}, \quad if \ n \ is \ even \\ \\ \frac{(n-1)}{2}, \quad if \ n \ is \ odd \end{cases}$ The minimum monopoly (MM) set is D = $\{v_1, v_2, ..., v_{\frac{n}{2}}\}$ , when n is even or

D $= \{v_1, v_2, ..., v_{\frac{n-1}{2}}\}$ , when n is odd.

**Case I:** When n is even
Then the MM matrix $A_{MM}(K_n) =$

$$\begin{bmatrix} 1 & 1 & 1 & .. & 1 & 1 & 1 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 1 & 1 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 1 & 1 & 1 & .. & 1 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 1 & 1 & 1 & .. & 1 & 1 & 1 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 0 & 1 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 1 & 0 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 1 & 1 & 0 & .. & 1 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 1 & 1 & 1 & .. & 1 & 1 & 1 & 1 & .. & 0 \end{bmatrix}_{n \times n}$$

Then the characteristic equation $|A_{MM}(K_n) - \mu I| = 0$ is

$\mu^{\frac{n-2}{2}} \ (\mu + 1)^{\frac{n-2}{2}} \ (\mu^2 - (n-1)\mu - \frac{n}{2}) = 0$

Then $\mu = 0$; $\frac{n-2}{2}$ times, $\mu =$ -1; $\frac{n-2}{2}$ times and $\mu = \frac{(n-1) \pm \sqrt{n^2+1}}{2}$ .

Mean $\bar{\mu} = \frac{1}{2}$ .

Minimum Monopoly Energy $= E_{MM}(K_n) = \frac{n-2}{2} + \sqrt{n^2+1}$ .

Minimum Mean Monopoly(MMM) Energy $= E^M_{MM}(K_n)$

$= \sum_{i=1}^{\frac{n-2}{2}} |0 - \frac{1}{2}| + \sum_{i=1}^{\frac{n-2}{2}} |-1 - \frac{1}{2}| + |\frac{(n-1) \pm \sqrt{n^2+1}}{2} - \frac{1}{2}|$

$= \frac{(n-2)}{4} + \frac{3(n-2)}{4} + |\frac{(n-2) \pm \sqrt{n^2+1}}{2}|$

$= \frac{(n-2)}{4} + \frac{3(n-2)}{4} + \sqrt{n^2+1}$

$=(n-2) + \sqrt{n^2+1}$ .
**Case II:** When n is odd

Then the MM matrix $A_{MM}(K_n) =$

$$\begin{bmatrix} 1 & 1 & 1 & .. & 1 & 1 & 1 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 1 & 1 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 1 & 1 & 1 & .. & 1 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 1 & 1 & 1 & .. & 1 & 1 & 1 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 0 & 1 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 1 & 0 & 1 & .. & 1 \\ 1 & 1 & 1 & .. & 1 & 1 & 1 & 0 & .. & 1 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 1 & 1 & 1 & .. & 1 & 1 & 1 & 1 & .. & 0 \end{bmatrix}_{n\times n}$$

Then the characteristic equation $|A_{MM}(K_n) - \mu I| = 0$ is

$\mu^{\frac{n-3}{2}}\ (\mu+1)^{\frac{n-1}{2}}\ (\mu^2 - (n-1)\mu - \frac{n-1}{2}) = 0.$

Then $\mu = 0$; $\frac{n-3}{2}$ times, $\mu =$ -1; $\frac{n-1}{2}$ times and $\mu = \frac{(n-1)\pm\sqrt{n^2-1}}{2}$ .

Mean $\bar{\mu} = \frac{n-1}{2n}$ .

Minimum Monopoly Energy $= E_{MM}(K_n) = \frac{n-1}{2} + \sqrt{n^2-1}$ .

Minimum Mean Monopoly(MMM) Energy $= E^M_{MM}(K_n)$

$= \sum_{i=1}^{\frac{n-3}{2}} |0 - \frac{n-1}{2n}| + \sum_{i=1}^{\frac{n-1}{2}} |-1 - \frac{n-1}{2n}| + |\frac{(n-1)\pm\sqrt{n^2-1}}{2} - \frac{n-1}{2n}|$

$= \frac{(n-1)(n-3)}{4n} + \frac{(3n-1)(n-1)}{4n} + |\frac{n(n-1)\pm n\sqrt{n^2-1}-(n-1)}{2n}|$

$= \frac{(n-1)(n-3)}{4n} + \frac{(3n-1)(n-1)}{4n} + |\frac{(n-1)^2\pm n\sqrt{n^2-1}}{2n}|$

$= \frac{(n-1)^2}{n} + \sqrt{n^2+1}$ .

□

**Theorem 3.2.** *For $n \geq 2$, the minimum mean monopoly energy of star graph $K_{1,n-1}$ is $\frac{n-2}{n} + \sqrt{4n-3}$ .*

*Proof.* $K_{1,n-1}$ is a star graph with vertex set V = $\{ v_1, v_2, ..., v_n \}$.
The MM set $= \{v_1\}$. (Assume that $v_1$ is the centre vertex).

$A_{MM}(K_{1,n-1}) =$

$$\begin{bmatrix} 1 & 1 & 1 \dots & 1 & 1 \\ 1 & 0 & 0 \dots & 0 & 0 \\ 1 & 0 & 0 \dots & 0 & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ 1 & 0 & 0. \dots & 0 & 0 \end{bmatrix}_{n\times n}$$

Then the characteristic equation $|A_{MM}(K_{1,n-1}) - \mu I| = 0$ is

$\mu^{n-2}\ (\mu^2 - \mu - (n-1)) = 0$

Then $\mu = 0$; (n-2) times, $\mu = \frac{1 \pm \sqrt{4n-3}}{2}$ .
Mean $\bar{\mu} = \frac{1}{n}$ .
The Minimum Monopoly(MM) Energy of star graph is

$E_{MM}(K_{1,n-1}) = \sum_{i=1}^{n-2} |0|(n-2) + |\frac{1 \pm \sqrt{4n-3}}{2}| = \sqrt{4n-3}.$
Minimum Mean Monopoly(MMM) Energy of star graph is $E_{MM}^{M}(K_{1,n-1})$

$$= \sum_{i=1}^{n-2} |0 - \tfrac{1}{n}| + |\tfrac{1 \pm \sqrt{4n-3}}{2} - \tfrac{1}{n}|$$

$$= \tfrac{(n-2)}{n} + |\tfrac{(n-2) \pm n\sqrt{4n-3}}{2n}|$$

$$= \tfrac{(n-2)}{n} + \sqrt{4n-3}.$$

□

**Theorem 3.3.** *For the complete bipartite graph $K_{m,n}$, for $m \leq n$ , the minimum mean monopoly energy is equal to* $\frac{2mn-(n+m)}{m+n} + \sqrt{4mn+1}$.

*Proof.* For the complete bipartite graph $K_{m,n}$, for m $\leq$ n, with vertex set
V = $\{ v_1, v_2,...,v_m , u_1, u_2,...,u_n\}$, the MM set = $\{ v_1,v_2,...,v_m \}$ .

Then the MM matrix is $A_{MM}(K_{m,n}) =$

$$\begin{bmatrix}
1 & 0 & 0 & .. & 0 & 1 & 1 & 1 & .. & 1 \\
0 & 1 & 0 & .. & 0 & 1 & 1 & 1 & .. & 1 \\
0 & 0 & 1 & .. & 0 & 1 & 1 & 1 & .. & 1 \\
. & . & . & .. & . & . & . & . & .. & . \\
. & . & . & .. & . & . & . & . & .. & . \\
0 & 0 & 0 & .. & 1 & 1 & 1 & 1 & .. & 1 \\
1 & 1 & 1 & .. & 1 & 0 & 0 & 0 & .. & 0 \\
1 & 1 & 1 & .. & 1 & 0 & 0 & 0 & .. & 0 \\
1 & 1 & 1 & .. & 1 & 0 & 0 & 0 & .. & 0 \\
. & . & . & .. & . & . & . & . & .. & . \\
. & . & . & .. & . & . & . & . & .. & . \\
1 & 1 & 1 & .. & 1 & 0 & 0 & 0 & .. & 0
\end{bmatrix}_{(m+n)\times(m+n)}$$

Then the characteristic equation $|A_{MM}(K_{m,n}) - \mu I| = 0$ is

$\mu^{m-1}(\mu - 1)^{n-1}(\mu^2 - \mu - mn) = 0$ .

Then $\mu = 0$; (m-1) times, $\mu$ =1 ; (n-1) times, $\mu = \frac{1 \pm \sqrt{4mn+1}}{2}$.

Mean $\bar{\mu} = \frac{n}{m+n}$.

Then, MM Energy = $E_{MM}(K_{m,n})$ = (m-1) + $\sqrt{4mn+1}$ .

MMM Energy = $E_{MM}^{M}(K_{m,n})$
$= \sum_{i=1}^{m-1} |(0 - \frac{n}{m+n})| + \sum_{i=1}^{n-1} |(1 - \frac{n}{m+n})| + |\frac{1 \pm \sqrt{4mn+1}}{2} - \frac{n}{m+n}|$

$= \frac{n(m-1)}{m+n} + \frac{m(n-1)}{m+n} + |\frac{(m-n)\pm(m+n)\sqrt{4mn+1}}{2(m+n)}|$

$=\frac{2mn-(m+n)}{m+n} + \sqrt{4mn+1}$ . □

**Theorem 3.4.** *For a double star graph $S_{n,n}$, the minimum mean monopoly energy is equal to $\frac{2(n-2)}{n} + 2(\sqrt{n} + \sqrt{n-1})$.*

*Proof.* For a double star graph $S_{n,n}$ with vertex set V $= \{v_0, v_1, v_2, ..., v_{n-1}, u_0, u_1, u_2, ..., u_{n-1}\}$, the MM set = { $u_0$,$v_0$ }.

The MM matrix $A_{MM}(S_{n,n}) =$

$$\begin{bmatrix} 1 & 1 & 1 & .. & 1 & 1 & 0 & 0 & .. & 0 \\ 1 & 0 & 0 & .. & 0 & 0 & 0 & 0 & .. & 0 \\ 1 & 0 & 0 & .. & 0 & 0 & 0 & 0 & .. & 0 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 1 & 0 & 0 & .. & 0 & 0 & 0 & 0 & .. & 0 \\ 1 & 0 & 0 & .. & 0 & 1 & 1 & 1 & .. & 1 \\ 0 & 0 & 0 & .. & 0 & 1 & 0 & 0 & .. & 0 \\ 0 & 0 & 0 & .. & 0 & 1 & 0 & 0 & .. & 0 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 0 & 0 & 0 & .. & 0 & 1 & 0 & 0 & .. & 0 \end{bmatrix}_{2n\times 2n}$$

The characteristic equation $|A_{MM}(S_{n,n}) - \mu I| = 0$ is

$\mu^{2n-4}(\mu^2 - (n-1))(\mu^2 - 2\mu - (n-1)) = 0.$

Then $\mu = 0$ ; $(2n-4)$ times , $\mu = \pm\sqrt{n-1}$ , $\mu = 1 \pm\sqrt{n}$.

Mean $\bar{\mu} = \frac{1}{n}$ .

Then MM Energy = $E_{MM}(S_{n,n}) = |0|(2n-4) + |\pm\sqrt{n-1}| + |1 \pm \sqrt{n}|$

= (2n-4) + 2 $(\sqrt{n} + \sqrt{n-1})$.

The MMM Energy = $E^M_{MM}(S_{n,n}) = \sum_{i=1}^{2n-4} |(0 - \frac{1}{n})| + |\pm\sqrt{n-1} - \frac{1}{n}| + |1 \pm \sqrt{n} - \frac{1}{n}|$

$= \frac{(2n-4)}{n} + |\frac{(-1\pm n\sqrt{n-1})}{n}| + |\frac{((n-1)\pm n\sqrt{n})}{n}|$

$= \frac{2(n-2)}{n} + 2(\sqrt{n} + \sqrt{n-1})$. □

**Theorem 3.5.** *For any integer $n \geq 3$, the minimum mean monopoly energy of the crown graph $S_n^0$ is $\sqrt{5}(n-1) + \sqrt{4n^2 - 8n + 5}$ .*

*Proof.* Let $S_n^0$ be the Crown graph with vertex set $V$= { $u_1$, $u_2$,...,$u_n$, $v_1$,$v_2$,...,$v_n$ }.

Then minimum monopoly set = { $u_1$, $u_2$,...,$u_n$ }.

Then the MMmatrix $A_{MM}(S_n^0) =$

$$\begin{bmatrix} 1 & 0 & 0 & .. & 0 & 0 & 1 & 1 & .. & 1 \\ 0 & 1 & 0 & .. & 0 & 1 & 0 & 1 & .. & 1 \\ 0 & 0 & 1 & .. & 0 & 1 & 1 & 0 & .. & 1 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 0 & 0 & 0 & .. & 1 & 1 & 1 & 1 & .. & 1 \\ 0 & 1 & 1 & .. & 1 & 0 & 0 & 0 & .. & 0 \\ 1 & 0 & 1 & .. & 1 & 0 & 0 & 0 & .. & 0 \\ 1 & 1 & 0 & .. & 1 & 0 & 0 & 0 & .. & 0 \\ . & . & . & .. & . & . & . & . & .. & . \\ . & . & . & .. & . & . & . & . & .. & . \\ 1 & 1 & 1 & .. & 0 & 0 & 0 & 0 & .. & 0 \end{bmatrix}_{2n\times 2n}$$

Then the characteristic equation $|A_{MM}(S_n^0) - \mu I| = 0$ is

$(\mu^2 - \mu - 1)^{n-1}(\mu^2 - \mu - (n-1)^2) = 0$ .

Then the Eigen values are $\mu = \frac{1\pm\sqrt{5}}{2}$ of multiplicity$(n-1)$ and $\mu = \frac{1\pm\sqrt{4n^2-8n+5}}{2}$ two simple roots.

Then, Mean $\bar{\mu} = \frac{1}{n}$ .

Then minimum monopoly energy is $E_{MM}(S_n^0) = \sqrt{5}(n-1) + \sqrt{4n^2-8n+5}$ .

Then minimum mean monopoly energy is

$E^M_{MM}(G) = \sum_{i=1}^{n-1} |\frac{1\pm\sqrt{5}}{2} - \frac{1}{n}| + |\frac{1\pm\sqrt{4n^2-8n+5}}{2} - \frac{1}{n}|$

$=$ (n-1) $|\frac{(n-2)\pm n\sqrt{5}}{2n}| + |\frac{(n-2)\pm n\sqrt{4n^2-8n+5}}{2n}|$

$= \sqrt{5}(n-1) + \sqrt{4n^2-8n+5}$ . □

## 4. Bounds of Minimum Mean Monopoly Energy

**Result 4.1.** *Let G be a connected graph with vertex set $V = \{v_1, v_2, ..., v_n\}$ and size m. Then $\sqrt{2m + mo(G)} \leq E_{MM}(G) \leq \sqrt{n(2m + mo(G))}$.*

For proof refer [7]

**Theorem 4.1.** *Let G be a connected graph with vertex set $V = \{v_1, v_2, ..., v_n\}$ and size m. Then $\sqrt{(2m + mo(G)) - 2|\bar{\mu}|\sqrt{n(2m + mo(G))}} \leq E^M_{MM}(G) \leq n\sqrt{(2m + mo(G)) + \bar{\mu}^2}$.*

*Proof.* By Cauchy Schwartz inequality

$(\sum_{i=1}^n |a_i b_i|)^2 = (\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2)$ .

Take $a_i = 1$ , $b_i$= $|\mu_i - \bar{\mu}|$ then,

$[E^M_{MM}(G)]^2 = (\sum_{i=1}^n 1)(\sum_{i=1}^n(|\mu_i - \bar{\mu}|)^2)$

$= (n)(\sum_{i=1}^{n}(|\mu_i - \bar{\mu}|)^2)$

$\leq n(\sum_{i=1}^{n}\mu_i^2 + \sum_{i=1}^{n}[\bar{\mu}]^2)$

$= n(\sum_{i=1}^{n}\mu_i^2 + n\ \bar{\mu}^2)$

$\leq n(n(2m + mo(G)) + n\bar{\mu}^2)$.

Then $E_{MM}^{M}(G) \leq n\sqrt{(2m + mo(G)) + \bar{\mu}^2}$.

Also $[E_{MM}^{M}(G)]^2 = (\sum_{i=1}^{n}|\mu_i - \bar{\mu}|)^2$

$\geq \sum_{i=1}^{n}|\mu_i - \bar{\mu}|^2$

$\geq \sum_{i=1}^{n}|\mu_i|^2 - 2|\bar{\mu}|\sum_{i=1}^{n}|\mu_i|$

$\geq (2m + mo(G)) - 2|\bar{\mu}|\sqrt{n(2m + mo(G))}$ .

Then $E_{MM}^{M}(G) \geq \sqrt{(2m + mo(G)) - 2|\bar{\mu}|\sqrt{n(2m + mo(G))}}$.
Hence the theorem. $\square$

**Result 4.2.** *Let $G$ be a connected graph with vertex set $V = \{v_1, v_2, ..., v_n\}$ and size $m$. Then $\sqrt{n+1} \leq E_{MM}(G) \leq n\sqrt{n}$.*

For proof refer [7]

**Theorem 4.2.** *Let $G$ be a connected graph with vertex set $V = \{v_1, v_2, ..., v_n\}$ and size $m$. Then $\sqrt{(n+1) - 2|\bar{\mu}|n\sqrt{n}} \leq E_{MM}^{M}(G) \leq \sqrt{n(n^3 + n\bar{\mu}^2)}$.*

*Proof.* By Cauchy Schwartz inequality

$(\sum_{i=1}^{n}|a_i b_i|)^2 = (\sum_{i=1}^{n}a_i^2)(\sum_{i=1}^{n}b_i^2)$ .

Take $a_i = 1$ , $b_i = |\mu_i - \bar{\mu}|$ then,

$[E_{MM}^{M}(G)]^2 = (\sum_{i=1}^{n}1)(\sum_{i=1}^{n}(|\mu_i - \bar{\mu}|)^2)$

$= (n)(\sum_{i=1}^{n}(|\mu_i - \bar{\mu}|)^2)$

$\leq n(\sum_{i=1}^{n}\mu_i^2 + \sum_{i=1}^{n}[\bar{\mu}]^2)$

$= n(\sum_{i=1}^{n}\mu_i^2 + n\ \bar{\mu}^2)$

$\leq n(n^3 + n\bar{\mu}^2)$

Then $E_{MM}^{M}(G) \leq \sqrt{n(n^3 + n\bar{\mu}^2)}$.

Also $[E_{MM}^{M}(G)]^2 = (\sum_{i=1}^{n}|\mu_i - \bar{\mu}|)^2$

$\geq \sum_{i=1}^{n}|\mu_i - \bar{\mu}|^2$

$\geq \sum_{i=1}^{n} |\mu_i|^2 - 2|\bar{\mu}| \sum_{i=1}^{n} |\mu_i|$

$\geq (n+1) - 2|\bar{\mu}|n\sqrt{n}.$

Then $E_{MM}^{M}(G) \geq \sqrt{(n+1) - 2|\bar{\mu}|n\sqrt{n}}.$

Hence the theorem. $\square$

**Result 4.3.** *Let $G$ be a connected graph with vertex set $V = \{v_1, v_2, ..., v_n\}$ , size $m$ and $D = det\ (A_{MM}(G))$. Then*

$$\sqrt{2m + mo(G) + n(n-1)D^{\frac{2}{n}}} \leq E_{MM}(G) \leq \frac{2m+mo(G)}{n} + \sqrt{(n-1)[2m + mo(G) - (\frac{2m+mo(G)}{n})^2]}.$$

For proof refer [7]

**Theorem 4.3.** *Let $G$ be a connected graph with vertex set $V = \{v_1, v_2, ..., v_n\}$ , size $m$ and $D = det\ (A_{MM}(G))$. Then*

$$\sqrt{2m + mo(G) + n(n-1)D^{\frac{2}{n}} - 2|\bar{\mu}|\frac{2m+mo(G)}{n} + \sqrt{(n-1)[2m + mo(G) - (\frac{2m+mo(G)}{n})^2]}} \leq$$
$$E_{MM}^{M}(G) \leq \sqrt{n[(\frac{2m+mo(G)}{n}) + \sqrt{(n-1)[2m + mo(G) - (\frac{2m+mo(G)}{n})^2]}]^2 + n^2\bar{\mu}^2}.$$

*Proof.* By Cauchy Schwartz inequality

$(\sum_{i=1}^{n} |a_i b_i|)^2 = (\sum_{i=1}^{n} a_i^2)(\sum_{i=1}^{n} b_i^2)$ .

Take $a_i = 1$ , $b_i = |\mu_i - \bar{\mu}|$ then,

$[E_{MM}^{M}(G)]^2 = (\sum_{i=1}^{n} 1)(\sum_{i=1}^{n}(|\mu_i - \bar{\mu}|)^2)$

$= (n)(\sum_{i=1}^{n}(|\mu_i - \bar{\mu}|)^2)$

$\leq n(\sum_{i=1}^{n} \mu_i^2 + \sum_{i=1}^{n}[\bar{\mu}]^2)$

$= n(\sum_{i=1}^{n} \mu_i^2 + n\ \bar{\mu}^2).$

$\leq n[[(\frac{2m+mo(G)}{n}) + \sqrt{(n-1)[2m + mo(G) - (\frac{2m+mo(G)}{n})^2]}]^2 + n\bar{\mu}^2].$

Then $E_{MM}^{M}(G) \leq \sqrt{n\ [(\frac{2m+mo(G)}{n}) + \sqrt{(n-1)[2m + mo(G) - (\frac{2m+mo(G)}{n})^2]}\ ]^2 + n^2\bar{\mu}^2}$

Also $[E_{MM}^{M}(G)]^2 = (\sum_{i=1}^{n} |\mu_i - \bar{\mu}|)^2$

$\geq \sum_{i=1}^{n} |\mu_i - \bar{\mu}|^2$

$\geq \sum_{i=1}^{n} |\mu_i|^2 - 2|\bar{\mu}| \sum_{i=1}^{n} |\mu_i|$

$\geq 2m + mo(G) + n(n-1)D^{\frac{2}{n}} - 2|\bar{\mu}|\frac{2m+mo(G)}{n} + \sqrt{(n-1)[2m + mo(G) - (\frac{2m+mo(G)}{n})^2]}$

Then $E^M_{MM}(G) \geq$

$$\sqrt{2m + mo(G) + n(n-1)D^{\frac{2}{n}} - 2|\bar{\mu}|\frac{2m+mo(G)}{n} + \sqrt{(n-1)[2m + mo(G) - (\frac{2m+mo(G)}{n})^2]}}.$$

Hence the theorem. □

## References


[1] Gutman. (1978), The energy of a graph, Ber. Math-Statist. Sekt. Forschungsz.Graz, 103, 1-22.
[2] C.Adiga, A.Bayad, I.Gutman and S.A.Srinivas. (2012), The minimum covering energy of a graph, Kragujevac Journal of Science,34, 39-56.
[3] R.B.Bapat. (2011), Graphs and Matrices, Hindustan Book Agency.
[4] I.Gutman, X.Li and J.Zhang. (2009), Graph Energy, (Ed-s: M. Dehmer, F. Em-mert), Streib., Analysis of Complex Networks,From Biology to Linguistics, Wiley-VCH, Weinheim, 145-174.
[5] F.Harary. (1969), Graph Theory, Addison Wesley, Massachusetts.
[6] X.Li, Y.Shi and Gutman. (2012), Graph energy, Springer, New York Heidelberg Dordrecht, London.
[7] Ahmed Mohammed Naji, N.D.Soner. (2015) The Minimum Monopoly Energy of a Graph, International Journal of Mathematics And its Applications, Volume 3, Issue 4 - B, 47-58.
[8] M.V.Chakradhara Rao, B.Satyanarayana, K.A.Venkatesh. (2017) The Minimum Mean Dominating Energy of Graphs, International Journal Of Computing Algorithm Vol 6(1), 23-26.
[9] LauraBuggy, Amaliailiuc, KatelynMccall, Duyguyen. The Energy of graphs and Matrices, Lecture notes, 1-27 .



**M. V. Chakradhara Rao** received his master's degree in the year 1995 from Banaras Hindu University, Varanasi, Ph.D in 2018 from Acharya NagarjunaUniversity, Guntur, India. He did his Ph.D. degree in the major research project entitled "Study of Minimum Mean Energy of Certain Domination of Graphs". He is currently a faculty member of Mathematics in Presidency University, Bangalore, India. His research interests focus mainly on Graph theory, Algebra and Analysis.



**K. A. Venkatesh** has completed his Ph.D from Alagappa University. He authored 5 book chapters. He is a member of Systems Society of India, RMS, Constrained Programming Association, USA, Academy of Discrete Mathematics and Applications. he is serving as Adjunct Professor at IIIT-B and Professor of Math and Computer Science, Myanmar Institute of Information Technology, Mandalay, Myanmar.



**D. V. Lakshmi** received her Master's degree in the year 1993, Ph. D degree in 2008 from University of Hyderabad, Hyderabad, India. She did her Ph.D. degree in the major research project entitled "On vector valued Amalgam Spaces". She is currently a faculty of member of Mathematics in Bapatla Engineering College, Bapatla, India. Her research interests focus mainly on Graph theory and Theoretical Computer Science.