Excursion Theory and Local Times for Bessel and Brownian Diffusions

with Applications to Credit Risk

Xiaolin Zhu

Department of Statistics

London School of Economics and Political Science

This thesis is submitted for the degree of

Doctor of Philosophy

March 2020

Dedicated to my 24-year-old self ...

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This thesis is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements.

Xiaolin Zhu

March 2020

Acknowledgements

First and foremost, I would like to express my deep gratitude to my supervisor, Professor Angelos Dassios, who has the attitude and the substance of a genius: he continually and convincingly conveyed a spirit of adventure in regard to research, and an excitement in regard to life. His immense knowledge, innovative ideas, and consistent enthusiasm in mathematics have been an invaluable source of motivation for me. Without his guidance and encouragement, this thesis would not have been possible.

In addition, I would like to thank my secondary supervisor, Professor Hao Xing, a person I could always turn to for affirmation and comfort. My sincere thanks also go to my examiners, Luciano Campi and Apostolos Papaioannou, for their insightful feedback and comments to improve my thesis.

Third, I gratefully acknowledge the Department of Statistics and the London School of Economics for providing a pleasurable and stimulating environment that allowed me to thrive. More importantly, my PhD literally could not have been a success without Ian Marshall and Penny Montague who supported academic staff and students beyond all measure.

To my fellow PhD students in the Columbia House - thank you all for making my time at the department exceptionally enjoyable, with special thanks to the officemates of my final year, Ragvir Sabharwal, Alice Pignatelli di Cerchiara, Filippo Pellegrino, Quoc Viet Dang and Junyi Zhang, for the alternately productive and unproductive chats and wonderful moments, which I will dearly miss. To my people with beautiful souls - Xiangyu Li, Weiguan Wang, Jing Han Tee, Jose Manuel Pedraza Ramirez, Jingshu Zhang and Xinyuan Zhang - thank you all for sharing with me all those joy and sadness moments in the past years.

Lastly, I am deeply indebted to my dearest parents for their endless love and encouragement over the years, and my little cat for offering as a companion to my parents in lieu of me.

Abstract

By means of excursion theory, the evolution of a continuous Markov process satisfying regularity assumptions is analysed in terms of its behaviour between visits to a recurrent point, for instance the point zero in the state space of Brownian and Bessel diffusions of type reflecting at the origin. As a preliminary conclusion, a sample path of the process can be reconstructed by the *excursions away from zero* of random finite lengths and the *time spent at visits to zero*. These two together constitute the core of the work in this thesis.

With respect to the zero-free intervals, we study the duration of the excursion in process away from zero by time t, namely *the age process*, of a Bessel process instantaneously reflected at the origin. The main contribution of our work is the development of a *hybrid structural-reduced form model* with an endogenous intensity defined by the age process. This model provides a framework for assessing default probabilities within a circumstance of very limited information, assuming that some statistics about a firm are not observable but the time points when they reach certain level are. Results presented include distributional properties for the default time and level as a joint stopping process, by which we discover a decomposition theorem that contributes to exact schemes for simulating the default process. A counting process for monitoring consecutive arrivals of some event driven by the same intensity is also established. Main aspects to be addressed are the properties and the derivations of distributional quantities concerning the interarrival times, the arrival of the nth event and the associated counting process.

With respect to the zero set, we construct a continuous family of functionals for the part of time spent at the origin by the age process, namely the local time at zero. It is a well known fact that there is no unified representation for the local time of Markov process, as it can be approximated as a limit of various processes describing the behaviour of trajectories of the underlying process. That being so, the focus and efforts are put on the certain properties of the limit processes served as the approximations, and on the first and second order limit theorems for the convergences to the local time.

Contents

1 Introduction

Following the profound study by P. Lévy and K. Itô on the excursion theory of Markov processes, the evolution of a continuous process satisfying regularity assumptions is analysed in terms of its behaviour between visits to a recurrent point in the state space (see Itô [54] for full details). A sample path of the process is then broken up into components, consisting of pieces of the path of random finite lengths starting and finishing at the recurrent point (namely the excursion process), and the time spent at all visits to the same point. This thesis proceeds in this spirit, studying and developing theories on the path behaviour of Bessel processes of dimension $(2-2\alpha)$ with $\alpha \in (0,1)$, denoted by $X = \{X_t,\ t \geq 0\}$ throughout the thesis. This type of diffusions has the feature of instantaneous reflection at zero, and for $\alpha = ½$ it coincides with a reflected Brownian motion. Partitioned by the reflection points, the core of this work points to the zero-free intervals and the set of all zeros.

1.1 Zero-Free Intervals

With respect to the zero-free intervals, we study the duration of the excursion in progress away from zero at time t of recurrent Bessel processes, namely *the age process of Bessel excursions*. Denoted by $U = \{U_t, t \geq 0\}$, the age of X straddling t is represented in the form:

$$U_t := t - \sup\{\ s < t \mid X_s = 0\ \}, \quad t \geq 0\ .$$

Excursion age along with relevant properties is well studied through working on various functionals of excursion processes (see Jeanblanc et al. [61, Chap 4.3] and Yen and Yor [89, Chap 7] for more details). Particular attention has been paid to the scaled/signed age, which leads to the development and extension of the famous Azéma martingale (see Azéma and Yor [2], Çetin [19] for reference). Literature addressing the mathematical finance aspect related to the age process is extensive. Considerable effort has been directed towards the study of the well-known "Parisian stopping time". It was introduced by Chesney et al. [21], who defined it as the first time the age of the underlying process exceeds a certain level. Topics regarding this stopping time include Parisian options with single/double barriers [30, 31, 36], Parisian ruin problems [4, 26, 33] and decision problems with implementation delay [24, 35, 70].

In this thesis, we extend the knowledge of Bessel age process to the area of default risk valuation with incomplete information, which has not received due attention in the current literature. As the main contribution of our work, we develop a *hybrid structural-reduced form model with an endogenously defined intensity* to assess default probabilities for the context within which some statistics associated with a firm's positive activities (e.g. revenue, cash inflow, investments or other forms of asset) and negative activities (e.g. debt, loan, cash outflow or other forms of liability) are not observable, but instead the time points when they reach zero are. The idea of adopting such a hybrid method into credit risk measurement arises primarily as a resolution to the defects existing in the classic structural and intensity models. For background references please see [9, 80] for the former and [41, 60] for the latter.

Turning first to the former, *structural models* assume complete knowledge of a very detailed information set, thereby implying a predictable default time. In reality, however, the information accessed from the public reports of a firm is certainly not exhaustive with noise, delay and omission (see Çetin et al. [20] in this regard). In response to these issues, we provide a framework for modelling default risk with a reduced information set, or a shrunk filtration from the perspective of filtration theory. More precisely, the available statistics for assessing default, modelled as the elapsed time U_t of the underlying process X_t since the last time visiting zero, are adapted to

$$\mathscr{G}_t = \sigma\{U_s,\ s \leq t\} \subset \sigma\{X_s,\ s \leq t\}.$$

As an immediate consequence, the default time turns inaccessible, which results in a transformation to the intensity-based approach (see Guo et al. [50], Jarrow and Protter [58] in this regard).

As opposed to the former, intensity-based models put no restriction to the completeness of the information set. The arrival of default is in general modelled as the first jump of a point process governed by an intensity process λ_t, either or not related to the assumed information set. The presumption of an utterly exogenous intensity appears prevalent in the early frameworks (see Hübner [53], Jarrow et al. [57], Jarrow and Turnbull [60] for instance), but becomes counterfactual as dealing with more practical cases [41]. It seems to us that the nature of default is inextricably tied to some sort of the performance of the underlying process, though it has been designed as an exogenous event coming by "surprise" in the traditional intensity-based models. The exogeneity assumption, disregarding the incentives to default and thus lacking fundamental interpretation in the economic context, makes itself obsolete. Given this belief, we further postulate that the likelihood of default is directly linked to the current age level by some function, e.g. $\lambda_t = \lambda(U_t)$.

Akin to our hybrid approach, various examples of model transformations as a result of imperfect information appeared in the recent literature. Interested readers can refer to Giesecke [47], Kusuoka [67] for coping with noisy information, to Collin-Dufresne et al. [23], Guo et al. [50] for deferred information and to Çetin et al. [20], Jarrow et al. [59] for shrinking information. In particular to the third case, it refers to modelling with a strict sub-filtration, to which our work belongs. The work of Çetin et al. [20] resembles closely to ours but within the context of Brownian motion, by assuming observables on both the age level and the sign changes and choosing a dependent intensity obtained from the Azéma martingale.

As a sequel to this model, a counting process for monitoring consecutive arrivals of some event driven by the same intensity is also established. Main aspects to be addressed are the properties and the derivations of distributional quantities concerning the interarrival times, the arrival of the nth event and the associated counting process.

Chapters related to the topic "Zero-Free Intervals" are 2, 3 and 4 with outlines as follows:

Chapter 2 Bessel Age Process. A formal introduction to the Bessel age process is given with mathematical definitions and a visual impression of this jump-linear Markov process. By means of a piecewise-deterministic Markov process framework, a *perturbed Bessel process* is constructed in order to resolve the problem of producing infinitely tiny jumps in the path of the age process. Accordingly, the distributional properties associated with the first stopping time problem are characterised with explicit results on the quantities like generating functions, Laplace transforms and densities. A notable result to be mentioned is that we derive a decomposition rule for the stopping time which in turn contributes to the development of simulation schemes.

Chapter 3 Application to Credit Risk Modelling with Exact Simulation Scheme. The realisation of the hybrid model is presented in this chapter. We provide precise simulation algorithm for the default time and level as a joint process by choosing a piecewise constant intensity function. To verify the accuracy of the algorithm and assess its performance in evaluating the default risk, a numerical study for the case of reflected Brownian motion is carried out.

Chapter 4 An Age-dependent Counting Process. A point process is established for monitoring consecutive arrivals of some event with an age-driven intensity over a finite time interval. Main aspects to be covered are the related properties and the derivation of distributional quantities concerning the interarrival time, the arrival of the nth event and the moments of the counting process.

1.2 The Set of Zeros

With respect to the zero set, we construct and study a continuous family of functionals in an integral form :

$$\int_0^t f(X_s)\,\mathrm{d}s, \qquad t \geq 0,$$

with f is a non-negative Borel measurable function. Great attention has been given to a (scaled) Lebesgue measure of the time spent by the age process under an arbitrary level ϖ up to time t. Denoted by $Z^p = \left\{Z_t^p, t \geq 0\right\}$ with

$$Z_t^p(\varpi) := \frac{1}{\varpi^p} \int_0^t \mathbb{1}_{\{U_s < \varpi\}}\,\mathrm{d}s, \qquad 0 < \varpi < t,$$

such measure is often known as *the occupation process at time t* of a Bessel age process. Of special interest to us is the asymptotic behaviour of $Z_t^p(\varpi)$ as ϖ approaches zero. With probability one for all $t \geq 0$, the existence of such limit is guaranteed for every regular point in the state space of U, according to the general theory of additive functionals [11]. In addition to the limit, this interest is also linked to the distributional properties of $\lim_{\varpi \to 0} Z_t^p(\varpi)$ as a limiting process, and to the convergence to the associated local time. By *the local time at zero* of a Bessel age process, we mean that there exists a continuous functional denoted by $L = \{L_t,\ t \geq 0\}$ that measures $\mathcal{Z} = \{\ t > 0 \mid U_t = 0\ \}$, i.e. the time set at which the age process visits zero.

The concept of "local times" originated from Lévy's study on the zero set of Brownian motion, at which early time it was described as an occupation density and named as "Mesure du Voisinage" (see [72] and [73] for details). It is a well known fact that there is no unified representation for the local time, as it can be approximated as a limit of various processes describing the behaviour of trajectories of the underlying process. Lévy gave several different definitions of Brownian local times in terms of the occupation measure, the number of downcrossings and the total length/number of zero-free intervals satisfying certain conditions. Interested readers are referred to Borodin [14], Itô and McKean [55] and Karatzsas and Shreve [62] for full accounts.

Apart from the occupation time as a continuous measure, we also take into account other equivalent representations for the local time. In the spirit of Lévy's downcrossing theorem, we construct a discontinuous one, defined as the number of times that U_t jumps down to zero by time t and denoted by $\{D_t,\ t \geq 0\}$. By virtue of the pathwise relation between U_t and X_t, D_t is equal in quantity to the number of completed excursions away from zero by the underlying Bessel path. Particular interests are given to the relevant properties of the limiting processes served as the approximations, and to the first and second-order limit theorems for the convergence to the local time.

Another important fact concerning the local time of Markov processes is that it is analogous to an inverse subordinator. Put another way, an inverse subordinator is the local time of some "well-behaved" Markov process [6, 69]. In this regard, the most famous case is the Brownian local time that is an inverse of a ½-stable subordinator; and a more general case is presented by Bingham [7] who showed the inverse of α-stable subordinators with $0 < \alpha < 1$ arising as a limit process of occupation times. Particularly relevant to our study is the correspondence of the local time to the inverse subordinator that leads us to the law of L_t by characterising the inverse of L_t to the family of α-stable processes. As an immediate consequence of this finding, L_t is further identified as a self-similar process, whose paths fulfil the *scale invariance* property such that for all $\lambda, t > 0$,

$$\left\{\lambda^{-\alpha} L_{\lambda t},\ t \geq 0\right\} \overset{\text{law}}{=} \left\{L_t,\ t \geq 0\right\}.$$

Real valued positive self-similar processes often arise as the limits of some rescaled processes. For instance, Brownian motion, Bessel processes, stable subordinators and the local time concerned in this thesis, etc.

Chapters related to the topic "The Set of Zeros" are 5, 6 and 7 with outlines as follows:

Chapter 5 Local Times related to Bessel Age Process. This chapter deals with equivalent approximations to the local time of a Bessel (age) process. Main attentions are put to the occupation time and the jump counting measures associated with the age process. Of great interests to us are the certain properties arising from their asymptotic behaviours, and the first and second-order limit theorems for the convergence to the local time. Furthermore, in favour of the fact that the inverse of local time is a subordinator, the law of the local time is determined with a closed-form expression.

Chapter 6 Scale Invariance about Local Times. This chapter further explores the local time of Bessel age process from a prospective of self-similar Markov processes. Specifically, we present some examples of basic scaling properties associated with a Bessel (age) process and show that the local time fulfils the *scale invariance* property. An interesting consequence is that the difference between two scaled local times follows a brand new time-changed Brownian motion.

Chapter 7 Local Times related to Brownian Motion. In this chapter, we restrict our attention to the zero set of a reflected Brownian motion and provide elementary proof for some celebrated results concerning a Brownian local time. For instance, Lévy’s "Mesure du Voisinage", Lévy’s downcrossing theorem and a central limit theorem as a sequel to the latter.

2

Bessel Age Process

Bessel processes have been intensively studied in the area of mathematical finance. Main focus has been put on the applications to the dynamics of asset prices (see [75, 77, 84] for examples), of interest rates (see [45, 51]) and of stochastic volatilities (see [74, 76]).

Of particular interest to us is the Bessel process of dimension $\delta \in (0, 2)$, whose path is continuous, non-negative and reflected instantaneously at zero. A special case is given to $\delta = 1$ as it corresponds to a standard Brownian motion. For such processes denoted by X_t with zero a regular point in the state space, we study the age process of its excursions away from zero. The *Bessel age process* defined by

$$U_t \;\equiv\; t - \sup\{\, s < t \mid X_s = 0 \,\}, \quad t \geq 0,$$

refers to the length up to time t of the excursion in progress, in line with the definitions appearing in the majority of the literature on excursion theory. To get a visual impression of this process, sample paths of the joint process (X_t, U_t) with $\delta = 0.5, 1.5$ are presented in Figure 2.1.

This chapter together with the next contributes to the relevant literature in two aspects. First, as a complement to the functionals of Bessel excursions, we develop distributional properties for joint process (U_t, t) stopped at the first jump time τ with a general intensity function of the form $\lambda(U_t)$. This is achieved by means of a piecewise-deterministic Markov process (PDMP) framework. Explicit formulae for the generating functions, Laplace transforms and marginal densities are derived. In addition to these, a notable result to be mentioned is that the stopping time τ is decomposable into two independent variables: the stopping level of the age process and a Lévy process stopped at a unit exponential time.

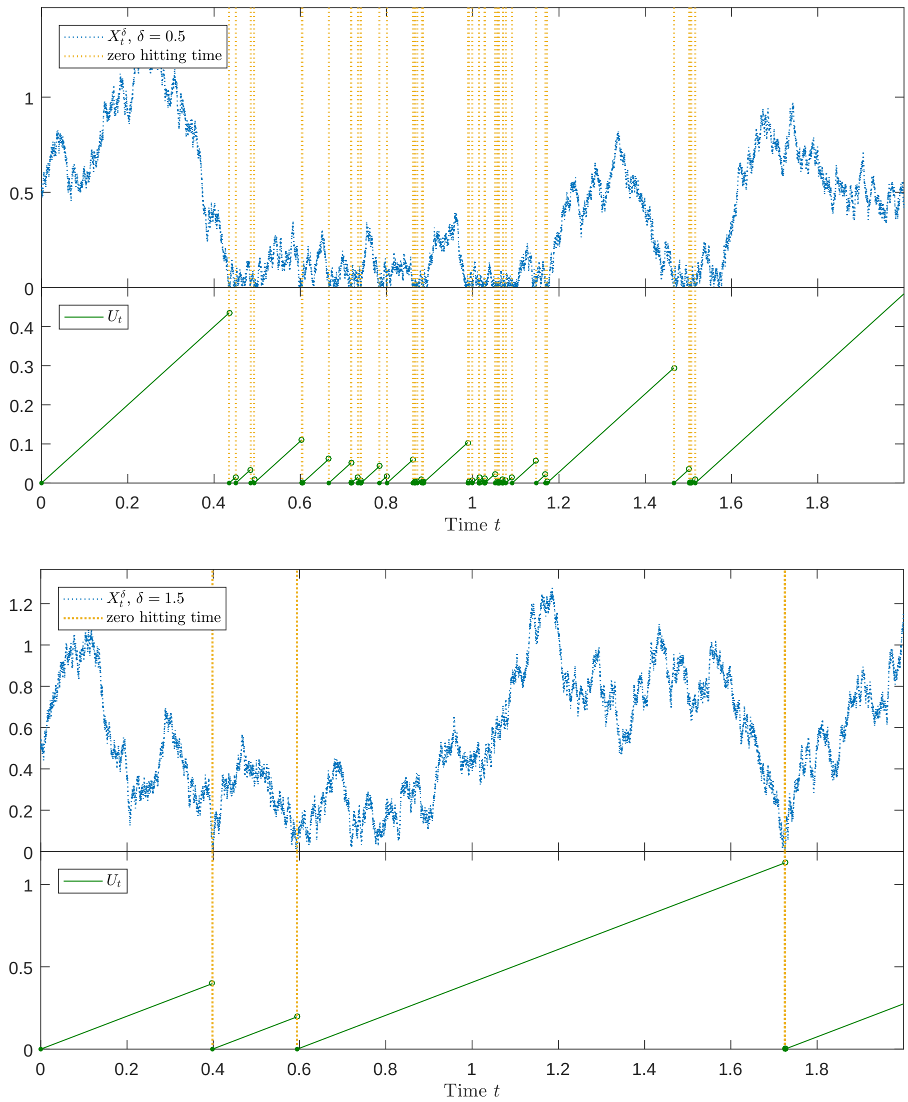


Fig. 2.1 A sample path of the joint process (X_t, U_t) with dimension 0.5 and 1.5 respectively.

Second, as a complement to the default risk measurement with limited information content, we set up the mathematical framework for a *hybrid structural-reduced form model* with an endogenous intensity $\lambda(U_t)$. This model provides a framework for assessing default probabilities within a circumstance of very limited information, assuming that some statistics about a firm are not observable but the time points when they reach certain level are. The distributional results associated with the default time achieved in the first part lead to the development of exact simulation schemes for the joint default process (U_τ, τ).

The realisation of this model awaits to be presented in the next chapter.

2.1 Age Process of Bessel Excursion

Characterised by its dimension (or index), a Bessel process behaves differently. The one of our concern is the so-called recurrent Bessel process. The following describes how this type of Bessel process is attractive to us.

2.1.1 Definitions

Given a filtered probability space $(\Omega, \mathscr{F}, \mathbb{P})$, let $X = \{X_t\}_{t\geq 0}$ adapted to $\{\mathscr{F}_t\}_{t\geq 0}$ be a Bessel process of dimension $\delta = 2(1-\alpha)$ or equivalently of index $(-\alpha)$ with $\alpha \in (0,1)$. This process, denoted as $\mathrm{BES}^{(\alpha)}$ throughout this chapter, is a $\mathbb{R}^+$-valued Feller diffusion whose infinitesimal generator coincides on $C^2(0,\infty)$ with :

$$\mathcal{G}^X = \frac{1-2\alpha}{2x}\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2}.$$

For an extensive study on the path properties for Bessel process, we refer to Borodin and Salminen [16, Chap IV-6] and Jeanblanc et al. [61, Chap 6].

We first recall some important characteristics of such a Bessel process:

(i) $\mathrm{BES}^{(\alpha)}$ *is a strong Markov process with continuous path taking values in* $[0,\infty)$*;*

(ii) For $\alpha = ½$, $\mathrm{BES}^{(\alpha)}$ *is identical in law to a reflected Brownian motion;*

(iii) (Existence of Excursions) α *restricted to* $(0,1)$ *ensures* $\mathrm{BES}^{(\alpha)}$ *recurrently and instantaneously reflecting at the boundary point zero, which implies the existence of excursions in its path.*

Then we shall give a precise definition of what we mean by the Bessel age process. For all $t > 0$, let

$$g_t \equiv \sup\{\, s < t \mid X_s = 0 \,\}; \qquad d_t \equiv \inf\{\, s > t \mid X_s = 0 \,\},$$

denote the last hitting time of zero before time t and the first hitting time after t, respectively. The Bessel excursion straddling t is the path $\{X_{g_t+s};\ 0 \leq s \leq d_t - g_t\}$, in which the portion up to t is often known as the Bessel meander. The age of Bessel excursion straddling t is defined by

$$0 \leq U_t = t - g_t \leq t \tag{2.1}$$

i.e. the length of the Bessel meander. In fact, it is right-continuous with left limits at every time point and jumps down to 0 at points in which $\mathrm{BES}^{(\alpha)}$ gets reflected as a result of reaching zero. Due to the Markov property of a recurrent Bessel process, $\{U_t\}_{t\geq 0}$ is thus a Markov process in the filtration $\{\mathscr{F}_{g_t}\}_{t\geq 0}$.

2.1.2 Some Preliminary Results

This section presents some preliminary results concerning the joint law of (X_t, U_t).

Proposition 2.1.1 (*First Passage Time*)**.** *Denote* $T_{x_0} := \inf\{\, t > 0 \mid X_t = 0 \,\}$ *as the first hitting time of zero for* $\mathrm{BES}^{(\alpha)}$ *starting from* $x_0 > 0$*. The density function of* T_{x_0} *is*

$$\mathbb{P}\big(T_{x_0} \in \mathrm{d}t\big) \;=\; \frac{x_0^{2\alpha}}{2^{\alpha}\Gamma(\alpha)} t^{-\alpha-1} e^{-\frac{{x_0}^2}{2t}}\,\mathrm{d}t, \qquad t > 0, \tag{2.2}$$

with $0 < \alpha < 1$*, i.e.* T_{x_0} *is indeed a multiple of the reciprocal Gamma variable.*

Proof. It is a well-known result in the area of first passage time of a Markov process, whose proof can be found in [18, 51, 52]. In the literature related, this result is mainly deduced by the time reversal property of Bessel process. Relying on the equality between the joint process of a recurrent Bessel with its first hitting time and that of a transient Bessel with its last exit time, they conclude that the first hitting time is identical in law to the last exit time. See examples in Göing-Jaeschke et al. [48] and time/path reversal properties in Nagasawa [81], Pitman and Yor [83], Williams [88].

We propose another approach by means of the martingale property associated with the infinitesimal generator. Interest readers may refer to **Appendix I**. □

In the following, we describe some measures with asymptotic analysis on the joint process (X_t, U_t). Let $p_t^{(\alpha)}(x, y)$, with $x \geq 0, y > 0$ and $t > 0$, denote the transition probability for $\mathrm{BES}^{(\alpha)}$. From Jeanblanc et al. [61, Chap 6.2.2], we have

$$\begin{aligned} p_t^{(\alpha)}(x, y) &= \frac{y}{t}\left(\frac{x}{y}\right)^{\alpha} \mathcal{I}_{\alpha}\left(\frac{xy}{t}\right) e^{-\frac{x^2+y^2}{2t}}, \\ p_t^{(\alpha)}(0, y) &= \frac{2^{\alpha}}{\Gamma(1-\alpha)} y^{1-2\alpha} t^{\alpha-1} e^{-\frac{y^2}{2t}}. \end{aligned}$$

where $\mathcal{I}_\nu(x)$ is the modified Bessel function with index ν of the first kind. The second result can be achieved by taking $x \to 0$ and applying L'Hôpital's Rule .

Lemma 2.1.2. *Let $(X_t)_{t\geq 0}$ be a Bessel process defined as before with starting level $\varepsilon > 0$ and $U_t = t - \sup\{\ s < t \mid X_s = 0\ \}$ be its age process. It holds for any $\alpha \in (0,1)$ that*

$$\lim_{\varepsilon\to 0} \varepsilon^{-2\alpha}\, \mathbb{P}(X_t \in \mathrm{d}x, U_t \in \mathrm{d}t \mid X_0 = \varepsilon) = \frac{xt^{-\alpha-1}}{2^\alpha \Gamma(1+\alpha)} e^{-\frac{x^2}{2t}}\, \mathrm{d}x\, \mathrm{d}t. \tag{2.3}$$

As a result,

$$\lim_{\varepsilon\to 0} \varepsilon^{-2\alpha}\, \mathbb{P}(U_t \in \mathrm{d}t \mid X_0 = \varepsilon) = \frac{t^{-\alpha}}{2^\alpha \Gamma(1+\alpha)}\, \mathrm{d}t, \tag{2.4}$$

and

$$\mathbb{P}(X_t \in \mathrm{d}x \mid U_t = t) = \frac{x}{t} e^{-\frac{x^2}{2t}}\, \mathrm{d}x. \tag{2.5}$$

Proof. Given T_ε the first time point in which the Bessel process X reaches zero, the probability in (2.3) is equivalent to

$$\begin{aligned} & \mathbb{P}(X_t \in \mathrm{d}x, X_s > 0, \forall s \in (0,t) \mid X_0 = \varepsilon) \\ = \ & \mathbb{P}(X_t \in \mathrm{d}x, T_\varepsilon \geq t \mid X_0 = \varepsilon) \\ = \ & \mathbb{P}(X_t \in \mathrm{d}x \mid X_0 = \varepsilon) - \mathbb{P}(X_t \in \mathrm{d}x, T_\varepsilon < t \mid X_0 = \varepsilon). \end{aligned} \tag{2.6}$$

The first probability follows $p_t^{(\alpha)}(\varepsilon, x)\mathrm{d}x$. Let $\tilde{X}$ be an independent duplicate of X starting from zero describing the motion by the underlying process after T_ε. We then calculate the second probability by

$$\begin{aligned} & \mathbb{P}(X_t \in \mathrm{d}x, T_\varepsilon < t \mid X_0 = \varepsilon) \\ = & \int_0^t \mathbb{P}(X_t \in \mathrm{d}x, T_\varepsilon \in \mathrm{d}s \mid X_0 = \varepsilon)\, \mathrm{d}s \\ = & \int_0^t \mathbb{P}(T_\varepsilon \in \mathrm{d}s) \cdot \mathbb{P}\Big(\tilde{X}_{t-s} \in \mathrm{d}x \,\Big|\, \tilde{X}_0 = 0\Big)\, \mathrm{d}s \\ = & \int_0^t \frac{\varepsilon^{2\alpha}}{2^\alpha \Gamma(\alpha)} s^{-\alpha-1} e^{-\frac{\varepsilon^2}{2s}} \cdot \frac{2^\alpha}{\Gamma(1-\alpha)} x^{1-2\alpha} (t-s)^{\alpha-1} e^{-\frac{x^2}{2(t-s)}}\, \mathrm{d}x\, \mathrm{d}s. \end{aligned}$$

This integral expression is obtained as a consequence of the strong Markov property of Bessel process and the first-entrance decomposition.

Taking Laplace transform over t of (2.6) gives,

$$\begin{aligned}
&\varepsilon^{\alpha} x^{1-\alpha} \int_0^{\infty} e^{-\beta t}\, t^{-1} \mathcal{I}_{\alpha}\left(\frac{\varepsilon x}{t}\right) \exp\left\{-\frac{\varepsilon^2 + x^2}{2t}\right\} \mathrm{d}x\, \mathrm{d}t \\
&\quad - \frac{\varepsilon^{2\alpha} x^{1-2\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^{\infty} e^{-\beta t} \int_0^t s^{-\alpha-1}(t-s)^{\alpha-1} \exp\left\{-\frac{\varepsilon^2}{2s} - \frac{x^2}{2(t-s)}\right\} \mathrm{d}x\, \mathrm{d}s\, \mathrm{d}t \\
&= 2\varepsilon^{\alpha} x^{1-\alpha} \mathcal{I}_{\alpha}\left(\varepsilon\sqrt{2\beta}\right) K_{\alpha}\left(x\sqrt{2\beta}\right) \mathrm{d}x \;-\; \frac{4\varepsilon^{\alpha} x^{1-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)} K_{\alpha}\left(x\sqrt{2\beta}\right) K_{\alpha}\left(\varepsilon\sqrt{2\beta}\right) \mathrm{d}x \\
&= 2^{1-\frac{\alpha}{2}} x^{1-\alpha} \beta^{-\frac{\alpha}{2}} K_{\alpha}\left(x\sqrt{2\beta}\right) \times \\
&\qquad \left\{ \left(\varepsilon\sqrt{2\beta}\right)^{\alpha} \mathcal{I}_{\alpha}\left(\varepsilon\sqrt{2\beta}\right) - \frac{2}{\Gamma(\alpha)\Gamma(1-\alpha)} \left(\varepsilon\sqrt{2\beta}\right)^{\alpha} K_{\alpha}\left(\varepsilon\sqrt{2\beta}\right) \right\} \mathrm{d}x \\
&= 2^{1-\frac{\alpha}{2}} x^{1-\alpha} \beta^{-\frac{\alpha}{2}} K_{\alpha}\left(x\sqrt{2\beta}\right) \left(\varepsilon\sqrt{2\beta}\right)^{\alpha} \mathcal{I}_{-\alpha}\left(\varepsilon\sqrt{2\beta}\right) \mathrm{d}x,
\end{aligned}$$

where we have used the following equation for the modified Bessel functions of first and second kind:

$$K_{\alpha}(v) \;=\; \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{2}\Big[\mathcal{I}_{-\alpha}(v) - \mathcal{I}_{\alpha}(v)\Big].$$

With regard to the limit behaviour as $\varepsilon \to 0$, the product of the last two terms involving ε grows at a rate of $\varepsilon^{2\alpha}$, i.e.

$$\left(\varepsilon\sqrt{2\beta}\right)^{\alpha} \mathcal{I}_{-\alpha}\left(\varepsilon\sqrt{2\beta}\right) \;\sim\; O\left(\varepsilon^{2\alpha}\right), \tag{2.7}$$

and thus

$$\lim_{\varepsilon\to 0} \varepsilon^{-2\alpha} \left(\varepsilon\sqrt{2\beta}\right)^{\alpha} \mathcal{I}_{-\alpha}\left(\varepsilon\sqrt{2\beta}\right) \;=\; \frac{\beta^{\alpha}}{\Gamma(1+\alpha)}. \tag{2.8}$$

Therefore, the limit of the Laplace transform as ε tends to zero is equal to

$$\frac{2^{1-\frac{\alpha}{2}}}{\Gamma(1+\alpha)} \beta^{\frac{\alpha}{2}} x^{1-\alpha} K_{\alpha}\left(x\sqrt{2\beta}\right). \tag{2.9}$$

Inverting it *w.r.t.* β produces (2.3). (2.4) follows immediately by integrating x from 0 to ∞ and the quotient of the two yields (2.5). □

Lemma 2.1.3. *For any* $x, x_0 \in \mathbb{R}^+$ *and* $0 < t < T$,

$$\mathbb{P}(X_T \in \mathrm{d}x \,,\, U_T \in \mathrm{d}t \mid X_0 = x_0) \;=\; \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} x t^{-\alpha-1} (T-t)^{\alpha-1} \exp\left\{-\frac{x^2}{2t} - \frac{{x_0}^2}{2(T-t)}\right\} \mathrm{d}x\,\mathrm{d}t, \tag{2.10}$$

and

$$\mathbb{P}(U_T \in \mathrm{d}t \mid X_0 = x_0) \;=\; \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} t^{-\alpha} (T-t)^{\alpha-1} \exp\left\{-\frac{{x_0}^2}{2(T-t)}\right\} \mathrm{d}t. \tag{2.11}$$

Proof. The idea of proving (2.10) is due to the path decomposition at last zero. Using the strong Markov property of Bessel (age) processes, the equality in probability of the following events is implied:

$$\begin{aligned} & \{X_T = x, U_T = t \mid X_0 = x_0\} \\ = \; & \{X_T = x, X_{T-t} = 0, U_T = t \mid X_0 = x_0\} \\ = \; & \{X_{T-t} = 0 \mid X_0 = x_0\} \cap \left\{\tilde{X}_t = x, \tilde{U}_t = t \;\middle|\; \tilde{X}_0 = 0\right\}, \end{aligned}$$

where $\tilde{X}$ is an independent duplicate of X describing the motion commencing from the last zero, referred by the time $T - t$, and $\tilde{U}$ is the age process defined based on $\tilde{X}$. By path analysis,

$$U_{(T-t)^+} \;=\; \tilde{U}_0 \;=\; X_{(T-t)^+} \;=\; \tilde{X}_0 \;=\; 0\,.$$

It allows us to compute its probability by weak convergence in distribution, i.e.

$$\mathbb{P}(X_T \in \mathrm{d}x, U_T \in \mathrm{d}t \mid X_0 = x_0) \;=\; \lim_{\varepsilon \to 0} \varepsilon^{-1} \mathbb{P}(X_{T-t} \in \mathrm{d}\varepsilon \mid X_0 = x_0) \cdot \mathbb{P}\left(\tilde{X}_t \in \mathrm{d}x, \tilde{U}_t \in \mathrm{d}t \;\middle|\; \tilde{X}_0 = \varepsilon\right). \tag{2.12}$$

Note that the right of the equation is constructed by the measure of the likelihood of X_{T-t} (or $\tilde{X}_0$) at ε-neighbourhood of zero. As $\varepsilon \to 0$, the result turns out to be the probability right at zero. The transition density of $\mathrm{BES}^{(\alpha)}$ gives

$$\mathbb{P}(X_{T-t} \in \mathrm{d}\varepsilon \mid X_0 = x_0) \;=$$

$$\varepsilon^{1-2\alpha}(T-t)^{\alpha-1}\left(\frac{x_0\varepsilon}{T-t}\right)^{\alpha}\mathcal{I}_{\alpha}\left(\frac{x_0\varepsilon}{T-t}\right)\exp\left\{-\frac{{x_0}^2+\varepsilon^2}{2(T-t)}\right\}\mathrm{d}\varepsilon.$$

Following this result and (2.3), (2.12) equates to

$$(T-t)^{\alpha-1}\lim_{\varepsilon\to 0}\left\{\left(\frac{x_0\varepsilon}{T-t}\right)^{\alpha}\mathcal{I}_{\alpha}\left(\frac{x_0\varepsilon}{T-t}\right)\exp\left\{-\frac{{x_0}^2+\varepsilon^2}{2(T-t)}\right\}\frac{\mathbb{P}\left(\tilde{X}_t\in\mathrm{d}x,\tilde{U}_t\in\mathrm{d}t\mid\tilde{X}_0=\varepsilon\right)}{\varepsilon^{2\alpha}}\right\}$$

$$=\frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)}xt^{-\alpha-1}(T-t)^{\alpha-1}\exp\left\{-\frac{{x_0}^2}{2(T-t)}-\frac{x^2}{2t}\right\}\mathrm{d}x\,\mathrm{d}t.$$

Integrating it over x from 0 to ∞ yields (2.11) . □

2.2 Mathematical Framework of Hybrid Model

As expounded in the introduction, we construct a hybrid structural-reduced form model based on the information described by $\mathscr{G}_t=\sigma\{U_s,s\leq t\}$ with an intensity defined endogenously by $\{U_t\}_{t\geq 0}$. This section establishes the mathematical framework underpinning the proposed model. To be more specific, we study the joint process (U_t,t) at the first jump time τ followed a general intensity function of the form $\lambda(U_t)$. Essentially the path of (U_t,t) admits two resources of jumps: one of which is *spontaneous jumps* coming from the self-reflection by $\mathrm{BES}^{(\alpha)}$ upon reaching zero (i.e. the time points when the statistics being observed attain zero), and the other a one-time *endogenous jump* occurring in a Poisson-like fashion with rate depending on the age of the excursion straddling t (i.e. the arrival of default). Within a piecewise-deterministic Markov process (PDMP) framework, a specific type of Markov process initially formalised by Davis [39] to deal with non-diffusion models, distributional properties concerning the stopping time τ and the stopping level U_τ, including the generating functions, Laplace transforms and marginal densities, have been derived.

2.2.1 Perturbed Bessel Process

A peculiar problem in considering the path of $\mathrm{BES}^{(\alpha)}$ is the frequent visits at the regular point zero that results in the occurrence of infinitely many small excursions, thereby producing also correspondingly infinite jumps in the path of the age process. To circumvent this problem, we construct a new process, namely *the perturbed Bessel process*, from the original process X defined in the space $(\Omega,\mathscr{F},\mathbb{P})$ such that for all $t \geq 0$

$$\{\, \omega \in \Omega \mid X_t(\omega) \in B \,\} \in \mathscr{F}_t, \quad \forall B \in \mathcal{B}(\mathbb{R}^+).$$

For $\varepsilon > 0$ and $n \in \mathbb{N}^+$, define a sequence of stopping times by

$$\begin{aligned} \delta_0 &= 0; \\ \sigma_n &= \inf\{\, t > \delta_n \mid X_t = \varepsilon \,\}; \\ \delta_{n+1} &= \inf\{\, t > \sigma_n \mid X_t = 0 \,\}, \end{aligned}$$

and then define the perturbed Bessel process, denoted as X^ε, by

$$X_t^\varepsilon = \begin{cases} \varepsilon - X_t, & \text{if } \delta_n \leq t < \sigma_n; \\ X_t, & \text{if } \sigma_n \leq t < \delta_{n+1}. \end{cases}$$

Accordingly, the age process for the new process is given by

$$U_t^\varepsilon = \begin{cases} t - \delta_n, & \text{if } \delta_n \leq t < \sigma_n; \\ t - \sigma_n, & \text{if } \sigma_n \leq t < \delta_{n+1}. \end{cases}$$

For a graphical illustration of this process, please refer to Figure 2.2 that shows a sample (original) path of X_t and the defined stopping times σ_n and δ_n, and to Figure 2.3 that demonstrates how the perturbation technique has been applied to the original path resulting in a clear pattern of the age process.

In words, the perturbation has been done by chopping up the original path into pieces according to the positions of σ_n and δ_n, and reversing all the parts of the periods $\delta_n \leq t < \sigma_n$ by $(\varepsilon - X_t)$. In this manner, we ensure the associated process U_t^ε conforms to our definition of age process since all the stopping times of σ_n and δ_n are actually describing the last zero points in the perturbed path.

This perturbation approach, aimed at achieving a clear structure of excursions around a regular point, was introduced by Dassios and Wu [36] in their study of the Parisian stopping time in the context of a drifted Brownian motion. Instead of doing path reversions, they do path movements by imposing a jump of ε-size on the process immediately after it reaches zero. In comparison, our method of path reversions is preferable in the respect of coordinating with a wider range of processes involving

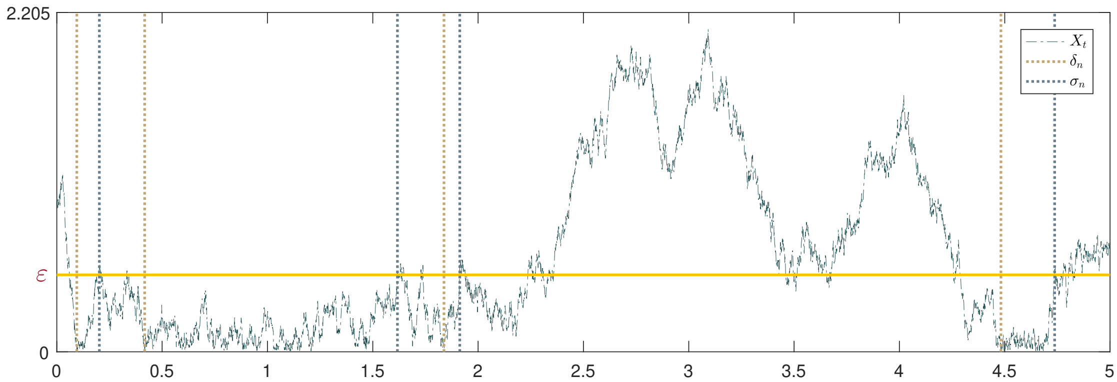


Fig. 2.2 A sample (original) path of X_t and the stopping times σ_n and δ_n.

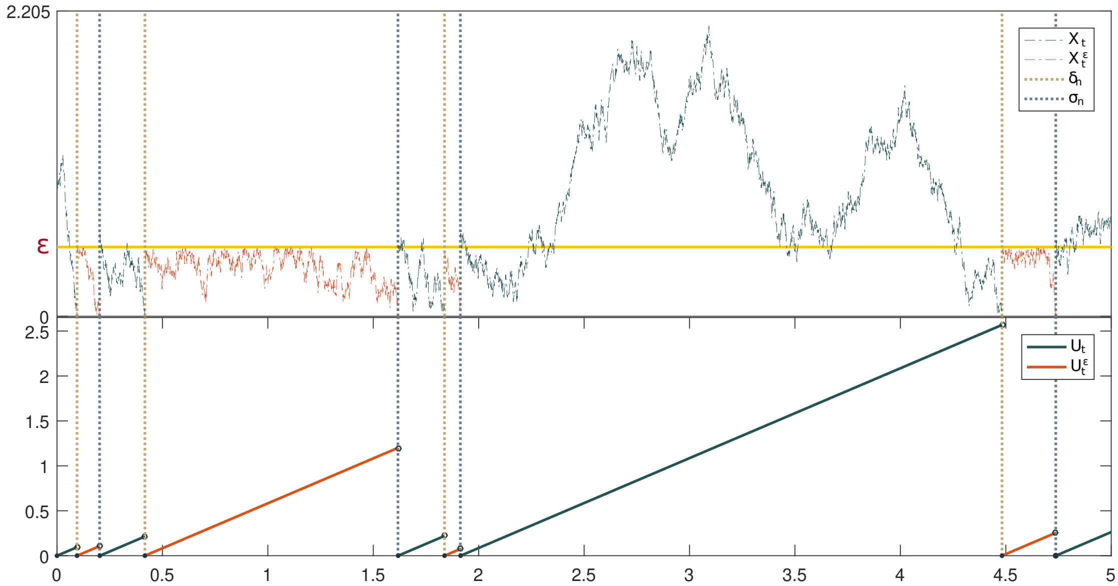


Fig. 2.3 Illustration of the perturbed path X_t^ε versus the associated age process U_t^ε.

reflected processes and non-negative processes. For more applications of this approach to the area of option pricing and barrier strategy, interested readers are referred to the subsequent papers of the same authors and in particular to Dassios and Wu [34, 35, 37, 38].

By such construction, it is clear that X^ε converges pointwise to X, often written as

$$\left\{ \omega \in \Omega \;\middle|\; \lim_{\varepsilon \downarrow 0} X_t^\varepsilon(\omega) = X_t(\omega), \forall t \geq 0 \right\} = \Omega.$$

This implies convergence between (i) variables defined on the basis of X^ε and X, in particular U^ε converges pointwise to U as ε approaches zero, and (ii) expectations of continuous bounded functions of the variables referred to in (i). In this regard, similar convergences for the case of a drifted Brownian motion have been proved in the mentioned [34, 38]. Following an analogous line of reasoning, one can find that it holds for a recurrent Bessel process as well. Consequently, we obtain the results with respect to U_t by carrying out the calculations for U_t^ε, in the course of which ε will be retained without taking the limit to zero until the very last step.

2.2.2 Martingale Problem and Generating Function

To find a martingale on the joint perturbed process $\left(U_t^\varepsilon, t\right)$, we refer to the framework of PDMP. According to it, $\left(U_t^\varepsilon, t\right)$ is characterised by two components: namely,

- *flow*: the motion between spontaneous jumps in the new age process U^ε is increasing at unit rate due to the time characteristics of age process; and
- *transition intensity*, the likelihood of observing a spontaneous jump in the next instant of time is formulated as

$$\frac{p_\varepsilon(u)}{\bar{P}_\varepsilon(u)} = \frac{\dfrac{\varepsilon^{2\alpha}}{2^\alpha \Gamma(\alpha)} u^{-\alpha-1} e^{-\frac{\varepsilon^2}{2u}}}{\displaystyle\int_u^\infty \frac{\varepsilon^{2\alpha}}{2^\alpha \Gamma(\alpha)} x^{-\alpha-1} e^{-\frac{\varepsilon^2}{2x}}\,\mathrm{d}x},$$

  where $p_\varepsilon(u)$ and $\bar{P}_\varepsilon(u)$ denote the density function and the tail distribution, respectively, of the first hitting time of zero for the perturbed Bessel process X^ε starting from ε.

The theory of PDMP provides a certain infinitesimal generator for $\left(U_t^\varepsilon, t\right)$. Consider a bounded function $f : \mathbb{R}^2 \to \mathbb{R}$, we define $\mathcal{A}$ as the operator making

$$f\left(U_t^\varepsilon, t\right) - \int_0^t \mathcal{A} f\left(U_s^\varepsilon, s\right) \mathrm{d}s,$$

a martingale and we have

$$\begin{aligned} \mathcal{A} f(u,t) &= \frac{\partial f}{\partial t} + \mathcal{A}_u f \\ &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial u} + \frac{p_\varepsilon(u)}{\bar{P}_\varepsilon(u)}\Big(f(0,t) - f(u,t)\Big). \end{aligned} \tag{2.13}$$

A suitable candidate of martingale of the form $f\left(U_t^\varepsilon, t\right)$ can be obtained from solving $\mathcal{A} f = 0$. In our hybrid model, the default time τ is defined in the way that the probability of a company to default in the infinitesimal interval $\left[t,\ t + \mathrm{d}t\right)$ having not defaulted before t is :

$$\mathbb{P}(\tau \in [\ t,\ t + \mathrm{d}t\ ) \mid \tau \geq t) \ = \ \lambda(U_t)\,\mathrm{d}t, \tag{2.14}$$

with $\lambda(U_t)$, a hazard rate, representing an instantaneous credit spread. Having constructed the above, we can find the generating functions for the joint stopping process $\left(U_\tau^\varepsilon, \tau^\varepsilon\right)$ and thus $(U_\tau,\ \tau)$ by taking ε to zero.

To simplify expressions, we adopt the following convention in subsequent calculations that for all $t \geq 0$

$$\Lambda(t) := \int_0^t \lambda(s)\,\mathrm{d}s, \qquad \text{and} \qquad \Lambda'(t) \ = \ \lambda(t). \tag{2.15}$$

Theorem 2.2.1 (Generating Functions)**.** *For $\kappa(\cdot)$ and $\lambda(\cdot)$ to be non-negative bounded functions, $\beta \in \mathbb{R}^+$ and $0 < \alpha < 1$ to be constants, and τ to be the first jump time with the intensity $\lambda(U^\varepsilon)$, the probability generating function of $\left(U_\tau^\varepsilon, \tau^\varepsilon\right)$ is given by*

$$\mathbb{E}\left[e^{-\beta\tau^\varepsilon}\kappa\left(U_\tau^\varepsilon\right)\right] \ = \ \frac{\displaystyle\int_0^\infty e^{-\beta v}\kappa(v)\lambda(v)e^{-\Lambda(v)}\bar{P}_\varepsilon(v)\,\mathrm{d}v}{\displaystyle\int_0^\infty (\beta + \lambda(v))e^{-\beta v}e^{-\Lambda(v)}\bar{P}_\varepsilon(v)\,\mathrm{d}v}. \tag{2.16}$$

Taking limit of ε to zero yields the generating function of (U_τ, τ) with τ being the default time dominated by the intensity $\lambda(U)$,

$$\mathbb{E}\left[e^{-\beta\tau}\kappa(U_\tau)\right] \ = \ \frac{\displaystyle\int_0^\infty e^{-\beta v}\kappa(v)\lambda(v)e^{-\Lambda(v)}v^{-\alpha}\,\mathrm{d}v}{\displaystyle\int_0^\infty (\beta + \lambda(v))e^{-\beta v}e^{-\Lambda(v)}v^{-\alpha}\,\mathrm{d}v}. \tag{2.17}$$

Proof. Let $\mathscr{G}_t^\varepsilon = \boldsymbol{\sigma}\left\{U_s^\varepsilon,\ s \leq t\right\}$ denote the σ-algebra generated by the information of U^ε up to time t. According to (2.14) the definition of τ , the probability of the endogenous jump occurring in $[\ t,\ t + \mathrm{d}t\ )$ is given by

$$\mathbb{P}\left(\tau^\varepsilon \in \mathrm{d}t \,\middle|\, \mathscr{G}_t^\varepsilon\right) \ = \ \lambda\left(U_t^\varepsilon\right)\ \exp\left\{-\int_0^t \lambda\left(U_s^\varepsilon\right)\mathrm{d}s\right\}\mathrm{d}t.$$

Based on this result, we formulate the generating function of $\big(U^{\varepsilon}_{\tau}, \tau^{\varepsilon}\big)$ as follows

$$\mathbb{E}\big[e^{-\beta\tau^{\varepsilon}}\kappa\big(U^{\varepsilon}_{\tau}\big)\big] \;=\; \int_0^{\infty} e^{-\beta t}\,\mathbb{E}\Big[\kappa\big(U^{\varepsilon}_t\big)\lambda\big(U^{\varepsilon}_t\big)\exp\Big\{-\int_0^t \lambda\big(U^{\varepsilon}_s\big)\mathrm{d}s\Big\}\Big]\,\mathrm{d}t. \tag{2.18}$$

To find the representation of (2.18), we apply the Feynman-Kac Theory to the PDMP framework constructed before for $\big(U^{\varepsilon}_t, t\big)$. First, we extend the joint process by adding two more components, the continuous processes

$$Y^{\varepsilon}_t \;=\; \int_0^t \lambda\big(U^{\varepsilon}_s\big)\,\mathrm{d}s \qquad \text{and} \qquad Z^{\varepsilon}_t \;=\; \int_0^t e^{-\beta s}\,\kappa\big(U^{\varepsilon}_s\big)\lambda\big(U^{\varepsilon}_s\big)e^{-Y^{\varepsilon}_s}\,\mathrm{d}s.$$

Please note that $\lambda(\cdot)$ and $\kappa(\cdot)$ are provisionally set to be arbitrary bounded functions and they will be chosen in the way that contributes to the achievement of various distributionals. We then define another generator $\mathcal{G}$ for the expanded process $\big(U^{\varepsilon}_t, Y^{\varepsilon}_t, Z^{\varepsilon}_t, t\big)$ acting on a function $f(u, y, z, t)$ in its domain as

$$\mathcal{G}f(u, y, z, t) \;=\; \frac{\partial f}{\partial t} + \lambda(u)\frac{\partial f}{\partial y} + e^{-\beta t}\kappa(u)\lambda(u)e^{-y}\frac{\partial f}{\partial z} + \mathcal{A}_u f.$$

Substituting $f(u, y, z, t) = z + e^{-\beta t}e^{-y}g(u)$ with $g(\cdot)$ assumed to be a bounded function into $\mathcal{G}f = 0$ generates

$$g'(u) - g(u)\bigg(\beta + \frac{p_{\varepsilon}(u)}{\bar{P}_{\varepsilon}(u)} + \lambda(u)\bigg) + g(0)\frac{p_{\varepsilon}(u)}{\bar{P}_{\varepsilon}(u)} + \kappa(u)\lambda(u) \;=\; 0.$$

Solving this differential equation gives,

$$g(u)e^{-\beta u}\bar{P}_{\varepsilon}(u)e^{-\Lambda(u)} \;=\;$$
$$g(0)\int_u^{\infty} e^{-\beta v}e^{-\Lambda(v)}p_{\varepsilon}(v)\,\mathrm{d}v \;+\; \int_u^{\infty} e^{-\beta v}\kappa(v)\lambda(v)e^{-\Lambda(v)}\bar{P}_{\varepsilon}(v)\,\mathrm{d}v.$$

Setting $u = 0$,

$$g(0) \;=\; \frac{\displaystyle\int_0^{\infty} e^{-\beta v}\kappa(v)\bar{P}_{\varepsilon}(v)\lambda(v)e^{-\Lambda(v)}\,\mathrm{d}v}{1 - \displaystyle\int_0^{\infty} e^{-\beta v}p_{\varepsilon}(v)e^{-\Lambda(v)}\,\mathrm{d}v}.$$

By the property of PDMP, we get a martingale in the form

$$f\left(U_t^{\varepsilon}, Y_t^{\varepsilon}, Z_t^{\varepsilon}, t\right) \;=\; Z_t^{\varepsilon} \;+\; e^{-\beta t} e^{-Y_t^{\varepsilon}} g\left(U_t^{\varepsilon}\right),$$

and therefore

$$\lim_{t\to\infty} \mathbb{E}\left[Z_t^{\varepsilon} \;+\; e^{-\beta t} e^{-Y_t^{\varepsilon}} g\left(U_t^{\varepsilon}\right)\right] \;=\; \lim_{t\to\infty} \mathbb{E}\left[Z_t^{\varepsilon}\right] \;=\; g(0).$$

This completes the proof of (2.16) and (2.17) follows immediately by taking ε to zero. □

2.2.3 Distributional Results

This section presents some other distributional results obtained from a proper choice of $\kappa(\cdot)$ in the generating functions. In particular, we find exact representations of the Laplace transforms of U_τ and τ, and get the marginal density of U_τ.

Theorem 2.2.2. *For $\beta, \varphi \in \mathbb{R}^+$ and $0 < \alpha < 1$, we find the joint Laplace transform of as*

$$\mathbb{E}\left[e^{-\beta\tau} e^{-\varphi U_\tau}\right] \;=\; \frac{\int_0^\infty e^{-\beta v} e^{-\varphi v} \lambda(v) e^{-\Lambda(v)} v^{-\alpha} \, \mathrm{d}v}{\int_0^\infty (\beta + \lambda(v)) e^{-\beta v} e^{-\Lambda(v)} v^{-\alpha} \, \mathrm{d}v}. \tag{2.19}$$

Proof. This result follows immediately by replacing $\kappa(u)$ in the generating function (2.17) with $e^{-\varphi u}$. □

Corollary 2.2.3. *Based on the joint Laplace transform, we have got the Laplace transform of U_τ,*

$$\mathbb{E}\left[e^{-\varphi U_\tau}\right] \;=\; \frac{\int_0^\infty e^{-\varphi v} \lambda(v) e^{-\Lambda(v)} v^{-\alpha} \, \mathrm{d}v}{\int_0^\infty \lambda(v) e^{-\Lambda(v)} v^{-\alpha} \, \mathrm{d}v}, \tag{2.20}$$

from which we extract the density function of U_τ,

$$f_{U_\tau}(u) = \frac{\lambda(u)e^{-\Lambda(u)}u^{-\alpha}}{\int_0^\infty \lambda(v)e^{-\Lambda(v)}v^{-\alpha}\,\mathrm{d}v}; \tag{2.21}$$

and the Laplace transform of τ,

$$\mathbb{E}\left[e^{-\beta\tau}\right] = \frac{\int_0^\infty e^{-\beta v}\lambda(v)e^{-\Lambda(v)}v^{-\alpha}\,\mathrm{d}v}{\int_0^\infty (\beta+\lambda(v))e^{-\beta v}e^{-\Lambda(v)}v^{-\alpha}\,\mathrm{d}v}. \tag{2.22}$$

For the conditional case, we also get

$$\mathbb{E}\left[e^{-\beta\tau}\,\middle|\,U_\tau\right] = \frac{e^{-\beta U_\tau}\int_0^\infty \lambda(v)e^{-\Lambda(v)}v^{-\alpha}\,\mathrm{d}v}{\int_0^\infty (\beta+\lambda(v))e^{-\beta v}e^{-\Lambda(v)}v^{-\alpha}\,\mathrm{d}v}. \tag{2.23}$$

Proof. (2.20) and (2.22) are obtained by setting β and φ zero, respectively, in the joint Laplace transform (2.19). The density of U_τ comes naturally from the representation of its Laplace transform. For the last result, it comes from the representation

$$\mathbb{E}\left[e^{-\beta\tau}\right] = \int_0^\infty \mathbb{E}\left[e^{-\beta\tau}\,\middle|\,U_\tau = u\right] f_{U_\tau}(u)\,\mathrm{d}u.$$

□

Upon realising that a closed-form solution to the density of τ is unattainable, we provide an alternative way of studying the randomness of τ. Specifically, we discover a decomposition rule for τ that it can be decomposed into U_τ, the stopping level of the age process and $\mathcal{V}_\varrho$, a Lévy process stopped at a unit exponential time.

Lemma 2.2.4. *For $0 < \alpha < 1$, the Laplace transform of τ can be decomposed as follows*

$$\mathcal{L}_\beta(\tau) \;=\; \mathcal{L}_\beta(U_\tau) \cdot \int_0^\infty e^{-v} \;\exp\left\{ -\frac{v}{\eta} \int_0^\infty \left(1 - e^{-\beta x}\right) e^{-\Lambda(x)} \alpha x^{-\alpha-1}\, \mathrm{d}x \right\} \mathrm{d}v, \tag{2.24}$$

with $\mathcal{L}_\beta(U_\tau)$ denotes the Laplace transform of U_τ with respect to β and

$$\eta \;=\; \int_0^\infty \lambda(v) e^{-\Lambda(v)} v^{-\alpha}\, \mathrm{d}v. \tag{2.25}$$

Proof. Multiplying the constant term $\int_0^\infty \lambda(v) e^{-\Lambda(v)} v^{-\alpha}\, \mathrm{d}v$ to both the numerator and the denominator of the result (2.22), we have

$$\mathbb{E}\left[e^{-\beta\tau}\right] \;=\; \frac{\int_0^\infty e^{-\beta v} \lambda(v) e^{-\Lambda(v)} v^{-\alpha}\, \mathrm{d}v}{\int_0^\infty \lambda(v) e^{-\Lambda(v)} v^{-\alpha}\, \mathrm{d}v} \frac{\int_0^\infty \lambda(v) e^{-\Lambda(v)} v^{-\alpha}\, \mathrm{d}v}{\int_0^\infty (\beta + \lambda(v)) e^{-\beta v} e^{-\Lambda(v)} v^{-\alpha}\, \mathrm{d}v}.$$

The first fraction recovers the Laplace transform of U_τ. Considering the denominator of the second fraction, we first rewrite it as

$$\int_0^\infty \beta e^{-\beta v} e^{-\Lambda(v)} v^{-\alpha}\, \mathrm{d}v \;+\; \int_0^\infty e^{-\beta v} \lambda(v) e^{-\Lambda(v)} v^{-\alpha}\, \mathrm{d}v, \tag{2.26}$$

and then substitute

$$e^{-\Lambda(v)} v^{-\alpha} \;=\; \int_v^\infty \lambda(x) e^{-\Lambda(x)} x^{-\alpha}\, \mathrm{d}x \;+\; \int_v^\infty e^{-\Lambda(x)} \alpha x^{-\alpha-1}\, \mathrm{d}x,$$

into the first integral in (2.26) and thus get

$$\int_0^\infty \beta e^{-\beta v} \int_v^\infty \lambda(x) e^{-\Lambda(x)} x^{-\alpha}\, \mathrm{d}x\, \mathrm{d}v \;+\; \int_0^\infty \beta e^{-\beta v} \int_v^\infty e^{-\Lambda(x)} \alpha x^{-\alpha-1}\, \mathrm{d}x\, \mathrm{d}v$$
$$+ \int_0^\infty e^{-\beta v} \lambda(v) e^{-\Lambda(v)} v^{-\alpha}\, \mathrm{d}v.$$

Changing the order of integrals gives

$$\int_0^\infty \lambda(x) e^{-\Lambda(x)} x^{-\alpha}\, \mathrm{d}x \; + \; \int_0^\infty \left(1 - e^{-\beta x}\right) e^{-\Lambda(x)} \alpha x^{-\alpha-1}\, \mathrm{d}x,$$

which completes the proof. □

The greater details on the decomposition along with necessary conditions are given in the following theorem.

Theorem 2.2.5 (*Decomposition Rule*)**.** *There exists a decomposition for τ in the form:*

$$\tau \overset{\mathcal{D}}{=} U_\tau \; + \; \mathcal{V}_\varrho, \tag{2.27}$$

where

- U_τ *is the stopping level of the age process ; and*
- $\mathcal{V}_\varrho$ *is a Lévy process stopped at a unit exponential time, i.e.* $\varrho \sim \exp(1)$*, whose measure is described by*
$$\Pi(\mathrm{d}x) \;=\; \frac{e^{-\Lambda(x)} \alpha x^{-\alpha-1}}{\eta}\, \mathrm{d}x, \quad x \;>\; 0 \tag{2.28}$$
with η defined by (2.25) *and satisfying* $\int_{\mathbb{R}^+} \left(1 \wedge x^2\right) \Pi(\mathrm{d}x) < \infty.$

Proof. Clearly, (2.24) is expressed as the product of the Laplace transforms of U_τ and $\mathcal{V}_\varrho$. By the convolution theorem, the result follows. □

At this stage, it is difficult to attribute $\mathcal{V}_\varrho$ further to any specific subclass, as one notice in the above that the choice of $\lambda(\cdot)$ acting on the Lévy measure determines the existence and characteristics of such process. Without loss of generality, we have selected a piecewise constant intensity, represented of the form:

$$\lambda(U_t) \;=\; \lambda_1 \mathbb{1}_{\{U_t \,<\, c\}} \;+\; \lambda_2 \mathbb{1}_{\{U_t \,\geq\, c\}}, \qquad \lambda_1 \geq 0,\; \lambda_2, c > 0\;.$$

to continue the study in terms of simulation. This appears in CHAPTER 3: Application to Credit Risk Modelling with Exact Simulation Scheme.

Appendix I : A Supplementary Proof for PROPOSITION 2.1.1.

Proof. With the above construction of the Bessel process, we define $Y = X^2$, a squared Bessel of the same dimension $2(1-\alpha)$ with $0 < \alpha < 1$ and starting value y. Considering a bounded function $f : \mathbb{R}^2 \to \mathbb{R}$, we define a generator $\mathcal{A}$ acting on the function $f(y,t)$ by,

$$\mathcal{A}f(y,t) = \frac{\partial f}{\partial t} + 2(1-\alpha)\frac{\partial f}{\partial y} + 2y\frac{\partial^2 f}{\partial y^2}. \tag{2.29}$$

Any candidate satisfying $\mathcal{A}f(y,t) = 0$ subject to certain conditions makes itself a martingale of the form $f(Y_t, t)$, to which allow the application of optional sampling theorem in order to obtain the Laplace transform of the stopping time/level.

Assuming f is of the following form,

$$e^{-\beta t}\int_0^\infty e^{-yu}h(u)\,\mathrm{d}u, \tag{2.30}$$

where $h(u)$ is a bounded function restricted to: $\lim_{u\to\infty} e^{-yu}u^2h(u) = 0$. Applying (2.30) to (2.29) and then equating the latter to zero give

$$-\beta\int_0^\infty e^{-yu}h(u)\,\mathrm{d}u - 2(1-\alpha)\int_0^\infty e^{-yu}uh(u)\,\mathrm{d}u + 2y\int_0^\infty e^{-yu}u^2h(u)\,\mathrm{d}u = 0,$$

from which we extract

$$-\beta h(u) - 2(1-\alpha)uh(u) + 2\tfrac{\partial}{\partial u}\left\{ u^2h(u) \right\} = 0. \tag{2.31}$$

By solving it we obtain, for c to be an arbitrary constant,

$$h(u) = cu^{-\alpha-1}e^{-\frac{\beta}{2u}}. \tag{2.32}$$

The martingale is thus found as,

$$f(Y_t,t) = ce^{-\beta t}\int_0^\infty e^{-Y_t u}u^{-\alpha-1}e^{-\frac{\beta}{2u}}\,\mathrm{d}u. \tag{2.33}$$

Define $T^Y_{y\to 0} \coloneqq \inf\{\, t > 0 \mid Y_t = 0 \,\}$ to be the first hitting time of level zero for the squared Bessel process. With $y = x^2$, $T^Y_{y\to 0}$ is identical in law to $T_{x\to 0}$. According to the optional sampling theorem, the Laplace transform of $T^Y_{y\to 0}$ is obtained by ,

$$\mathbb{E}\left[e^{-\beta T^Y_{y\to 0}}\right] = \frac{\beta^\alpha}{2^\alpha \Gamma(\alpha)} \int_0^\infty e^{-yu} u^{-\alpha-1} e^{-\frac{\beta}{2u}} \,\mathrm{d}u.$$

Change of variable $u = \frac{\beta t}{y}$,

$$\mathbb{E}\left[e^{-\beta T^Y_{y\to 0}}\right] = \frac{y^\alpha}{2^\alpha \Gamma(\alpha)} \int_0^\infty e^{-\beta t} t^{-\alpha-1} e^{-\frac{y}{2t}} \,\mathrm{d}t.$$

Replacing y with x^2 completes the proof. □

3

Application to Credit Risk Modelling with Exact Simulation Scheme

Following the *Decomposition Rule* (THEOREM 2.2.5) in Chapter 2, we compute default probabilities by developing exact simulation schemes for the joint stopping process (U_τ, τ). The intensity at this stage has been chosen in the form of a piecewise constant function, whose description along with the implication is given in Section 3.1. The motivation of this choice, in a nutshell, is that an intensity (or a hazard rate) in such form has been widely used in survival analysis due to its strength in coordinating with recurrent event processes, e.g. visiting zero by U_t in our case, while achieving a high degree of flexibility and robustness [71]. The simulation algorithm for the joint process is presented in Section 3.3. To assess the accuracy of the algorithm and its performance in evaluating default risk, a numerical study for the case of reflected Brownian motion (i.e. the Bessel of dimension $\delta = 1$) is carried out in Section 3.4.

3.1 Piecewise constant Intensity Function

What we have achieved before is conducive to investigating default time from the perspective of simulation. To demonstrate this, particular attention is given to a default intensity defined by a two-step function such that for all $\lambda_1 \geq 0,\ \lambda_2, c > 0$,

$$\lambda^*(U_t) \ = \ \lambda_1 \mathbb{1}_{\{U_t < c\}} \ + \ \lambda_2 \mathbb{1}_{\{U_t \geq c\}} \ . \tag{3.1}$$

That is to say, given the current age smaller than a predetermined level c, the conditional probability of default arriving within $\triangle t$-period of time is

$$\mathbb{P}(\tau \leq t + \triangle t \mid \tau > t, U_t < c) = \lambda_1 \triangle t + o(\triangle t),$$

otherwise, the probability is

$$\mathbb{P}(\tau \leq t + \triangle t \mid \tau > t, U_t \geq c) = \lambda_2 \triangle t + o(\triangle t).$$

We presume that conditioned on whether the age level exceeds c at time t, a firm is afflicted with different loads of default risk. In terms of economic interpretation, the case $\lambda_1 > \lambda_2$ is applicable to the scenario where the underlying process X_t represents some unobservable *positive* statistics and U_t is constructed from the known points of time when they fall down to zero. Following it, if the firm's revenue (e.g.) remains above zero for a period longer than c, the intensity of default is thought to downgrade from λ_1 to λ_2. Conversely, the case $\lambda_1 < \lambda_2$ applies to that where X_t represents some unobservable *negative* statistics and U_t records the time elapsed since the last time they are cleared to zero. If a period of length c has passed and the debt (e.g.) remains outstanding, the default intensity as a consequence upgrades from λ_1 to λ_2.

3.2 Distributional Results

In the sequel, for simplicity in presenting results we denote by

$$\begin{aligned} \gamma(\alpha, z) &= \int_0^z e^{-t}\, t^{\alpha-1}\, \mathrm{d}t; \\ \Gamma(\alpha, z) &= \int_z^\infty e^{-t}\, t^{\alpha-1}\, \mathrm{d}t, \end{aligned}$$

the lower and upper incomplete gamma functions.

We first update THEOREM 2.2.2 and its COROLLARY 2.2.3 with the chosen intensity (3.1). New results are given explicitly in closed form.

Theorem 3.2.1. *Given τ^* the default time driven by $\lambda^*(U)$, the joint Laplace transform of (U_{τ^*}, τ^*) is obtained by*

$$\mathbb{E}\left[e^{-\beta\tau^*} e^{-\varphi U_{\tau^*}}\right] = \frac{\dfrac{\lambda_1}{\beta+\varphi+\lambda_1}\eta^-(\beta+\varphi+\lambda_1) + \dfrac{\lambda_2}{\beta+\varphi+\lambda_2}\eta^+(\beta+\varphi+\lambda_2)}{\eta^-(\beta+\lambda_1)+\eta^+(\beta+\lambda_2)},$$

and we have defined functions η^- and η^+ to be

$$\eta^-(s) = e^{\lambda_1 c} s^{\alpha} \gamma(1-\alpha, sc); \tag{3.2}$$

$$\eta^+(s) = e^{\lambda_2 c} s^{\alpha} \Gamma(1-\alpha, sc). \tag{3.3}$$

Corollary 3.2.2. *We hence have the Laplace transform of U_{τ^*},*

$$\mathbb{E}\left[e^{-\varphi U_{\tau^*}}\right] = \frac{\dfrac{\lambda_1}{\varphi+\lambda_1}\eta^-(\varphi+\lambda_1) + \dfrac{\lambda_2}{\varphi+\lambda_2}\eta^+(\varphi+\lambda_2)}{\eta^-(\lambda_1)+\eta^+(\lambda_2)}; \tag{3.4}$$

the density function of U_{τ^*},

$$f_{U_{\tau^*}}(u) = \frac{\lambda_1 e^{-\lambda_1(u-c)} u^{-\alpha} \mathbb{1}_{\{u<c\}} + \lambda_2 e^{-\lambda_2(u-c)} u^{-\alpha} \mathbb{1}_{\{u \geq c\}}}{\eta^-(\lambda_1)+\eta^+(\lambda_2)};$$

the Laplace transform of τ^*,

$$\mathbb{E}\left[e^{-\beta\tau^*}\right] = \frac{\dfrac{\lambda_1}{\beta+\lambda_1}\eta^-(\beta+\lambda_1) + \dfrac{\lambda_2}{\beta+\lambda_2}\eta^+(\beta+\lambda_2)}{\eta^-(\beta+\lambda_1)+\eta^+(\beta+\lambda_2)};$$

and the Laplace transform of τ conditioned on U_{τ^*},

$$\mathbb{E}\left[e^{-\beta\tau^*} \,\middle|\, U_{\tau^*}\right] = e^{-\beta U_{\tau^*}} \frac{\eta^-(\lambda_1)+\eta^+(\lambda_2)}{\eta^-(\beta+\lambda_1)+\eta^+(\beta+\lambda_2)}. \tag{3.5}$$

Proof. The results from **Theorem** 3.2.1 and **Corollary** 3.2.2 are simply obtained by substituting $\lambda(u) = \lambda_1 \mathbb{1}_{\{u<c\}} + \lambda_2 \mathbb{1}_{\{u \geq c\}}$ into the results (2.19) – (2.23) from the last chapter. □

3.3 Exact Simulation Algorithm

This section is devoted to developing simulation schemes for the default time τ^* and the default level U_{τ^*} governed by the new intensity measure. Following the τ-decomposition in THEOREM 2.2.5, we impose an upgrade on the Lévy part by decomposing it to an exhaustive extent, with each component corresponding to a well-defined random variable that allows exact simulation. The simulation method is straightforward and fast in implementation and avoids generating sample paths in the entire process thus eliminating discretization bias.

3.3.1 Simulation Scheme for the Stopping Level

The algorithm for generating N samples of U_{τ^*} is given as follows:

Algorithm 1 : Generating N samples of U_{τ^*}

(I) Generate n indicator variates, $\mathbf{u} = \{ u_1, u_2, \cdots, u_n \}$, from a Bernoulli distribution with success rate

$$p = \frac{\eta^{-}(\lambda_1)}{\eta^{-}(\lambda_1) + \eta^{+}(\lambda_2)} \tag{3.6}$$

where $\eta^{-}(\cdot)$ and $\eta^{+}(\cdot)$ are given by (3.2) and (3.3).

(II) Compute two parameters: $l_1 = \sum_{i=1}^{n} u_i$ and $l_2 = n - l_1$.

(III) Generate l_1 variates, $\mathbf{x} = \left\{ x_1, x_2, \cdots, x_{l_1} \right\}$, from a left-truncated gamma distribution with density:

$$\tilde{f}_1(x) = \frac{e^{-x} x^{-\alpha}}{\gamma(1-\alpha, c\lambda_1)} \mathbb{1}_{\{x < c\lambda_1\}}; \tag{3.7}$$

and rescale them by the factor $\frac{1}{\lambda_1}$, i.e. $\tilde{\mathbf{x}} = \frac{\mathbf{x}}{\lambda_1}$.

(IV) Generate l_2 variates, $\mathbf{y} = \left\{ y_1, y_2, \cdots, y_{l_2} \right\}$, from a right-truncated gamma distribution with density:

$$\tilde{f}_2(x) = \frac{e^{-x} x^{-\alpha}}{\Gamma(1-\alpha, c\lambda_2)} \mathbb{1}_{\{x \geq c\lambda_2\}}. \tag{3.8}$$

and rescale them by $\frac{1}{\lambda_2}$, i.e. $\tilde{\mathbf{y}} = \frac{\mathbf{y}}{\lambda_2}$.

(V) Combine $\tilde{\mathbf{x}}$ with $\tilde{\mathbf{y}}$ in order of the one-zero distribution of $\mathbf{u}$.

Proof. We start by rewriting the Laplace transform of U_{τ^*} in (3.4) into the following form

$$\mathbb{E}\left[e^{-\varphi U_{\tau^*}}\right] = \frac{\lambda_1 \int\limits_0^c e^{-\lambda_1 v} e^{-\varphi v} v^{-\alpha}\, \mathrm{d}v + \lambda_2 e^{-(\lambda_1-\lambda_2)c} \int\limits_c^\infty e^{-\lambda_2 v} e^{-\varphi v} v^{-\alpha}\, \mathrm{d}v}{\lambda_1 \int\limits_0^c e^{-\lambda_1 v} v^{-\alpha}\, \mathrm{d}v + \lambda_2 e^{-(\lambda_1-\lambda_2)c} \int\limits_c^\infty e^{-\lambda_2 v} v^{-\alpha}\, \mathrm{d}v}$$

$$= \int\limits_0^\infty e^{-\varphi v} \frac{\lambda_1 e^{\lambda_1 c} e^{-\lambda_1 v} v^{-\alpha} \mathbb{1}_{\{v\,<\,c\}} + \lambda_2 e^{\lambda_2 c} e^{-\lambda_2 v} v^{-\alpha} \mathbb{1}_{\{v\,\geq\,c\}}}{\lambda_1 e^{\lambda_1 c} \int\limits_0^c e^{-\lambda_1 v} v^{-\alpha}\, \mathrm{d}v + \lambda_2 e^{\lambda_2 c} \int\limits_c^\infty e^{-\lambda_2 v} v^{-\alpha}\, \mathrm{d}v}\, \mathrm{d}v,$$

from which we extract the density of U_{τ^*},

$$f_{U_{\tau^*}}(u) = \frac{\lambda_1 e^{\lambda_1 c} e^{-\lambda_1 u} u^{-\alpha}}{\lambda_1 e^{\lambda_1 c} \int\limits_0^c e^{-\lambda_1 v} v^{-\alpha}\, \mathrm{d}v + \lambda_2 e^{\lambda_2 c} \int\limits_c^\infty e^{-\lambda_2 v} v^{-\alpha}\, \mathrm{d}v} \mathbb{1}_{\{u\,<\,c\}}$$
$$+ \frac{\lambda_2 e^{\lambda_2 c} e^{-\lambda_2 u} u^{-\alpha}}{\lambda_1 e^{\lambda_1 c} \int\limits_0^c e^{-\lambda_1 v} v^{-\alpha}\, \mathrm{d}v + \lambda_2 e^{\lambda_2 c} \int\limits_c^\infty e^{-\lambda_2 v} v^{-\alpha}\, \mathrm{d}v} \mathbb{1}_{\{u\,\geq\,c\}}.$$

Multiplying the top and bottom of the first fraction with $\int_0^c e^{-\lambda_1 v} v^{-\alpha}\, \mathrm{d}v$ and of the second with $\int_c^\infty e^{-\lambda_2 v} v^{-\alpha}\, \mathrm{d}v$, we get

$$\frac{\lambda_1 e^{\lambda_1 c} \int\limits_0^c e^{-\lambda_1 v} v^{-\alpha}\, \mathrm{d}v}{\lambda_1 e^{\lambda_1 c} \int\limits_0^c e^{-\lambda_1 v} v^{-\alpha}\, \mathrm{d}v + \lambda_2 e^{\lambda_2 c} \int\limits_c^\infty e^{-\lambda_2 v} v^{-\alpha}\, \mathrm{d}v} \frac{e^{-\lambda_1 u} u^{-\alpha}}{\int\limits_0^c e^{-\lambda_1 v} v^{-\alpha}\, \mathrm{d}v} \mathbb{1}_{\{u\,<\,c\}}$$
$$+ \frac{\lambda_2 e^{\lambda_2 c} \int\limits_c^\infty e^{-\lambda_2 v} v^{-\alpha}\, \mathrm{d}v}{\lambda_1 e^{\lambda_1 c} \int\limits_0^c e^{-\lambda_1 v} v^{-\alpha}\, \mathrm{d}v + \lambda_2 e^{\lambda_2 c} \int\limits_c^\infty e^{-\lambda_2 v} v^{-\alpha}\, \mathrm{d}v} \frac{e^{-\lambda_2 u} u^{-\alpha}}{\int\limits_c^\infty e^{-\lambda_2 v} v^{-\alpha}\, \mathrm{d}v} \mathbb{1}_{\{u\,\geq\,c\}}$$
$$= p \cdot f_1(u) + (1-p) \cdot f_2(u),$$

where

$$p = \frac{\lambda_1 e^{\lambda_1 c} \int\limits_0^c e^{-\lambda_1 v} v^{-\alpha} \, \mathrm{d}v}{\lambda_1 e^{\lambda_1 c} \int\limits_0^c e^{-\lambda_1 v} v^{-\alpha} \, \mathrm{d}v + \lambda_2 e^{\lambda_2 c} \int\limits_c^\infty e^{-\lambda_2 v} v^{-\alpha} \, \mathrm{d}v},$$

$$f_1(u) = \frac{e^{-\lambda_1 u} u^{-\alpha}}{\int\limits_0^c e^{-\lambda_1 v} v^{-\alpha} \, \mathrm{d}v} \mathbb{1}_{\{u < c\}} \qquad \text{and} \qquad f_2(u) = \frac{e^{-\lambda_2 u} u^{-\alpha}}{\int\limits_c^\infty e^{-\lambda_2 v} v^{-\alpha} \, \mathrm{d}v} \mathbb{1}_{\{u \geq c\}}.$$

It is apparent that the result of the above form is a convex combination of two density functions: f_1 (a left-truncated gamma) and f_2 (a right-truncated gamma). The simulation of one variate from such a mixed distribution with weights specified as p and $1-p$ can be undertaken in two steps. We first generate an indicator variate from a Bernoulli distribution with success rate p and indicated by which we then generate another variate from either f_1 or f_2. Note that in order to improve the efficiency of simulating the two truncated gamma functions, we have rescaled the random variable by defining $x = \lambda_1 u$ for its support in $(0, c)$ and $x = \lambda_2 u$ for $u \in (c, \infty)$. The resulting density functions are the results shown by (3.7) and (3.8). □

3.3.2 Simulation Scheme for the Stopping Time

According to the decomposition rule in THEOREM 2.2.5, the difference by subtracting τ^* from U_{τ^*} has been clarified as a Lévy process stopped at unit exponential time, denoted as $\mathcal{V}^*_\varrho$, with measure $\Pi^*(\mathrm{d}x)$. As for our choice of $\lambda^*(U)$, one can easily check that $\int_{\mathbb{R}^+} \left(1 \wedge x^2\right) \Pi^*(\mathrm{d}x) < \infty$ holds where

$$\Pi^*(\mathrm{d}x) = \frac{e^{\lambda_1 c} \int\limits_0^c x^{-\frac{3}{2}} e^{-\lambda_1 x} \, \mathrm{d}x + e^{\lambda_2 c} \int\limits_c^\infty x^{-\frac{3}{2}} e^{-\lambda_2 x} \, \mathrm{d}x}{2\left(\eta^-(\lambda_1) + \eta^+(\lambda_2)\right)}.$$

However, it does not provide any insight into the implementation of its simulation. Towards a better understanding of $\mathcal{V}^*_\varrho$, following the same spirit of decomposition we subdivide it into two independent components, each corresponding to a precise and efficient simulation scheme. More precisely, we find on the whole that a random variate of $\mathcal{V}^*_\varrho$ can be obtained by generating one variate from α-stable process and one from compound Poisson. Due to the difference in quantity between λ_1 and λ_2, there will be a slight difference in the parametrization.

Case I : $\lambda_1 > \lambda_2$

We find an integral representation for the Laplace transform of $\mathcal{V}_\varrho^*$ in the case $\lambda_1 > \lambda_2$.

Theorem 3.3.1. *For any $\lambda_1 > \lambda_2$, the Laplace transform of $\mathcal{V}_\varrho^*$ has an integral representation of the form*

$$\mathbb{E}\left[e^{-\beta \mathcal{V}_\varrho^*}\right] = \int_0^\infty e^{-t} \cdot \exp\left\{-\frac{t}{\eta^*} e^{\lambda_1 c} \int_0^\infty \left(1 - e^{-\beta x}\right) e^{-\lambda_1 x} \alpha x^{-\alpha-1} \, \mathrm{d}x\right\} \cdot \exp\left\{-\frac{t}{\eta^*} \int_c^\infty \left(1 - e^{-\beta v}\right)\left(e^{-\lambda_2(v-c)} - e^{-\lambda_1(v-c)}\right) \alpha v^{-\alpha-1} \, \mathrm{d}v\right\} \mathrm{d}t. \tag{3.9}$$

with $\eta^* = \eta^-(\lambda_1) + \eta^+(\lambda_2)$.

Accordingly, there exists an explicit and exhaustive decomposition for $\mathcal{V}_\varrho^*$ such that

$$\mathcal{V}_\varrho^* \overset{\mathcal{D}}{=} T_s(\varrho) + T_{\mathrm{cp}}(\varrho), \qquad \varrho \sim \mathrm{Exp}(1), \tag{3.10}$$

where $T_s(\varrho)$ and $T_{\mathrm{cp}}(\varrho)$ denote respectively the realisations at unit exponential time ϱ of

- *α-stable subordinator T_s described by Lévy measure*

$$\frac{\alpha x^{-\alpha-1} e^{-\lambda_1(x-c)}}{\eta^*}; \tag{3.11}$$

- *and compound Poisson variable T_{cp} with rate*

$$\frac{\Gamma(1-\alpha)\lambda_1^\alpha e^{\lambda_1 c}}{\eta^*} - 1,$$

and jump size density

$$\frac{\alpha v^{-\alpha-1}\left(e^{-\lambda_2(v-c)} - e^{-\lambda_1(v-c)}\right)}{\Gamma(1-\alpha)\lambda_1^\alpha e^{\lambda_1 c} - \eta^*} \mathbb{1}_{\{v > c\}}.$$

Proof. We first give out the derivation of (3.9). The Laplace transform of $\tau^* \mid U_{\tau^*}$ is given in (3.5), to which we multiply both sides with $e^{\beta U_{\tau^*}}$ to get the Laplace transform of $\mathcal{V}^*_\varrho$,

$$\mathbb{E}\left[e^{-\beta \mathcal{V}^*_\varrho}\right] = \frac{\lambda_1 e^{\lambda_1 c} \int_0^c e^{-\lambda_1 x} x^{-\alpha}\,\mathrm{d}x \; + \; \lambda_2 e^{\lambda_2 c} \int_c^\infty e^{-\lambda_2 x} x^{-\alpha}\,\mathrm{d}x}{(\beta+\lambda_1) e^{\lambda_1 c} \int_0^c e^{-(\beta+\lambda_1) x} x^{-\alpha}\,\mathrm{d}x \; + \; (\beta+\lambda_2) e^{\lambda_2 c} \int_c^\infty e^{-(\beta+\lambda_2) x} x^{-\alpha}\,\mathrm{d}x}. \tag{3.12}$$

The numerator, computed and expressed as $\left(\eta^-(\lambda_1) + \eta^+(\lambda_2)\right)$, is constant of no effect. For simplicity in notation, we denote this quantity by η^*. Considering the denominator, we rewrite the lower incomplete gamma function in the first integral as a subtraction of the upper function from the corresponding complete function and thus obtain

$$\Gamma(1-\alpha)(\beta+\lambda_1)^\alpha e^{\lambda_1 c} \quad + \tag{3.13}$$

$$\int_c^\infty e^{-\beta v} e^{-\lambda_1 (v-c)} \alpha v^{-\alpha-1}\,\mathrm{d}v \; - \; \int_c^\infty e^{-\beta v} e^{-\lambda_2 (v-c)} \alpha v^{-\alpha-1}\,\mathrm{d}v. \tag{3.14}$$

Regarding the first term (3.13), with $\alpha \in (0,1)$ we expand it to

$$\begin{aligned} &\Gamma(1-\alpha)\big((\beta+\lambda_1)^\alpha - \lambda_1^\alpha\big) e^{\lambda_1 c} \; + \; \Gamma(1-\alpha)\lambda_1^\alpha e^{\lambda_1 c} \\ = \; &\int_0^\infty \left(1 - e^{-\beta x}\right) e^{-\lambda_1 (x-c)} \alpha x^{-\alpha-1}\,\mathrm{d}x \; + \; \Gamma(1-\alpha)\lambda_1^\alpha e^{\lambda_1 c}. \end{aligned} \tag{3.15}$$

Moreover, the two integrals in (3.14) can be transformed as follows,

$$\begin{aligned} &- e^{\lambda_1 c} \int_c^\infty \left(e^{-\lambda_1 v} - e^{-(\beta+\lambda_1) v} - e^{-\lambda_1 v}\right) \alpha v^{-\alpha-1}\,\mathrm{d}v \\ &\qquad + \; e^{\lambda_2 c} \int_c^\infty \left(e^{-\lambda_2 v} - e^{-(\beta+\lambda_2) v} - e^{-\lambda_2 v}\right) \alpha v^{-\alpha-1}\,\mathrm{d}v \\ = \; &- \int_c^\infty \left(e^{-\lambda_2 (v-c)} - e^{-\lambda_1 (v-c)}\right) \alpha v^{-\alpha-1}\,\mathrm{d}v \\ &\qquad + \; \int_c^\infty \left(1 - e^{-\beta v}\right)\left(e^{-\lambda_2 (v-c)} - e^{-\lambda_1 (v-c)}\right) \alpha v^{-\alpha-1}\,\mathrm{d}v. \end{aligned} \tag{3.16}$$

Following (3.15) and (3.16), the denominator is equal to

$$\Gamma(1-\alpha)\lambda_1^{\alpha}e^{\lambda_1 c} - \int_c^{\infty}\Big(e^{-\lambda_2(v-c)} - e^{-\lambda_1(v-c)}\Big)\alpha v^{-\alpha-1}\,\mathrm{d}v \tag{3.17}$$

$$+ \int_0^{\infty}\Big(1-e^{-\beta x}\Big)e^{-\lambda_1(x-c)}\alpha x^{-\alpha-1}\,\mathrm{d}x + \int_c^{\infty}\Big(1-e^{-\beta v}\Big)\Big(e^{-\lambda_2(v-c)} - e^{-\lambda_1(v-c)}\Big)\alpha v^{-\alpha-1}\,\mathrm{d}v,$$

and (3.17) is further found matching the numerator,

$$\begin{aligned}
&\Gamma(1-\alpha)\lambda_1^{\alpha}e^{\lambda_1 c} - \int_c^{\infty}\Big(e^{-\lambda_2(v-c)} - e^{-\lambda_1(v-c)}\Big)\alpha v^{-\alpha-1}\,\mathrm{d}v \\
&= \lambda_1 e^{\lambda_1 c}\int_0^{\infty} e^{-\lambda_1 x}x^{-\alpha}\,\mathrm{d}x + \int_c^{\infty}\Big(e^{-\lambda_2(v-c)} - e^{-\lambda_1(v-c)}\Big)\tfrac{\partial}{\partial v}\{v^{-\alpha}\}\,\mathrm{d}v \\
&= \lambda_1 e^{\lambda_1 c}\int_0^{c} e^{-\lambda_1 x}x^{-\alpha}\,\mathrm{d}x + \lambda_2 e^{\lambda_2 c}\int_c^{\infty} e^{-\lambda_2 x}x^{-\alpha}\,\mathrm{d}x.
\end{aligned}$$

Gathering all the results obtained, the Laplace transform of $\mathcal{V}_{\varrho}^{*}$, after dividing both the numerator and the denominator by η^{*}, is represented as

$$\mathbb{E}\Big[e^{-\beta\mathcal{V}_{\varrho}^{*}}\Big] = \left(\begin{aligned}&1 + \frac{1}{\eta^{*}}\int_0^{\infty}\Big(1-e^{-\beta x}\Big)e^{-\lambda_1(x-c)}\alpha x^{-\alpha-1}\,\mathrm{d}x \\ &\quad + \frac{1}{\eta^{*}}\int_c^{\infty}\Big(1-e^{-\beta v}\Big)\Big(e^{-\lambda_2(v-c)} - e^{-\lambda_1(v-c)}\Big)\alpha v^{-\alpha-1}\,\mathrm{d}v\end{aligned}\right)^{-1}.$$

Note that the two integrals, disregarding the parametrization, are the characteristic exponents of a α-stable subordinator and a compound Poisson process, respectively. Taking this fraction as a whole, we rewrite it as an integral form as (3.9). Following the integral representation whose integrand contains a product of the two characteristic functions, the decomposition (3.10) comes by the convolution theorem. □

This theorem leads immediately to the simulation algorithm of $\mathcal{V}_{\varrho}^{*}$.

Algorithm 2 : Generating one sample of $\mathcal{V}_{\varrho}^{*}$ with $\lambda_1 > \lambda_2$.

(I) Generate a unit exponential variable $\varrho \sim \text{Exp}(1)$.

(II) Generate one α-stable subordinator (at time ϱ) T_s with measure given by (3.11).

(III) Generate one Poisson random variable N with rate :

$$\frac{\varrho}{\eta^{*}}\Big(\Gamma(1-\alpha)\lambda_1^{\alpha}e^{\lambda_1 c} - \eta^{*}\Big).$$

(IV) Generate N independent jump variables $\{Y_i\}_{i=1,2,\cdots N}$ using Acceptance - Rejection (A-R) scheme. For each jump Y_i, take the following steps:

(i) Generate a random variable Y from an envelope density

$$f_Y(y) = \frac{\alpha e^{\lambda_2 c} e^{-\lambda_2 y} y^{-\alpha-1}}{c^{-\alpha} - \eta^{+}(\lambda_2)} \mathbb{1}_{\{y > c\}}; \tag{3.18}$$

(ii) Generate a standard uniform variate $U \sim \mathcal{U}(0,1)$;

(iii) Set $Y_i \leftarrow Y$ if $U < 1 - e^{-(\lambda_1-\lambda_2)(Y-c)}$; otherwise, return to step (i).

Then, a sample of the compound Poisson is obtained by $T_{\text{cp}} = \sum_{i=1}^{N} Y_i$.

(V) Set $\mathcal{V}_{\varrho}^{*} = T_s + T_{\text{cp}}$.

Remark 3.3.2. To be more precise on simulating the α-stable subordinator, there seems no problem for the case $\alpha = \frac{1}{2}$, as it is the well-known inverse Gaussian process with ready-made simulation packages. In general situation when $\alpha \neq \frac{1}{2}$, we refer to Dassios et al. [32] who develops a so-called backward recursive scheme for exactly simulating a class of tempered stable distribution with stability index $\frac{q}{2n} \in (0,1)$ and $q, n \in \mathbb{N}^{+}$.

Case II : $\lambda_1 < \lambda_2$

Within a manner analogous to the previous case, we find a modified integral representation for the Laplace transform of $\mathcal{V}_{\varrho}^{*}$ in the case $\lambda_1 < \lambda_2$.

Theorem 3.3.3. *For any $\lambda_1 < \lambda_2$, the Laplace transform of $\mathcal{V}_\varrho^*$ can be represented in the following integral form*

$$\mathbb{E}\Big[e^{-\beta \mathcal{V}_\varrho^*}\Big] = \int_0^\infty e^{-t} \cdot \exp\Bigg\{-\frac{t}{\eta^*} e^{\lambda_1 c} \int_0^\infty \Big(1 - e^{-\beta v}\Big)\, e^{-\lambda_2 v} \alpha v^{-\alpha-1}\, \mathrm{d}v\Bigg\} \cdot$$
$$\exp\Bigg\{-t\Lambda \int_0^\infty \Big(1 - e^{-\beta v}\Big) \Big[q \cdot Q_1(v) + (1-q) \cdot Q_2(v)\Big]\, \mathrm{d}v\Bigg\}\, \mathrm{d}t. \quad (3.19)$$

Accordingly, the decomposition is given by

$$\mathcal{V}_\varrho^* \overset{\mathcal{D}}{=} T_s(\varrho) + T_{\mathrm{cp}}(\varrho), \qquad \varrho \sim \mathrm{Exp}(1), \quad (3.20)$$

where $T_s(\varrho)$ and $T_{\mathrm{cp}}(\varrho)$ denote the realisations at unit exponential time ϱ of :

- *α-stable subordinator T_s described by Lévy measure*

$$\frac{e^{\lambda_1 c} e^{-\lambda_2 v} \alpha v^{-\alpha-1}}{\eta^*}\, \mathrm{d}v; \quad (3.21)$$

- *and compound Poisson variable T_{cp} with rate Λ equal to*

$$\frac{\Gamma(1-\alpha)\lambda_2^\alpha e^{\lambda_1 c}}{\eta^*} - 1; \quad (3.22)$$

and jump size density $\Big(q \cdot Q_1(v) + (1-q) \cdot Q_2(v)\Big)$ where

$$q = \frac{\eta^-(\lambda_2) - \eta^-(\lambda_1) - \dfrac{1 - e^{-(\lambda_2-\lambda_1)c}}{c^\alpha}}{\Gamma(1-\alpha)\lambda_2^\alpha e^{\lambda_1 c} - \eta^*}; \quad (3.23)$$

$$Q_1(v) = \frac{e^{\lambda_1 c}\Big(e^{-\lambda_1 v} - e^{-\lambda_2 v}\Big)\alpha v^{-\alpha-1}}{\eta^-(\lambda_2) - \eta^-(\lambda_1) - \dfrac{1 - e^{-(\lambda_2-\lambda_1)c}}{c^\alpha}} \mathbb{1}_{\{v < c\}}; \quad (3.24)$$

$$Q_2(v) = \frac{e^{\lambda_2 c} e^{-\lambda_2 v} \alpha v^{-\alpha-1}}{c^{-\alpha} - \eta^+(\lambda_2)} \mathbb{1}_{\{v > c\}}, \quad (3.25)$$

with $\eta^* = \eta^-(\lambda_1) + \eta^+(\lambda_2)$. $\eta^-(\cdot)$ and $\eta^+(\cdot)$ are defined before in (3.2) and (3.3).

Proof. The core of this proof is to show the formation of the integral result (3.19), during which course the detailed components in the decomposition are identified. We proceed as in the last proof with minor modification on the representation of the denominator in (3.12), i.e.

$$\int_0^c \beta e^{-\beta x} e^{-\lambda_1(x-c)} x^{-\alpha}\,\mathrm{d}x \;+\; \int_c^\infty \beta e^{-\beta x} e^{-\lambda_2(x-c)} x^{-\alpha}\,\mathrm{d}x$$

$$+\;\lambda_1 e^{\lambda_1 c}\int_0^c e^{-\beta x} e^{-\lambda_1 x} x^{-\alpha}\,\mathrm{d}x \;+\; \lambda_2 e^{\lambda_2 c}\int_c^\infty e^{-\beta x} e^{-\lambda_2 x} x^{-\alpha}\,\mathrm{d}x.$$

Substituting for $i = 1,2$

$$e^{-\lambda_i x} x^{-\alpha} \;=\; \int_x^\infty \lambda_i e^{-\lambda_i v} v^{-\alpha}\,\mathrm{d}v \;+\; \int_x^\infty e^{-\lambda_i v}\alpha v^{-\alpha-1}\,\mathrm{d}v,$$

into the first two integrals and then interchanging the order of each yields, we have

$$e^{\lambda_1 c}\int_0^c \left(1-e^{-\beta v}\right)\left(\lambda_1 e^{-\lambda_1 v} v^{-\alpha} + e^{-\lambda_1 v}\alpha v^{-\alpha-1}\right)\mathrm{d}v$$

$$+\; e^{\lambda_1 c}\int_c^\infty \left(1-e^{-\beta c}\right)\left(\lambda_1 e^{-\lambda_1 v} v^{-\alpha} + e^{-\lambda_1 v}\alpha v^{-\alpha-1}\right)\mathrm{d}v$$

$$+\; e^{\lambda_2 c}\int_c^\infty \left(e^{-\beta c}-e^{-\beta v}\right)\left(\lambda_2 e^{-\lambda_2 v} v^{-\alpha} + e^{-\lambda_2 v}\alpha v^{-\alpha-1}\right)\mathrm{d}v$$

$$+\;\lambda_1 e^{\lambda_1 c}\int_0^c e^{-(\beta+\lambda_1)x} x^{-\alpha}\,\mathrm{d}x \;+\; \lambda_2 e^{\lambda_2 c}\int_c^\infty e^{-(\beta+\lambda_2)x} x^{-\alpha}\,\mathrm{d}x.$$

These integrals are computed to

$$\lambda_1 e^{\lambda_1 c}\int_0^c e^{-\lambda_1 v} v^{-\alpha}\,\mathrm{d}v \;+\; c^{-\alpha} \;-\; \int_c^\infty e^{-\beta v} e^{-\lambda_2(v-c)}\alpha v^{-\alpha-1}\,\mathrm{d}v$$

$$+\;\int_0^c \left(1-e^{-\beta v}\right) e^{-\lambda_1(v-c)}\alpha v^{-\alpha-1}\,\mathrm{d}v$$

$$=\;\eta^* + \int_0^c \left(1-e^{-\beta v}\right) e^{-\lambda_1(v-c)}\alpha v^{-\alpha-1}\,\mathrm{d}v \;+\; \int_c^\infty \left(1-e^{-\beta v}\right) e^{-\lambda_2(v-c)}\alpha v^{-\alpha-1}\,\mathrm{d}v.$$

With $\lambda_1 < \lambda_2$, this can be rewritten as follows,

$$
\begin{aligned}
&\eta^* + \int_0^c \left(1 - e^{-\beta v}\right)\left(e^{-\lambda_1(v-c)} - e^{-\lambda_2(v-c)}\right)\alpha v^{-\alpha-1}\,\mathrm{d}v \;+\; \int_0^\infty \left(1 - e^{-\beta v}\right)e^{-\lambda_2(v-c)}\alpha v^{-\alpha-1}\,\mathrm{d}v \\
&= \eta^* \;+\; \int_0^c \left(1 - e^{-\beta v}\right)e^{-\lambda_1(v-c)}\alpha v^{-\alpha-1}\,\mathrm{d}v \;-\; \int_0^c \left(1 - e^{-\beta v}\right)e^{-\lambda_2(v-c)}\alpha v^{-\alpha-1}\,\mathrm{d}v \\
&\quad + \; e^{\lambda_1 c}\int_0^\infty \left(1 - e^{-\beta v}\right)e^{-\lambda_2 v}\alpha v^{-\alpha-1}\,\mathrm{d}v \;+\; \left(e^{\lambda_2 c} - e^{\lambda_1 c}\right)\int_0^\infty \left(1 - e^{-\beta v}\right)e^{-\lambda_2 v}\alpha v^{-\alpha-1}\,\mathrm{d}v \\
&= \eta^* \;+\; e^{\lambda_1 c}\int_0^\infty \left(1 - e^{-\beta v}\right)e^{-\lambda_2 v}\alpha v^{-\alpha-1}\,\mathrm{d}v \;+\; e^{\lambda_1 c}\int_0^c \left(1 - e^{-\beta v}\right)\left(e^{-\lambda_1 v} - e^{-\lambda_2 v}\right)\alpha v^{-\alpha-1}\,\mathrm{d}v \\
&\quad + \; \left(e^{\lambda_2 c} - e^{\lambda_1 c}\right)\int_c^\infty \left(1 - e^{-\beta v}\right)e^{-\lambda_2 v}\alpha v^{-\alpha-1}\,\mathrm{d}v.
\end{aligned}
$$

Therefore, a modified version of the Laplace transform of $\mathcal{V}_\varrho^*$ has been achieved as

$$
\mathbb{E}\left[e^{-\beta \mathcal{V}_\varrho^*}\right] \;=\; \begin{pmatrix} 1 \;+\; \dfrac{e^{\lambda_1 c}}{\eta^*}\displaystyle\int_0^\infty \left(1 - e^{-\beta v}\right)e^{-\lambda_2 v}\alpha v^{-\alpha-1}\,\mathrm{d}v \\ \quad + \; \dfrac{e^{\lambda_1 c}}{\eta^*}\displaystyle\int_0^c \left(1 - e^{-\beta v}\right)\left(e^{-\lambda_1 v} - e^{-\lambda_2 v}\right)\alpha v^{-\alpha-1}\,\mathrm{d}v \\ \quad + \; \dfrac{\left(e^{\lambda_2 c} - e^{\lambda_1 c}\right)}{\eta^*}\displaystyle\int_c^\infty \left(1 - e^{-\beta v}\right)e^{-\lambda_2 v}\alpha v^{-\alpha-1}\,\mathrm{d}v \end{pmatrix}^{-1}. \tag{3.26}
$$

It is apparent that the first integral is related to the characteristic exponent of α-stable subordinator. Considering the second and the third integral in terms of a proper Lévy density, we find that they combined together constitute the jump measure of a compound Poisson process. That is to say we can express the sum of them as in the form

$$
\begin{aligned}
&\frac{e^{\lambda_1 c}}{\eta^*}\int_0^c \left(1 - e^{-\beta v}\right)\left(e^{-\lambda_1 v} - e^{-\lambda_2 v}\right)\alpha v^{-\alpha-1}\,\mathrm{d}v \;+\; \frac{e^{\lambda_2 c} - e^{\lambda_1 c}}{\eta^*}\int_c^\infty \left(1 - e^{-\beta v}\right)e^{-\lambda_2 v}\alpha v^{-\alpha-1}\,\mathrm{d}v \\
&= \Lambda \int_0^\infty \left(1 - e^{-\beta v}\right)\left(q \cdot Q_1(v) + (1-q)\cdot Q_2(v)\right)\mathrm{d}v.
\end{aligned}
$$

Λ denotes the Poisson jump rate and equates to

$$
\begin{aligned}
& \frac{e^{\lambda_1 c}}{\eta^*} \int_0^c \Big(e^{-\lambda_1 v} - e^{-\lambda_2 v}\Big)\alpha v^{-\alpha-1}\,\mathrm{d}v \;+\; \frac{e^{\lambda_2 c} - e^{\lambda_1 c}}{\eta^*} \int_c^\infty e^{-\lambda_2 v}\alpha v^{-\alpha-1}\,\mathrm{d}v \\
= \;& \frac{e^{\lambda_1 c}}{\eta^*}\left(\int_0^c \lambda_2 e^{-\lambda_2 v} v^{-\alpha}\,\mathrm{d}v \;-\; \int_0^c \lambda_1 e^{-\lambda_1 v} v^{-\alpha}\,\mathrm{d}v \;-\; \frac{e^{-\lambda_1 c} - e^{-\lambda_2 c}}{c^\alpha} \right) \\
& \qquad + \frac{e^{\lambda_2 c} - e^{\lambda_1 c}}{\eta^*}\left(\frac{e^{-\lambda_2 c}}{c^\alpha} - \int_c^\infty \lambda_2 e^{-\lambda_2 v} v^{-\alpha}\,\mathrm{d}v \right) \\
= \;& \frac{\Gamma(1-\alpha)\lambda_2^\alpha e^{\lambda_1 c}}{\eta^*} - 1.
\end{aligned}
$$

$\Big(q \cdot Q_1(v) + (1-q)\cdot Q_2(v)\Big)$ is a convex combination of two density functions $Q_1(v)$ and $Q_2(v)$ weighted by $q \in (0,1)$ with

$$
q \;=\; \frac{\lambda_2 e^{\lambda_1 c} \int_0^c e^{-\lambda_2 v} v^{-\alpha}\,\mathrm{d}v - \lambda_1 e^{\lambda_1 c} \int_0^c e^{-\lambda_1 v} v^{-\alpha}\,\mathrm{d}v - \frac{1 - e^{-(\lambda_2 - \lambda_1)c}}{c^\alpha}}{\Gamma(1-\alpha)\lambda_2^\alpha e^{\lambda_1 c} - \eta^*}.
$$

$Q_1(v)$ and $Q_2(v)$ have been found by

$$
\begin{aligned}
Q_1(v) \;&=\; \frac{\Big(e^{-\lambda_1 v} - e^{-\lambda_2 v}\Big)\alpha v^{-\alpha-1}\,\mathbb{1}_{\{v\,<\,c\}}}{\int_0^c \lambda_2 e^{-\lambda_2 v} v^{-\alpha}\,\mathrm{d}v - \int_0^c \lambda_1 e^{-\lambda_1 v} v^{-\alpha}\,\mathrm{d}v - \frac{e^{-\lambda_1 c} - e^{-\lambda_2 c}}{c^\alpha}}; \\
Q_2(v) \;&=\; \frac{e^{-\lambda_2 v}\alpha v^{-\alpha-1}\,\mathbb{1}_{\{v\,>\,c\}}}{\frac{e^{-\lambda_2 c}}{c^\alpha} - \int_c^\infty \lambda_2 e^{-\lambda_2 v} v^{-\alpha}\,\mathrm{d}v}.
\end{aligned}
$$

It is easy to prove that they are proper density functions.

To conclude this proof, we rewrite (3.26) into an integral form incorporating the above results to get (3.19). This leads immediately to the decomposition result (3.20). □

The simulation algorithm for one sample of $\mathcal{V}_\varrho^*$ is given as follows:

Algorithm 3 : Generating one sample of $\mathcal{V}_{\varrho}^{*}$ with $\lambda_1 < \lambda_2$.

(I) Generate a unit exponential variable $\varrho \sim \text{Exp}(1)$;

(II) Generate a α-stable subordinator (at time ϱ) T_s with measure given by (3.21) ;

(III) Generate one Poisson random variable N with rate :

$$\frac{\varrho}{\eta^*}\Big(\Gamma(1-\alpha)\lambda_2^{\alpha} e^{\lambda_1 c} - \eta^* \Big) \, ;$$

(IV) Generate N independent jump variables $\{Y_i\}_{i=1,2,\cdots N}$ using Acceptance - Rejection (A-R) scheme. For each jump Y_i, take the following steps:

(i) Generate an indicator variable $R \sim \text{Bernoulli}(q)$ with q given by (3.23) ;

(ii)

- If $R = 1$, generate two independent standard uniform variables $\{U_1, U_2\} \sim \mathcal{U}(0,1)$ and compute $Y = cU_1^{\frac{1}{1-\alpha}}$. Set $Y_i \leftarrow Y$ if $U_2 < \dfrac{e^{-\lambda_1 Y} - e^{-\lambda_2 Y}}{(\lambda_2 - \lambda_1)Y}$; otherwise repeat this point;
- If $R = 0$, generate a random variable Y from the density given by (3.25) and set $Y_i \leftarrow Y$;

Then, a sample of the compound Poisson $T_{\text{cp}} = \sum_{i=1}^{N} Y_i$.

(V) Set $\mathcal{V}_{\varrho}^{*} = T_s + T_{\text{cp}}$.

The decomposition (3.20) has been found in the same form as (3.10) so the algorithm for the case $\lambda_1 < \lambda_2$ is expected to follow a similar rule to that for $\lambda_1 > \lambda_2$. In particular, the simulation of T_s from the α-stable family would not be hard by referring to the literature mentioned before. Moreover, the methodology of generating T_{cp}, represented as a linearly combined density, has been introduced in ALGORITHM 1 : **Generating N samples of U_{τ^*}** . Exactly the same procedures can be applied here.

3.4 Numerical Studies

This section illustrates the accuracy of our simulations schemes and their performance in assessing default risk. We assume $\alpha = ½$ that reduces the underlying process, originally being a recurrent Bessel process, to a reflected Brownian motion. As a result, the α-stable subordinator appearing in the $\mathcal{V}_{\varrho}^{*}$-decomposition turns into an inverse Gaussian process.

Examination of Simulation Algorithms. To verify the accuracy of our algorithms by means of error analysis, we compare the estimated mean values of U_{τ^*}, $\mathcal{V}_{\varrho}^{*}$ and τ^* with the corresponding theoretical results evaluated from their Laplace transforms obtained previously. For $\alpha \in (0,1)$ and $\lambda_1, \lambda_2 > 0$, we get

$$\left.\begin{aligned}
\mathbb{E}[U_{\tau^*}] &= \frac{1}{\eta^*}\bigg(e^{\lambda_1 c}\lambda_1^{\alpha-1}\gamma(2-\alpha,\lambda_1 c) \;+\; e^{\lambda_2 c}\lambda_2^{\alpha-1}\Gamma(2-\alpha,\lambda_2 c)\bigg)\\
\mathbb{E}\big[\mathcal{V}_{\varrho}^{*}\big] &= \frac{1}{\eta^*}\bigg(e^{\lambda_1 c}\lambda_1^{\alpha-1}\Big(\gamma(1-\alpha,\lambda_1 c) \;-\; \gamma(2-\alpha,\lambda_1 c)\Big)\\
&\qquad\quad + e^{\lambda_2 c}\lambda_2^{\alpha-1}\Big(\Gamma(1-\alpha,\lambda_2 c) \;-\; \Gamma(2-\alpha,\lambda_2 c)\Big)\bigg)\\
\mathbb{E}[\tau^*] &= \frac{1}{\eta^*}\bigg(e^{\lambda_1 c}\lambda_1^{\alpha-1}\gamma(1-\alpha,\lambda_1 c) \;+\; e^{\lambda_2 c}\lambda_2^{\alpha-1}\Gamma(1-\alpha,\lambda_2 c)\bigg)
\end{aligned}\right\}, \qquad (3.27)$$

with

$$\eta^* \;=\; e^{\lambda_1 c}\lambda_1^{\alpha}\gamma(1-\alpha,\lambda_1 c) \;+\; e^{\lambda_2 c}\lambda_2^{\alpha}\Gamma(1-\alpha,\lambda_2 c).$$

In the special case of $\alpha = ½$, $\mathbb{E}[U_{\tau^*}] = \mathbb{E}\big[\mathcal{V}_{\varrho}^{*}\big] = ½\,\mathbb{E}[\tau^*]$ holds.

The associated discrepancies are measured by *relative error*, expressed as the percentage of estimates relative to the true values. To implement the simulation, we set parameters

$$\lambda_1 = (\,0, 0.5, 1, 1.5, 2, 2.5, 3\,),$$

one-to-one corresponding to

$$\lambda_2 = (\,3.5, 3, 2.5, 2, 1.5, 1, 0.5\,),$$

to be combined with each in $c = (0.5, 1, 1.5, 2)$. Each estimate is obtained based on a size of 10^5 samples. The comparison results are listed in TABLE 3.1, from which we conclude that demonstrated by the close agreement between the true means and the estimated values, the proposed algorithms provide consistent estimators for U_{τ^*}, $\mathcal{V}_\varrho^*$ and τ^* with an achievement of high-level accuracy. The associated histograms for the chosen cases $\Theta := (c, \lambda_1, \lambda_2) = (0.5, 0.5, 3)_i, (1, 0, 3.5)_{ii}, (1.5, 2, 1.5)_{iii}$ and $(2, 3, 0.5)_{iv}$ are provided in FIGURE 3.1.

Application to Default Risk Assessment. Looking into the comparison table, we are indicated that with an increase in the level c, the expected U_{τ^*}, $\mathcal{V}_\varrho^*$ and τ^* (both true and estimated results) moves progressively upwards for the case $\lambda_1 > \lambda_2$ but downwards for the opposite case. Once studying further the role of $(c, \lambda_1, \lambda_2)$ in determining the behaviour of default risk, we understand that the default probability $\mathbb{P}(\tau^* < T \mid c, \lambda_1, \lambda_2)$ and the default level $\mathbb{P}(U_{\tau^*} < T \mid c, \lambda_1, \lambda_2)$ do not always follow the same rule. To demonstrate this, we conduct a sensitivity analysis by generating a table for the cumulative distribution function of τ^* and U_{τ^*} within a range of parameter sets representing the following three scenarios,

$$\text{(I)}\ \ (\lambda_1, \lambda_2) = (0, 3) \quad \text{with } c = \{\, 0.5, 1.0, 1.5, 2.0 \,\} \,;$$

$$\text{(II)}\ \ (\lambda_1, \lambda_2) = (0.5, 3) \text{ with } c = \{\, 0.5, 1.0, 1.5, 2.0 \,\} \,;$$

$$\text{(III)}\ \ (\lambda_1, \lambda_2) = (2, 0.3) \text{ with } c = \{\, 0.2, 0.4, 0.6, 0.8 \,\} \,.$$

The numerical results are presented in TABLE 3.2 and the associated plots in FIGURE 3.2.

It can be observed from the sub-plots in FIGURE 3.2 that in general the level c imposes a negative effect on the possibility of default for the case $\lambda_1 < \lambda_2$ and a positive effect for the opposite case. This means that as c increases, the probability of a default happening before time T is descending under the case $\lambda_1 < \lambda_2$ and ascending under the opposite case. The situation becomes a bit complicated for the default level U_τ. The extreme case $\lambda_1 = 0$ suggests a strictly negative relationship in the effect of c to the probability of U_τ before T. For the rest two cases, there exist transitions in their CDF-plots that turns the negative relationship towards positive referring to the case $\lambda_1 < \lambda_2$ or the other way around to the case $\lambda_1 > \lambda_2$.

The intuition behind these findings is straightforward. Depending on the age of the excursion straddling t, the underlying process is subjected to and switching between different exposures to risk measured by λ_1 within the space $\{U_t < c\}$ and λ_2 within $\{U_t \geq c\}$. With a rise in the value of c, it is increasingly unlikely for the process to complete an excursion with a length exceeding c so as to enter into the region $\{U_t \geq c\}$. As for the case $\lambda_1 < \lambda_2$, the process is expected to suffer more of the time from the relatively lower risk than higher. However, each time it succeeds in escaping from the lower risk region into that of higher risk, a boost in the possibility of default is ensured. Moreover, the bigger value in c the less possibility of a boost. As a result, once a default happens it is ended up with a higher value. This explains the finding mentioned at the beginning that the expected τ and U_τ become smaller as c increases. It has also been reflected in the associated plot as the conspicuous increase in the slope of the curve happening immediately after c. Then for the other case $\lambda_1 > \lambda_2$, following a similar line of reasoning, the underlying process is expected to remain more of the time in the higher-risk region and highly likely the event of default will arrive before c. Due to the increasing difficulty of entering into the lower-risk region, i.e. completing an excursion of length over c, the default is less likely to happen after c. In the associated plot, this has been captured by the drop in the slope right after c.

Table 3.1 Comparison between the true means (3.27) and the simulated results based on varying parameter settings of $(c, \lambda_1, \lambda_2)$; the associated histograms are provided in FIGURE 3.1.

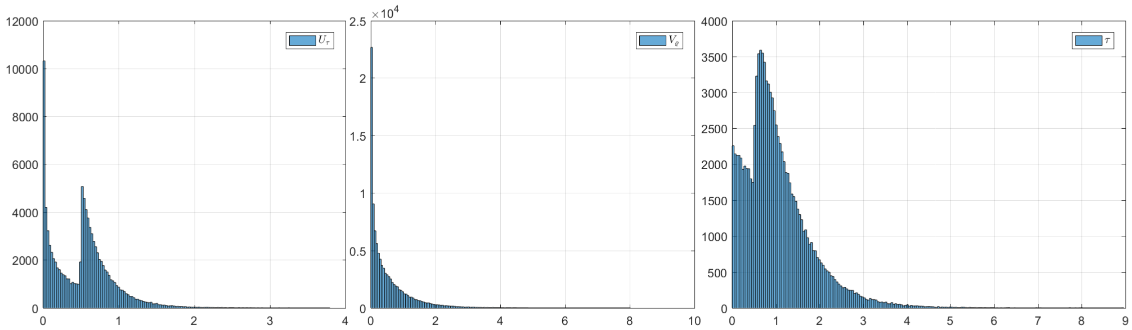
c	λ_1	λ_2	$\mathbb{E}[U_{\tau^*}]$	*Estimates*	*Error(%)*	$\mathbb{E}[\mathcal{V}_\varrho^*]$	*Estimates*	*Error(%)*	$\mathbb{E}[\tau^*]$	*Estimates*	*Error(%)*
	0.0	3.5	0.7467	0.7461	-0.07	0.7467	0.7475	0.10	1.4934	1.4936	0.02
	0.5	3.0	0.5186	0.5193	0.13	0.5186	0.5183	-0.06	1.0373	1.0377	0.04
	1.0	2.5	0.3925	0.3928	0.07	0.3925	0.3918	-0.20	0.7851	0.7845	-0.07
0.5	1.5	2.0	0.3141	0.3146	0.14	0.3141	0.3136	-0.17	0.6283	0.6282	-0.02
	2.0	1.5	0.2625	0.2624	-0.05	0.2625	0.2629	0.14	0.5250	0.5253	0.04
	2.5	1.0	0.2290	0.2293	0.13	0.2290	0.2285	-0.21	0.4580	0.4578	-0.04
	3.0	0.5	0.2147	0.2145	-0.10	0.2147	0.2147	-0.02	0.4294	0.4292	-0.06
	0.0	3.5	1.2601	1.2607	0.05	1.2601	1.2597	-0.04	2.5202	2.5204	0.01
	0.5	3.0	0.6794	0.6791	-0.04	0.6794	0.6805	0.17	1.3588	1.3596	0.06
	1.0	2.5	0.4473	0.4476	0.06	0.4473	0.4479	0.14	0.8946	0.8955	0.10
1.0	1.5	2.0	0.3261	0.3266	0.14	0.3261	0.3259	-0.09	0.6523	0.6524	0.02
	2.0	1.5	0.2537	0.2532	-0.19	0.2537	0.2539	0.10	0.5073	0.5071	-0.05
	2.5	1.0	0.2067	0.2069	0.13	0.2067	0.2062	-0.25	0.4134	0.4131	-0.06
	3.0	0.5	0.1756	0.1754	-0.10	0.1756	0.1758	0.11	0.3511	0.3511	0.00
	0.0	3.5	1.7665	1.7663	-0.01	1.7665	1.7655	-0.05	3.5329	3.5318	-0.03
	0.5	3.0	0.7790	0.7797	0.09	0.7790	0.7794	0.05	1.5581	1.5591	0.07
	1.0	2.5	0.4725	0.4734	0.18	0.4725	0.4721	-0.09	0.9450	0.9454	0.05
1.5	1.5	2.0	0.3304	0.3301	-0.10	0.3304	0.3312	0.23	0.6608	0.6613	0.06
	2.0	1.5	0.2512	0.2508	-0.13	0.2512	0.2513	0.05	0.5023	0.5021	-0.04
	2.5	1.0	0.2017	0.2019	0.13	0.2017	0.2016	-0.03	0.4033	0.4035	0.05
	3.0	0.5	0.1684	0.1686	0.10	0.1684	0.1682	-0.11	0.3369	0.3369	0.00
	0.0	3.5	2.2702	2.2713	0.05	2.2702	2.2684	-0.08	4.5405	4.5397	-0.02
	0.5	3.0	0.8444	0.8449	0.05	0.8444	0.8454	0.11	1.6889	1.6903	0.08
	1.0	2.5	0.4851	0.4849	-0.06	0.4851	0.4858	0.14	0.9703	0.9707	0.04
2.0	1.5	2.0	0.3321	0.3325	0.11	0.3321	0.3316	-0.14	0.6642	0.6641	-0.02
	2.0	1.5	0.2504	0.2501	-0.12	0.2504	0.2506	0.11	0.5008	0.5007	-0.01
	2.5	1.0	0.2004	0.2000	0.20	0.2004	0.2006	0.06	0.4009	0.4006	-0.07
	3.0	0.5	0.1670	0.1669	-0.06	0.1670	0.1674	0.17	0.3341	0.3343	0.07

Table 3.2 Estimated CDFs of τ^* and U_{τ^*} for the Scenario (I), (II), and (III) with each based on a sample size of 10^5; the associated plots are provided in FIGURE 3.2.

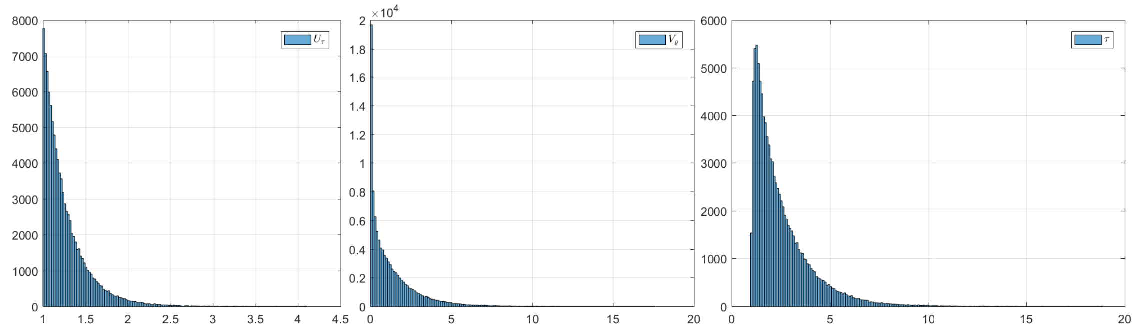
Scenario	Time T	$\mathbb{P}(\tau^* < T \mid c, \lambda_1, \lambda_2)$				Time T	$\mathbb{P}(U_{\tau^*} < T \mid c, \lambda_1, \lambda_2)$			
	$c =$	0.5	1.0	1.5	2.0	$c =$	0.5	1.0	1.5	2.0
I: $(\lambda_1, \lambda_2) = (0, 3)$	1	0.3310	0	0	0	0.6	0.3032	0	0	0
	2	0.7593	0.4572	0.2068	0	0.8	0.6590	0	0	0
	3	0.9169	0.7134	0.5104	0.3367	1.1	0.8776	0.2863	0	0
	4	0.9715	0.8501	0.6936	0.5411	1.3	0.9367	0.6352	0	0
	5	0.9899	0.9213	0.8076	0.6809	1.6	0.9763	0.8645	0.2772	0
	6	0.9964	0.9592	0.8785	0.7792	1.8	0.9882	0.9290	0.6234	0
	7	0.9986	0.9787	0.9237	0.8475	2.1	0.9953	0.9732	0.8578	0.2744
	8	0.9996	0.9890	0.9520	0.8937	2.5	0.9988	0.9923	0.9600	0.7983
	9	0.9999	0.9943	0.9702	0.9271	3.0	0.9998	0.9984	0.9919	0.9582
	10	1	0.9972	0.9817	0.9496	3.5	1	0.9997	0.9984	0.9913
	$c =$	0.5	1.0	1.5	2.0	$c =$	0.5	1.0	1.5	2.0
II: $(\lambda_1, \lambda_2) = (0.5, 3)$	0.5	0.2227	0.2226	0.2205	0.2217	0.3	0.3374	0.3745	0.3922	0.4031
	1.0	0.5739	0.3935	0.3930	0.3942	0.6	0.5976	0.5053	0.5290	0.5442
	1.5	0.7899	0.6316	0.5280	0.5291	0.9	0.8588	0.5919	0.6183	0.6376
	2.0	0.9001	0.7875	0.7013	0.6343	1.2	0.9489	0.8028	0.6853	0.7051
	2.5	0.9524	0.8777	0.8191	0.7612	1.5	0.9816	0.9264	0.7357	0.7552
	3.0	0.9774	0.9304	0.8890	0.8481	1.8	0.9928	0.9723	0.8998	0.7937
	3.5	0.9893	0.9602	0.9303	0.9019	2.1	0.9973	0.9894	0.9620	0.8659
	4.0	0.9949	0.9771	0.9573	0.9358	2.4	0.9990	0.9960	0.9857	0.9480
	4.5	0.9976	0.9867	0.9732	0.9586	2.7	0.9996	0.9985	0.9947	0.9799
	5.0	0.9989	0.9923	0.9835	0.9725	3.0	0.9998	0.9994	0.9978	0.9922
	$c =$	0.2	0.4	0.6	0.8	$c =$	0.2	0.4	0.6	0.8
III: $(\lambda_1, \lambda_2) = (2, 0.3)$	0.2	0.3291	0.3301	0.3299	0.3288	0.1	0.5699	0.5178	0.4934	0.4844
	0.4	0.4868	0.5518	0.5510	0.5505	0.2	0.7581	0.6866	0.6574	0.6460
	0.6	0.5744	0.6660	0.6987	0.6991	0.3	0.7772	0.7922	0.7598	0.7457
	0.8	0.6357	0.7331	0.7808	0.7988	0.4	0.7931	0.8654	0.8309	0.8157
	1.0	0.6838	0.7778	0.8273	0.8558	0.5	0.8057	0.8742	0.8820	0.8652
	1.2	0.7209	0.8096	0.8585	0.8883	0.6	0.8174	0.8821	0.9200	0.9032
	1.4	0.7517	0.8346	0.8800	0.9095	0.7	0.8280	0.8889	0.9247	0.9305
	1.6	0.7777	0.8542	0.8969	0.9244	0.8	0.8373	0.8948	0.9287	0.9512
	1.8	0.8010	0.8714	0.9098	0.9361	0.9	0.8461	0.9006	0.9325	0.9541
	2.0	0.8204	0.8853	0.9207	0.9441	1.0	0.8541	0.9056	0.9356	0.9565

Fig. 3.1 Histograms of the simulated results of $\left(U_{\tau^*}, \mathcal{V}_{\varrho}^*, \tau^*\right)$ with each based on a sample size of 10^5 for four sets of parameters: $\Theta_i = (0.5, 0.5, 3), \Theta_{ii} = (1, 0, 3.5), \Theta_{iii} = (1.5, 2, 1.5)$ and $\Theta_{iv} = (2, 3, 0.5)$, respectively.

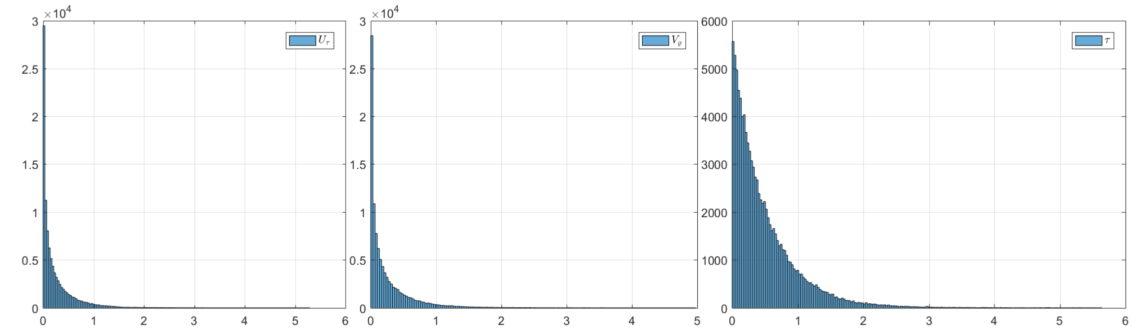
Case i: $(c, \lambda_1, \lambda_2) = (0.5, 0.5, 3)$



Case ii: $(c, \lambda_1, \lambda_2) = (1, 0, 3.5)$



Case iii: $(c, \lambda_1, \lambda_2) = (1.5, 2, 1.5)$



Case iv: $(c, \lambda_1, \lambda_2) = (2, 3, 0.5)$

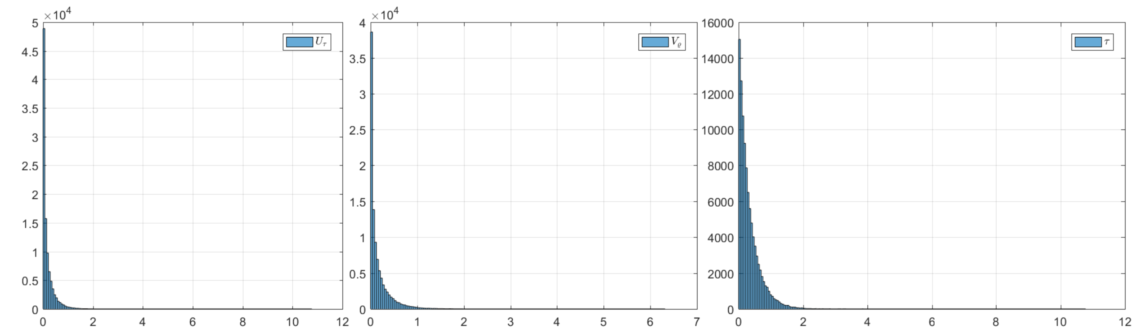
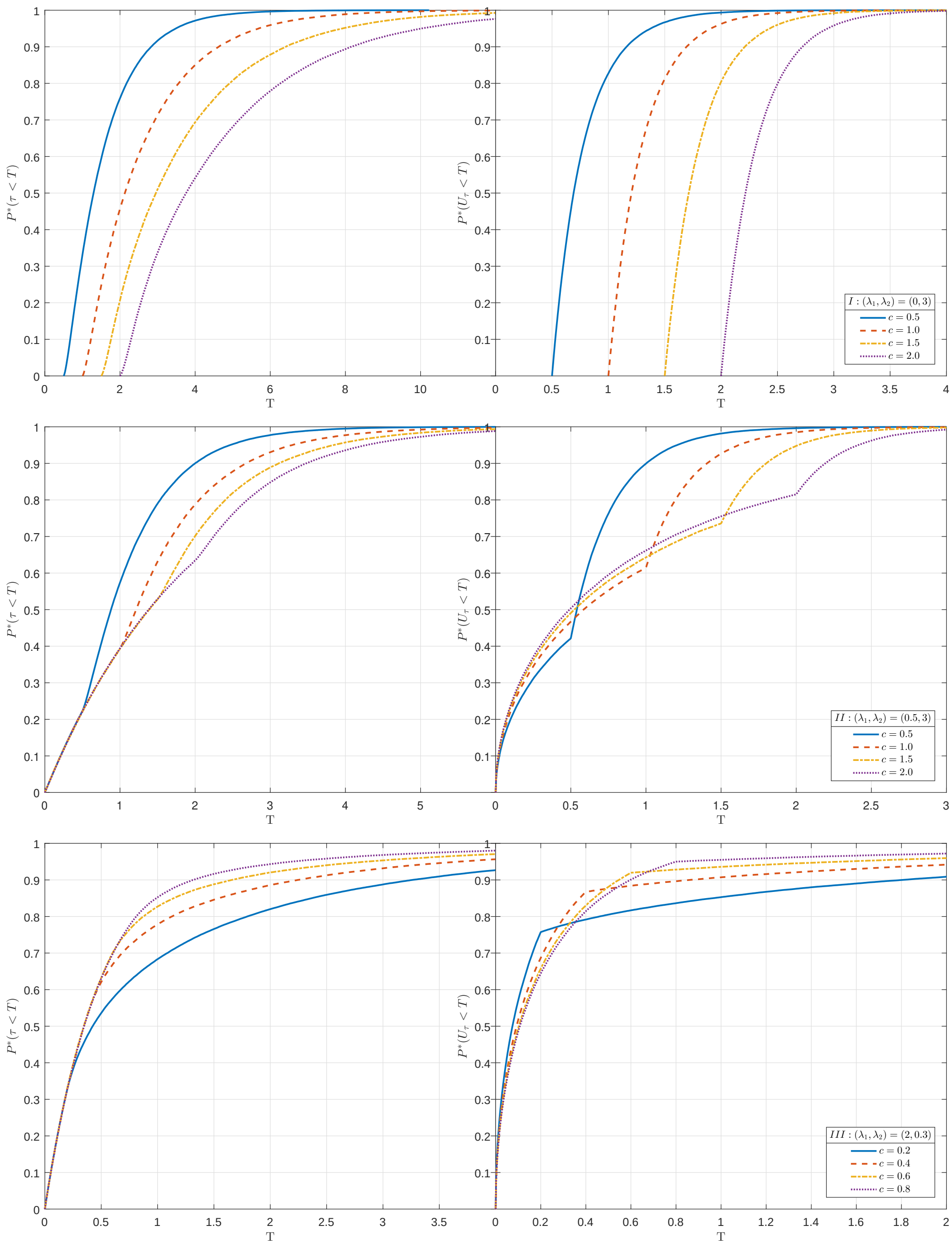


Fig. 3.2 The CDFs of τ^* (the default time) and U_{τ^*} (the default level) at time T based on a sample size of 10^5 within the Scenario (I), (II) and (III), respectively .



4

An Age-dependent Counting Process

A point process is said to be a *counting process* if it is non-negative, integer-valued and right-continuous with left limits and upward jumps of magnitude one [17]. It is often used for monitoring event occurrences with an arrival rate modulated by a random process. In the previous chapters, we introduce and study an endogenous intensity process defined on the age of the current excursion of the underlying process, and produced distributional results concerning the arrival time of the first event only. As a succeeding development to the work done before, this chapter deals with consecutive arrivals over a finite time interval. Main aspects to be considered are the study of related properties and the derivation of distributional quantities concerning the interarrival times, the arrival of the nth event and the associated counting process.

Organised in the following manner, the first SECTION 4.1 presents definitions, notations and some preliminary results for such a counting process, which will be needed in the coming proofs. Using the martingale approach within the framework of PDMP, we obtain some probabilistic results associated with the counting process in SECTION 4.3.1 . Main results pertaining to the moments are delivered with proof in SECTION 4.4. A further discussion on finding the distribution of the counting process is included in the final section.

4.1 Definition and Preliminary

On a filtered probability space $\left(\Omega, \mathscr{F}^U, \{\mathscr{F}_t^U\}_{t\geq 0}, \mathbb{P}\right)$, there is an age process adapted to $\mathscr{F}_t^U$ for all $t \geq 0$, given by

$$U_t \ = \ t - \sup\{\ s < t \mid X_s = 0\ \},$$

for a recurrent Bessel process X with index $(-\alpha) \in (-1, 0)$. Within the same space, we construct a framework for modelling event arrivals characterised by an intensity of the form $\lambda(U_t)$, where $\lambda : [0, t) \to \mathbb{R}^+$ is a non-negative measurable function. On this basis, we denote by $\{T_i\}_{i\in\mathbb{N}^0}$ with $T_0 = 0$, a non-decreasing sequence of random times, representing the arrival time of the ith event, and define the associated point process $\boldsymbol{N} = (N_t)_{t\geq 0}$ by

$$N_t \ = \ \sum_{i\geq 1} \mathbb{1}_{\{T_i \ \leq \ t\}}, \qquad N_0 = 0, \tag{4.1}$$

counting the number of events that has arrived in the time interval $(0, t]$.

We now recall some relevant concepts and properties of such point process:

(i) Given the arrival times $\{T_i\}_{i\in\mathbb{N}^0}$, the quantity

$$\tau_i \ = \ T_i - T_{i-1}, \qquad i = 1, 2, 3, \cdots$$

refers to the waiting time between two successive arrivals. Due to the fact that the chosen intensity is stochastically varying with time, the sequence of interarrival times $\{\tau_i\}_{i\in\mathbb{N}^+}$ is generally not independent. Let $\mathscr{G}_{T_{i-1}} = \boldsymbol{\sigma}\{U_s,\ s \leq T_{i-1}\}$ denote the information of U up to the $(i-1)$th arrival time. It follows for $t > 0$ that

$$\mathbb{P}\big(\tau_i \in [\ t,\ t + \mathrm{d}t) \,\big|\, \mathscr{G}_{T_{i-1}}\big) \ = \ \mathbb{E}\left[\lambda\big(U_{T_{i-1}+t}\big) \exp\left\{-\int_{T_{i-1}}^{T_{i-1}+t} \lambda(U_s)\, \mathrm{d}s\right\}\right] \mathrm{d}t.$$

(ii) N is a stochastic process taking values in $\mathbb{N}^0$. For all $\omega \in \Omega$, it is càdlàg (right continuous with left limits), i.e. $\lim_{s\uparrow t} N_t(\omega)$ exists and finite for all $t \geq 0$. Assuming that no two jumps occur simultaneously, the sample path $t \to N_t(\omega)$ is piecewise constant with jumps of size $+1$.

(iii) $\lambda(U_t)$ is a $\mathbb{R}^+$-valued stochastic process such that for any non-negative measurable function $\lambda : [0,t) \to \mathbb{R}^+$, with probability one, the intensity measure

$$\Lambda_t \;=\; \int_0^t \lambda(U_s)\,\mathrm{d}s < \infty, \qquad t \geq 0. \tag{4.2}$$

This measure is often called as the *cumulative intensity process* up to time t.

(iv) Consider an augmented filtration $\mathscr{G}_t = \mathscr{F}_t^N \vee \mathscr{F}_t^U$, where $\mathscr{F}_t^N = \boldsymbol{\sigma}\{N_s,\ 0 \leq s \leq t\}$ and $\mathscr{F}_t^U = \boldsymbol{\sigma}\{U_s,\ 0 \leq s \leq t\}$, are the natural filtrations generated by gathering the information up to time t of the counting and the age processes. Then N_t is called a *doubly stochastic point process* with $\mathscr{G}_t$-intensity $\lambda(U_t)$ in the sense that

$$\mathbb{E}[N_{t+d} - N_t \mid \mathscr{G}_t] \;=\; \mathbb{E}\bigg[\int_t^{t+d} \lambda(U_s)\,\mathrm{d}s\bigg], \qquad d \geq 0, \tag{4.3}$$

which implies

- the integrability of $\int_0^t \lambda(U_s)\mathrm{d}s$ and U_t;
- the independence of $(N_{t+d} - N_t)$ and $(\mathscr{G}_s)_{s\leq t}$; and
- the existence of a $\mathscr{G}_t$-martingale in the form

$$N_t - \int_0^t \lambda(U_s)\,\mathrm{d}s, \qquad t \geq 0. \tag{4.4}$$

Hence, Λ_t is also called the *compensator process* for N_t.

(v) Intuitively, given the information up to time t, the probability of m jumps occurring within the next infinitesimal time interval $\triangle t \to 0$ is equal to

$$\mathbb{P}(N_{t+\triangle t} - N_t = m \mid \mathscr{G}_t) \;=\; \begin{cases} 1 - \lambda(U_t)\triangle t + o(\triangle t), & m = 0; \\ \lambda(U_t)\triangle t + o(\triangle t), & m = 1; \\ o(\triangle t), & m > 1. \end{cases} \tag{4.5}$$

Furthermore, it follows from (4.3) that the distribution of N over the time interval (s,t) with $0 \leq s < t$ is given by

$$\mathbb{P}(N_t - N_s = n \mid \mathscr{G}_s) \;=\; \frac{\mathbb{E}\bigg[\exp\Big\{-\int_s^t \lambda(U_x)\,\mathrm{d}x\Big\}\Big(\int_s^t \lambda(U_x)\,\mathrm{d}x\Big)^n\bigg]}{n!}, \quad n \in \mathbb{N}^0. \tag{4.6}$$

For a more detailed and extensive treatment on the doubly stochastic processes, interested readers are referred to the books by Brémaud [17], Björk [8], Grandell [49] and Jacobsen [56].

4.2 Mathematical Framework of the Counting Model

For the same reason that there are infinitely tiny jumps in the path associated with a Bessel age process of index $0 < \alpha < 1$, we will keep working with the ε-perturbed Bessel process, formulated in CHAPTER 2.2.1 Perturbed Bessel Process as

$$X_t^{\varepsilon} \;=\; \begin{cases} \varepsilon - X_t, & \text{if } \delta_n \le t < \sigma_n; \\ \\ X_t, & \text{if } \sigma_n \le t < \delta_{n+1}, \end{cases} \qquad\qquad n = 1, 2, 3, \cdots,$$

with $\delta_0 \;=\; 0$,

$$\sigma_n \;=\; \inf\{\, t > \delta_n \mid X_t = \varepsilon \,\}, \qquad \delta_{n+1} \;=\; \inf\{\, t > \sigma_n \mid X_t = 0 \,\},$$

denoting an alternating sequence of the stopping times passing through ε and zero by the original process X_t. We define the corresponding age process by

$$U_t^{\varepsilon} \;=\; \begin{cases} t - \delta_n, & \text{if } \delta_n \le t < \sigma_n; \\ \\ t - \sigma_n, & \text{if } \sigma_n \le t < \delta_{n+1}. \end{cases}$$

Accordingly, we denote by $\{T_i^{\varepsilon}\}_{i \in \mathbb{N}^+}$ with $T_0^{\varepsilon} \;=\; 0$, the sequence of ordered arrival times, and by $\{\tau_i^{\varepsilon}\}_{i \in \mathbb{N}^+}$, the corresponding interarrival times. We then define

$$N_t^{\varepsilon} \;=\; \sum_{i \ge 1} \mathbb{1}_{\{T_i^{\varepsilon} \,\le\, t\}}, \qquad N_0^{\varepsilon} = 0,$$

to be the counting process generated by $\lambda\big(U_t^{\varepsilon}\big)$ acting as its intensity.

The joint process we concerned in this chapter is a mixture of deterministic motion and random jumps. As a powerful mathematical tool in dealing with such non-diffusion process, we construct a martingale in the form $f\big(U_t^{\varepsilon}, N_t^{\varepsilon}\big)$ and obtain an explicit representation of f with the aid of the piecewise-deterministic Markov process (PDMP) theory. This result produces immediately the (joint)

Laplace transforms and the generating functions of the variables of interest, from which we derive out more distributional quantities concerning the Nth arrival. As previously justified, the joint process $f\big(N_t^\varepsilon, U_t^\varepsilon\big)$ converges pointwisely to $f(N_t, U_t)$ as ε approaches zero. Calculations with respect to the path of the original process are actually carried out with the corresponding perturbed process.

On the basis of the above construction, the infinitesimal generator for $\big(U_t^\varepsilon, N_t^\varepsilon\big)$ acting on a bounded function $f : [0,t) \times \mathbb{N}^0 \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is given as

$$\begin{aligned} \mathcal{A} f(u,n,t) &= \frac{\partial f}{\partial t} + \mathcal{A}_{u,n} f; \\ \mathcal{A}_{u,n} f &= \frac{\partial f}{\partial u} + \frac{p_\varepsilon(u)}{\bar{P}_\varepsilon(u)}\Big(f(0,n) - f(u,n)\Big) + \lambda(u)\Big(f(u,n+1) - f(u,n)\Big), \end{aligned}$$

with

$$\begin{aligned} p_\varepsilon(u) &= \frac{\varepsilon^{2\alpha}}{2^\alpha \Gamma(\alpha)} u^{-\alpha-1} e^{-\frac{\varepsilon^2}{2u}}; \\ \bar{P}_\varepsilon(u) &= \int_u^\infty \frac{\varepsilon^{2\alpha}}{2^\alpha \Gamma(\alpha)} x^{-\alpha-1} e^{-\frac{\varepsilon^2}{2x}}\, \mathrm{d}x, \end{aligned}$$

where $p_\varepsilon(u)$ and $\bar{P}_\varepsilon(u)$ denote the density function and the tail distribution, respectively, of the first hitting time of zero for a recurrent Bessel process starting from ε. In particular to the time domain $t \in \big[\, T_{i-1},\ T_{i-1} + \tau_i\, \big)_{i \in \mathbb{N}^+}$, the main concern associated with this period of time is the interarrival process, denoted by $\big(U_{T_i}, \tau_i\big)$, such that for $\nu > 0$,

$$\begin{aligned} &\mathbb{P}\Big(\tau_i \in \mathrm{d}\nu \,\Big|\, \mathcal{G}_{T_{i-1}} \ \vee\ \boldsymbol{\sigma}\Big\{U_s,\ s \in \Big[\, T_{i-1},\ T_{i-1} + \nu\, \Big)\Big\}\Big) \\ &= \lambda\big(U_{T_{i-1}+\nu}\big) \exp\Big\{-\int_{T_{i-1}}^{T_{i-1}+\nu} \lambda(U_s)\, \mathrm{d}s\Big\}\, \mathrm{d}\nu. \end{aligned} \tag{4.7}$$

4.3 Distributional Results concerning the Counting Process

For better presenting the results, we adopt the following conventions

$$\Lambda(t) := \int_0^t \lambda(s)\, \mathrm{d}s, \qquad \Lambda'(t) = \lambda(t)\,.$$

To proceed, we define a pair of complementary probabilities denoted by $\Theta_\varepsilon\big(U_{T_{i-1}}\big)$ and $\bar{\Theta}_\varepsilon\big(U_{T_{i-1}}\big)$ such that

$$\Theta_\varepsilon(U) \;=\; \frac{e^{\Lambda(U)}}{\bar{P}_\varepsilon(U)} \int_U^\infty \lambda(x) e^{-\Lambda(x)} \bar{P}_\varepsilon(x)\,\mathrm{d}x;$$

$$\bar{\Theta}_\varepsilon(U) \;=\; \frac{e^{\Lambda(U)}}{\bar{P}_\varepsilon(U)} \int_U^\infty e^{-\Lambda(x)} p_\varepsilon(x)\,\mathrm{d}x.$$

It holds for any proper choice of $\lambda(\cdot)$ that: for any $0 < U < \infty$

- $\Theta_\varepsilon(U) \in (0,1), \quad \bar{\Theta}_\varepsilon(U) \in (0,1);$
- $\lim\limits_{U\to 0} \Theta_\varepsilon(U) = 0 \quad \text{and} \quad \lim\limits_{U\to\infty} \Theta_\varepsilon(U) = 1\,;$

  $\lim\limits_{U\to 0} \bar{\Theta}_\varepsilon(U) = 1 \quad \text{and} \quad \lim\limits_{U\to\infty} \bar{\Theta}_\varepsilon(U) = 0\,;$
- $\Theta_\varepsilon(U) \;+\; \bar{\Theta}_\varepsilon(U) \;=\; 1$

Intuitively, the quantity $\Theta_\varepsilon\big(U_{T_{i-1}}\big)$ measures the possibility that the i-th event arrives before the first reflection at zero by the age process since the last arrival $U_{T_{i-1}}$, i.e.

$$\Theta_\varepsilon\big(U_{T_{i-1}}\big) \;=\; \mathbb{P}\bigg(T_i \;<\; \inf\big\{\, t > T_{i-1} \,\big|\, U_t^\varepsilon = 0 \,\big\} \bigg), \qquad i \in \mathbb{N}^+\,,$$

and $\bar{\Theta}_\varepsilon\big(U_{T_{i-1}}\big)$ stands as the contrary. Within each circumstance, the probability densities corresponding respectively to Θ_ε and $\bar{\Theta}_\varepsilon$ are denoted and given by

$$\vartheta_\varepsilon(x;U) \;=\; \frac{\lambda(x) e^{-\Lambda(x)} \bar{P}_\varepsilon(x)}{\int_U^\infty \lambda(x) e^{-\Lambda(x)} \bar{P}_\varepsilon(x)\,\mathrm{d}x} \mathbb{1}_{\{x \,\geq\, U\}};$$

$$\bar{\vartheta}_\varepsilon(x;U) \;=\; \frac{e^{-\Lambda(x)} p_\varepsilon(x)}{\int_U^\infty e^{-\Lambda(x)} p_\varepsilon(x)\,\mathrm{d}x} \mathbb{1}_{\{x \,\geq\, U\}}\,.$$

After taking ε to zero, the above quantities turn into

$$\Theta(U) \;=\; \lim_{\varepsilon\to 0}\Theta_\varepsilon(U) \;=\; e^{\Lambda(U)}U^{\alpha}\int\limits_{U}^{\infty}\lambda(x)e^{-\Lambda(x)}x^{-\alpha}\,\mathrm{d}x;$$

$$\bar{\Theta}(U) \;=\; \lim_{\varepsilon\to 0}\bar{\Theta}_\varepsilon(U) \;=\; e^{\Lambda(U)}U^{\alpha}\int\limits_{U}^{\infty}e^{-\Lambda(x)}\alpha x^{-\alpha-1}\,\mathrm{d}x;$$

$$\vartheta(x;U) \;=\; \lim_{\varepsilon\to 0}\vartheta_\varepsilon(x;U) \;=\; \frac{\lambda(x)e^{-\Lambda(x)}x^{-\alpha}}{\int\limits_{U}^{\infty}\lambda(x)e^{-\Lambda(x)}x^{-\alpha}\,\mathrm{d}x}\mathbb{1}_{\{x\,\geq\,U\}};$$

$$\bar{\vartheta}(x;U) \;=\; \lim_{\varepsilon\to 0}\bar{\vartheta}_\varepsilon(x;U) \;=\; \frac{e^{-\Lambda(x)}\alpha x^{-\alpha-1}}{\int\limits_{U}^{\infty}e^{-\Lambda(x)}\alpha x^{-\alpha-1}\,\mathrm{d}x}\mathbb{1}_{\{x\,\geq\,U\}}\,.$$

Please note these notations will be utilised throughout the rest of this chapter.

4.3.1 Generating Functions

It can be seen immediately that due to the Markov property of the age process, results associated with the i-th arrival are mostly representable of that with the first denoted in this chapter by $\tau_1 = T_1$.

Theorem 4.3.1 (Generating Functions for the Interarrival Time). *Let $\kappa(\cdot)$ be a non-negative bounded function and $\alpha \in (0,1)$ and $\beta \in \mathbb{R}^+$ be constants, the generating function for the conditional process $\left(U^\varepsilon_{T_i}, \tau^\varepsilon_i \,\middle|\, \mathscr{G}_{T^\varepsilon_{i-1}}\right)_{i\in\mathbb{N}^+}$ is evaluated as*

$$\mathbb{E}\Big[e^{-\beta\tau^\varepsilon_i}\kappa\Big(U^\varepsilon_{T_i}\Big)\,\Big|\,\mathscr{G}_{T^\varepsilon_{i-1}}\Big]\cdot e^{-\beta U^\varepsilon_{T_{i-1}}} \;=$$

$$\bar{\Theta}_\varepsilon\Big(U^\varepsilon_{T_{i-1}}\Big)\int\limits_{U^\varepsilon_{T_{i-1}}}^{\infty} e^{-\beta v}\;\bar{\vartheta}_\varepsilon\Big(v;U^\varepsilon_{T_{i-1}}\Big)\,\mathrm{d}v\cdot\mathbb{E}\big[e^{-\beta\tau^\varepsilon_1}\kappa\big(U^\varepsilon_{\tau_1}\big)\big] \;+$$

$$\Theta_\varepsilon\Big(U^\varepsilon_{T_{i-1}}\Big)\int\limits_{U^\varepsilon_{T_{i-1}}}^{\infty} e^{-\beta v}\;\kappa(v)\vartheta_\varepsilon\Big(v;U^\varepsilon_{T_{i-1}}\Big)\,\mathrm{d}v. \tag{4.8}$$

Taking limit of ε to zero yields the generating function for $\big(U_{T_i},\ \tau_i \,\big|\, \mathscr{G}_{T_{i-1}}\big)_{i\in\mathbb{N}^+}$

$$\begin{aligned}
&\mathbb{E}\big[e^{-\beta\tau_i}\kappa\big(U_{T_i}\big) \,\big|\, \mathscr{G}_{T_{i-1}}\big] \cdot e^{-\beta U_{T_{i-1}}} \ = \\
&\bar{\Theta}\big(U_{T_{i-1}}\big) \int\limits_{U_{T_{i-1}}}^{\infty} e^{-\beta v}\ \bar{\vartheta}\big(v; U_{T_{i-1}}\big)\,\mathrm{d}v \cdot \mathbb{E}\big[e^{-\beta\tau_1}\kappa\big(U_{\tau_1}\big)\big] \ + \\
&\Theta_\varepsilon\big(U_{T_{i-1}}\big) \int\limits_{U_{T_{i-1}}}^{\infty} e^{-\beta v}\ \kappa(v)\vartheta\big(v; U_{T_{i-1}}\big)\,\mathrm{d}v.
\end{aligned} \tag{4.9}$$

In the above expectations, $\mathbb{E}\big[e^{-\beta\tau_1^\varepsilon}\kappa\big(U^\varepsilon_{\tau_1}\big)\big]$ *and* $\mathbb{E}\big[e^{-\beta\tau_1}\kappa\big(U_{\tau_1}\big)\big]$ *are the results corresponding to* τ_1*, which have been derived by* (2.16) *and* (2.17) *respectively of Theorem 2.2.1 (in Chapter 2).*

Proof. Given the representation in (4.7), the following probabilities are equal in quantity,

$$\begin{aligned}
&\mathbb{P}\Big(\tau_i^\varepsilon \in \mathrm{d}v \,\Big|\, \mathscr{G}_{T^\varepsilon_{i-1}} \vee \boldsymbol{\sigma}\Big\{U^\varepsilon_s,\ s\in\Big[\,T^\varepsilon_{i-1},\ T^\varepsilon_{i-1}+v\,\Big)\Big\}\Big) \\
=\ &\mathbb{P}\Big(T_i^\varepsilon \in \mathrm{d}t \,\Big|\, \mathscr{G}_{T^\varepsilon_{i-1}} \vee \boldsymbol{\sigma}\big\{U^\varepsilon_s,\ s\in\big[\,T^\varepsilon_{i-1},\ t\,\big)\big\}\Big) \\
=\ &\lambda\big(U^\varepsilon_t\big)\exp\bigg\{-\int_{T^\varepsilon_{i-1}}^{t}\lambda\big(U^\varepsilon_s\big)\,\mathrm{d}s\bigg\}\,\mathrm{d}t,
\end{aligned}$$

with $0<v<\infty$ and $t = v + T^\varepsilon_{i-1} \in \big[\,T^\varepsilon_{i-1},\ \infty\,\big)$.

The generating function for $\Big(U^\varepsilon_{T_i}, \tau^\varepsilon_i \,\Big|\, \mathscr{G}_{T^\varepsilon_{i-1}}\Big)$ is formulated as a Laplace transform of the following form

$$\begin{aligned}
&e^{\beta T^\varepsilon_{i-1}} \cdot \mathbb{E}\Big[e^{-\beta T_i^\varepsilon}\kappa\Big(U^\varepsilon_{T_i}\Big) \,\Big|\, \mathscr{G}_{T^\varepsilon_{i-1}}\Big] \\
=\ &e^{\beta T^\varepsilon_{i-1}} \int\limits_{T^\varepsilon_{i-1}}^{\infty} \mathbb{E}\Big[e^{-\beta t}\kappa\big(U^\varepsilon_t\big)\mathbb{P}\Big(T_i^\varepsilon\in\mathrm{d}t \,\Big|\, \mathscr{G}_{T^\varepsilon_{i-1}} \vee \boldsymbol{\sigma}\Big\{U^\varepsilon_s,\ s\in\Big[\,T^\varepsilon_{i-1},\ t\,\Big)\Big\}\Big) \,\Big|\, \mathscr{G}_{T^\varepsilon_{i-1}}\Big]\,\mathrm{d}t \\
=\ &e^{\beta T^\varepsilon_{i-1}} \int\limits_{T^\varepsilon_{i-1}}^{\infty} e^{-\beta t}\ \mathbb{E}\Bigg[\kappa\big(U^\varepsilon_t\big)\lambda\big(U^\varepsilon_t\big)\exp\bigg\{-\int_{T^\varepsilon_{i-1}}^{t}\lambda\big(U^\varepsilon_s\big)\mathrm{d}s\bigg\}\,\mathrm{d}t \,\Bigg|\, \mathscr{G}_{T^\varepsilon_{i-1}}\Bigg]\,\mathrm{d}t.
\end{aligned} \tag{4.10}$$

To find a representation of this formula, we augment the original process by adding two continuous processes, read as

$$Y_t^\varepsilon \;=\; \int\limits_{T_{i-1}^\varepsilon}^{t} \lambda\big(U_s^\varepsilon\big)\,\mathrm{d}s, \qquad Z_t^\varepsilon \;=\; \int\limits_{T_{i-1}^\varepsilon}^{t} e^{-\beta s}\,\kappa\big(U_s^\varepsilon\big)\lambda\big(U_s^\varepsilon\big)e^{-Y_s^\varepsilon}\,\mathrm{d}s.$$

The generator written by the bounded function f for the augmented process $\big(U_t^\varepsilon, Y_t^\varepsilon, Z_t^\varepsilon, t\big)$ in its domain is:

$$\mathcal{G}f(u,y,z,t) \;=\; \frac{\partial f}{\partial t} + \mathcal{A}_u f + \lambda(u)\frac{\partial f}{\partial y} + e^{-\beta t}\kappa(u)\lambda(u)e^{-y}\frac{\partial f}{\partial z}.$$

Substituting $f(u,y,z,t) \;=\; z + e^{-\beta t}e^{-y}g(u)$ with $g(\cdot)$ assumed to be a bounded function into $\mathcal{G}f = 0$ generates

$$g'(u) - g(u)\tfrac{\partial}{\partial u}\Big\{\,\beta u - \ln\bar{P}_\varepsilon(u) + \Lambda(u)\,\Big\} \;=\; -\left(\frac{p_\varepsilon(u)}{\bar{P}_\varepsilon(u)}g(0) + \lambda(u)\kappa(u)\right).$$

The solution to this differential equation is obtained as

$$e^{-\beta u}g(u) \;=\; \frac{e^{\Lambda(u)}}{\bar{P}_\varepsilon(u)}\int\limits_u^\infty e^{-\beta v}e^{-\Lambda(v)}p_\varepsilon(v)\,\mathrm{d}v \cdot \frac{\int\limits_0^\infty e^{-\beta v}\kappa(v)\lambda(v)e^{-\Lambda(v)}\bar{P}_\varepsilon(v)\,\mathrm{d}v}{\int\limits_0^\infty (\beta + \lambda(v))e^{-\beta v}e^{-\Lambda(v)}\bar{P}_\varepsilon(v)\,\mathrm{d}v}$$
$$+\;\frac{e^{\Lambda(u)}}{\bar{P}_\varepsilon(u)}\int\limits_u^\infty e^{-\beta v}\kappa(v)\lambda(v)e^{-\Lambda(v)}\bar{P}_\varepsilon(v)\,\mathrm{d}v.$$

According to the theory of PDMP, we prove a martingale of the form,

$$f\big(U_t^\varepsilon, Y_t^\varepsilon, Z_t^\varepsilon, t\big) \;=\; Z_t^\varepsilon \;+\; e^{-\beta t}e^{-Y_t^\varepsilon}g\big(U_t^\varepsilon\big),$$

and further by the property of Markov process we get,

$$\begin{aligned}
&\lim_{t\to\infty}\mathbb{E}\Big[Z_t^\varepsilon \;+\; e^{-\beta t}e^{-Y_t^\varepsilon}g\big(U_t^\varepsilon\big)\;\Big|\;\mathscr{G}_{T_{i-1}^\varepsilon}\Big]\\
=\;&\lim_{t\to\infty}\mathbb{E}\Big[Z_t^\varepsilon\;\Big|\;\mathscr{G}_{T_{i-1}^\varepsilon}\Big]\\
=\;&e^{-\beta T_{i-1}^\varepsilon}g\Big(U_{T_{i-1}}^\varepsilon\Big).
\end{aligned}$$

This leads immediately to the result of the Laplace transform in (4.10) and thus to the generating function in (4.8). So the other function (4.9) is simply obtained by taking ε to zero. □

Theorem 4.3.2 (Generating Functions for the Counting Process). *Let $h(\cdot)$ be a non-negative bounded function and $\alpha, \phi \in (0,1)$, $\beta \in \mathbb{R}^+$ be constants, the Laplace transform (with respect to time t) of the generating function for the counting process $\big(N_t^\varepsilon,\ U_t^\varepsilon \,\big|\, U_0\big)$ is defined and evaluated by*

$$\begin{aligned}
&\mathcal{L}_t\Big\{\ \mathbb{E}\big[\phi^{N_t^\varepsilon} h\big(U_t^\varepsilon\big) \,\big|\, U_0\big],\ \beta\ \Big\} \cdot e^{-\beta U_0} \ = \\
&\int_{U_0}^{\infty} e^{-\beta v} e^{-(1-\phi)\big(\Lambda(v)-\Lambda(U_0)\big)} \frac{p_\varepsilon(v)}{\bar{P}_\varepsilon(U_0)}\, \mathrm{d}v \cdot \mathcal{L}_t\Big\{\ \mathbb{E}\big[\phi^{N_t^\varepsilon} h\big(U_t^\varepsilon\big)\big],\ \beta\ \Big\} \ + \\
&\int_{U_0}^{\infty} e^{-\beta v} h(v) e^{-(1-\phi)\big(\Lambda(v)-\Lambda(U_0)\big)} \frac{\bar{P}_\varepsilon(v)}{\bar{P}_\varepsilon(U_0)}\, \mathrm{d}v.
\end{aligned} \tag{4.11}$$

Taking limit of ε to zero yields the generating function for $(N_t,\ U_t \mid U_0)$,

$$\begin{aligned}
&\mathcal{L}_t\Big\{\ \mathbb{E}\big[\phi^{N_t} h(U_t) \,\big|\, U_0\big],\ \beta\ \Big\} \cdot e^{-\beta U_0} \ = \\
&{U_0}^{\alpha} \int_{U_0}^{\infty} e^{-\beta v} e^{-(1-\phi)\big(\Lambda(v)-\Lambda(U_0)\big)} \alpha v^{-\alpha-1}\, \mathrm{d}v \cdot \mathcal{L}_t\Big\{\ \mathbb{E}\big[\phi^{N_t} h(U_t)\big],\ \beta\ \Big\} \ + \\
&{U_0}^{\alpha} \int_{U_0}^{\infty} e^{-\beta v} h(v) e^{-(1-\phi)\big(\Lambda(v)-\Lambda(U_0)\big)} v^{-\alpha}\, \mathrm{d}v.
\end{aligned} \tag{4.12}$$

In the above results, we have defined

$$\mathcal{L}_t\Big\{\ \mathbb{E}\big[\phi^{N_t^\varepsilon} h\big(U_t^\varepsilon\big)\big],\ \beta\ \Big\} \ = \ \frac{\displaystyle\int_0^\infty e^{-\beta v} h(v) e^{-(1-\phi)\Lambda(v)} \bar{P}_\varepsilon(v)\, \mathrm{d}v}{\displaystyle\int_0^\infty \Big(\beta + (1-\phi)\lambda(v)\Big) e^{-\beta v} e^{-(1-\phi)\Lambda(v)} \bar{P}_\varepsilon(v)\, \mathrm{d}v}.$$

It represents a limited case for the counting model as taking U_0 equal to zero, in which circumstance the underlying process is reset to zero.

Then the limit of $\varepsilon \to 0$ is given by

$$\mathcal{L}_t\left\{ \mathbb{E}\left[\phi^{N_t} h(U_t)\right],\ \beta \right\} \ = \ \frac{\displaystyle\int_0^\infty e^{-\beta v} h(v) e^{-(1-\phi)\Lambda(v)} v^{-\alpha}\ \mathrm{d}v}{\displaystyle\int_0^\infty \Big(\beta + (1-\phi)\lambda(v)\Big) e^{-\beta v} e^{-(1-\phi)\Lambda(v)} v^{-\alpha}\ \mathrm{d}v}.$$

Proof. We start by adding

$$Z_t^\varepsilon \ = \ \int_0^t e^{-\beta s} \phi^{N_s^\varepsilon} h\big(U_s^\varepsilon\big)\ \mathrm{d}s, \qquad \phi \in (0,1),\ t \geq 0,$$

to the joint process $\big(N_t^\varepsilon, U_t^\varepsilon\big)$. Then the new generator for the extended process $\big(Z_t^\varepsilon, N_t^\varepsilon, U_t^\varepsilon, t\big)$ in the domain $t \geq 0$ is written by

$$\mathcal{G} f(z,u,n,t) \ = \ \frac{\partial f}{\partial t} + e^{-\beta t}\phi^n h(u)\frac{\partial f}{\partial z} + \mathcal{A}_{u,n} f, \tag{4.13}$$

where $\mathcal{A}_{u,n} f$ has been given in (4.2). We propose a suitable solution to $\mathcal{G} f = 0$ in the form

$$f(z,u,n,t) \ = \ z + e^{-\beta t}\phi^n g(u),$$

with $g(\cdot)$ assumed to be a bounded function. Substituting it into (4.13) generates an equation,

$$-\beta g(u) + g'(u) + h(u) + \frac{p_\varepsilon(u)}{\bar{P}_\varepsilon(u)}\Big(g(0) - g(u)\Big) - (1-\phi)\lambda(u) g(u) \ = \ 0.$$

The solution to this differential equation is found as

$$g(u) \ = \ g(0)\frac{\displaystyle\int_u^\infty e^{-\beta v} e^{-(1-\phi)\Lambda(v)} p_\varepsilon(v)\ \mathrm{d}v}{e^{-\beta u} e^{-(1-\phi)\Lambda(u)} \bar{P}_\varepsilon(u)} \ + \ \frac{\displaystyle\int_u^\infty e^{-\beta v} h(v) e^{-(1-\phi)\Lambda(v)} \bar{P}_\varepsilon(v)\ \mathrm{d}v}{e^{-\beta u} e^{-(1-\phi)\Lambda(u)} \bar{P}_\varepsilon(u)},$$

with

$$g(0) \ = \ \frac{\displaystyle\int_0^\infty e^{-\beta v} h(v) e^{-(1-\phi)\Lambda(v)} \bar{P}_\varepsilon(v)\ \mathrm{d}v}{\displaystyle\int_0^\infty \beta e^{-\beta v} e^{-(1-\phi)\Lambda(v)} \bar{P}_\varepsilon(v)\ \mathrm{d}v + \int_0^\infty (1-\phi) e^{-\beta v} \lambda(v) e^{-(1-\phi)\Lambda(v)} \bar{P}_\varepsilon(v)\ \mathrm{d}v}.$$

By the property of PDMP, we find a martingale of the form

$$f\big(Z_t^{\varepsilon}, N_t^{\varepsilon}, U_t^{\varepsilon}, t\big) \;=\; Z_t^{\varepsilon} \;+\; e^{-\beta t}\phi^{N_t^{\varepsilon}} g\big(U_t^{\varepsilon}\big).$$

Given that both $g(u)$ and $h(u)$ are bounded functions, we have

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}\big[Z_t^{\varepsilon} \;+\; e^{-\beta t}\phi^{N_t^{\varepsilon}} g\big(U_t^{\varepsilon}\big)\big] \\
&= \int_0^{\infty} e^{-\beta t}\, \mathbb{E}\big[\phi^{N_t^{\varepsilon}} h\big(U_t^{\varepsilon}\big)\big]\, \mathrm{d}t \\
&= g(U_0).
\end{aligned}$$

The proof of (4.11) is then completed and (4.12) follows immediately by taking ε to zero. □

4.3.2 Laplace Transforms

On the basis of these generating functions, they produce distributional results describing each of the concerned processes over time, or conditioned on the others.

Corollary 4.3.3 (Laplace Transforms for the Interarrival Times). *Regarding the interarrival process, the Laplace transform of* $U_{T_i} \mid \mathscr{G}_{T_{i-1}}$, $i \in \mathbb{N}^+$, *with* $\varphi > 0$ *is given by,*

$$\begin{aligned}
&\mathbb{E}\big[e^{-\varphi U_{T_i}} \,\big|\, \mathscr{G}_{T_{i-1}}\big] \;= \\
&\bar{\Theta}\big(U_{T_{i-1}}\big)\cdot \mathbb{E}\big[e^{-\varphi U_{\tau_1}}\big] \;+\; \Theta\big(U_{T_{i-1}}\big)\cdot \int_{U_{T_{i-1}}}^{\infty} e^{-\varphi v}\; \vartheta\big(v; U_{T_{i-1}}\big)\, \mathrm{d}v,
\end{aligned}$$

from which we get the density function of $U_{T_i} \mid \mathscr{G}_{T_{i-1}}$,

$$\begin{aligned}
&\mathbb{P}\big(U_{T_i} \in \mathrm{d}u \,\big|\, \mathscr{G}_{T_{i-1}}\big) \;= \\
&\bar{\Theta}\big(U_{T_{i-1}}\big)\cdot \mathbb{P}\big(U_{\tau_1} \in \mathrm{d}u\big) \;+\; \Theta\big(U_{T_{i-1}}\big)\cdot \vartheta\big(u; U_{T_{i-1}}\big)\, \mathrm{d}u.
\end{aligned}$$

Furthermore, the Laplace transform of $\tau_i \mid \mathscr{G}_{T_{i-1}}$ *is given by,*

$$\mathbb{E}\big[e^{-\beta\tau_i} \,\big|\, \mathscr{G}_{T_{i-1}}\big]\cdot e^{-\beta U_{T_{i-1}}} \;=$$

$$\bar{\Theta}\big(U_{T_{i-1}}\big) \int_{U_{T_{i-1}}}^{\infty} e^{-\beta v}\ \bar{\vartheta}\big(v; U_{T_{i-1}}\big)\,\mathrm{d}v \cdot \mathbb{E}\big[e^{-\beta \tau_1}\big] \ + \ \Theta\big(U_{T_{i-1}}\big) \int_{U_{T_{i-1}}}^{\infty} e^{-\beta v}\ \vartheta\big(v; U_{T_{i-1}}\big)\,\mathrm{d}v.$$

In the results above, $\mathbb{E}\big[e^{-\varphi U_{\tau_1}}\big]$, $\mathbb{E}\big[e^{-\beta \tau_1}\big]$ *and* $\mathbb{P}\big(U_{\tau_1} \in \mathrm{d}u\big)$ *are obtained before in Corollary 2.2.3 of Chapter 2.*

Proof. These results are obtained as a direct consequence of setting $\beta = 0$ and $\kappa(v) = 1$ in (4.9). □

Corollary 4.3.4 (Laplace Transform concerning the Counting Process). *Regarding the counting process, the Laplace transform of the generating function of* $N_t \mid U_0$ *is given by*

$$\begin{aligned}
&\mathcal{L}_t\big\{\, \mathbb{E}\big[\phi^{N_t} \,\big|\, U_0\big],\, \beta \,\big\} \ = \\
&{U_0}^{\alpha} \int_{U_0}^{\infty} e^{-\beta v} e^{-(1-\phi)\big(\Lambda(v)-\Lambda(U_0)\big)} \alpha v^{-\alpha-1}\,\mathrm{d}v \cdot \mathcal{L}_t\big\{\, \mathbb{E}\big[\phi^{N_t}\big],\, \beta \,\big\} \ + \\
&{U_0}^{\alpha} \int_{U_0}^{\infty} e^{-\beta v} e^{-(1-\phi)\big(\Lambda(v)-\Lambda(U_0)\big)} v^{-\alpha}\,\mathrm{d}v,
\end{aligned} \tag{4.14}$$

with

$$\mathcal{L}_t\big\{\, \mathbb{E}\big[\phi^{N_t}\big],\, \beta \,\big\} \ = \ \frac{\displaystyle\int_0^{\infty} e^{-\beta v} e^{-(1-\phi)\Lambda(v)} v^{-\alpha}\,\mathrm{d}v}{\displaystyle\int_0^{\infty} \Big(\beta + (1-\phi)\lambda(v)\Big) e^{-\beta v} e^{-(1-\phi)\Lambda(v)} v^{-\alpha}\,\mathrm{d}v}.$$

Proof. This result follows immediately by setting $h(v) = 1$ of the generating function in (4.12). □

4.4 The First Two Moments for the Counting Process

This section is concerned with the derivation of first two moments of the counting process with a general intensity function. The moments are derived from the distributional results just solved in SECTION 4.3.2 Laplace Transforms, and expressed explicitly as integrals.

Theorem 4.4.1 (First Moment). *The first moment (expectation) of the counting process $N_t \mid U_0 > 0$ with a density defined by $\lambda(U_t)$ is given by*

$$\mathbb{E}[N_t \mid U_0] = {U_0}^{\alpha} \int_0^t \left\{ \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_0^v (t - v + U_0)^{-\alpha-1} (v-x)^{\alpha} x^{-\alpha} \lambda(x)\, \mathrm{d}x \right. \\ \left. + \; (U_0 + v)^{-\alpha} \lambda(v + U_0) \right\} \mathrm{d}v, \tag{4.15}$$

where $\lambda(\cdot)$ is any non-negative function satisfying $\int_0^t \lambda(s)\mathrm{d}s < \infty$ for $t > 0$.

Proof. To avoid tediously long expressions, we temporarily set

$$R(\phi) = \frac{\int_0^{\infty} e^{-\beta v} e^{-(1-\phi)\Lambda(v)} v^{-\alpha}\, \mathrm{d}v}{\int_0^{\infty} \left(1 - e^{-\beta x} e^{-(1-\phi)\Lambda(x)}\right) \alpha x^{-\alpha-1}\, \mathrm{d}x}, \tag{4.16}$$

and then

$$R'(\phi) = \frac{\int_0^{\infty} e^{-\beta v} \Lambda(v) e^{-(1-\phi)\Lambda(v)} v^{-\alpha}\, \mathrm{d}v + R(\phi) \int_0^{\infty} e^{-\beta x} \Lambda(x) e^{-(1-\phi)\Lambda(x)} \alpha x^{-\alpha-1}\, \mathrm{d}x}{\int_0^{\infty} \left(1 - e^{-\beta x} e^{-(1-\phi)\Lambda(x)}\right) \alpha x^{-\alpha-1}\, \mathrm{d}x}. \tag{4.17}$$

Taking the derivative of (4.14) with respect to ϕ, we obtain

$$\begin{aligned} & \tfrac{\partial}{\partial \phi} \mathcal{L}_t\left\{ \mathbb{E}\left[\phi^{N_t} \mid U_0\right], \beta \right\} \cdot e^{-\beta U_0} {U_0}^{-\alpha} \\ &= R(\phi) \cdot \int_{U_0}^{\infty} e^{-\beta v} \Big(\Lambda(v) - \Lambda(U_0)\Big) e^{-(1-\phi)\left(\Lambda(v) - \Lambda(U_0)\right)} \alpha v^{-\alpha-1}\, \mathrm{d}v \\ &\quad + R'(\phi) \cdot \int_{U_0}^{\infty} e^{-\beta v} e^{-(1-\phi)\left(\Lambda(v) - \Lambda(U_0)\right)} \alpha v^{-\alpha-1}\, \mathrm{d}v \\ &\quad + \int_{U_0}^{\infty} e^{-\beta v} \Big(\Lambda(v) - \Lambda(U_0)\Big) e^{-(1-\phi)\left(\Lambda(v) - \Lambda(U_0)\right)} v^{-\alpha}\, \mathrm{d}v. \end{aligned}$$

Setting $\phi = 1$, we get the Laplace transform for the first moment

$$\mathcal{L}_t\{\, \mathbb{E}[N_t \mid U_0],\ \beta\,\} \ = $$

$$\frac{{U_0}^{\alpha}}{\Gamma(1-\alpha)\Gamma(1+\alpha)} \int_0^{\infty} e^{-\beta t} \int_0^{t} \alpha(v+U_0)^{-\alpha-1} \int_0^{t-v} (t-v-x)^{\alpha} x^{-\alpha} \lambda(x)\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t$$

$$+\ {U_0}^{\alpha} \int_0^{\infty} e^{-\beta t} \int_0^{t} \lambda(v+U_0)(v+U_0)^{-\alpha}\,\mathrm{d}v\,\mathrm{d}t.$$

By the property of Laplace transform, we get an exact expression for $\mathbb{E}[N_t \mid U_0]$. □

Corollary 4.4.2. *The conditioned age process* $\{U_t \mid U_0 > 0\}_{t>0}$ *follows a generalized arcsine law with parameter* $\alpha \in (0,1)$*, read as*

$$\mathbb{P}(U_t \in \mathrm{d}x \mid U_0) \ = \begin{cases} \left(1 - \dfrac{U_0}{t-x+U_0}\right)\dfrac{(t-x)^{\alpha-1}x^{-\alpha}}{\Gamma(1-\alpha)\Gamma(\alpha)}\,\mathrm{d}x, & 0 < x < t; \\ \left(1 + \dfrac{t}{U_0}\right)^{-\alpha}\mathrm{d}x, & x = U_0 + t, \end{cases} \tag{4.18}$$

i.e. a Beta distribution with parameter α*, denoted by* $U \sim \mathrm{Beta}(\alpha, 1-\alpha)$*.*

Proof. First we rewrite the expectation (4.15) as

$$\frac{{U_0}^{\alpha}}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_0^{t} \int_0^{v} x^{-\alpha}\lambda(x) \int_x^{v} \alpha(v-z+U_0)^{-\alpha-1}(z-x)^{\alpha-1}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}v$$

$$+\ {U_0}^{\alpha} \int_0^{t} \lambda(v+U_0)(v+U_0)^{-\alpha}\,\mathrm{d}v$$

$$= \int_0^{t} \left\{ \int_0^{v} \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}(v-x+U_0)^{-1}(v-x)^{\alpha}x^{-\alpha}\lambda(x)\,\mathrm{d}x \right.$$

$$\left. +\ {U_0}^{\alpha}(U_0+v)^{-\alpha}\lambda(v+U_0) \right\} \mathrm{d}v$$

$$= \int_0^{t} \mathbb{E}[\lambda(U_v) \mid U_0]\,\mathrm{d}v.$$

As mentioned earlier, it holds for every Borel function $\lambda : (0,t] \to \mathbb{R}^+$ such that $\int_0^t \lambda(U_s)\,\mathrm{d}s < \infty$. The inner integral contains a proper density function, verified by the Euler's reflection formula [1],

$$\int_0^t (t-s)^{\alpha-1} s^{-\alpha}\,\mathrm{d}s \;=\; \Gamma(1-\alpha)\Gamma(\alpha) \;=\; \frac{\pi}{\sin(\alpha\pi)}.$$

□

Remark 4.4.3. There is a special case for the expectation. By taking the limit of U_0 to zero, we get

$$\lim_{U_0\to 0} \mathbb{E}[N_t \mid U_0] \;=\; \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} \int_0^t (t-x)^{\alpha} x^{-\alpha} \lambda(x)\,\mathrm{d}x, \qquad t > 0,$$

and then

$$\mathbb{P}(U_t \in \mathrm{d}x) \;=\; \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} (t-x)^{\alpha-1} x^{-\alpha}\,\mathrm{d}x, \qquad 0 < x < t.$$

The result above is an extension of the classical Lévy's arcsine law to the context of Bessel processes, thus recovering the results proved already in Dynkin [42], Getoor and Sharpe [46] and Nikeghbali [82] by means of excursion theory.

Theorem 4.4.4 (Second Moment). *The second moment of the counting process $N_t \mid U_0 > 0$ with the density $\lambda(U_t)$ is given by*

$$\mathbb{E}\Big[{N_t}^2 \;\Big|\; U_0\Big] \;=\; \mathbb{E}[N_t \mid U_0] \;+\; 2\int_0^t \int_0^v \mathbb{E}\Big[\lambda(U_x)\lambda\Big(\tilde{U}_{v-x}\Big) \;\Big|\; U_0\Big]\,\mathrm{d}x\,\mathrm{d}v, \tag{4.19}$$

where $\lambda(\cdot)$ is any non-negative function satisfying $\int_0^t \lambda(s)\mathrm{d}s < \infty$ for $t > 0$. $\tilde{U}$ is a duplicate age process with a time-varying initial level depending on $U \mid U_0$ such that,

$$\mathbb{P}\Big(U_s \in \mathrm{d}x,\; \tilde{U}_{t-s} \in \mathrm{d}y \;\Big|\; U_0\Big) \;=$$

$$
\begin{cases}
\dfrac{1}{\Gamma(1-\alpha)^2\Gamma(\alpha)^2}\dfrac{(t-s-y)^{\alpha}y^{-\alpha}(s-x)^{\alpha}x^{-\alpha}}{(t-s-y+x)(s-x+U_0)}\,\mathrm{d}x\,\mathrm{d}y, & \\
 & \textit{for}\ \ x\in(0,\ s),\ y\in(\ 0,\ t-s\ ); \\
\dfrac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\dfrac{(t-s+x)^{-\alpha}(s-x)^{\alpha}}{s-x+U_0}\,\mathrm{d}x, & \\
 & \textit{for}\ \ x\in(0,\ s),\ y=t-s+x; \\
\dfrac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\dfrac{(t-s-y)^{\alpha}y^{-\alpha}}{t-y+U_0}\left(1+\dfrac{s}{U_0}\right)^{-\alpha}\mathrm{d}y, & \\
 & \textit{for}\ \ x=s+U_0,\ y\in(\ 0,\ t-s\ ); \\
\left(1+\dfrac{t}{U_0}\right)^{-\alpha}, & \\
 & \textit{for}\ \ x=s+U_0,\ y=t+U_0.
\end{cases}
$$

Proof. In a nutshell, the second moment of N_t is obtained from the 2^{nd}-derivative of the moment generating function, given in (4.14), evaluated at $\phi^* = 0$. That is to say,

$$
\mathcal{L}_t\Big\{\ \mathbb{E}\big[{N_t}^2\ \big|\ U_0\big],\ \beta\ \Big\}\ =\ \frac{\partial^2}{\partial{\phi^*}^2}\left[\mathcal{L}_t\Big\{\ \mathbb{E}\Big[e^{\phi^* N_t}\ \Big|\ U_0\Big],\ \beta\ \Big\}\right]_{\phi^*=0}.
$$

Due to the complexity involved in the calculations, detailed steps of this proof are left to **Appendix I**. □

Here comes the special case when taking the limit of U_0 to zero.

Corollary 4.4.5. *The second moment of the counting process N_t with $U_0 = 0$ is*

$$
\mathbb{E}\big[{N_t}^2\big]\ =\ \mathbb{E}[N_t]+2\int_0^t\int_0^v\mathbb{E}\Big[\lambda(U_x)\lambda\Big(\tilde{U}_{v-x}\Big)\Big]\,\mathrm{d}x\,\mathrm{d}v. \tag{4.20}
$$

Likewise, the joint density of $\big\{U_s,\ \tilde{U}_{t-s}\big\}_{s\in(0,t)}$ is given by

$$
\mathbb{P}\Big(U_s\in\ \mathrm{d}x,\ \tilde{U}_{t-s}\in\ \mathrm{d}y\Big)\ =
$$

$$\begin{cases} \left(1 - \dfrac{x}{t-s-y+x}\right) \dfrac{(t-s-y)^{\alpha-1} y^{-\alpha} (s-x)^{\alpha-1} x^{-\alpha}}{\Gamma(1-\alpha)^2 \Gamma(\alpha)^2} \, \mathrm{d}x \, \mathrm{d}y, & \\ & \textit{for } x \in (0,\ s),\ y \in (\ 0,\ t-s\ ); \\ \left(1 + \dfrac{t-s}{x}\right)^{-\alpha} \dfrac{(s-x)^{\alpha-1} x^{-\alpha}}{\Gamma(1-\alpha)\Gamma(\alpha)} \, \mathrm{d}x, & \\ & \textit{for } x \in (0,\ s),\ y = t-s+x. \end{cases}$$

Proof. By taking limit of U_0, we get

$$\begin{aligned} \mathbb{E}\left[{N_t}^2\right] &= \lim_{U_0 \to 0} \mathbb{E}\left[{N_t}^2 \,\middle|\, U_0\right] \\ &= \mathbb{E}[N_t] + \int_0^t \frac{(t-v)^{\alpha} v^{-\alpha}}{\Gamma(1-\alpha)\Gamma(1+\alpha)} 2\lambda(v)\Lambda(v) \, \mathrm{d}v \\ &\quad + \int_0^t \frac{(t-v)^{2\alpha}}{\Gamma(1-\alpha)^2 \Gamma(1+2\alpha)} \int_0^v x^{-\alpha} \alpha (v-x)^{-\alpha-1} 2\lambda(x)\Lambda(v-x) \, \mathrm{d}x \, \mathrm{d}v. \end{aligned}$$

The two integrals are further transformed as follows

- for the first integral:

$$\begin{aligned} & \frac{2}{\Gamma(1-\alpha)\Gamma(1+\alpha)} \int_0^t (t-v)^{\alpha} v^{-\alpha} \lambda(v)\Lambda(v) \, \mathrm{d}v \\ =\ & \frac{2}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_0^t \int_0^v \int_0^z \left(1 + \frac{v-z}{x}\right)^{-\alpha} \lambda(v-z+x)(z-x)^{\alpha-1} x^{-\alpha} \lambda(x) \, \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}v \\ =\ & 2 \int_0^t \int_0^v \int_0^z \mathbb{P}\left(\tilde{U}_{v-z} \ =\ v-z+U_z,\ U_z \in\ \mathrm{d}x\right) \lambda(v-z+x)\lambda(x) \, \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}v \\ =\ & 2 \int_0^t \int_0^v \mathbb{E}\left[\lambda(U_z)\lambda(\tilde{U}_{v-z}) \, \mathbb{1}_{\{\tilde{U}_{v-z} \ =\ v-z+U_z\}}\right] \mathrm{d}z \, \mathrm{d}v; \end{aligned}$$

- for the second,

$$\frac{1}{\Gamma(1-\alpha)^2 \Gamma(1+2\alpha)} \int_0^t (t-v)^{2\alpha} \int_0^v x^{-\alpha} \alpha (v-x)^{-\alpha-1} 2\lambda(x)\Lambda(v-x) \, \mathrm{d}x \, \mathrm{d}v$$

$$= \frac{2}{\Gamma(1-\alpha)^2\Gamma(\alpha)^2} \int_0^t \int_0^v \int_0^{v-x} \int_0^x \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}v$$

$$\left(1 - \frac{z}{v-x-y+z}\right)(v-x-y)^{\alpha-1} y^{-\alpha} (x-z)^{\alpha-1} z^{-\alpha} \lambda(y)\lambda(z)$$

$$= 2 \int_0^t \int_0^v \int_0^{v-x} \int_0^x \mathbb{P}\Big(U_x \in \mathrm{d}z, \, \tilde{U}_{v-x} \in \mathrm{d}y\Big) \, \lambda(z)\lambda(y) \, \mathrm{d}x \, \mathrm{d}v$$

$$= 2 \int_0^t \int_0^v \mathbb{E}\Big[\lambda(U_x)\lambda\Big(\tilde{U}_{v-x}\Big) \, \mathbb{1}_{\{\tilde{U}_{v-x} \in (0, \, v-x)\}}\Big] \, \mathrm{d}x \, \mathrm{d}v.$$

This proof is completed by adding the two terms up, which equates to

$$2 \int_0^t \int_0^v \mathbb{E}\Big[\lambda(U_x)\lambda\Big(\tilde{U}_{v-x}\Big)\Big] \, \mathrm{d}x \, \mathrm{d}v.$$

□

Appendix I : A Supplementary Proof for THEOREM 4.4.4

Proof. Replacing ϕ with e^{ϕ^*} to the Laplace transform in (4.14), we get

$$\mathcal{L}_t\Big\{ \mathbb{E}\Big[e^{\phi^* N_t} \,\Big|\, U_0\Big],\, \beta \Big\} \cdot e^{-\beta U_0} \;=$$

$$\frac{\bar{\Theta}(U_0)}{\int_{U_0}^{\infty} e^{-\Lambda(x)} \alpha x^{-\alpha-1}\,\mathrm{d}x} \int_{U_0}^{\infty} e^{-\beta v} e^{e^{\phi^*}\left(\Lambda(v)-\Lambda(U_0)\right)} e^{-\Lambda(v)} \alpha v^{-\alpha-1}\,\mathrm{d}v \cdot \tilde{R}(\phi^*)$$

$$+\; \frac{\Theta(U_0)}{\int_{U_0}^{\infty} \lambda(x) e^{-\Lambda(x)} x^{-\alpha}\,\mathrm{d}x} \int_{U_0}^{\infty} e^{-\beta v} e^{e^{\phi^*}\left(\Lambda(v)-\Lambda(U_0)\right)} e^{-\Lambda(v)} v^{-\alpha}\,\mathrm{d}v, \tag{4.21}$$

with

$$\tilde{R}(\phi^*) \;=\; \frac{\int_0^{\infty} e^{-\beta v} e^{-\left(1-e^{\phi^*}\right)\Lambda(v)} v^{-\alpha}\,\mathrm{d}v}{\int_0^{\infty} \left(1 - e^{-\beta x} e^{-\left(1-e^{\phi^*}\right)\Lambda(x)}\right) \alpha x^{-\alpha-1}\,\mathrm{d}x}.$$

The first and the second derivatives of $\tilde{R}(\phi^*)$ are obtained as follows

$$\tilde{R}'(\phi^*) \;=\; \frac{e^{\phi^*} \int_0^{\infty} e^{-\beta v} \Lambda(v) e^{-\left(1-e^{\phi^*}\right)\Lambda(v)} \left(v^{-\alpha} + \tilde{R}(\phi^*) \alpha v^{-\alpha-1} \right) \mathrm{d}v}{\int_0^{\infty} \left(1 - e^{-\beta x} e^{-\left(1-e^{\phi^*}\right)\Lambda(x)}\right) \alpha x^{-\alpha-1}\,\mathrm{d}x};$$

$$\tilde{R}''(\phi^*) \;=\; \tilde{R}'(\phi^*) \;+\; \frac{e^{2\phi^*} \int_0^{\infty} e^{-\beta v} \Lambda(v) e^{-\left(1-e^{\phi^*}\right)\Lambda(v)} \left\{ \begin{array}{c} \Lambda(v) v^{-\alpha} + \tilde{R}(\phi^*) \Lambda(v) \alpha v^{-\alpha-1} \\ + \left(1 + e^{-\phi^*}\right) \tilde{R}'(\phi^*) \alpha v^{-\alpha-1} \end{array} \right\} \mathrm{d}v}{\int_0^{\infty} \left(1 - e^{-\beta x} e^{-\left(1-e^{\phi^*}\right)\Lambda(x)}\right) \alpha x^{-\alpha-1}\,\mathrm{d}x}.$$

Now we take a derivative w.r.t. ϕ^* of (4.21) and obtain,

$$\frac{\partial}{\partial \phi^*}\, \mathcal{L}_t\Big\{ \mathbb{E}\Big[e^{\phi^* N_t} \,\Big|\, U_0\Big],\, \beta \Big\} \cdot e^{-\beta U_0} \;=$$

$$
\frac{\bar{\Theta}(U_0)}{\int\limits_{U_0}^{\infty} e^{-\Lambda(x)}\alpha x^{-\alpha-1}\,\mathrm{d}x} \int\limits_{U_0}^{\infty} e^{-\beta v} e^{e^{\phi^*}\left(\Lambda(v)-\Lambda(U_0)\right)} \left\{ e^{\phi^*}\tilde{R}(\phi^*)\Big(\Lambda(v)-\Lambda(U_0)\Big) + \tilde{R}'(\phi^*) \right\} \times e^{-\Lambda(v)}\alpha v^{-\alpha-1}\,\mathrm{d}v
$$

$$
+ \frac{\Theta(U_0)}{\int\limits_{U_0}^{\infty} \lambda(x) e^{-\Lambda(x)} x^{-\alpha}\,\mathrm{d}x} \int\limits_{U_0}^{\infty} e^{-\beta v} e^{e^{\phi^*}\left(\Lambda(v)-\Lambda(U_0)\right)} e^{\phi^*}\Big(\Lambda(v)-\Lambda(U_0)\Big) e^{-\Lambda(v)} v^{-\alpha}\,\mathrm{d}v.
$$

Taking further derivative over the same gives,

$$
\frac{\partial^2}{\partial \phi^{*2}} \mathcal{L}_t\left\{ \mathbb{E}\left[e^{\phi^* N_t} \,\Big|\, U_0\right],\, \beta \right\} \cdot e^{-\beta U_0} \;=
$$

$$
\frac{\bar{\Theta}(U_0)}{\int\limits_{U_0}^{\infty} e^{-\Lambda(x)}\alpha x^{-\alpha-1}\,\mathrm{d}x} \int\limits_{U_0}^{\infty} e^{-\beta v} e^{e^{\phi^*}\left(\Lambda(v)-\Lambda(U_0)\right)} \left\{ \begin{array}{l} \tilde{R}''(\phi^*) + 2\tilde{R}'(\phi^*)e^{\phi^*}\Big(\Lambda(v)-\Lambda(U_0)\Big) \\ +\ \tilde{R}(\phi^*) \sum\limits_{j=1}^{2} \left[e^{\phi^*}\Big(\Lambda(v)-\Lambda(U_0)\Big)\right]^j \end{array} \right\} \times e^{-\Lambda(v)}\alpha v^{-\alpha-1}\,\mathrm{d}v
$$

$$
+ \frac{\Theta(U_0)}{\int\limits_{U_0}^{\infty} \lambda(x) e^{-\Lambda(x)} x^{-\alpha}\,\mathrm{d}x} \int\limits_{U_0}^{\infty} e^{-\beta v} e^{e^{\phi^*}\left(\Lambda(v)-\Lambda(U_0)\right)} \sum_{j=1}^{2} \left[e^{\phi^*}\Big(\Lambda(v)-\Lambda(U_0)\Big)\right]^j e^{-\Lambda(v)} v^{-\alpha}\,\mathrm{d}v.
$$

To get the Laplace transform of $\mathbb{E}\left[{N_t}^2 \,\middle|\, U_0\right]$ we set $\phi^* = 0$, then

$$
\mathcal{L}_t\left\{ \mathbb{E}\left[{N_t}^2 \,\middle|\, U_0\right],\, \beta \right\} \;=
$$

$$
{U_0}^{\alpha} \int\limits_{0}^{\infty} e^{-\beta v} \int_0^v \lambda(x+U_0)\Big(1 + 2\Big(\Lambda(x+U_0) - \Lambda(U_0)\Big)\Big)(x+U_0)^{-\alpha}\,\mathrm{d}x\,\mathrm{d}v
$$

$$
+\ {U_0}^{\alpha} \int\limits_{0}^{\infty} e^{-\beta v} \Big(\tilde{R}''(0) + 2\tilde{R}'(0)\Big(\Lambda(v+U_0) - \Lambda(U_0)\Big)\Big)\alpha(v+U_0)^{-\alpha-1}\,\mathrm{d}v,
$$

with

$$
\tilde{R}'(0) \;=\; \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} \int\limits_{0}^{\infty} e^{-\beta t} \int_0^t (t-x)^{\alpha} x^{-\alpha} \lambda(x)\,\mathrm{d}x\,\mathrm{d}t;
$$

$$\tilde{R}''(0) = \tilde{R}'(0) + \int_0^\infty e^{-\beta t} \int_0^t \left\{ \frac{(t-v)^\alpha v^{-\alpha}}{\Gamma(1-\alpha)\Gamma(1+\alpha)} 2\lambda(v)\Lambda(v) \right. +$$

$$\left. \int_0^v \frac{(t-v)^{2\alpha} x^{-\alpha} \alpha (v-x)^{-\alpha-1}}{\Gamma(1-\alpha)^2 \Gamma(2\alpha+1)} 2\lambda(x)\Lambda(v-x)\, \mathrm{d}x \right\} \mathrm{d}v\, \mathrm{d}t.$$

Thus,

$$\mathbb{E}\left[{N_t}^2 \,\middle|\, U_0\right] =$$

$$\int_0^t \left(1 - \left(1 + \frac{v}{U_0}\right)^{-\alpha}\right) \cdot \left\{ \int_0^{t-v} \frac{(t-v-x)^{\alpha-1} x^{-\alpha}}{\Gamma(1-\alpha)\Gamma(\alpha)} \lambda(x)\Big(1 + 2\Lambda(x)\Big)\, \mathrm{d}x \right.$$

$$\left. + \int_0^{t-v} \int_0^x \frac{(t-v-x)^{2\alpha-1} z^{-\alpha} \alpha (x-z)^{-\alpha-1}}{\Gamma(1-\alpha)^2 \Gamma(2\alpha)} 2\lambda(z)\Lambda(x-z)\, \mathrm{d}z\, \mathrm{d}x \right\} \mathrm{d}v$$

$$+ \int_0^t \left(1 + \frac{v}{U_0}\right)^{-\alpha} \cdot \left\{ \lambda(v+U_0)\Big(1 + 2\Big(\Lambda(v+U_0) - \Lambda(U_0)\Big)\Big) \right.$$

$$\left. + \frac{2\Big(\Lambda(v+U_0) - \Lambda(U_0)\Big)}{v+U_0} \int_v^t \frac{(t-x)^\alpha (x-v)^{-\alpha} \lambda(x-v)}{\Gamma(1-\alpha)\Gamma(1+\alpha)}\, \mathrm{d}x \right\} \mathrm{d}v.$$

It can be further transformed as follows

$$\mathbb{E}\left[{N_t}^2 \,\middle|\, U_0\right] = \mathbb{E}[N_t \mid U_0]$$

$$+ \frac{2{U_0}^\alpha}{\Gamma(1-\alpha)\Gamma(1+\alpha)} \int_0^t \alpha (t-v+U_0)^{-\alpha-1} \int_0^v (v-x)^\alpha x^{-\alpha} \lambda(x)\Lambda(x)\, \mathrm{d}x\, \mathrm{d}v \tag{4.22}$$

$$+ \frac{2{U_0}^\alpha}{\Gamma(1-\alpha)^2 \Gamma(1+2\alpha)} \int_0^t \int_0^v (v-x)^{2\alpha} \int_0^x (x-z)^{-\alpha} \lambda(x-z) \alpha z^{-\alpha-1} \Lambda(z)\, \mathrm{d}z\, \mathrm{d}x$$

$$\alpha (t-v+U_0)^{-\alpha-1}\, \mathrm{d}v \tag{4.23}$$

$$+ 2{U_0}^\alpha \int_0^t \Big(\Lambda(v+U_0) - \Lambda(U_0)\Big) \left\{ \alpha (v+U_0)^{-\alpha-1} \int_0^{t-v} \frac{(t-v-x)^\alpha x^{-\alpha} \lambda(x)}{\Gamma(1-\alpha)\Gamma(1+\alpha)}\, \mathrm{d}x \right.$$

$$\left. + (v+U_0)^{-\alpha} \lambda(v+U_0) \right\} \mathrm{d}v. \tag{4.24}$$

Respectively expressing (4.22), (4.23) and (4.24) in terms of some expectations of $\lambda(U)$ and $\lambda(\tilde{U})$, where $\tilde{U}$ is defined to be an age process whose initial level is dependent on U. With the density of $\{U_t \mid U_0\}_{t>0}$ obtained in COROLLARY 4.4.2, we have

- for (4.22) :

$$\frac{2{U_0}^{\alpha}}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^t\int_0^v \alpha(z+U_0)^{-\alpha-1}\int_0^{v-z}(v-z-x)^{\alpha-1}x^{-\alpha}\lambda(x)\Lambda(x)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}v$$

$$= \frac{2}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^t\int_0^v\int_0^x\Big(1-\frac{U_0}{v-x+U_0}\Big)(v-x)^{\alpha-1}x^{-\alpha}\lambda(x)\lambda(z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}v$$

$$= \frac{2}{\Gamma(1-\alpha)\Gamma(\alpha)}\int_0^t\int_0^v\int_0^z\Big(1-\frac{U_0}{z-x+U_0}\Big)(z-x)^{\alpha-1}x^{-\alpha}\lambda(x)$$

$$\Big(1+\frac{v-z}{x}\Big)^{-\alpha}\lambda(v-z+x)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}v$$

$$= 2\int_0^t\int_0^v\int_0^z \mathbb{P}\Big(\tilde{U}_{v-z}=v-z+U_z,\ U_z\in\ \mathrm{d}x \,\Big|\, U_0\Big)\lambda(v-z+x)\lambda(x)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}v$$

$$= 2\int_0^t\int_0^v \mathbb{E}\Big[\lambda(U_z)\lambda(\tilde{U}_{v-z})\mathbb{1}_{\{U_z\in(0,\,z),\ \tilde{U}_{v-z}\,=\,v-z+U_z\}}\,\Big|\,U_0\Big]\,\mathrm{d}z\,\mathrm{d}v;$$

- for (4.23) :

$$\frac{2{U_0}^{\alpha}}{\Gamma(1-\alpha)^2\Gamma(2\alpha)}\int_0^t\int_0^v \alpha(w+U_0)^{-\alpha-1}\int_0^{v-w}(v-w-x)^{2\alpha-1}$$

$$\int_0^x (x-z)^{-\alpha}\lambda(x-z)\alpha z^{-\alpha-1}\Lambda(z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}w\,\mathrm{d}v$$

$$= \frac{2}{\Gamma(1-\alpha)^2\Gamma(\alpha)^2}\int_0^t\int_0^v\int_0^w\Big(1-\frac{U_0}{w-x+U_0}\Big)(w-x)^{\alpha-1}x^{-\alpha}\lambda(x)\,\mathrm{d}x$$

$$\int_0^{v-w}\Big(1-\frac{y}{v-w}\Big)(v-w-y)^{\alpha-1}y^{-\alpha}\lambda(y)\,\mathrm{d}y\,\mathrm{d}w\,\mathrm{d}v$$

$$= \frac{2}{\Gamma(1-\alpha)^2\Gamma(\alpha)^2}\int_0^t\int_0^v\int_x^v\int_x^y\Big(1-\frac{U_0}{w-x+U_0}\Big)(w-x)^{\alpha-1}x^{-\alpha}\lambda(x)$$

$$\Big(1-\frac{y-w}{v-w}\Big)(v-y)^{\alpha-1}(y-w)^{-\alpha}\lambda(y-w)\,\mathrm{d}w\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}v$$

$$= \frac{2}{\Gamma(1-\alpha)^2\Gamma(\alpha)^2}\int_0^t\int_0^v\int_0^{v-x}\int_0^x\left(1-\frac{w}{v-x-y+w}\right)(v-x-y)^{\alpha-1}y^{-\alpha}\lambda(y)$$

$$\left(1-\frac{U_0}{x-w+U_0}\right)(x-w)^{\alpha-1}w^{-\alpha}\lambda(w)\,\mathrm{d}w\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}v$$

$$= 2\int_0^t\int_0^v\int_0^{v-x}\int_0^x\mathbb{P}\Big(U_x\in\,\mathrm{d}w,\ \tilde{U}_{v-x}\in\,\mathrm{d}y\,\Big|\,U_0\Big)\,\lambda(w)\lambda(y)\,\mathrm{d}x\,\mathrm{d}v$$

$$= 2\int_0^t\int_0^v\mathbb{E}\Big[\lambda(U_x)\lambda\Big(\tilde{U}_{v-x}\Big)\,\mathbb{1}_{\{U_x\in(0,\ x),\ \tilde{U}_{v-x}\in(0,\ v-x)\}}\,\Big|\,U_0\Big]\,\mathrm{d}x\,\mathrm{d}v;$$

- for (4.24) :

$$2U_0{}^{\alpha}\int_0^t\int_0^v(\Lambda(x+U_0)-\Lambda(U_0))\alpha(x+U_0)^{-\alpha-1}\int_0^{v-x}\frac{(v-x-z)^{\alpha-1}z^{-\alpha}}{\Gamma(1-\alpha)\Gamma(\alpha)}\lambda(z)\,\mathrm{d}z\,\mathrm{d}v\,\mathrm{d}x$$

$$+\ 2U_0{}^{\alpha}\int_0^t\int_0^v\lambda(x+U_0)\,\mathrm{d}x\,(v+U_0)^{-\alpha}\lambda(v+U_0)\,\mathrm{d}v$$

$$= 2\int_0^t\int_0^v\left(1+\frac{x}{U_0}\right)^{-\alpha}\lambda(x+U_0)\left\{\int_0^{v-x}\left(1-\frac{x+U_0}{v-z+U_0}\right)\frac{(v-x-z)^{\alpha-1}z^{-\alpha}}{\Gamma(1-\alpha)\Gamma(\alpha)}\lambda(z)\,\mathrm{d}z\right.$$

$$\left.+\ \left(1+\frac{v-x}{x+U_0}\right)^{-\alpha}\lambda(v+U_0)\right\}\mathrm{d}x\,\mathrm{d}v$$

$$= 2\int_0^t\int_0^v\mathbb{E}\Big[\lambda(U_x)\lambda\Big(\tilde{U}_{v-x}\Big)\mathbb{1}_{\{U_x\,=\,x+U_0\}}\,\Big|\,U_0\Big]\,\mathrm{d}x\,\mathrm{d}v.$$

It is quite obvious to see that

$$\mathbb{1}_{\{U_x\in(0,\ x),\ \tilde{U}_{v-x}\,=\,v-x+U_x\}}+\mathbb{1}_{\{U_x\in(0,\ x),\ \tilde{U}_{v-x}\in(0,\ v-x)\}}+\mathbb{1}_{\{U_x\,=\,x+U_0\}}\ =\ 1,$$

so putting together the results of (4.22) – (4.24) leads to

$$2\int_0^t\int_0^v\mathbb{E}\Big[\lambda(U_x)\lambda\Big(\tilde{U}_{v-x}\Big)\,\Big|\,U_0\Big]\,\mathrm{d}x\,\mathrm{d}v.$$

This completes the proof. □

5

Local Times related to Bessel Age Process

For a standard Markov process with zero a regular point for itself, the excursions of such processes are described as a sequence of independent and identically distributed pieces of path (each with finite length), glued together by the time points of visiting zero. The behaviour between visits has been well characterised except in the vicinity of zero. The measure in time of the zero set is called the "local time". More precisely, the local time at zero is an additive functional of the Markov process, measuring how often the process visits $\{0\}$ thus evolving only on the random set of zeros. According to the theory of Blumenthal and Getoor [11], such a functional exists on the premise of $\{0\}$ a suitable point in the state space. For background knowledge on local times, we refer to Blumenthal [10], Borodin [14], Marcus and Rosen [78], McKean [79].

5.1 Introduction

The remaining chapters are devoted to the study of the local time at zero of Bessel age process, denoted by $L = \{L_t, t \geq 0\}$ throughout. Focusing on this subject, there are many attractive aspects to discover. Among them to be considered in the present chapter are the following:

- ♦ There are various ways of approximating a local time, mainly by limiting processes describing the sample path properties of the underlying process. Above all, an intrinsic definition of L_t is given as the derivative of an occupation measure of $(0, \varepsilon)$, representable of the form:

$$L_t \;=\; \lim_{\varepsilon \downarrow 0} \nu_1(\varepsilon)^{-1}\, \textbf{measure}\Big\{s \in [0,t) \mid U_s < \varepsilon\Big\}, \qquad \varepsilon > 0 \,.$$

$\{U_t, t \geq 0\}$ remains unchanged, the Bessel age process. $\nu_1(\varepsilon)$ is the speed measure function and determined uniquely up to multiplication by a constant.

- Another way of constructing a local time is based on the theory of Poisson point process established by Itô [54], who presented an idea of characterising a Markov process in terms of its excursions away from a fixed point in the state space. One merit of this theory is the concept of "excursion point processes" that recovers a local time from the behaviour of the associated excursions (see [62, Chap. 6] for full information). To illustrate, let M_t denote the number of jumps in the path of U_t. According to Itô's idea, $M_{\ell(t)}(\varepsilon)$ is a Poisson process with a proper intensity $\nu_2(\varepsilon)^{-1}$, which leads to a weak convergence in distribution :

$$L_t \overset{\mathcal{D}}{\to} \nu_2(\varepsilon)\, M_t(\varepsilon), \qquad \text{as} \quad \varepsilon \to 0,$$

  where ℓ denotes the right-continuous inverse of L. It is worth mentioning that the limit relation stems from a famous conjecture namely "Lévy's downcrossing theorem", proved by Itô and McKean [55, Chap 2, 6] in the case of reflecting Brownian motion. Depending on the path features of the underlying process, Itô's way of approaching local times is applicable to a wider class of Markov processes. Please refer to Blumenthal [10], Fristedt and Taylor [44], Itô and McKean [55], McKean [79] and Karatzsas and Shreve [62, Chap 6] for other constructions of local times.

- There is a well-known property about the local time of a standard Markov process. That is, the inverse process of the local time at a regular point a is a subordinator whose jumps correspond to the lengths of the excursions away from a (see [64, 66, 78] and [5, Chap 4] for references in this regard). As a direct consequence, the law of the local time L_t can be determined from the fact that $\ell(t)$ coincides with a stable process of index $\alpha \in (0, 1)$.

- Applying the central limit theorem, it is of great importance to show that the difference between the local time L_t and the limiting process, defined either by the occupation or the counting measures introduced above, is closely related to a zero-mean Gaussian process subordinated by L_t. This problem is motivated by the work of Kasahara [63], who following the downcrossing theorem obtained an independent time-changed Brownian motion from the associated difference. Extensions of this result to the Brownian excursions and general Lévy processes are discussed in [25] and [64] by the same author.

5.2 Definitions and Notations

On a filtered probability space $(\Omega, \mathscr{F}, \mathbb{P})$, there is a $(\mathscr{F}_t)$-adapted age process $U = \{U_t, t \geq 0\}$ given by

$$U_t \;=\; t - \sup\{\; s < t \mid X_s = 0 \;\},$$

for X to be a $(2-2\alpha)$-dimensional Bessel process instantaneously reflected at the origin with $\alpha \in (0,1)$. Within the space, the *occupation time* of the Bessel age on the time interval $[0,t)$ is the measure defined by

$$Z_t(\varpi) := \int_0^t \mathbb{1}_{\{U_s < \varpi\}} \,\mathrm{d}s, \qquad 0 < \varpi < t. \tag{5.1}$$

Technically speaking, the quantity $Z_t(\varpi)$ is equal to the Lebesgue measure of the time spent by the age process under the level ϖ before t. Obviously $t \to Z_t(\varpi)$ is continuous, non-decreasing and $Z_0(\varpi) = 0$.

Integral functionals for Markov processes Y of type $\left\{\int_0^t f(Y_s)\,\mathrm{d}s, t \geq 0\right\}$ are being widely explored, where f is a non-negative Borel function. Please refer to Blumenthal and Getoor [12] for general knowledge on (additive) functionals. Great attention has been given to the perpetual integral functionals, i.e. the limiting process by taking t to infinity (see [40, 85, 86] for examples); to the distributional properties related to α-quantiles (see [27, 28, 29]); and to the convergence to some other functionals (see [13, 15, 65]), in particular to the local time concerned here.

The probabilistic behaviour of U is well clarified in previous chapters except when it is approaching the set of zeros given by $\mathscr{Z} = \{\; t : U_t = 0 \;\}$. By the *local time at zero* of a Bessel age process, we mean that there exists a continuous family of functionals denoted by $L = \{L_t,\; t \geq 0\}$ and L_t is measurable relative to $\sigma\{U_s,\; s \leq t\}$ for each t. It measures the part of time spent at zero by the age process over the time interval $[0,t)$. This makes it clear that L is a non-decreasing process and it increases strictly on the zero set $\mathscr{Z}$. The notion of local times for general Markov processes was due to Blumenthal and Getoor [11], who by means of the potential theory clarified the existence of such additive functionals. Precisely, the local time L is constructed on the fact that zero is a regular point in the state space of U, i.e.

$$\mathbb{P}\Big(\inf\{\; t > 0 \mid U_t = 0 \;\} = 0\Big) = 1\;.$$

It is a well known fact that there is no unified representation for the local time of Markov process, as it can be established as a limit of various processes describing the behaviour of trajectories of the underlying process. As mentioned in the introduction, we are concerned with two ways of characterising L_t, in terms of the occupation time $Z_t(\varpi)$ and a counting measure. The former provides an intrinsic definition of L_t, expressed as a limit of $Z_t(\varpi)$,

$$L_t \;=\; \lim_{\varpi\downarrow 0} \nu_1(\varpi)^{-1}\, Z_t(\varpi), \tag{5.2}$$

under a certain manner of normalisation on $\nu_1(\varpi)$. The latter arises from the Poisson random measure attached to Markov processes.

To get a precise definition of this counting measure, let U_t^ε be a perturbed age process by size ε. Please refer to CHAPTER 2.2.1 Perturbed Bessel Process for full accounts on perturbation. The zero set up to time t of U_t^ε is given by

$$\mathscr{Z}_t(\varepsilon) \;:=\; \left\{\, s \in [0,t) \mid U_s^\varepsilon = 0 \,\right\}.$$

The complement $\mathscr{Z}_t^c(\varepsilon) = [0,t) \setminus \mathscr{Z}_t$ is then a countable union of disjoint open intervals of lengths $\rho_i > 0$ such that $\sum_{i=1}^{\infty} \rho_i = t$. Let $M_t(\varepsilon) := \#\mathscr{Z}_t(\varepsilon)$ denote the number of points in the zero set, which in quantity equal to the jumps in the path of U_t^ε represented of the form:

$$M_t(\varepsilon) \;=\; \sum_{i=1}^{\infty} \mathbb{1}_{\{T_i^\varepsilon \,\le\, t\}}, \qquad M_0(\varepsilon) \;=\; 0\,,$$

where $\left\{T_i^\varepsilon\right\}_{i\in\mathbb{N}^+}$ is a sequence of random times defined by

$$T_i^\varepsilon \;:=\; \inf\left\{\, s > T_{i-1}^\varepsilon \,\middle|\, U_s^\varepsilon = 0 \,\right\}, \qquad \text{with} \quad T_0^\varepsilon = 0.$$

According to Itô's idea, if $\mathscr{Z}_t(\varepsilon)$ is parametrized in terms of the local time at zero, there implies a point process describing the pieces of path corresponding to $\mathscr{Z}_t^c(\varepsilon)$ with values in $\{\rho_i\}_{i\in\mathbb{N}^+}$ (see [10]). To be more precise, suppose ℓ is the right continuous inverse of L such that

$$\ell(t) \;=\; \inf\{\, s \mid L_s > t \,\}, \qquad 0 \;\le\; t \;<\; \infty\,.$$

$\{M_{\ell(t)}(\varepsilon)\}$ follows a Poisson process with a proper intensity $\nu_2(\varepsilon)^{-1}$ satisfying $\lim_{\varepsilon\to 0} \nu_2(\varepsilon) = 0$. Then for all $t > 0$, there should be a relation like this:

$$M_{\ell(t)}(\varepsilon)\nu_2(\varepsilon) \overset{\mathcal{D}}{\to} t, \qquad \text{as} \quad \varepsilon \to 0,$$

and by changing $t \to L_t$, it leads to

$$\nu_2(\varepsilon)M_t(\varepsilon) \overset{\mathcal{D}}{\to} L_t, \qquad \text{as} \quad \varepsilon \to 0, \tag{5.3}$$

where $\overset{\mathcal{D}}{\to}$ denotes the convergence in distribution.

An important consequence of (5.2) and (5.3) is to present a central limit theorem (CLT) that relates the local time of Bessel age process to a time-changed Gaussian process with mean zero. Many interesting features of the Gaussian process are described on the basis of the characteristic measures $\nu_1(\varepsilon)$ and $\nu_2(\varpi)$. The two functions are uniquely determined by

$$\lim_{\varpi\to 0} \mathbb{E}\Big[\int_0^\infty e^{-\beta t} f\Big(\nu_1(\varpi)^{-1}\, Z_t(\varpi)\Big)\,\mathrm{d}t\Big] = \lim_{\varepsilon\to 0} \mathbb{E}\Big[\int_0^\infty e^{-\beta t} f(\nu_2(\varepsilon)M_t(\varepsilon))\,\mathrm{d}t\Big], \tag{5.4}$$

for every $\beta > 0$ and Borel f on $[0,\infty)$. By means of martingale approach, ν_1 and ν_2 are dealt with in Limit Theorem I & II respectively.

While performing calculations, to avoid confusions in presenting results we define a double Laplace transform for a real-valued random variable X_t by

$$\mathcal{L}^{\beta,\xi}\{X_t\} := \mathbb{E}\Big[\int_0^\infty e^{-\beta t} e^{-\xi X_t}\,\mathrm{d}t\Big] = \int_0^\infty e^{-\beta t}\,\mathbb{E}\big[e^{-\xi X_t}\big]\,\mathrm{d}t,$$

where the last equality follows from Fubini's theorem. We further adopt the following notations

$$\mathcal{L}_t^{\beta}\{\varphi(t)\} = \int_{\mathbb{R}^+} e^{-\beta t}\,\varphi(t)\,\mathrm{d}t,$$

the Laplace transform of some function $\varphi(t)$ defined for all real numbers $t \geq 0$, and

$$\begin{aligned} \gamma(\alpha, z) &= \int_0^z e^{-t}\, t^{\alpha-1}\,\mathrm{d}t; \\ \Gamma(\alpha, z) &= \int_z^\infty e^{-t}\, t^{\alpha-1}\,\mathrm{d}t, \end{aligned}$$

for the lower and upper incomplete gamma functions, and thus $\gamma(\alpha, z) + \Gamma(\alpha, z) \; = \; \Gamma(\alpha)$.

Note: the symbols and notations defined above will be used throughout in this chapter.

5.3 Limit Theorem I : Local Time as a Limit of Occupation Time

By describing our approach as martingale, we meant to consider a continuous function $f : [0, t)^2 \to \mathbb{R}^+$ arising as the solution to $\mathcal{A}_z f(u, z) = 0$ such that $f(U_t, Z_t(\varpi))$ is a martingale. $\mathcal{A}_z$ is a generator used to analyse the infinitesimal movements of the Markov process $\Big\{\big(U_t, Z_t(\varpi)\big)\Big\}$, and obtained from:

$$\mathcal{A}_z f(u, z) \; = \; \lim_{t \to 0} \frac{\mathbb{E}\Big[f\Big(U_t, Z_t(\varpi)\Big)\Big] - f(U_0, 0)}{t} \, .$$

The martingale properties are then of great help in investigating $f(U_t, Z_t(\varpi))$ and its asymptotic behaviour as $\varpi \to 0$. Furthermore, this section develops some distributional properties and facts that are relevant to the occupation process, and preliminary to the limit theorem for the local time.

5.3.1 A Scaled Occupation Time Process

Given $Z_t(\varpi)$ an occupation time of level ϖ, we consider a *scaled occupation measure* in the manner,

$$\varpi^{-p} Z_t(\varpi), \qquad 0 < \varpi < t, \tag{5.5}$$

with p a scaler taking non-negative values. Within the scheme of perturbation, the process used to perform calculations is $\big(U_t^\varepsilon, Z_t^\varepsilon(\varpi)\big)$. U^ε is the age process defined upon a perturbed Bessel process of size $\varepsilon > 0$, and then

$$Z_t^\varepsilon(\varpi) \; = \; \int_0^t \mathbb{1}_{\{U_s^\varepsilon \, < \, \varpi\}} \, \mathrm{d}s, \qquad Z_0^\varepsilon(\varpi) = 0.$$

As explained in CHAPTER 2.2.1, U^ε converges pointwise to U as ε approaches zero. By similar construction, it implies another convergence of $\big(U_t^\varepsilon, Z_t^\varepsilon(\varpi)\big)$ to $(U_t, Z_t(\varpi))$. The infinitesimal generator of $\big(U_t^\varepsilon, Z_t^\varepsilon(\varpi)\big)$ acting on bounded functions $f \in C^1$ is given by

$$\mathcal{A}_z f(u, z) \; = \; \frac{\partial f}{\partial u} \; + \; \frac{\mathbb{1}_{\{u \, < \, \varpi\}}}{\varpi^p} \frac{\partial f}{\partial z} \; + \; \frac{p_\varepsilon(u)}{\bar{P}_\varepsilon(u)} \Big[\, f(0, z) - f(u, z) \, \Big], \tag{5.6}$$

with

$$p_\varepsilon(u) \;=\; \frac{\varepsilon^{2\alpha}}{2^\alpha \Gamma(\alpha)} u^{-\alpha-1} e^{-\frac{\varepsilon^2}{2u}}; \tag{5.7}$$

$$\bar{P}_\varepsilon(u) \;=\; \int_u^\infty \frac{\varepsilon^{2\alpha}}{2^\alpha \Gamma(\alpha)} v^{-\alpha-1} e^{-\frac{\varepsilon^2}{2v}} \,\mathrm{d}v. \tag{5.8}$$

Applying the martingale techniques to (5.6), for every suitable choice of f satisfying $\mathscr{A}_Z f = 0$,

$$f\big(U_t^\varepsilon, Z_t^\varepsilon(\varpi)\big) - f(U_0, 0)$$

is an $(\mathscr{F}_t)$-martingale for all $t > 0$. An immediate consequence of this result is some characteristic function of the (scaled) occupation time in (5.5).

Lemma 5.3.1. *We denote the double Laplace transform for $\varpi^{-p} Z_t(\varpi)$ with $U_0 \geq 0$ by*

$$\mathcal{L}_{U_0}^{\beta,\zeta}\{\, \varpi^{-p} Z_t(\varpi) \,\} \;=\; \mathbb{E}\left[\int_0^\infty e^{-\beta t}\; e^{-\zeta \varpi^{-p} Z_t(\varpi)} \,\mathrm{d}t \;\middle|\; U_0 \right].$$

For the case $0 < \varpi < U_0$,

$$\begin{aligned} &\mathcal{L}_{U_0}^{\beta,\zeta}\{\, \varpi^{-p} Z_t(\varpi) \,\} \;= \\ &{U_0}^\alpha \beta^{\alpha-1} e^{\beta U_0} \Gamma(1-\alpha, \beta U_0) \;+\; \mathcal{L}^{\beta,\zeta}\{\, \varpi^{-p} Z_t(\varpi) \,\} \cdot \Big(1 - {U_0}^\alpha \beta^\alpha e^{\beta U_0} \Gamma(1-\alpha, \beta U_0) \Big), \end{aligned}$$

and the other case $\varpi \geq U_0$,

$$\begin{aligned} \mathcal{L}_{U_0}^{\beta,\zeta}\{\, \varpi^{-p} Z_t(\varpi) \,\} \;=\; & {U_0}^\alpha e^{\left(\beta + \frac{\zeta}{\varpi^p}\right) U_0} \cdot \Bigg(\mathcal{L}_t^\beta \bigg\{ \exp\Big\{ -\frac{\zeta}{\varpi^p} \min\{t,\, \varpi\} \Big\} t^{-\alpha} \mathbb{1}_{\{t \,>\, U_0\}} \bigg\} \\ & + \mathcal{L}^{\beta,\zeta}\{\, \varpi^{-p} Z_t(\varpi) \,\} \cdot \mathcal{L}_t^\beta \bigg\{ \exp\Big\{ -\frac{\zeta}{\varpi^p} \min\{t,\, \varpi\} \Big\} \alpha t^{-\alpha-1} \mathbb{1}_{\{t \,>\, U_0\}} \bigg\} \Bigg), \end{aligned}$$

where in both equations, the quantity $\mathcal{L}^{\beta,\zeta}\{\, \varpi^{-p} Z_t(\varpi) \,\}$ represents the Laplace transform of the special case $U_0 \to 0$ with

$$\mathcal{L}^{\beta,\zeta}\{\, \varpi^{-p} Z_t(\varpi) \,\} \;=\; \lim_{U_0 \downarrow 0} \mathcal{L}_{U_0}^{\beta,\zeta}\{\, \varpi^{-p} Z_t(\varpi) \,\} \;=$$

$$\left(\beta+\frac{\zeta}{\varpi^p}\right)^{-1}\left(1+\frac{\frac{\zeta}{\varpi^p}e^{-\zeta\varpi^{1-p}}\beta^{\alpha-1}\Gamma(1-\alpha,\beta\varpi)}{\left(\beta+\frac{\zeta}{\varpi^p}\right)^{\alpha}\gamma\big(1-\alpha,\beta\varpi+\zeta\varpi^{1-p}\big)+e^{-\zeta\varpi^{1-p}}\beta^{\alpha}\Gamma(1-\alpha,\beta\varpi)}\right). \tag{5.9}$$

Proof. Based on what we have structured, we are keen to find a martingale of the form

$$\int_0^t e^{-\beta s}e^{-\frac{\zeta}{\varpi^p}Z_s^{\varepsilon}(\varpi)}\,\mathrm{d}s \;+\; e^{-\beta t}e^{-\frac{\zeta}{\varpi^p}Z_t^{\varepsilon}(\varpi)}g\big(U_t^{\varepsilon}\big),$$

with $g(\cdot)$ assumed to be a bounded function. By the martingale property,

$$\begin{aligned}
&\lim_{t\to 0}\ \mathbb{E}\left[\int_0^t e^{-\beta s}e^{-\frac{\zeta}{\varpi^p}Z_s^{\varepsilon}(\varpi)}\,\mathrm{d}s \;+\; e^{-\beta t}e^{-\frac{\zeta}{\varpi^p}Z_t^{\varepsilon}(\varpi)}g\big(U_t^{\varepsilon}\big)\;\middle|\;U_0\right]\\
=&\ \int_0^\infty e^{-\beta s}\,\mathbb{E}\Big[e^{-\frac{\zeta}{\varpi^p}Z_s^{\varepsilon}(\varpi)}\;\Big|\;U_0\Big]\,\mathrm{d}s\\
=&\ g(U_0).
\end{aligned}$$

To find the expression of $g(U_0)$, we first expand the Markov process by one more element,

$$W_t^{\varepsilon} := \int_0^t e^{-\beta s}e^{-\frac{\zeta}{\varpi^p}Z_s^{\varepsilon}(\varpi)}\,\mathrm{d}s, \qquad \beta,\zeta>0.$$

As a result, the infinitesimal generator for the expanded process $\big(U_t^{\varepsilon},Z_t^{\varepsilon},W_t^{\varepsilon},t\big)$ is written as

$$\mathcal{A}f(u,z,w,t) \;=\; \mathcal{A}_z f+\frac{\partial f}{\partial t}+e^{-\beta t}e^{-\frac{\zeta}{\varpi^p}z}\frac{\partial f}{\partial w}.$$

Consider that $f(u,z,w,t)$ is of an exponential form

$$w \;+\; e^{-\beta t}e^{-\frac{\zeta}{\varpi^p}z}g(u).$$

Substituting it into $\mathcal{A}f=0$, we have

$$-\beta g(u)+g'(u)-\frac{\zeta}{\varpi^p}g(u)\mathbb{1}_{\{u\,<\,\varpi\}}+1+\frac{p_{\varepsilon}(u)}{\bar{P}_{\varepsilon}(u)}\Big(g(0)-g(u)\Big) \;=\; 0.$$

We find the unique solution to the differential equation, read as

$$g(u)e^{-\beta u}e^{-\frac{\zeta}{\varpi^p}\min\{u,\,\varpi\}}\bar{P}_\varepsilon(u) \quad = $$
$$g(0)\int_u^\infty e^{-\beta v}e^{-\frac{\zeta}{\varpi^p}\min\{v,\,\varpi\}}p_\varepsilon(v)\,\mathrm{d}v \;+\; \int_u^\infty e^{-\beta v}e^{-\frac{\zeta}{\varpi^p}\min\{v,\,\varpi\}}\bar{P}_\varepsilon(v)\,\mathrm{d}v,$$

with

$$g(0) \;=\; \frac{\displaystyle\int_0^\infty e^{-\beta v}e^{-\frac{\zeta}{\varpi^p}\min\{v,\,\varpi\}}\bar{P}_\varepsilon(v)\,\mathrm{d}v}{\displaystyle\int_0^\infty\left(\beta+\mathbb{1}_{\{v\,<\,\varpi\}}\frac{\zeta}{\varpi^p}\right)e^{-\beta v}e^{-\frac{\zeta}{\varpi^p}\min\{v,\,\varpi\}}\bar{P}_\varepsilon(v)\,\mathrm{d}v}.$$

Assuming $0<\varpi<U_0$,

$$g(U_0) \;=\; \frac{e^{\beta U_0}}{\bar{P}_\varepsilon(U_0)}\left(\int_{U_0}^\infty e^{-\beta v}\bar{P}_\varepsilon(v)\,\mathrm{d}v \;+\; g(0)\int_{U_0}^\infty e^{-\beta v}p_\varepsilon(v)\,\mathrm{d}v\right).$$

Otherwise with $\varpi \geq U_0$

$$g(U_0) \;=\; \frac{e^{\beta U_0+\frac{\zeta}{\varpi^p}U_0}}{\bar{P}_\varepsilon(U_0)}\times$$
$$\left\{\int_{U_0}^\infty e^{-\beta v}e^{-\frac{\zeta}{\varpi^p}\min\{v,\,\varpi\}}\bar{P}_\varepsilon(v)\,\mathrm{d}v \;+\; g(0)\int_{U_0}^\infty e^{-\beta v}e^{-\frac{\zeta}{\varpi^p}\min\{v,\,\varpi\}}p_\varepsilon(v)\,\mathrm{d}v\right\}.$$

Then the results follow immediately by taking $\varepsilon \to 0$. □

There is an important remark on the initial value U_0.

Remark 5.3.2. *The main goal of this section can be achieved by studying the limiting behaviour of* $\mathcal{L}^{\beta,\zeta}\{\,\varpi^{-p}Z_t(\varpi)\,\}$*. Specifically, we investigate on the scaler* p *that admits a limit at* $\varpi \to 0$*. As obviously being of no use, results associated with the case* $0 \leq U_0 \leq \varpi$ *will be left out of consideration at all stages of limits taking. That being said, for the sake of completeness we yet provide results for both cases where possible.*

Having the Laplace transforms in hand, we proceed to the limit of $\mathcal{L}^{\beta,\zeta}\{\,\varpi^{-p}Z_t(\varpi)\,\}$. As the main results to the whole work, the choice of p and the corresponding limit are determined in the following lemma by using L'Hôpital's Rule .

Lemma 5.3.3. *For every* $0<\alpha<1$ *and* $p>0$*, it holds that*

$$\lim_{\varpi\to 0}\ \mathcal{L}^{\beta,\zeta}\{\,\varpi^{-p}Z_t(\varpi)\,\}\ =\ \mathcal{L}^{\beta,\zeta}\Big\{\lim_{\varpi\to 0}\ \varpi^{-p}Z_t(\varpi)\Big\};$$

and in particular for $\boldsymbol{p=1-\alpha}$*,*

$$\mathcal{L}^{\beta,\zeta}\Big\{\lim_{\varpi\to 0}\ \varpi^{\alpha-1}Z_t(\varpi)\Big\}\ =\ \left(\beta+\frac{\zeta\beta^{1-\alpha}}{\Gamma(2-\alpha)}\right)^{-1}. \tag{5.10}$$

Proof. We begin by writing the representation of $\mathcal{L}^{\beta,\zeta}\{\,\varpi^{-p}Z_t(\varpi)\,\}$ in (5.9) into the following form,

$$\frac{\left(\beta+\frac{\zeta}{\varpi^p}\right)^{\alpha-1}\gamma\left(1-\alpha,\left(\beta+\frac{\zeta}{\varpi^p}\right)\varpi\right)+\beta^{\alpha-1}e^{-\zeta\varpi^{1-p}}\Gamma(1-\alpha,\beta\varpi)}{\left(\beta+\frac{\zeta}{\varpi^p}\right)^{\alpha}\gamma\left(1-\alpha,\left(\beta+\frac{\zeta}{\varpi^p}\right)\varpi\right)+\beta^{\alpha}e^{-\zeta\varpi^{1-p}}\Gamma(1-\alpha,\beta\varpi)}.$$

Given $0<\alpha<1$, it is obvious that the term $\left(\beta+\frac{\zeta}{\varpi^p}\right)^{\alpha-1}\gamma\left(1-\alpha,\left(\beta+\frac{\zeta}{\varpi^p}\right)\varpi\right)$ is dominated by $e^{-\zeta\varpi^{1-p}}\Gamma(1-\alpha,\beta\varpi)$ in all possible values of ϖ, and

$$\lim_{\varpi\to 0}\ e^{-\zeta\varpi^{1-p}}\beta^{\alpha-1}\Gamma(1-\alpha,\beta\varpi)\ =\ \begin{cases}\Gamma(1-\alpha)\beta^{\alpha-1}, & p\ \in\ (\,0,1\,);\\ \Gamma(1-\alpha)\beta^{\alpha-1}e^{-\zeta}, & p\ =\ 1;\\ 0, & p\ \in\ (\,1,\infty\,).\end{cases}$$

Further for the denominator,

$$\lim_{\varpi\to 0}\ \frac{\zeta}{\varpi^p}\left(\beta+\frac{\zeta}{\varpi^p}\right)^{\alpha-1}\gamma\left(1-\alpha,\left(\beta+\frac{\zeta}{\varpi^p}\right)\varpi\right)\ =\ \begin{cases}0, & p\ \in\ (\,0,1-\alpha\,);\\ \dfrac{\zeta}{1-\alpha}, & p\ =\ 1-\alpha;\\ \infty, & p\ \in\ (\,1-\alpha,\infty\,).\end{cases}$$

Putting them together, we get

$$\lim_{\varpi\to 0}\ \mathcal{L}^{\beta,\zeta}\{\ \varpi^{-p} Z_t(\varpi)\ \} \ = \begin{cases} \dfrac{1}{\beta}, & p\ \in\ (\ 0, 1-\alpha\ ); \\[2ex] \dfrac{\Gamma(1-\alpha)\beta^{\alpha-1}}{\Gamma(1-\alpha)\beta^{\alpha} + \dfrac{\zeta}{1-\alpha}}, & p\ =\ 1-\alpha; \\[4ex] 0, & p\ \in\ (\ 1-\alpha, \infty\ ). \end{cases}$$

□

In conjunction with Lévy's continuity theorem, the transform function in (5.10) is of great help in studying the asymptotics of $\{\varpi^{-p} Z_t(\varpi)\}$, particularly the convergence to the local time. Once the limit relation between the occupation and local times is clarified, the probability density of the latter is determined from the inverse Laplace transform method. Since most inversions are effected by recognition, it leads to a better understanding towards the local time by identifying with a class of variables in our knowledge. In this regard, we refer to the α-stable subordinators as mentioned in the introduction. All these aspects will be dealt with in more detail below.

5.3.2 Convergence of Occupation Time to Local Time

Recall that we denote by L_t the local time at zero of a Bessel age process over interval $[0, t)$. In terms of the occupation time, it can be approximated by the form :

$$\nu_1(\varpi)^{-1}\ Z_t(\varpi)\ \stackrel{\mathcal{D}}{\to}\ L_t, \qquad \text{as} \quad \varpi \to 0.$$

It turns out, by LEMMA 5.3.3, that $\nu_1(\varpi)$ is a multiplication of $\varpi^{1-\alpha}$ by some constant that is determined in the course of normalising the density of L_t.

A well-known fact about the local times for a wide class of Markov processes is associated with inverse subordinators. Put another way, an inverse subordinator is the local time of some "well-behaved" Markov process (see [6, 69] for details). The most famous example is the Brownian local time whose inverse is described as a ½-stable process. A more general case is due to Bingham [7] who showed the inverse of α-stable subordinators with $0 < \alpha < 1$ arising as limiting processes of some occupation times.

Particularly relevant to the present study is the connection of stable subordinators to the local times of Bessel related processes. Let us pause briefly to recall some basic facts about stable subordinators of index $\alpha \in (0,1)$. For more detailed study on (inverse) subordinators, please refer to Bertoin [5] and Kyprianou [68].

Definition 5.3.4 (**Lévy process**). *On the probability space* $(\Omega, \mathscr{F}, \mathbb{P})$, *a real-valued process* $S = \{S_t, t \geq 0\}$ *is said to be a Lévy process if it satisfies the following properties:*

(i) $\mathbb{P}(S_0 = 0) = 1$ *almost surely (a.s.) ;*

(ii) As a function of t, S_t *is right-continuous with left limits a.s ;*

(iii) S *has independent increments. For any choice of* $n \geq 1$ *and* $0 \leq t_0 < t_1 < \cdots < t_n$, *the random variables* $S_{t_0}, S_{t_1} - S_{t_0}, S_{t_2} - S_{t_1}, \cdots S_{t_n} - S_{t_{n-1}}$ *are independent ;*

(iv) S *is time homogeneous. The distribution of* $\{S_{t+v} - S_t, t \geq 0\}$ *does not depend on* t.

Lévy processes have infinitely divisible distributions. Following this fact, a Lévy process can be uniquely defined via the Lévy-Khintchine formula. Suppose that $a, \sigma \in \mathbb{R}$ and Π is a measure concentrated on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}} (1 \wedge x^2) \Pi(\mathrm{d}x) < \infty$. The probability law of a Lévy process is determined with a unique characteristic exponent Ψ, such that

$$\mathbb{E}\big[e^{i\lambda S_t}\big] = \exp\big\{ - t\Psi(\lambda)\big\}, \qquad t \geq 0,\ \lambda \in \mathbb{R},$$

where

$$\Psi(\lambda) := ia\lambda + \frac{\sigma^2 \lambda^2}{2} + \int_{\mathbb{R}\setminus\{0\}} \Big(1 - e^{i\lambda x} + i\lambda x \mathbb{1}_{\{|x|<1\}}\Big) \Pi(\mathrm{d}x).$$

By looking into this formula, one can deduce that the corresponding Lévy process is equal in probability to the sum of three terms: a deterministic linear drift, a Gaussian process, and a jump process whose size and frequency are characterised by Π, namely Lévy measure.

Definition 5.3.5 (**Lévy subordinator**). *On the same space, the Lévy process* S *is said to be a subordinator if it has non-decreasing sample paths and can be defined with its Laplace transform*

$$\mathbb{E}\big[e^{-\theta S_t}\big] = \exp\big\{ - t\Psi_S(\theta)\big\}, \qquad \theta \in \mathbb{R}^+,$$

with $\Psi_S(\theta)$ a Laplace exponent of the form

$$\Psi_S(\theta) \;=\; \int_{\mathbb{R}^+} \left(1 - e^{-\theta x}\right) \Pi(\mathrm{d}x),$$

and Π a Lévy measure on $\mathbb{R}^+ \setminus \{0\}$ satisfying $\int_{\mathbb{R}^+}(1 \wedge x)\Pi(\mathrm{d}x) \;<\; \infty$.

By the inverse subordinator, we mean a process $\left\{\ell_t^S, t > 0\right\}$ defined by

$$\ell_t^S \;=\; \inf\{\, x \;>\; 0 \mid S_x \;>\; t \,\},$$

i.e. the right-continuous inverse of S_x.

Proposition 5.3.6. *Due to the relation between ℓ_t^S and S_x, the law of the former represented by the latter has the form*

$$\mathbb{P}\Big(\ell_t^S \in \mathrm{d}x\Big) \;=\; -\,\tfrac{\partial}{\partial x}\mathbb{P}(S_x \le t)\,\mathrm{d}x\;.$$

For all $\beta, \theta > 0$, the double Laplace transform of ℓ_t^S is then given by

$$\mathcal{L}^{\beta,\theta}\left\{\ell_t^S\right\} \;=\; \frac{\beta^{-1}\Psi_S(\beta)}{\theta + \Psi_S(\beta)}.$$

Proof. By the definition of ℓ_t^S, we have an equality $\left\{\;\ell_t^S \le x\;\right\} \;=\; \{\;S_x \ge t\;\}$. Then with $\mathbb{E}\left[e^{-\theta S_x}\right] = e^{-x\Psi_S(\theta)}$,

$$\begin{aligned}
\mathcal{L}^{\beta,\theta}\left\{\ell_t^S\right\} &= \int_0^\infty e^{-\beta t} \int_0^\infty e^{-\theta x}\, \mathbb{P}\Big(\ell_t^S \in \mathrm{d}x\Big)\,\mathrm{d}t \\
&= \int_0^\infty e^{-\beta t} \int_0^\infty e^{-\theta x}\, \mathrm{d}\Big\{\,\mathbb{P}(S_x \ge t)\,\Big\}\,\mathrm{d}t \\
&= \theta \int_0^\infty e^{-\beta t} \int_0^\infty e^{-\theta x}\, \mathbb{P}(S_x \ge t)\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{1}{\beta}\frac{\Psi_S(\beta)}{\theta + \Psi_S(\beta)}\;.
\end{aligned}$$

□

Definition 5.3.7. *In particular for* $\Psi_S(\theta) = c\theta^{\alpha}$*, or equivalently*

$$\Pi(\mathrm{d}x) = \frac{c}{\Gamma(1-\alpha)}\alpha x^{-\alpha-1}\,\mathrm{d}x,$$

with $c > 0$ *a scale constant and* $\alpha \in (0,1)$ *a stability index, such a process is called a* ***α-stable subordinator***.

To get an explicit expression of the stable law, we adopt an integral representation provided by Zolotarev [90] for one-sided stable random variables. Let s_α denote a stable r.v. of index $0 < \alpha < 1$, whose distribution according to [90] is given by

$$\mathbb{P}\left(s_\alpha^{\frac{\alpha}{1-\alpha}} \leq x\right) = \frac{1}{\pi}\int_0^{\pi} e^{-\frac{A(v)}{x}}\,\mathrm{d}v, \qquad x \geq 0,$$

where A is the Zolotarev's function, read as

$$A(v) \stackrel{\text{def}}{=\!=} \left(\frac{(\sin(\alpha v))^{\alpha}(\sin((1-\alpha)v))^{1-\alpha}}{\sin(v)}\right)^{\frac{1}{1-\alpha}}.$$

By a simple change of variable, the stable law is written by

$$f_{s_\alpha}(x) = \frac{\alpha}{(1-\alpha)\pi}\int_0^{\pi} x^{-\frac{1}{1-\alpha}}A(v)e^{-A(v)x^{-\frac{\alpha}{1-\alpha}}}\,\mathrm{d}v. \tag{5.11}$$

The above definitions and results concerning stable subordinators play an important role in delivering the subsequent theorems.

Theorem 5.3.8. *By the continuity theorem, there is a limit relation holding for any* $0 < \alpha < 1$ *that,*

$$\frac{\Gamma(2-\alpha)}{\varpi^{1-\alpha}}Z_t(\varpi) \stackrel{\mathcal{D}}{\to} L_t, \qquad \text{as} \quad \varpi \to 0. \tag{5.12}$$

That is to say, $\nu_1(\varpi) = \frac{\varpi^{1-\alpha}}{\Gamma(2-\alpha)}$*. It also implies a characterisation of* $\{L_t\}$ *to the inverse of a* α*-stable subordinator, whose law is thus determined by*

$$\mathbb{P}(L_t \in \mathrm{d}z) = \frac{tz^{-\frac{1}{\alpha}-1}}{\alpha}f_{s_\alpha}\left(tz^{-\frac{1}{\alpha}}\right)\mathrm{d}z, \qquad z \in \mathbb{R}^+. \tag{5.13}$$

f_{s_α} *is the density function of a stable r.v. given in* (5.11).

Proof. By Lévy's continuity theorem, the pointwise limit of characteristic functions is a characteristic function provided that it is continuous at zero. That is to say, since we have shown that

$$\lim_{\varpi\to 0} \mathcal{L}^{\beta,\zeta}\left\{ \varpi^{\alpha-1} Z_t(\varpi) \right\} = \mathcal{L}^{\beta,\zeta}\left\{ \lim_{\varpi\to 0} \varpi^{\alpha-1} Z_t(\varpi) \right\}$$

holds for all $\beta, \xi \in \mathbb{R}^+$ and $\mathcal{L}^{\beta,\zeta}$ is an operator defined continuous at zero, there exists a family of random variables (clearly it refers to $\{L_t\}$), such that normalised by $\Gamma(2-\alpha)$ we have the equality in distribution:

$$L_t = \lim_{\varpi\to 0} \frac{\Gamma(2-\alpha)}{\varpi^{1-\alpha}} Z_t(\varpi) .$$

To find the law of L_t, we start by writing the Laplace transform (5.10) in LEMMA 5.3.3 as

$$\mathcal{L}^{\beta,\zeta'}\{L_t\} = -\int_0^\infty e^{-\zeta' y} \frac{\partial}{\partial\beta}\left\{ e^{-\beta^\alpha y} \right\} \frac{\mathrm{d}y}{\alpha y}$$

with $\zeta' = \frac{\zeta}{\Gamma(2-\alpha)}$. Obviously $e^{-\beta^\alpha y}$ coincides with the Laplace transform of s_α (see DEFINITION 5.3.7). Then the stable law in (5.11) contributes to the equation

$$-\frac{\partial}{\partial\beta}\left\{ e^{-\beta^\alpha y} \right\} = \frac{\alpha}{(1-\alpha)\pi} \int_0^\infty e^{-\beta t} \int_0^\pi y^{\frac{1}{1-\alpha}} t^{-\frac{\alpha}{1-\alpha}} A(x) e^{-A(x) y^{\frac{1}{1-\alpha}} t^{-\frac{\alpha}{1-\alpha}}} \,\mathrm{d}x\, \mathrm{d}t. \tag{5.14}$$

We thus have

$$\mathcal{L}^{\beta,\zeta'}\{L_t\} = \frac{1}{(1-\alpha)\pi} \int_0^\infty \int_0^\infty e^{-\beta t} e^{-\zeta' y} \int_0^\pi y^{\frac{\alpha}{1-\alpha}} t^{-\frac{\alpha}{1-\alpha}} A(x) e^{-A(x) y^{\frac{1}{1-\alpha}} t^{-\frac{\alpha}{1-\alpha}}} \,\mathrm{d}x\, \mathrm{d}t\, \mathrm{d}y.$$

This proof is completed by checking the validity of the density function involved,

$$\begin{aligned}
&\frac{1}{(1-\alpha)\pi} \int_0^\infty \int_0^\pi y^{\frac{\alpha}{1-\alpha}} t^{-\frac{\alpha}{1-\alpha}} A(x) e^{-A(x) y^{\frac{1}{1-\alpha}} t^{-\frac{\alpha}{1-\alpha}}} \,\mathrm{d}x\, \mathrm{d}y \\
&= -\frac{1}{\pi} \int_0^\infty \frac{\partial}{\partial y}\left\{ \int_0^\pi e^{-A(x) y^{\frac{1}{1-\alpha}} t^{-\frac{\alpha}{1-\alpha}}} \,\mathrm{d}x \right\} \mathrm{d}y \\
&= 1.
\end{aligned}$$

□

In line with Paul Lévy's manner, we provide another representation for L_t, expressed as a limit of some quantity that measures the amount of time spent at zero over $(0, \varphi)$ by a powered age process.

Definition 5.3.9. It is called the Lévy's local time of Bessel age process, written by

$$L_t = \lim_{\varphi \to 0} \frac{\Gamma(2-\alpha)}{\varphi} \int_0^t \mathbb{1}_{\{U_s{}^{1-\alpha} < \varphi\}} \, \mathrm{d}s. \tag{5.15}$$

THEOREM 5.3.3 states that the limit of the occupation time exists if it is scaled by $\varpi^{\alpha-1}$. By transforming $\varpi^{1-\alpha}$ with φ, we get this representation.

5.4 Limit Theorem II : Local Time as Limit of Jump Counting

This section is devoted to another approximation of the local time on the basis of Poisson random measure. It is constructed with a counting measure on the number of jumps in the sample path of age process. Following an analogous approach to that presented in Limit Theorem I, we shall develop a set of results concerning the asymptotics of the new measure and their applications to the local time.

5.4.1 A Scaled Jump Counting Process

Recall that $M(\varepsilon)$ denotes the counting process of jumps from the paths of U^ε, whose underlying process is a perturbed Bessel denoted by X^ε. By virtue of the correspondence between U_t^ε and X_t^ε for every $t \in \mathbb{R}^+$, $M_t(\varepsilon)$ is equal to the times that X_t^ε drops down to zero from $\varepsilon > 0$.

Of particular interest is a *scaled counting measure* in the form:

$$\varepsilon^\kappa M_t(\varepsilon), \qquad \varepsilon > 0,$$

with κ a scaler taking non-negative values. In some sense, ε^κ is regarded as a scaled unit increase per occurrence of a jump arriving at rate $\frac{p_\varepsilon(u)}{\bar{P}_\varepsilon(u)}$. Accordingly, the infinitesimal generator of $\left(U_t^\varepsilon, M_t(\varepsilon)\right)$, defined by its action on C^1 functions $f : [0,t) \times \mathbb{N}^0 \to \mathbb{R}^+$ as

$$\mathcal{A}_m f(u,m) = \frac{\partial f}{\partial u} + \frac{p_\varepsilon(u)}{\bar{P}_\varepsilon(u)} \Big[f(0, m+1) - f(u,m) \Big], \tag{5.16}$$

where $p_\varepsilon(u)$ & $\bar{P}_\varepsilon(u)$ are defined before by (5.7) & (5.8). Any solution to $\mathcal{A}_m f = 0$ is an $(\mathscr{F}_t)$-martingale represented by $f\big(U_t^\varepsilon, M_t(\varepsilon)\big) - f(U_0, 0)$. To examine the asymptotic behaviour of this measure, we derive the characteristic function of $M_t(\varepsilon)$ and decide a suitable choice of κ upon that for any $t \geq 0$

$$\lim_{\varepsilon \to 0} \mathbb{E}\big[f\big(U_t^\varepsilon, M_t(\varepsilon)\big)\big] < \infty.$$

The associated results are presented below.

Lemma 5.4.1. *We denote the double Laplace transform for $\varepsilon^\kappa M_t(\varepsilon)$ with $U_0 \geq 0$ by*

$$\mathcal{L}_{U_0}^{\beta,\xi}\{\varepsilon^\kappa M_t(\varepsilon)\} = \mathbb{E}\left[\int_0^\infty e^{-\beta t}\, e^{-\xi \varepsilon^\kappa M_t(\varepsilon)}\, \mathrm{d}t \,\middle|\, U_0\right].$$

For every $0 < \alpha < 1$ and $\kappa > 0$, it holds that

$$\lim_{\varepsilon \to 0} \mathcal{L}_{U_0}^{\beta,\xi}\{\, \varepsilon^\kappa M_t(\varepsilon)\,\} = \mathcal{L}_{U_0}^{\beta,\xi}\Big\{ \lim_{\varepsilon \to 0} \varepsilon^\kappa M_t(\varepsilon) \Big\},$$

and in particular for $\boldsymbol{\kappa = 2\alpha}$

$$\mathcal{L}_{U_0}^{\beta,\xi}\Big\{ \lim_{\varepsilon \to 0} \varepsilon^{2\alpha} M_t(\varepsilon) \Big\} = \mathcal{L}^{\beta,\xi}\Big\{ \lim_{\varepsilon \to 0} \varepsilon^{2\alpha} M_t(\varepsilon) \Big\}\Big(1 - \beta^\alpha {U_0}^\alpha e^{\beta U_0} \Gamma(1-\alpha, \beta U_0) \Big) + \beta^{\alpha-1} {U_0}^\alpha e^{\beta U_0} \Gamma(1-\alpha, \beta U_0).$$

where

$$\begin{aligned} \mathcal{L}^{\beta,\xi}\Big\{ \lim_{\varepsilon \to 0} \varepsilon^{2\alpha} M_t(\varepsilon) \Big\} &= \lim_{U_0 \downarrow 0} \mathcal{L}_{U_0}^{\beta,\xi}\Big\{ \lim_{\varepsilon \to 0} \varepsilon^{2\alpha} M_t(\varepsilon) \Big\} \\ &= \left(\beta + \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)} 2^\alpha \xi \beta^{1-\alpha} \right)^{-1}. \end{aligned} \tag{5.17}$$

Proof. It is our aim to find a martingale of the form

$$\int_0^t e^{-\beta s} e^{-\xi \varepsilon^\kappa M_s(\varepsilon)}\, \mathrm{d}s + e^{-\beta t} e^{-\xi \varepsilon^\kappa M_t(\varepsilon)} g\big(U_t^\varepsilon\big),$$

with g a bounded function. By the martingale property, we have

$$\lim_{t \to 0} \mathbb{E}\left[\int_0^t e^{-\beta s} e^{-\xi \varepsilon^\kappa M_s(\varepsilon)}\, \mathrm{d}s + e^{-\beta t} e^{-\xi \varepsilon^\kappa M_t(\varepsilon)} g\big(U_t^\varepsilon\big) \,\middle|\, U_0\right]$$

$$= \int_0^\infty e^{-\beta s}\, \mathbb{E}\Big[e^{-\xi\varepsilon^\kappa M_s(\varepsilon)} \,\Big|\, U_0\Big]\, \mathrm{d}s$$

$$= g(U_0).$$

To find a suitable solution of $g(U_0)$, we insert an auxiliary process defined by

$$W_t^\varepsilon := \int_0^t e^{-\beta s} e^{-\xi\varepsilon^\kappa M_s(\varepsilon)}\, \mathrm{d}s$$

into $\big(U_t^\varepsilon, M_t(\varepsilon)\big)$. Accordingly, the generator acting on $f(u,m,w,t)$ is given by

$$\mathcal{A} f(u,m,w,t) = \mathcal{A}_m f + \frac{\partial f}{\partial t} + e^{-\beta t} e^{-\xi\varepsilon^\kappa m} \frac{\partial f}{\partial w}.$$

Substituting $f(u,m,w,t) = w + e^{-\beta t} e^{-\xi\varepsilon^\kappa m} g(u)$ into $\mathcal{A} f = 0$, we get an equation

$$-\beta g(u) + g'(u) + 1 + \frac{p_\varepsilon(u)}{\bar{P}_\varepsilon(u)}\Big(e^{-\xi\varepsilon^\kappa} g(0) - g(u) \Big) = 0,$$

and the solution is obtained as

$$g(U_0) = \frac{e^{-\xi\varepsilon^\kappa} \int_0^\infty e^{-\beta v} \bar{P}_\varepsilon(v)\, \mathrm{d}v}{1 - e^{-\xi\varepsilon^\kappa} + e^{-\xi\varepsilon^\kappa} \int_0^\infty \beta e^{-\beta v} \bar{P}_\varepsilon(v)\, \mathrm{d}v} \frac{e^{\beta u_0}}{\bar{P}_\varepsilon(u_0)} \int_{u_0}^\infty e^{-\beta v} p_\varepsilon(v)\, \mathrm{d}v + \frac{e^{\beta u_0}}{\bar{P}_\varepsilon(u_0)} \int_{u_0}^\infty e^{-\beta v} \bar{P}_\varepsilon(v)\, \mathrm{d}v.$$

Implied by $\bar{P}_\varepsilon(v)$ a suitable choice of κ is determined as 2α. This leads immediately to

$$g(U_0) = \frac{\Gamma(1-\alpha)\beta^{\alpha-1}}{\Gamma(1-\alpha)\beta^\alpha + 2^\alpha \Gamma(1+\alpha)\xi} \left(1 - \beta {U_0}^\alpha e^{\beta U_0} \int_{U_0}^\infty e^{-\beta v} v^{-\alpha}\, \mathrm{d}v \right) + {U_0}^\alpha e^{\beta U_0} \int_{U_0}^\infty e^{-\beta v} v^{-\alpha}\, \mathrm{d}v.$$

□

5.4.2 Convergence of Jump Counting to Local Time

Following the same reasoning as in the proof of THEOREM 5.3.8, it turns out that the local time L_t arises as some limit of $\varepsilon^{2\alpha}M_t(\varepsilon)$.

Theorem 5.4.2. *There is an identity in distribution between the counting measure and the local time, such that for any* $0 < \alpha < 1$,

$$\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)2^{\alpha}}\varepsilon^{2\alpha}M_t(\varepsilon) \overset{\mathcal{D}}{\to} L_t, \qquad \textit{as} \quad \varepsilon \to 0. \tag{5.18}$$

Proof. To identify the limit, we compare the characteristic function in (5.10) with that in (5.17). Obviously it holds by setting $c(\alpha) = \Big[(1-\alpha)\Gamma(1+\alpha)2^{\alpha}\Big]^{-1}$ that

$$\mathbb{E}\Big[\exp\Big\{-\zeta c(\alpha)\lim_{\varepsilon\to 0}\ \varepsilon^{2\alpha}M_t(\varepsilon)\Big\}\Big] = \mathbb{E}\Big[\exp\Big\{-\zeta\lim_{\varpi\to 0}\ \varpi^{\alpha-1}Z_t(\varpi)\Big\}\Big].$$

This in turn by the continuity theorem proves the limit relation between $\varepsilon^{2\alpha}M_t(\varepsilon)$ and L_t. □

5.5 Limit Theorem III : A Central Limit Theorem for Local Time

In Limit Theorem I & II, we develop concepts for the local time at zero of Bessel age process and characterise its law as the inverse of a α-stable subordinator. In terms of convergence, we prove two equalities in distribution with the associated occupation and counting processes. Another concern around the local time arises as an interesting consequence of the weak convergences. In brief, the central limit theorem regarded as a corollary of the continuity theorem indicates that the limiting distribution of the difference between the local time and the approximations is *Gaussian*.

5.5.1 A Scaled Difference about Local Time

Following the relations among $Z(\varpi)$, $M(\varepsilon)$ and L, the stochastic process describing the difference is defined by

$$V_t(\varepsilon,\varpi) = \varpi^{-q}\left(\frac{\Gamma(2-\alpha)}{\varpi^{1-\alpha}}Z_t(\varpi) - \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)2^{\alpha}}\varepsilon^{2\alpha}M_t(\varepsilon)\right), \qquad t \geq 0, \tag{5.19}$$

with q a scaler taking non-negative values. In particular by allowing $\varepsilon \to 0$,

$$V_t(\varpi) \;=\; \lim_{\varepsilon \to 0} V_t(\varepsilon, \varpi) \;=\; \varpi^{-q}\left(\frac{\Gamma(2-\alpha)}{\varpi^{1-\alpha}} Z_t(\varpi) - L_t \right).$$

We are keen to find asymptotics of $V_t(\varepsilon, \varpi)$ as ε and ϖ approach zero, normalized by ϖ^q for measuring the speed that makes the convergence takes place. To this end, a martingale framework is established on the Markov process $(U_t, Z_t(\varpi), M_t(\varepsilon))$ to study the distributional properties of $V_t(\varepsilon, \varpi)$.

Let U_t^ε, $Z_t^\varepsilon(\varpi)$ and $M_t(\varepsilon)$ remain unchanged. The infinitesimal generator $\mathcal{A}_v$ acting on C^1-functions $f : [0,t)^2 \times \mathbb{N}^0 \to \mathbb{R}^+$ is simply obtained by merging (5.6) with (5.16). We thus have

$$\mathcal{A}_v f(u,z,m) \;=\; \frac{\partial f}{\partial u} + \frac{\mathbb{1}_{\{u\,<\,\varpi\}}}{\varpi^{1-\alpha}} \frac{\partial f}{\partial z} + \frac{p_\varepsilon(u)}{\bar{P}_\varepsilon(u)} \Big[f(0,z,m+1) - f(u,z,m) \Big].$$

The characteristic function of $V_t(\varepsilon, \varpi)$ is then derived from any f satisfying $\mathcal{A}_v f = 0$. Please see below for this result.

Lemma 5.5.1. *We denote the double Laplace transforms of $V_t(\varpi)$ by $\mathcal{L}^{\beta,\vartheta}\{V_t(\varpi)\}$. For $0 < \alpha < 1$ and $\beta, \vartheta > 0$*

$$\mathcal{L}^{\beta,\vartheta}\{V_t(\varpi)\} \;=\; \frac{R_1(\beta,\vartheta;\varpi) \;+\; R_2(\beta,\vartheta;\varpi)}{\left(\beta \;+\; \dfrac{\Gamma(2-\alpha)}{\varpi^{1-\alpha+q}}\vartheta\right) R_1(\beta,\vartheta;\varpi) \;+\; \beta R_2(\beta,\vartheta;\varpi) \;-\; \dfrac{\Gamma(1-\alpha)}{\varpi^q}\vartheta}. \tag{5.20}$$

R_1 and R_2 are define by

$$R_1(\beta,\vartheta;\varpi) \;=\; \left(\beta \;+\; \frac{\Gamma(2-\alpha)}{\varpi^{1-\alpha+q}}\vartheta\right)^{\alpha-1} \gamma\left(1-\alpha, \beta\varpi \;+\; \frac{\Gamma(2-\alpha)}{\varpi^{q-\alpha}}\vartheta\right);$$
$$R_2(\beta,\vartheta;\varpi) \;=\; \beta^{\alpha-1}\Gamma(1-\alpha,\beta\varpi)\exp\left\{-\frac{\Gamma(2-\alpha)}{\varpi^{q-\alpha}}\vartheta\right\}.$$

Proof. Denoted by $\mathcal{L}^{\beta,\vartheta}\{V_t(\varepsilon,\varpi)\}$ the double transform of $V_t(\varepsilon, \varpi)$, it can be obtained from computing

$$\lim_{t\to 0} \mathbb{E}\left[\int_0^t e^{-\beta s} e^{-\tilde{\zeta} Z_s^\varepsilon(\varpi) - \tilde{\xi}\varepsilon^{2\alpha} M_s(\varepsilon)}\, \mathrm{d}s \;+\; e^{-\beta t} e^{-\tilde{\zeta} Z_t(\varepsilon)} e^{-\tilde{\xi}\varepsilon^{2\alpha} M_s^\varepsilon} g\big(U_t^\varepsilon\big) \right]$$
$$= \int_0^\infty e^{-\beta s}\, \mathbb{E}\Big[e^{-\tilde{\zeta} Z_s^\varepsilon(\varpi)} e^{-\tilde{\xi}\varepsilon^{2\alpha} M_s(\varepsilon)} \Big]\, \mathrm{d}s$$
$$= g(0),$$

with substitution

$$\tilde{\zeta} \ := \ \tilde{\zeta}(\vartheta) \ = \ \frac{\Gamma(2-\alpha)}{\varpi^{1-\alpha+q}}\vartheta; \qquad \tilde{\xi} \ := \ \tilde{\xi}(\vartheta) \ = \ -\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)2^{\alpha}\varpi^{q}}\vartheta.$$

Considering $f\big(U_t, Z_t(\varpi), M_t(\varepsilon), W_t^{\varepsilon}\big)$ with

$$W_t^{\varepsilon} \ := \ \int_0^t e^{-\beta s} e^{-\tilde{\zeta} Z_s^{\varepsilon}(\varpi) - \tilde{\xi}\varepsilon^{2\alpha} M_s(\varepsilon)} \, \mathrm{d}s$$

is a martingale of the form: $f(u,z,m,w,t) \ = \ w + e^{-\beta t} e^{-\tilde{\zeta} z} e^{-\tilde{\xi}\varepsilon^{2\alpha} m} g(u)$. g is a non-negative bounded function. To prove it so as to get the representation of $g(0)$, we show that

$$\begin{aligned} \mathcal{A} f(u,z,m,w,t) \ &= \ \frac{\partial f}{\partial t} \ + \ \frac{\partial f}{\partial u} \ + \ \mathbb{1}_{\{u\,<\,\varpi\}} \frac{\partial f}{\partial z} \ + \ e^{-\beta t} e^{-\tilde{\zeta} z} e^{-\tilde{\xi}\varepsilon^{2\alpha} m} \frac{\partial f}{\partial w} \\ &\quad + \ \frac{p_{\varepsilon}(u)}{\bar{P}_{\varepsilon}(u)} \Big[f(\,0,z,m+1,w,t\,) - f(\,u,z,m,w,t\,) \Big] \\ &= \ 0\,. \end{aligned}$$

By solving $\mathcal{A}\Big(w \ + \ e^{-\beta t} e^{-\tilde{\zeta} z} e^{-\tilde{\xi}\varepsilon^{2\alpha} m} g(u)\Big) = 0$, we get

$$g(0) \ = \ \frac{\displaystyle\int_0^{\infty} e^{-\beta v} e^{-\tilde{\zeta}\min\{v,\,\varpi\}} \bar{P}_{\varepsilon}(v) \,\mathrm{d}v}{\displaystyle e^{\xi\varepsilon^{2\alpha}} - 1 + \int_0^{\infty} \Big(\beta + \mathbb{1}_{\{v\,<\,\varpi\}}\tilde{\zeta} \Big) e^{-\beta v} e^{-\tilde{\zeta}\min\{v,\,\varpi\}} \bar{P}_{\varepsilon}(v) \,\mathrm{d}v}\,.$$

Then (5.20) follows immediately by taking $\varepsilon \to 0$ and replacing $\tilde{\zeta} \to \tilde{\zeta}(\vartheta)$ and $\tilde{\xi} \to \tilde{\xi}(\vartheta)$. □

To find a suitable normalizing factor ϖ^q that makes the difference having a non-trivial limit distribution, we investigate the asymptotic behaviour of $V_t(\varpi)$ as $\varpi \to 0$. The associated results are presented below.

Lemma 5.5.2. *For every* $0 < \alpha < 1$ *and* $q > 0$*, it holds that*

$$\lim_{\varpi\to 0} \mathcal{L}^{\beta,\vartheta}\{V_t(\varpi)\} \ = \ \mathcal{L}^{\beta,\vartheta}\Big\{ \lim_{\varpi\to 0} V_t(\varpi) \Big\}, \tag{5.21}$$

and in particular for $q = \frac{\alpha}{2}$,

$$\mathcal{L}^{\beta,\vartheta}\Big\{ \lim_{\varpi\to 0} V_t(\varpi) \Big\} \;=\; \left(\beta - \frac{(1-\alpha)\Gamma(2-\alpha)}{2-\alpha} \vartheta^2 \beta^{1-\alpha} \right)^{-1}. \tag{5.22}$$

Proof. Considering the numerator in (5.20), define $\tilde{\vartheta}(\varpi) = \Gamma(2-\alpha)\vartheta\varpi^{\alpha-q}$

$$\left(\beta + \frac{\tilde{\vartheta}(\varpi)}{\varpi}\right)^{\alpha-1} \gamma\Big(1-\alpha, \beta\varpi + \tilde{\vartheta}(\varpi)\Big) \;+\; \beta^{\alpha-1} e^{-\tilde{\vartheta}(\varpi)}\, \Gamma(1-\alpha, \beta\varpi)$$

$$= \; \varpi^{(1-\alpha)^2 + q(1-\alpha)} \int_0^{\varpi^{\alpha-q}} e^{-\beta\varpi^{1-\alpha+q} z} e^{-\Gamma(2-\alpha)\vartheta z} z^{-\alpha}\, \mathrm{d}z \;+\; e^{-\tilde{\vartheta}(\varpi)} \int_{\varpi}^{\infty} e^{-\beta v} v^{-\alpha}\, \mathrm{d}v.$$

With $0 < \alpha < 1$, it can be checked that for all values of q

$$\lim_{\varpi\to 0} \varpi^{(1-\alpha)^2 + q(1-\alpha)} \int_0^{\varpi^{\alpha-q}} e^{-\beta\varpi^{1-\alpha+q} z} e^{-\Gamma(2-\alpha)\vartheta z} z^{-\alpha}\, \mathrm{d}z \;=\; 0,$$

and the rest term is evaluated to

$$\lim_{\varpi\to 0} e^{-\tilde{\vartheta}(\varpi)} \int_{\varpi}^{\infty} e^{-\beta v} v^{-\alpha}\, \mathrm{d}v \;=\; \begin{cases} \Gamma(1-\alpha)\beta^{\alpha-1}, & q \in (\,0, \alpha\,); \\ \Gamma(1-\alpha)\beta^{\alpha-1} e^{-\Gamma(2-\alpha)\vartheta}, & q = \alpha; \\ 0, & q \in (\,\alpha, \infty\,). \end{cases}$$

Then for the denominator

$$\left(\beta + \frac{\tilde{\vartheta}(\varpi)}{\varpi}\right)^{\alpha} \gamma\Big(1-\alpha, \beta\varpi + \tilde{\vartheta}(\varpi)\Big) \;+\; \beta^{\alpha} e^{-\tilde{\vartheta}(\varpi)} \Gamma(1-\alpha, \beta\varpi) - \Gamma(1-\alpha)\vartheta\varpi^{-q}$$

$$= \; \beta\left[\left(\beta + \frac{\tilde{\vartheta}(\varpi)}{\varpi}\right)^{\alpha-1} \gamma\Big(1-\alpha, \beta\varpi + \tilde{\vartheta}(\varpi)\Big) \;+\; \beta^{\alpha-1} e^{-\tilde{\vartheta}(\varpi)}\, \Gamma(1-\alpha, \beta\varpi) \right] \tag{5.23}$$

$$+ \; \Gamma(2-\alpha)\vartheta\varpi^{-\alpha(1-\alpha+q)} \left(\int_0^{\varpi^{\alpha-q}} e^{-\beta\varpi^{1-\alpha+q} z} e^{-\Gamma(2-\alpha)\vartheta z} z^{-\alpha}\, \mathrm{d}z \;-\; \frac{\varpi^{(\alpha-q)(1-\alpha)}}{1-\alpha} \right). \tag{5.24}$$

(5.23) coincides with the numerator. Applying L'Hôpital's Rule to (5.24),

$$\Gamma(2-\alpha)\vartheta \lim_{\varpi\to 0} \frac{\displaystyle\int_0^{\varpi^{\alpha-q}} e^{-\beta\varpi^{1-\alpha+q} z} e^{-\Gamma(2-\alpha)\vartheta z} z^{-\alpha}\, \mathrm{d}z - \frac{\varpi^{(\alpha-q)(1-\alpha)}}{1-\alpha}}{\varpi^{\alpha(1-\alpha+q)}}$$

$$= -\frac{\Gamma(2-\alpha)\vartheta(\alpha-q)}{\alpha q(1-\alpha+q)} \lim_{\varpi\to 0} \left(\beta\varpi^{1-q} + \Gamma(2-\alpha)\vartheta(\alpha-q)\varpi^{\alpha-2q}\right) e^{-\beta\varpi} e^{-\tilde{\vartheta}(\varpi)}$$

$$= \begin{cases} 0, & q \in \left(0, \frac{\alpha}{2}\right); \\ -\frac{(\Gamma(2-\alpha)\vartheta)^2}{2-\alpha}, & q = \frac{\alpha}{2}; \\ -\infty, & q \in \left(\frac{\alpha}{2}, \infty\right). \end{cases}$$

Putting them together, we get

$$\mathcal{L}^{\beta,\vartheta}\left\{ \lim_{\varpi\to 0} V_t(\varpi) \right\} = \begin{cases} \frac{1}{\beta}, & q \in \left(0, \frac{\alpha}{2}\right); \\ \dfrac{\Gamma(1-\alpha)\beta^{\alpha-1}}{\Gamma(1-\alpha)\beta^{\alpha} - \dfrac{(\Gamma(2-\alpha)\vartheta)^2}{2-\alpha}}, & q = \frac{\alpha}{2}; \\ 0, & q \in \left(\frac{\alpha}{2}, \infty\right). \end{cases}$$

□

5.5.2 Convergence of the Difference to Gaussian Process

The aim of what follows is to show a Gaussian process for the limiting difference, to which the normality is shown by the central limit theorem. It is oriented by the work of Kasahara [63] who showed that the difference between the Brownian local time and the normalised number of downcrossings follows a brand new Brownian motion.

Theorem 5.5.3. *By central limit theorem it holds for any* $0 < \alpha < 1$ *that*

$$\varpi^{-\frac{\alpha}{2}} \left(\frac{\Gamma(2-\alpha)}{\varpi^{1-\alpha}} Z_t(\varpi) - L_t \right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma_\alpha^2(t)\right), \qquad \text{as} \quad \varpi \to 0. \tag{5.25}$$

$\mathcal{N}$ represents a normal distribution whose variance $\sigma_\alpha^2(t)$ is equal to

$$\sigma_\alpha^2(t) = \frac{2(1-\alpha)\Gamma(2-\alpha)}{2-\alpha} L_t. \tag{5.26}$$

Proof. We start by writing (5.22) in the form

$$\mathcal{L}^{\beta,\vartheta}\Big\{ \lim_{\varpi\to 0} V_t(\varpi) \Big\} = -\int_0^\infty \frac{\partial}{\partial\beta}\big\{ e^{-\beta^\alpha y} \big\} \exp\Big\{\frac{\phi^2\vartheta^2 y}{2}\Big\}\frac{\mathrm{d}y}{\alpha y},$$

with $\phi^2 = \frac{2(1-\alpha)\Gamma(2-\alpha)}{2-\alpha}$. A proper integral representation of $-\frac{\partial}{\partial\beta}\big\{ e^{-\beta^\alpha y} \big\}$ has been derived in (5.14). Moreover, the exponential term involving ϑ^2 can be expanded in terms of the moment generating function of a normal distribution, such that

$$\exp\Big\{\frac{\vartheta^2\phi^2 y}{2}\Big\} = \int_{-\infty}^\infty e^{-\vartheta v}\, \frac{1}{\phi\sqrt{2\pi y}} \exp\Big\{-\frac{v^2}{2y\phi^2}\Big\}\,\mathrm{d}v.$$

This leads to

$$\begin{aligned}\mathcal{L}^{\beta,\vartheta}\Big\{ \lim_{\varpi\to 0} V_t(\varpi) \Big\} = &\int_0^\infty e^{-\beta t} \int_{-\infty}^\infty e^{-\vartheta v}\,\mathrm{d}v\,\mathrm{d}t \\ &\frac{1}{(1-\alpha)\phi\sqrt{2\pi^3}} \int_0^\infty y^{\frac{1}{1-\alpha}-\frac{3}{2}} t^{-\frac{\alpha}{1-\alpha}} e^{-\frac{v^2}{2y\phi^2}} \int_0^\pi A(x) e^{-A(x) y^{\frac{1}{1-\alpha}} t^{-\frac{\alpha}{1-\alpha}}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

We thus obtain the density of $\lim_{\varpi\to 0} V_t(\varpi)$. With $\mathbb{P}(L_t \in \mathrm{d}y)$ given by (5.13), it can then be expressed in the form of a normal distribution subordinated by L_t acting on the variance, such that

$$\mathbb{P}\Big(\lim_{\varpi\to 0} V_t(\varpi) \in \mathrm{d}v\Big) = \int_0^\infty \frac{1}{\sqrt{2\pi y\phi^2}} \exp\Big\{-\frac{v^2}{2y\phi^2}\Big\}\cdot \mathbb{P}(L_t \in \mathrm{d}y)\,\mathrm{d}v.$$

The proof is completed by this representation. □

6

Scale Invariance about Local Times

By *self-similar Markov processes*, we mean a $\mathbb{R}^+$-valued strong Markov process $X = \{X_t, t \geq 0\}$ with $X_0 = 0$, whose paths are almost surely right-continuous with left limits such that,

$$\{ c^{-a} X_{ct},\ t \geq 0 \} \overset{\text{law}}{=} \{ X_t,\ t \geq 0 \}, \qquad a,\ c > 0.$$

That is to say, X fulfils the *scaling invariant* property (please refer to Kyprianou [68, Chap 13] for accounts on this process and to Embrechts and Maejima [43] for more general cases). Some known examples having the same path property are subordinators, Brownian motions, Bessel processes and the local time for Bessel age processes constructed in the last chapter.

This chapter concerns with the scaling property for some processes introduced before, involving Bessel(Brownian) process, Bessel age process and occupation time of the age process. As a sequel to the study on local times, we put an emphasis on exploring path properties of a time-scaled process represented by $\{L_{\lambda t}\}$ with λ a scaler taking non-negative values. Main questions to be discussed are the distributional results associated with $L_{\lambda t}$ and scale invariance for the local time along with a limit theorem characterising the difference between L_t and $L_{\lambda t}$ to a Gaussian process.

To avoid confusion on notations, we will keep using the symbols defined in previous chapters as much as possible.

6.1 Definitions and Notations

On the filtered probability space $(\Omega, \mathscr{F}, \mathbb{P})$, adapted to $\{\mathscr{F}_t\}$ let $U = \{U_t,\ t \geq 0\}$ be the age process defined by

$$U_t \ = \ t - \sup\{\ s < t \mid X_s = 0\ \}. \tag{6.1}$$

The underlying process $X = \{X_t,\ t \geq 0\}$ is Bessel of dimension $(2 - 2\alpha)$ with $\alpha \in (0,1)$, thus instantaneously reflected at the origin. Within the space, there is an occupation time of level $\varpi > 0$ expressed in the form

$$Z_t(\varpi) \ = \ \int_0^t \mathbb{1}_{\{U_s\ <\ \varpi\}}\, \mathrm{d}s, \qquad 0 < \varpi < t. \tag{6.2}$$

It has been proved with a scaler of $\varpi^{\alpha-1}$ that the limiting process of $Z_t(\varpi)$ measures the local time at zero of U up to time t. Denoted by $\{L_t, t \geq 0\}$, the local time process admits the following representation :

$$L_t \ = \ \lim_{\varpi \to 0} \Gamma(2-\alpha)\varpi^{\alpha-1} Z_t(\varpi). \tag{6.3}$$

While performing calculations, to avoid confusions in presenting results we define a double Laplace transform for a real-valued random variable X_t by

$$\mathcal{L}^{\beta,\xi}\{X_t\} \ \coloneqq \ \mathbb{E}\Big[\int_0^\infty e^{-\beta t} e^{-\xi X_t}\ \mathrm{d}t\Big] \ = \ \int_0^\infty e^{-\beta t}\ \mathbb{E}\big[e^{-\xi X_t}\big]\ \mathrm{d}t,$$

where the last equality follows from Fubini's theorem. We further adopt the following notations

$$\mathcal{L}_t^\beta\{\varphi(t)\} \ = \ \int_{\mathbb{R}^+} e^{-\beta t}\ \varphi(t)\ \mathrm{d}t,$$

the Laplace transform of some function $\varphi(t)$ defined for all real numbers $t \geq 0$, and

$$\gamma(\alpha, z) \ = \ \int_0^z e^{-t}\ t^{\alpha-1}\ \mathrm{d}t;$$
$$\Gamma(\alpha, z) \ = \ \int_z^\infty e^{-t}\ t^{\alpha-1}\ \mathrm{d}t,$$

for the lower and upper incomplete gamma functions, and thus $\gamma(\alpha, z) + \Gamma(\alpha, z) \ = \ \Gamma(\alpha)$.

6.2 Scaling Property associated to Bessel (Age) Process

This section presents some examples of basic scaling properties associated to a Bessel (age) process, which will in turn facilitate as guidelines towards developing the main theorem appearing in SECTION 6.4.

Denoted by $\mathrm{BES}^{(\alpha)}$, the Bessel process X with index $\alpha \in (0,1)$ is the solution of

$$\mathrm{d}X_t \;=\; \frac{1-2\alpha}{2X_t}\,\mathrm{d}t + \,\mathrm{d}W_t,$$

with W a standard Brownian motion. Let us consider a transformation in time of a general form: $t \to \ell(t)$, for $\ell \in C^1$ a non-negative Borel function with $\ell(0) = 0$. Assuming ℓ is continuously differentiable on $(0,\infty)$,

$$X_{\ell(t)} \;=\; X_0 + \int_0^{\ell(t)} \frac{1-2\alpha}{2X_s}\,\mathrm{d}s + W_{\ell(t)}.$$

By a simple change of variable,

$$\mathrm{d}X_{\ell(t)} \;=\; \frac{1-2\alpha}{2X_{\ell(t)}}\,\ell'(t)\,\mathrm{d}t + \sqrt{\ell'(t)}\,\mathrm{d}W_t. \tag{6.4}$$

Let $Y = X^2$ be the squared Bessel process $\mathrm{BESQ}^{(\alpha)}$, whose path is described by

$$\mathrm{d}Y_t \;=\; 2(1-\alpha)\,\mathrm{d}t + 2\sqrt{Y_t}\,\mathrm{d}W_t.$$

Following the same method,

$$\mathrm{d}Y_{\ell(t)} \;=\; 2(1-\alpha)\ell'(t)\,\mathrm{d}t + 2\sqrt{Y_{\ell(t)}\ell'(t)}\,\mathrm{d}W_t. \tag{6.5}$$

(Squared) Bessel diffusions lie in the intersection of several important classes of processes. Below are a couple of well-known examples.

Example 6.2.1. Not all scale transforms have the feature of invariance.

- *Scaling Invariance*. When $\alpha = 1/2$, $\mathrm{d}X_t = \mathrm{d}W_t$. For any $c \in \mathbb{R}^+$, it follows immediately from (6.4),

$$\left\{ c^{-\frac{1}{2}} W_{ct},\; t \geq 0 \right\} \overset{\text{law}}{=} \{\, W_t,\; t \geq 0 \,\}.$$

This is the famous Brownian scaling property and it can be further extended to Bessel diffusions. For other choices of α, from (6.4) and (6.5),

$$\left\{ c^{-\frac{1}{2}} X_{ct},\ t \geq 0 \right\} \overset{\text{law}}{=} \{ X_t,\ t \geq 0 \};$$
$$\left\{ c^{-1} Y_{ct},\ t \geq 0 \right\} \overset{\text{law}}{=} \{ Y_t,\ t \geq 0 \}.$$

- *Other Scalings.* Consider a scaled Bessel process at exponential times, say $R_t = e^{-2kt} Y_{\ell(t)}$ with $\ell(t) = \frac{e^{2kt}-1}{2k}$ and $k > 0$. We obtain by using Itô formula,

$$\begin{aligned} \mathrm{d}R_t &= -2ke^{-2kt} Y_{\ell(t)}\,\mathrm{d}t + e^{-2kt}\,\mathrm{d}Y_{\ell(t)} \\ &= 2\Big((1-\alpha) - kR_t\Big)\,\mathrm{d}t + 2\sqrt{R_t}\,\mathrm{d}W_t, \end{aligned}$$

  a space-time changed $\mathrm{BESQ}^{(\alpha)}$, namely the Cox-Ingersoll-Ross process.

The invariant scaling of X immediately carries over to the Bessel age process U.

Proposition 6.2.2. *Given U the age process defined as* (6.1)*, for any $c \in \mathbb{R}^+$, it enjoys the scaling property such that*

$$\left\{ c^{-1} U_{ct},\ t \geq 0 \right\} \overset{\text{law}}{=} \{ U_t,\ t \geq 0 \}\,. \tag{6.6}$$

As an immediate consequence,

$$\{ Z_{ct}(\varpi),\ t \geq 0 \} \overset{\text{law}}{=} \left\{ cZ_t\Big(c^{-1}\varpi\Big),\ t \geq 0 \right\}. \tag{6.7}$$

Proof. From the scaling property of X, for $c > 0$,

$$\begin{aligned} U_{ct} &= ct - \sup\left\{ cv < ct \,\middle|\, c^{-\frac{1}{2}} X_{cv} = 0 \right\} \\ &\overset{\text{law}}{=} c\left(t - \sup\left\{ v < t \,\middle|\, \tilde{X}_v = 0 \right\} \right), \end{aligned}$$

where $\tilde{X}$ is an independent replicate of X. By the same token with a simple change of variable

$$Z_{ct}(\varpi) = c\int_0^t \mathbb{1}_{\{c^{-1}U_{cx} < c^{-1}\varpi\}}\,\mathrm{d}x$$

$$\overset{\text{law}}{=} \; c \int_0^t \mathbb{1}_{\{\tilde{U}_x \,<\, c^{-1}\varpi\}} \,\mathrm{d}x,$$

and $\tilde{U}$ is an independent replicate of U. □

6.3 Distributional Results for Time-scaled Occupation /Local Time

This section proceeds with the distributional study on a scaled occupation measure in the form $Z_t(c\varpi)$ with $0 < c < 1$, and on its association to the local time defined as (6.3). We shall construct a martingale for the process $(\, U_t, Z_t(\varpi), Z_t(c\varpi)\,)$, from which distributional properties for the limiting difference between $Z_t(\varpi)$ and $Z_t(c\varpi)$ are developed .

Denoted by $\big(U_t^{\varepsilon}, Z_t^{(1)}, Z_t^{(2)}\big)$ the perturbed Markov process used to perform calculations, the infinitesimal generator acting on a non-negative bounded Borel function $f \in C^1$ is given by

$$\begin{aligned}
\mathcal{A}_{Z^c} f\big(\, u, z^{(1)}, z^{(2)}\,\big) \;&=\; \frac{\partial f}{\partial u} \;+\; \frac{p_{\varepsilon}(u)}{\bar{P}_{\varepsilon}(u)}\Big[\, f\big(0, z^{(1)}, z^{(2)}\big) - f\big(u, z^{(1)}, z^{(2)}\big)\,\Big] \\
&\quad + \; \mathbb{1}_{\{u \,<\, \varpi\}}\frac{\partial f}{\partial z^{(1)}} \;+\; \mathbb{1}_{\{u \,<\, c\varpi\}}\frac{\partial f}{\partial z^{(2)}},
\end{aligned}$$

where $Z_t^{(1)} \;:=\; Z_t(\varpi)$ and $Z_t^{(2)} \;:=\; Z_t(c\varpi)$, and

$$\begin{aligned}
p_{\varepsilon}(u) \;&=\; \frac{\varepsilon^{2\alpha}}{2^{\alpha}\Gamma(\alpha)} u^{-\alpha-1} e^{-\frac{\varepsilon^2}{2u}}; \\
\bar{P}_{\varepsilon}(u) \;&=\; \int_u^{\infty} \frac{\varepsilon^{2\alpha}}{2^{\alpha}\Gamma(\alpha)} v^{-\alpha-1} e^{-\frac{\varepsilon^2}{2v}} \,\mathrm{d}v.
\end{aligned}$$

Within the framework of infinitesimal generator, a martingale meeting our needs is obtained from solving $\mathcal{A}_{Z^c} f(\, u, z^{(1)}, z^{(2)}\,) = 0$.

Lemma 6.3.1. *We find a representation for the double Laplace transform of* $\{Z_t(\varpi), Z_t(c\varpi)\}$*, denoted by*

$$\hat{Z}(\beta, \zeta_1, \zeta_2; \varpi) \;=\; \mathcal{L}_t^{\beta}\bigg\{\, \mathbb{E}\bigg[\exp\bigg\{ -\frac{\zeta_1}{\varpi^p} Z_t(\varpi) - \frac{\zeta_2}{(c\varpi)^p} Z_t(c\varpi) \bigg\}\bigg]\,\bigg\}.$$

For $\beta > 0$ and $0 < \alpha < 1$,

$$\hat{Z}(\beta,\zeta_1,\zeta_2;\varpi) = \frac{\displaystyle\int_0^\infty e^{-\beta v} E(v,\varpi) v^{-\alpha}\,\mathrm{d}v}{\displaystyle\int_0^\infty \left(\beta + \mathbb{1}_{\{v<\varpi\}}\frac{\zeta_1}{\varpi^p} + \mathbb{1}_{\{v<c\varpi\}}\frac{\zeta_2}{(c\varpi)^p}\right) e^{-\beta v} E(v,\varpi) v^{-\alpha}\,\mathrm{d}v}, \tag{6.8}$$

with

$$E(v,\varpi) = \exp\left\{ -\frac{\zeta_1}{\varpi^p}\min\{v,\ \varpi\} - \frac{\zeta_2}{(c\varpi)^p}\min\{v,\ c\varpi\} \right\}.$$

Proof. The Laplace transform is implied by a martingale of the form

$$\int_0^t e^{-\beta s}\, e^{-\zeta_1 Z_s^{(1)}} e^{-\zeta_2 Z_s^{(2)}}\,\mathrm{d}s + e^{-\beta t} e^{-\zeta_1 Z_t^{(1)}} e^{-\zeta_2 Z_t^{(2)}} g\big(U_t^\varepsilon\big),$$

with g a non-negative Borel function bounded by one. By the martingale property, it follows immediately

$$\begin{aligned}
&\lim_{t\to 0} \mathbb{E}\left[\int_0^t e^{-\beta s}\, e^{-\zeta_1 Z_s^{(1)}} e^{-\zeta_2 Z_s^{(2)}}\,\mathrm{d}s + e^{-\beta t} e^{-\zeta_1 Z_t^{(1)}} e^{-\zeta_2 Z_t^{(2)}} g\big(U_t^\varepsilon\big)\right] \\
&= \int_0^\infty e^{-\beta s}\, \mathbb{E}\left[e^{-\zeta_1 Z_s^{(1)}} e^{-\zeta_2 Z_s^{(2)}}\right]\mathrm{d}s \\
&= g(0).
\end{aligned}$$

To find a solution of $g(0)$, we insert an auxiliary process W_t^ε of the form

$$W_t^\varepsilon := \int_0^t e^{-\beta s}\, e^{-\zeta_1 Z_s^{(1)}} e^{-\zeta_2 Z_s^{(2)}}\,\mathrm{d}s$$

into the Markov process. Accordingly, a new generator acting on $f(\,u, z^{(1)}, z^{(2)}, w, t\,)$ is written by

$$\mathcal{A} f\big(u, z^{(1)}, z^{(2)}, w, t\big) = \mathcal{A}_{z^c} f + \frac{\partial f}{\partial t} + e^{-\beta t} e^{-\zeta_1 z^{(1)}} e^{-\zeta_2 z^{(2)}} \frac{\partial f}{\partial w}.$$

We propose a solution to be $\Big(w + e^{-\beta t} e^{-\zeta_1 z^{(1)}} e^{-\zeta_2 z^{(2)}} g(u) \Big)$. Substituting it into $\mathcal{A} f = 0$, we have an equation

$$0 \;=\; -\beta g(u) \;+\; g'(u) - \Big(\zeta_1 \mathbb{1}_{\{u \,<\, \varpi\}} + \zeta_2 \mathbb{1}_{\{u \,<\, c\varpi\}} \Big) g(u) \\ \qquad + \; 1 \; + \; \frac{p_\varepsilon(u)}{\bar{P}_\varepsilon(u)} \Big(g(0) - g(u) \Big) \tag{6.9}$$

To get the solution to (6.9), we multiply both sides with the term below

$$e^{-\beta u}\; e^{-\zeta_1 m(\varpi)} e^{-\zeta_2 m(c\varpi)} \bar{P}_\varepsilon(u),$$

with $m(w) = \min\{u,\ w\}$. As a result, we get

$$g(u) e^{-\beta u} e^{-\zeta_1 m(\varpi)} e^{-\zeta_2 m(c\varpi)} \bar{P}_\varepsilon(u) \quad = \\ g(0) \int_u^\infty e^{-\beta v} e^{-\zeta_1 m(\varpi)} e^{-\zeta_2 m(c\varpi)} p_\varepsilon(v)\,\mathrm{d}v \;+\; \int_u^\infty e^{-\beta v} e^{-\zeta_1 m(\varpi)} e^{-\zeta_2 m(c\varpi)} \bar{P}_\varepsilon(v)\,\mathrm{d}v.$$

Taking $\varepsilon \longrightarrow 0$,

$$g(0) \;=\; \frac{\displaystyle\int_0^\infty e^{-\beta v} e^{-\zeta_1 m(\varpi)} e^{-\zeta_2 m(c\varpi)} v^{-\alpha}\,\mathrm{d}v}{\displaystyle\int_0^\infty \big(\beta + \zeta_1 \mathbb{1}_{\{v \,<\, \varpi\}} + \zeta_2 \mathbb{1}_{\{v \,<\, c\varpi\}}\big) e^{-\beta v} e^{-\zeta_1 m(\varpi)} e^{-\zeta_2 m(c\varpi)} v^{-\alpha}\,\mathrm{d}v}.$$

Replacing $\zeta_1 \longrightarrow \frac{\zeta_1}{\varpi^p}$ and $\zeta_2 \longrightarrow \frac{\zeta_2}{(c\varpi)^p}$ completes this proof. □

Next we present the limit of $\hat{Z}(\beta, \zeta_1, \zeta_2; \varpi)$ at $\varpi \longrightarrow 0$.

Lemma 6.3.2. *For any $c > 0$, we show that the limit exists when $\boldsymbol{p = 1 - \alpha}$ and*

$$\lim_{\varpi \to 0} \hat{Z}(\beta, \zeta_1, \zeta_2; \varpi) \;=\; \left(\beta + \frac{\zeta_1 + \zeta_2}{\Gamma(2-\alpha)} \beta^{1-\alpha} \right)^{-1}.$$

Proof. We restrict the calculations to $c \in (0, 1)$. Other choices can be done by taking $c^{-1} \in (1, \infty)$. However, it will be later shown that c has no effect on the final results.

Considering the numerator of (6.8),

$$\int_0^\infty e^{-\beta v} e^{-\frac{\zeta_1}{\varpi^p}\min\{u,\,\varpi\}-\frac{\zeta_2}{(c\varpi)^p}\min\{u,\,c\varpi\}} v^{-\alpha}\,\mathrm{d}v \tag{6.10}$$

$$= \varpi^{p(1-\alpha)} \int_0^{c\varpi^{1-p}} e^{-\beta\varpi^p z} e^{-\left(\zeta_1+\frac{\zeta_2}{c^p}\right)z} z^{-\alpha}\,\mathrm{d}z \;+\; \varpi^{p(1-\alpha)} e^{-\frac{\zeta_2}{(c\varpi)^{p-1}}} \int_{c\varpi^{1-p}}^{\varpi^{1-p}} e^{-\beta\varpi^p z} e^{-\zeta_1 z} z^{-\alpha}\,\mathrm{d}z$$

$$+\; e^{-\left(\zeta_1+\frac{\zeta_2}{c^{p-1}}\right)\varpi^{1-p}} \int_{\varpi}^{\infty} e^{-\beta v} v^{-\alpha}\,\mathrm{d}v.$$

The limit of (6.10) by taking $\varpi \to 0$ is obtained as

$$\begin{cases} \dfrac{\Gamma(1-\alpha)}{\beta^{1-\alpha}}, & p \in (0,1); \\ e^{-(\zeta_1+\zeta_2)}\dfrac{\Gamma(1-\alpha)}{\beta^{1-\alpha}}, & p = 1; \\ 0, & p \in (1,\infty). \end{cases}$$

Then for the denominator,

$$\int_0^\infty \left(\frac{\zeta_1}{\varpi^p}\mathbb{1}_{\{v\,<\,\varpi\}} + \frac{\zeta_2}{(c\varpi)^p}\mathbb{1}_{\{v\,<\,c\varpi\}} \right) e^{-\beta v} e^{-\frac{\zeta_1}{\varpi^p}\min\{u,\,\varpi\}-\frac{\zeta_2}{(c\varpi)^p}\min\{u,\,c\varpi\}} v^{-\alpha}\,\mathrm{d}v \tag{6.11}$$

$$= \frac{\zeta_1}{\varpi^p}\int_0^{\varpi} e^{-\beta v} e^{-\frac{\zeta_1}{\varpi^p}\min\{v,\,\varpi\}-\frac{\zeta_2}{(c\varpi)^p}\min\{v,\,c\varpi\}} v^{-\alpha}\,\mathrm{d}v$$

$$+\; \frac{\zeta_2}{(c\varpi)^p}\int_0^{c\varpi} e^{-\beta v} e^{-\frac{\zeta_1}{\varpi^p}\min\{v,\,\varpi\}-\frac{\zeta_2}{(c\varpi)^p}\min\{v,\,c\varpi\}} v^{-\alpha}\,\mathrm{d}v$$

$$= \zeta_1\varpi^{-\alpha p}\int_0^{\varpi^{1-p}} e^{-\beta\varpi^p z} e^{-\left(\zeta_1+\frac{\zeta_2}{c^p}\right)z} z^{-\alpha}\,\mathrm{d}z \;+\; \zeta_2 c^{-p}\varpi^{-\alpha p}\int_0^{c\varpi^{1-p}} e^{-\beta\varpi^p z} e^{-\left(\zeta_1+\frac{\zeta_2}{c^p}\right)z} z^{-\alpha}\,\mathrm{d}z.$$

Taking $\varpi \to 0$ of (6.11) gives,

$$\begin{cases} 0, & p \in (0,1-\alpha); \\ \dfrac{\zeta_1+\zeta_2}{1-\alpha}, & p = 1-\alpha; \\ \infty, & p \in (1-\alpha,\infty). \end{cases}$$

Putting them together yields the results. □

It is not surprising to notice that the rescaled process $(c\varpi)^{-p}Z_t(c\varpi)$ tends to behave exactly as $\varpi^{-p}Z_t(\varpi)$ in the sense of weak convergence. That means, by choosing $\zeta_1 + \zeta_2 = 0$, it will holds for any $c \in (\,0, \infty\,)$ that

$$\lim_{\varpi\to 0} \varpi^{\alpha-1}Z_t(\varpi) \overset{\mathfrak{D}}{=} \lim_{\varpi\to 0}(c\varpi)^{\alpha-1}Z_t(c\varpi).$$

In relation to the main objective, due to the scaling property of Z_t, this result will contribute in deriving the law of the time-scaled local time and associated properties.

6.4 Scaling Property for Local Times

By a *time scaled local time* process, we mean the quantity $L_{\lambda t}$ defined via

$$\frac{L_{\lambda t}}{\Gamma(2-\alpha)} = \lim_{\varpi\to 0} \varpi^{\alpha-1}\int_0^{\lambda t} \mathbb{1}_{\{U_s\,<\,\varpi\}}\,\mathrm{d}s = \lim_{\varpi\to 0} \varpi^{\alpha-1}Z_{\lambda t}(\varpi)\,,$$

where $\lambda > 1$ is a scaler on time t. This section puts emphasis on the relation between $L_{\lambda t}$ and L_t which enables to determine the self-similarity of the local time, and on the limit theorems associated with the difference between them.

Theorem 6.4.1. *Let L_t be the local time admitting the representation of* (6.3)*. With $\lambda \in (1, \infty)$, it can be shown that inheriting from the scaling property of U the following two identities hold for all $t \geq 0$:*

$$\Gamma(2-\alpha)\lambda\varpi^{\alpha-1}Z_t\left(\lambda^{-1}\varpi\right) \overset{\mathfrak{D}}{\to} L_{\lambda t}, \qquad \textit{as} \qquad \varpi \to 0\,; \tag{6.12}$$

$$\lambda^{-\alpha}L_{\lambda t} \overset{\mathfrak{D}}{=} L_t. \tag{6.13}$$

These then prove $\{L_t\}$ a ***self-similar*** *process with index $\alpha \in (0, 1)$.*

Proof. As shown in (6.7), the occupation time Z_t is scaling invariant. That is to say,

$$\varpi^{\alpha-1}Z_{\lambda t}(\varpi) \overset{\text{law}}{=} \lambda\varpi^{\alpha-1}Z_t\left(\lambda^{-1}\varpi\right).$$

Taking limits on both sides yields (6.12). It then follows immediately

$$\frac{L_t}{\Gamma(2-\alpha)} \stackrel{\mathcal{D}}{=} \lim_{\varpi\to 0} \varpi^{\alpha-1} Z_t(\varpi) \stackrel{\mathcal{D}}{=} \lim_{\varpi\to 0} \lambda^{1-\alpha}\varpi^{\alpha-1} Z_t\Big(\lambda^{-1}\varpi\Big) \stackrel{\mathcal{D}}{=} \frac{\lambda^{-\alpha}L_{\lambda t}}{\Gamma(2-\alpha)} \,.$$

□

Following the scaling invariant relation between $L_{\lambda t}$ and L_t, the attention is then turned to the stochastic analysis of their difference denoted by $W_t(\varpi)$. We define the stochastic process describing the difference scaled by a factor ϖ^q as

$$\varpi^{-q} W_t(\varpi) \;=\; \frac{\Gamma(2-\alpha)}{\varpi^{1-\alpha+q}}\bigg(\, Z_t(\varpi) \;-\; \lambda^{1-\alpha} Z_t\Big(\lambda^{-1}\varpi\Big)\bigg), \qquad q \;\geq\; 0. \tag{6.14}$$

The distributional characteristics of $\{(L_t - \lambda^{-\alpha}L_{\lambda t}),\ t \geq 0\}$ are then derived from the Markov process $\big(U_t, Z_t(\varpi), Z_t\left(\lambda^{-1}\varpi\right)\big)$, whose limiting behaviours corresponding to needs are studied in SECTION 6.3.

Corollary 6.4.2. *The double Laplace transform of $\varpi^{-q}W_t(\varpi)$, denoted by $\mathcal{L}^{\beta,\vartheta}\{\varpi^{-q}W_t(\varpi)\}$, is obtained as*

$$\mathcal{L}^{\beta,\vartheta}\{\, \varpi^{-q}W_t(\varpi)\,\} \;=\; \frac{\displaystyle\int_0^\infty e^{-\beta v} v^{-\alpha} \tilde{e}(v,\varpi)\,\mathrm{d}v}{\displaystyle\int_0^\infty \bigg(\beta + \tilde{\vartheta}(\varpi)\Big(\mathbb{1}_{\{v\,<\,\varpi\}} - \lambda^{1-\alpha}\mathbb{1}_{\{v\,<\,\lambda^{-1}\varpi\}}\Big)\bigg)\, e^{-\beta v} v^{-\alpha}\tilde{e}(v,\varpi)\,\mathrm{d}v},$$

where

$$\tilde{e}(v,\varpi) \;=\; \exp\bigg\{-\tilde{\vartheta}(\varpi)\bigg(\min\{v,\ \varpi\} - \lambda^{1-\alpha}\min\big\{v,\ \lambda^{-1}\varpi\big\}\bigg)\bigg\}\;;\textit{and}$$

$$\tilde{\vartheta}(\varpi) \;=\; \frac{\Gamma(2-\alpha)}{\varpi^{1-\alpha+q}}\vartheta\;.$$

Proof. This is simply obtained by substituting $\zeta_1 = \frac{\Gamma(2-\alpha)}{\varpi^q}\vartheta$ and $\zeta_2 = -\zeta_1$ to the transform by (6.8). □

The limiting behaviour of $\mathcal{L}^{\beta,\vartheta}\{\varpi^{-q}W_t(\varpi)\}$ as $\varpi \to 0$ is presented below.

Corollary 6.4.3. *For every* $0 < \alpha < 1$ *and* $q > 0$*, it holds that*

$$\lim_{\varpi\to 0} \mathcal{L}^{\beta,\vartheta}\{\varpi^{-q} W_t(\varpi)\} = \mathcal{L}^{\beta,\vartheta}\Big\{ \lim_{\varpi\to 0} \varpi^{-q} W_t(\varpi) \Big\};$$

and in particular for $\boldsymbol{q = \frac{\alpha}{2}}$*,*

$$\mathcal{L}^{\beta,\vartheta}\Big\{ \lim_{\varpi\to 0} \varpi^{-\frac{\alpha}{2}} W_t(\varpi) \Big\} = \left(\beta - \left(1-\lambda^{-1}\right)\frac{(1-\alpha)\Gamma(2-\alpha)}{(2-\alpha)}\vartheta^2\beta^{1-\alpha}\right)^{-1}. \tag{6.15}$$

Proof. First considering the numerator,

$$\int_0^\infty e^{-\beta v} \exp\Big\{-\tilde{\vartheta}(\varpi)\Big(\min\{v,\ \varpi\} - \lambda^{1-\alpha}\min\{v,\ \lambda^{-1}\varpi\}\Big)\Big\} v^{-\alpha}\,\mathrm{d}v$$

$$= \int_0^{\lambda^{-1}\varpi} e^{-\beta v} e^{\left(\lambda^{1-\alpha}-1\right)\tilde{\vartheta}(\varpi)v} v^{-\alpha}\,\mathrm{d}v + \frac{e^{\lambda^{-\alpha}\tilde{\vartheta}(\varpi)\varpi} \displaystyle\int_{\lambda^{-1}\varpi^{\alpha-q}}^{\varpi^{\alpha-q}} e^{-\left(\beta\varpi^{1-\alpha+q} + \vartheta\Gamma(2-\alpha)\right)z} z^{-\alpha}\,\mathrm{d}z}{\varpi^{(\alpha-q-1)(1-\alpha)}}.$$

Given $0 < \alpha < 1$ and $\lambda > 1$, the first integral is equal to zero by taking $\varpi \to 0$ for all value of $q > 0$. The rest is calculated as follows

$$\lim_{\varpi\to 0} \frac{e^{\lambda^{-\alpha}\tilde{\vartheta}(\varpi)\varpi} \displaystyle\int_{\lambda^{-1}\varpi^{\alpha-q}}^{\varpi^{\alpha-q}} e^{-\left(\beta\varpi^{1-\alpha+q} + \vartheta\Gamma(2-\alpha)\right)z} z^{-\alpha}\,\mathrm{d}z}{\varpi^{(\alpha-q-1)(1-\alpha)}}$$

$$= \begin{cases} \dfrac{\Gamma(1-\alpha)}{\beta^{1-\alpha}}, & q \in (0,\alpha); \\ \dfrac{\Gamma(1-\alpha)}{\beta^{1-\alpha}} e^{-(1-\lambda^{-\alpha})\Gamma(2-\alpha)\vartheta}, & q = \alpha; \\ 0, & q \in (\alpha,\infty). \end{cases}$$

Then consider the denominator,

$$\tilde{\vartheta}(\varpi)\int_0^\infty \left(\mathbb{1}_{\{v < \varpi\}} - \lambda^{1-\alpha}\mathbb{1}_{\{v < \lambda^{-1}\varpi\}}\right) e^{-\beta v} e^{-\tilde{\vartheta}(\varpi)\left(\min\{v,\ \varpi\} - \lambda^{1-\alpha}\min\{v,\ \lambda^{-1}\varpi\}\right)} v^{-\alpha}\,\mathrm{d}v$$

$$= \left(1-\lambda^{1-\alpha}\right)\tilde{\vartheta}(\varpi) \int_0^{\lambda^{-1}\varpi} e^{-\beta v} e^{-\left(1-\lambda^{1-\alpha}\right)\tilde{\vartheta}(\varpi)v} v^{-\alpha}\,\mathrm{d}v$$

$$+ \left(1-\lambda^{1-\alpha}\right)\tilde{\vartheta}(\varpi)e^{\lambda^{-\alpha}\tilde{\vartheta}(\varpi)\varpi}\int_{\lambda^{-1}\varpi}^{\varpi} e^{-\beta v}e^{-\tilde{\vartheta}(\varpi)v}v^{-\alpha}\,\mathrm{d}v$$

$$= \frac{\Gamma(2-\alpha)\vartheta}{\varpi^{\alpha(1-\alpha+q)}}\left[\left(1-\lambda^{1-\alpha}\right)\int_0^{\lambda^{-1}\varpi^{\alpha-q}} e^{-\beta\varpi^{1-\alpha+q}z}e^{-\left(1-\lambda^{1-\alpha}\right)\Gamma(2-\alpha)\vartheta z}z^{-\alpha}\,\mathrm{d}z\right.$$

$$\left. + e^{\lambda^{-\alpha}\tilde{\vartheta}(\varpi)\varpi}\int_{\lambda^{-1}\varpi^{\alpha-q}}^{\varpi^{\alpha-q}} e^{-\beta\varpi^{1-\alpha+q}z}e^{-\Gamma(2-\alpha)\vartheta z}z^{-\alpha}\,\mathrm{d}z\right].$$

Assuming $q < \alpha$, it is easy to verify that

$$\lim_{\varpi\to 0}\frac{\left(1-\lambda^{1-\alpha}\right)\Gamma(2-\alpha)\vartheta}{\varpi^{\alpha(1-\alpha+q)}}\int_0^{\lambda^{-1}\varpi^{\alpha-q}} e^{-\beta\varpi^{1-\alpha+q}z}e^{-\left(1-\lambda^{1-\alpha}\right)\Gamma(2-\alpha)\vartheta z}z^{-\alpha}\,\mathrm{d}z = 0.$$

Applying the L'Hôpital's Rule to the integral in the above term,

$$\frac{(\alpha-q)\Gamma(2-\alpha)\vartheta}{\alpha(1-\alpha+q)}\lim_{\varpi\to 0}\left\{\left(\lambda^{\alpha-1}-1\right)\frac{e^{(1-\lambda^{-1})\beta\varpi}e^{-\lambda^{-1}\Gamma(2-\alpha)\vartheta\varpi^{\alpha-q}}}{\varpi^{q}} - \right.$$

$$\left.\lambda^{\alpha-1}\frac{e^{-\lambda^{-1}\Gamma(2-\alpha)\vartheta\varpi^{\alpha-q}} - e^{-\Gamma(2-\alpha)\vartheta\varpi^{\alpha-q}}}{\varpi^{q}}\right\}$$

$$= \begin{cases} 0, & q \in \left(0, \frac{\alpha}{2}\right); \\ -\frac{1-\lambda^{-1}}{2-\alpha}\left(\Gamma(2-\alpha)\vartheta\right)^2, & q = \frac{\alpha}{2}; \\ -\infty, & q \in \left(\frac{\alpha}{2}, \infty\right). \end{cases}$$

Putting them together gives immediately the results. □

In the last chapter, we show that the difference of the local time L_t to the limit process converging to it follows a Gaussian process subordinated by L_t. Concerning $\{L_t\}$ a self-similar process, we present a similar result with the rescaled local time $\left\{L_{\lambda_t}\right\}$.

Theorem 6.4.4. *Given $W_t(\varpi)$ defined by (6.14), we shown that the limiting difference of $(\ L_t - \lambda^{-\alpha}L_{\lambda t}\ )$ can be described by*

$$V_t^{\lambda} := \lim_{\varpi\to 0}\varpi^{-\frac{\alpha}{2}}W_t(\varpi), \qquad t > 0\ .$$

By central limit theorem, for every $\lambda > 1$,

$$V_t^{\lambda} \;\sim\; \mathcal{N}\Big(\, 0, \sigma_\alpha^2(\lambda) L_t \,\Big)\,, \tag{6.16}$$

where $\mathcal{N}$ represents a normal distribution and

$$\sigma_\alpha^2(\lambda) \;=\; \frac{2\big(1-\lambda^{-1}\big)(1-\alpha)\Gamma(2-\alpha)}{2-\alpha}\,. \tag{6.17}$$

Proof. By inverting the Laplace transform in (6.15), we have

$$\mathcal{L}^{\beta,\vartheta}\Big\{\lim_{\varpi\to 0}\varpi^{-\frac{\alpha}{2}}W_t(\varpi)\Big\} \;=\; -\int_0^\infty \frac{\partial}{\partial\beta}\big\{e^{-\beta^\alpha y}\big\}\;\exp\left\{\frac{\vartheta^2\sigma_\alpha^2(\lambda)y}{2}\right\}\frac{\mathrm{d}y}{\alpha y}\,.$$

Recall that $e^{-\beta^\alpha y}$ characterises a one-sided stable r.v. who admits an integral representation, and the exponential term involving ϑ^2 can be written into the moment generating function of a normal distribution with mean zero and variance described by $\sigma_\alpha^2(\lambda)L_t$. Accordingly, we have

$$\mathcal{L}^{\beta,\vartheta}\big\{\,V_t^\lambda\,\big\} = \frac{1}{(1-\alpha)\sigma_\alpha(\lambda)\sqrt{2\pi^3}} \times$$
$$\int_0^\infty e^{-\beta t}\int_{-\infty}^\infty e^{-\vartheta v}\int_0^\infty y^{\frac{1}{1-\alpha}-\frac{3}{2}}t^{-\frac{\alpha}{1-\alpha}}e^{-\frac{v^2}{2y\sigma_\alpha^2(\lambda)}}\int_0^\pi A(x)e^{-A(x)y^{\frac{1}{1-\alpha}}t^{-\frac{\alpha}{1-\alpha}}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}v\,\mathrm{d}t\,,$$

with

$$A(x) \overset{\text{def}}{=\!=} \left\{\frac{(\sin(\alpha x))^\alpha(\sin((1-\alpha)x))^{1-\alpha}}{\sin(x)}\right\}^{\frac{1}{1-\alpha}}.$$

We thus obtain the density of V_t^λ. It is expressed in the form of a normal distribution subordinated by L_t acting on the part of variance, i.e.

$$\mathbb{P}\Big(V_t^\lambda \in\, \mathrm{d}v\Big) \;=\; \int_0^\infty \frac{1}{\sqrt{2\pi\sigma_\alpha^2(\lambda)y}}\,e^{-\frac{v^2}{2y\sigma_\alpha^2(\lambda)}}\cdot\mathbb{P}(L_t \in\, \mathrm{d}y)\,\mathrm{d}v.$$

The proof is completed by this representation. □

7

Local Times related to Brownian Motion

With respect to a reflected Bessel process, approximations for the associated local time are established from various paths describing the trajectories of the underlying process. In previous study, the whole work is formulated with an age process recording the time the Bessel process has been away from last zero. Making extensive use of martingale approach, we construct two versions arising as limiting processes of the occupation time and the jump counting for approximating the local time at zero. By means of path analysis, some results derived within the Bessel structure generalise easily to provide similar results for the other diffusions [87], e.g. the Brownian local time at zero.

The purpose of this chapter is two-fold. First, we provide elementary proof for some celebrated results on Brownian local times. Second, comparing with the local time constructed by the age process, we shall develop relations between different choices of local times for Bessel path.

7.1 Definitions and Notations

We are concerned with a reflected process arising from setting the index $\alpha = ½$ of a $(2 - 2\alpha)$-dimensional Bessel process with $0 < \alpha < 1$. Denote it by $Y = \{Y_t,\ t \geq 0\}$ with $Y_0 = 0$, it has a non-negative Brownian path reflected instantaneously at zero, and thus $Y_t = |B_t|$ where B_t is a standard Brownian motion (SBM). On a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, Y is an $\mathbb{R}^+ \cup \{0\}$-valued continuous process adapted to $\{\mathscr{F}_t\}_{t\geq 0}$.

For every Borel set $A \in \mathcal{B}(\mathbb{R}^+)$, the *occupation time* of Y by the Brownian path up to time $t \geq 0$ is defined by the Lebesgue measure

$$Z_t(A) \ := \ \text{measure}\{ \ 0 \ \leq \ s \ \leq \ t \mid Y_s \ \in \ A \ \} \ = \ \int_0^t \mathbb{1}_{\{Y_s \ \in \ A\}} \, \mathrm{d}s.$$

The resulting process $Z(A) = \{Z_t(A),\ t \geq 0\}$ is continuous, non-decreasing and adapted to $\{\mathscr{F}_t\}$, and it belongs to the family of additional functionals. Denoted by $L_t^B(s)$ the Brownian local time at level s up to time t, it intrinsically serves as the density with respect to Lebesgue measure for occupation times, representable of the form:

$$Z_t(A) \ = \ \int_A L_t^B(s) \, \mathrm{d}s.$$

Of special interest to us is the random measure $L_t^B(0)$, the local time at zero increases only on the zero set $\mathcal{Z} = \{ \ t \geq 0 \mid Y_t = 0 \ \}$, such that

$$\int_0^t f(Y_s) \, \mathrm{d}L_t^B(0) \ = \ f(0) L_t^B(0).$$

for every non-negative Borel bounded function f. In the sequel, we will write L^B simply for $L^B(0)$.

There are a couple of equivalent ways of constructing this measure. Among them, the following representations are found significantly related to our work (see [62, Chapter 6] for reference) ;

(1) $\mathbb{P}\left(\lim_{\epsilon \downarrow 0} \dfrac{1}{2\epsilon} \ \text{measure}\{ \ s < t \mid Y_s < \epsilon \ \} \ = \ L_t^+ \ , \ t \geq 0 \right) \ = \ 1$.

(2) $\mathbb{P}\left(\lim_{\epsilon \downarrow 0} \sqrt{2}\epsilon \ \boldsymbol{\nu_1}\Big((0, \infty) \times [\ 0, \epsilon) \Big) \ = \ L_t^+ \ , \ t \geq 0 \right) \ = \ 1$.

$*$ $\boldsymbol{\nu_1}$ = The number of times $\{Y_s; 0 \ \leq \ s \ \leq \ t\}$ crosses down from ϵ to 0.

(3) $\mathbb{P}\left(\lim_{\epsilon \downarrow 0} \sqrt{\dfrac{\pi}{4\epsilon}} \ \boldsymbol{\nu_2}\Big((0, t) \times [\ 0, \epsilon) \Big) \ = \ L_t^+ \ , \ t \geq 0 \right) \ = \ 1$.

$*$ $\boldsymbol{\nu_2}$ = Total duration of all excursion intervals away from the origin of individual duration less than ϵ, completed by $\{Y_s; 0 \ \leq \ s \ \leq \ t\}$.

Note: L_t^+ denotes the local time defined in the mentioned reference.

(1) is the remarkable Lévy's "Mesure du Voisinage", which initiates the subsequent study on the limiting behaviour in the neighbourhood of zero of Markov processes, for instance (2) and (3). (2) is the downcrossing theorem, conjectured by Lévy and proved by Chung and Durrett [22], Itô and

McKean [55], Kasahara [63] and Csáki et al. [25] with various methods. Results corresponding to (3) have been exploited by CHAPTER 5, in which $\boldsymbol{\nu_2}$ is measured as $\{s < t \mid U_s^Y < \epsilon\}$ with $U_t^Y = t - \sup\{ s < t \mid Y_t = 0 \}$ a Brownian age process. For an in-depth study on Brownian local times, please refer to Jeanblanc et al. [61, Chapter 4], Marcus and Rosen [78] and Itô and McKean [55].

Within the framework of martingale, Limit Theorem I & II are devoted to the refinements of some known results about Brownian local times, with a special emphasis on the reconstructions of the above representations(1) and (2).

In the sequel, to avoid confusions in presenting results we define a double Laplace transform for a real-valued random variable X_t by

$$\mathcal{L}^{\beta,\xi}\{X_t\} \;:=\; \mathbb{E}\left[\int_0^\infty e^{-\beta t} e^{-\xi X_t}\, \mathrm{d}t\right] \;=\; \int_0^\infty e^{-\beta t}\, \mathbb{E}\left[e^{-\xi X_t}\right] \mathrm{d}t,$$

where the last equality follows from Fubini's theorem. We further adopt the following notations

$$\mathcal{L}_t^{\beta}\{\varphi(t)\} \;=\; \int_{\mathbb{R}^+} e^{-\beta t}\, \varphi(t)\, \mathrm{d}t,$$

the Laplace transform of some function $\varphi(t)$ defined for all real numbers $t \geq 0$, and

$$\gamma(\alpha, z) \;=\; \int_0^z e^{-t}\, t^{\alpha-1}\, \mathrm{d}t;$$
$$\Gamma(\alpha, z) \;=\; \int_z^\infty e^{-t}\, t^{\alpha-1}\, \mathrm{d}t,$$

for the lower and upper incomplete gamma functions, and thus $\gamma(\alpha, z) + \Gamma(\alpha, z) \;=\; \Gamma(\alpha)$.

7.2 Limit Theorem I : Local Time as Limit of Occupation Time

The martingale approach is particularly helpful in dealing with approximation and convergence in distribution for Markov processes with continuous sample paths that can be characterised in terms of its infinitesimal generator.

To fix ideas, we consider a continuous function $f : \mathbb{R}^+ \times [0, t) \to \mathbb{R}^+$ arising as the solution to $\mathcal{A}_z f(y, z) = 0$ such that $f(Y_t, Z_t(\varpi))$ is a martingale. $\mathcal{A}_z$ is the infinitesimal generator characterising the Markov process $(Y_t, Z_t(\varpi))$, and obtained from computing

$$\mathcal{A}_z f(y,z) \;=\; \lim_{t\to 0} t^{-1}\mathbb{E}\Big[f\Big(Y_t, Z_t(\varpi)\Big)\Big].$$

The martingale properties are then of great help in investigating $f(Y_t, Z_t(\varpi))$ and its asymptotic behaviour as $\varpi \to 0$. Furthermore, this section develops some distributional properties and facts that are relevant to the occupation process, and preliminary to the limit theorem for the local time.

7.2.1 A Scaled Occupation Time Process

We first restrict ourselves to an occupation measure in the manner:

$$Z_t(\varpi) \;=\; \int_0^t \mathbb{1}_{\{Y_s \,<\, \varpi\}}\, \mathrm{d}s, \qquad \varpi \;>\; 0.$$

The infinitesimal generator of $(Y_t,\ Z_t(\varpi))$ acting on bounded functions $f \in C^2$ is obtained as

$$\mathcal{A}_z f(y,z) \;=\; \frac{1}{2}\frac{\partial^2 f}{\partial y^2} \;+\; \mathbb{1}_{\{y \,<\, \varpi\}}\frac{\partial f}{\partial z}\,. \tag{7.1}$$

Every suitable choice of f satisfying $\mathcal{A}_z f = 0$ is an $(\mathscr{F}_t)$-martingale for all $t \geq 0$.

Lemma 7.2.1. *Let $\beta, \zeta \in \mathbb{R}^+$ and $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous bounded function in C^2 . Then the process $f(Y_t, Z_t(\varpi))$ is a martingale of the following form*

$$\int_0^t e^{-\beta s} e^{-\zeta Z_s(\varpi)}\, \mathrm{d}s \;+\; e^{-\beta t} e^{-\zeta Z_t(\varpi)} h(Y_t),$$

where depending on the domain of Y_t, the function $h(y)$ has representations

$$h(y) \;=\; \begin{cases} \dfrac{1}{\beta+\zeta} \;+\; 2C_1 \cosh\Big(\, y\sqrt{2(\beta+\zeta)}\,\Big), & y \;\leq\; \varpi\;; \\ \dfrac{1}{\beta} \;+\; C_3 e^{-y\sqrt{2\beta}}, & y \;>\; \varpi\,. \end{cases} \tag{7.2}$$

C_1 and C_3 are defined via

$$2C_1 \;=\; \frac{\Big(\frac{1}{\beta} - \frac{1}{\beta+\zeta}\Big)\sqrt{\beta}}{\sqrt{\beta}\cosh\Big(\varpi\sqrt{2(\beta+\zeta)}\Big) + \sqrt{\beta+\zeta}\,\sinh\Big(\varpi\sqrt{2(\beta+\zeta)}\Big)}\;; \textit{and}$$

$$C_3 \;=\; -\,\frac{e^{\varpi\sqrt{2\beta}}\Big(\frac{1}{\beta}-\frac{1}{\beta+\zeta}\Big)\sqrt{\beta+\zeta}}{\sqrt{\beta+\zeta}+\sqrt{\beta}\coth\Big(\varpi\sqrt{2(\beta+\zeta)}\Big)}\,,$$

in which **sinh, cosh** *and* **coth** *are hyperbolic functions, such that*

$$\sinh(x) \;=\; \frac{e^x - e^{-x}}{2} \quad \& \quad \cosh(x) \;=\; \frac{e^x + e^{-x}}{2} \quad \& \quad \coth(x) \;=\; \frac{\cosh(x)}{\sinh(x)}\,.$$

Proof. Assuming that $f(y,z)$ is of an exponential form like this

$$\int_0^t e^{-\beta s} e^{-\zeta Z_s}\,\mathrm{d}s \;+\; e^{-\beta t} e^{-\zeta z} h(y)\,.$$

Expanding $f(y,z)$ with $W_t = \int_0^t e^{-\beta s} e^{-\zeta Z_s}\,\mathrm{d}s$ leads to

$$\mathcal{A} f(y,z,w,t) \;=\; \mathcal{A}_z f \;+\; \frac{\partial f}{\partial t} \;+\; e^{-\beta t} e^{-\zeta z}\frac{\partial f}{\partial w} \;=\; 0\,.$$

By substitution,

$$\begin{cases} h''(y) - 2(\beta+\zeta)h(y) + 2 \;=\; 0, & y \;<\; \varpi; \\ h''(y) - 2\beta h(y) + 2 \;=\; 0, & y \;>\; \varpi. \end{cases} \tag{7.3}$$

Referring to the Feynman-Kac Formula [61, p112], bounded and continuous solutions to (7.3) are given by

$$h(y) \;=\; \begin{cases} \dfrac{1}{\beta+\zeta} \;+\; C_1 e^{y\sqrt{2(\beta+\zeta)}} \;+\; C_2 e^{-y\sqrt{2(\beta+\zeta)}}, & 0 \;\le\; y \;<\; \varpi; \\ \dfrac{1}{\beta} \;+\; C_3 e^{-y\sqrt{2\beta}}, & y \;\ge\; \varpi. \end{cases}$$

To decide the values of $C_i,\ i = 1,2,3$, we rely on the continuity of h and h' at ϖ, the boundedness of $h(y)$ as $y < \infty$ and $h'(0) \;=\; 0$. These produce $C_1 = C_2$ and then

$$C_1\Big(e^{\varpi\sqrt{2(\beta+\zeta)}} + e^{-\varpi\sqrt{2(\beta+\zeta)}}\Big) \;-\; C_3 e^{-\varpi\sqrt{2\beta}} \;=\; \frac{1}{\beta} - \frac{1}{\beta+\zeta}\,;$$

$$C_1\sqrt{\beta+\zeta}\Big(e^{\varpi\sqrt{2(\beta+\zeta)}} - e^{-\varpi\sqrt{2(\beta+\zeta)}}\Big) \;+\; C_3\sqrt{\beta}e^{-\varpi\sqrt{2\beta}} \;=\; 0\,.$$

As a result, the unique solutions are completed with

$$
\begin{aligned}
C_1 &= \frac{\dfrac{1}{\beta} - \dfrac{1}{\beta+\zeta}}{\left(e^{\varpi\sqrt{2(\beta+\zeta)}} + e^{-\varpi\sqrt{2(\beta+\zeta)}}\right) + \left(e^{\varpi\sqrt{2(\beta+\zeta)}} - e^{-\varpi\sqrt{2(\beta+\zeta)}}\right)\dfrac{\sqrt{\beta+\zeta}}{\sqrt{\beta}}}\ ; \\
C_3 &= -C_1 e^{\varpi\sqrt{2\beta}}\left(e^{\varpi\sqrt{2(\beta+\zeta)}} - e^{-\varpi\sqrt{2(\beta+\zeta)}}\right)\frac{\sqrt{\beta+\zeta}}{\sqrt{\beta}}.
\end{aligned}
$$

□

A main contribution of the martingale process is to produce the characteristic function of $Z_t(\varpi)$.

Corollary 7.2.2. *Let $\beta, \zeta > 0$, the double Laplace transform of $Z_t(\varpi)$ is given by*

$$
\mathcal{L}^{\beta,\zeta}\{Z_t(\varpi)\} = \frac{1}{\beta+\zeta} + \zeta\, \mathcal{L}^{\beta,\zeta}\left\{ \mathbb{P}(Z_t(\varpi) \le z)\right\},
$$

where for all $0 < z < t$,

$$
\mathcal{L}^{\beta,\zeta}\left\{ \mathbb{P}(Z_t(\varpi) \le z)\right\} = \\
\left[\sqrt{\beta}(\beta+\zeta)\left(\sqrt{\beta}\cosh\left(\varpi\sqrt{2(\beta+\zeta)}\right) + \sqrt{\beta+\zeta}\sinh\left(\varpi\sqrt{2(\beta+\zeta)}\right)\right)\right]^{-1}. \tag{7.4}
$$

Proof. By the property of a martingale process, having $h(y)$ a bounded function on all support of y gives

$$
\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}\left[\int_0^t e^{-\beta s} e^{-\zeta Z_s(\varpi)}\, \mathrm{d}s + e^{-\beta t} e^{-\zeta Z_t(\varpi)} h(Y_t)\right] \\
&= \int_0^\infty e^{-\beta s}\, \mathbb{E}\left[e^{-\zeta Z_s(\varpi)}\right] \mathrm{d}s \\
&= h(0).
\end{aligned}
$$

On the other hand, given $Z_t(\varpi) \in (0,t)$ for any $\varpi > 0$,

$$
\int_0^\infty e^{-\beta t} \int_0^t e^{-\zeta z}\, \mathbb{P}(Z_t(\varpi) \in \mathrm{d}z)\, \mathrm{d}t
$$

$$= \int_0^\infty e^{-\beta t}\Big(e^{-\zeta t} + \zeta \int_0^t e^{-\zeta z}\, \mathbb{P}(Z_t(\varpi) \le z)\, \mathrm{d}z \Big)\, \mathrm{d}t$$

$$= \frac{1}{\beta+\zeta} + \zeta \int_0^\infty e^{-\beta t} \int_0^t e^{-\zeta z}\, \mathbb{P}(Z_t(\varpi) \le z)\, \mathrm{d}z\, \mathrm{d}t.$$

This completes the proof. □

By means of inverse Laplace transform, the law of $Z_t(\varpi)$ is obtained in an explicit form.

Theorem 7.2.3. *Given $\varpi > 0$, we define $\psi(z,t;\varpi)$ by*

$$\mathbb{P}(Z_t(\varpi) \in \mathrm{d}z) = \psi(z,t\,;\varpi)\, \mathrm{d}z\,.$$

Then for all $\zeta > 0$ and $0 < z < t$,

$$\int_0^t e^{-\zeta z} \int_0^z \psi(x,t;\varpi)\, \mathrm{d}x\, \mathrm{d}z = \frac{4\varpi}{\pi^2} \int_0^t e^{-\zeta z} \int_0^z \psi_1(x,t)\, \psi_2(z-x,t)\, \mathrm{d}x\, \mathrm{d}z\,, \tag{7.5}$$

with

$$\psi_1(x,t) = \int_0^x \Big((t-x)^{-\frac{1}{2}} - (t-v)^{-\frac{1}{2}}\Big) v^{-\frac{3}{2}} e^{-\frac{\varpi^2}{2v}}\, \mathrm{d}v\,,$$

$$\psi_2(x,t) = \int_0^x \Big[(t-x)(x-v)\Big]^{-\frac{1}{2}} (t-v)^{-1} v^{-\frac{3}{2}} \sum_{k=0}^{\infty} k^2 e^{-\frac{2k^2\varpi^2}{v}} \cos\Big(k \cos^{-1}\Big(\frac{2x-t-v}{t-v}\Big)\Big)\, \mathrm{d}v\,,$$

and thus

$$\psi(z,t\,;\varpi) = \frac{4\varpi}{\pi^2} \int_0^z \psi_2(z-x,t)(t-x)^{-\frac{3}{2}} \Big(1 - \Phi\Big[\frac{\varpi}{\sqrt{x}}\Big]\Big)\, \mathrm{d}x\,, \tag{7.6}$$

where $\Phi[x]$ is the CDF *of a standard normal distribution.*

Proof. To invert (7.4), we write it into

$$\frac{2}{(\beta+\zeta)\sqrt{\beta}} \frac{1}{\Big(\sqrt{\beta+\zeta}+\sqrt{\beta}\Big) e^{\varpi\sqrt{2(\beta+\zeta)}} - \zeta \dfrac{e^{-\varpi\sqrt{2(\beta+\zeta)}}}{\Big(\sqrt{\beta+\zeta}+\sqrt{\beta}\Big)}}$$

$$= \frac{2}{(\beta+\zeta)\sqrt{\beta}} \frac{e^{-\varpi\sqrt{2(\beta+\zeta)}}}{\left(\sqrt{\beta+\zeta}+\sqrt{\beta}\right)} \frac{1}{1-\dfrac{\zeta e^{-2\varpi\sqrt{2(\beta+\zeta)}}}{\left(\sqrt{\beta+\zeta}+\sqrt{\beta}\right)^2}}$$

$$= \frac{2}{(\beta+\zeta)} \frac{e^{-\varpi\sqrt{2(\beta+\zeta)}}}{\sqrt{\beta}\left(\sqrt{\beta+\zeta}+\sqrt{\beta}\right)} \sum_{k=0}^{\infty} \frac{\zeta^k}{\left(\sqrt{\beta+\zeta}+\sqrt{\beta}\right)^{2k}} e^{-2k\varpi\sqrt{2(\beta+\zeta)}}.$$

First the part before Σ can be expressed as some proper transform,

$$\frac{2}{(\beta+\zeta)} \frac{e^{-\varpi\sqrt{2(\beta+\zeta)}}}{\sqrt{\beta}\left(\sqrt{\beta+\zeta}+\sqrt{\beta}\right)}$$

$$= \frac{\varpi}{\pi} \int_0^{\infty} e^{-\beta v} e^{-\zeta v} \int_0^v x^{-\frac{3}{2}} e^{-\frac{\varpi^2}{2x}}\, \mathrm{d}x\, \mathrm{d}v \int_0^{\infty} e^{-\beta t} t^{-\frac{3}{2}} \frac{1-e^{-\zeta t}}{\zeta}\, \mathrm{d}t$$

$$= \frac{\varpi}{\pi} \int_0^{\infty} e^{-\beta t} \int_0^t (t-v)^{-\frac{3}{2}} \frac{e^{-\zeta v}-e^{-\zeta t}}{\zeta} \int_0^v x^{-\frac{3}{2}} e^{-\frac{\varpi^2}{2x}}\, \mathrm{d}x\, \mathrm{d}v\, \mathrm{d}t$$

$$= \frac{2\varpi}{\pi} \int_0^{\infty} e^{-\beta t} \int_0^t e^{-\zeta z} \int_0^z \left((t-z)^{-\frac{1}{2}} - (t-x)^{-\frac{1}{2}}\right) x^{-\frac{3}{2}} e^{-\frac{\varpi^2}{2x}}\, \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}t.$$

With reference to Bateman [3], it can be checked that

$$\frac{\zeta^k}{\left(\sqrt{\beta+\zeta}+\sqrt{\beta}\right)^{2k}} = k \int_0^{\infty} e^{-\beta t}\, t^{-1} e^{-\frac{1}{2}\zeta t} I_k\left(\frac{\zeta t}{2}\right) \mathrm{d}t$$

$$e^{-\zeta t} I_k(\zeta t) = \frac{1}{\pi} \int_0^{2t} e^{-\zeta x} \left((2t-x)x\right)^{-\frac{1}{2}} \cos\left(k \cos^{-1}\left(\frac{x-t}{t}\right)\right) \mathrm{d}x.$$

Then the Σ part is calculated to

$$\sum_{k=0}^{\infty} \frac{\zeta^k}{\left(\sqrt{\beta+\zeta}+\sqrt{\beta}\right)^{2k}} e^{-2k\varpi\sqrt{2(\beta+\zeta)}}$$

$$= \frac{\sqrt{2}\varpi}{\sqrt{\pi}} \sum_{k=0}^{\infty} k^2 \int_0^{\infty} e^{-\beta t}\, t^{-1} e^{-\frac{1}{2}\zeta t} I_k\left(\frac{1}{2}\zeta t\right) \mathrm{d}t \int_0^{\infty} e^{-\beta x}\, e^{-\zeta x} x^{-\frac{3}{2}} e^{-\frac{2k^2\varpi^2}{x}}\, \mathrm{d}x$$

$$= \frac{\sqrt{2}\varpi}{\pi\sqrt{\pi}} \sum_{k=0}^{\infty} k^2 \int_0^{\infty} e^{-\beta v} \int_0^v (v-t)^{-\frac{3}{2}} e^{-\frac{2k^2\varpi^2}{v-t}} t^{-1} \quad \times$$

$$e^{-\zeta(v-t)} \int_0^t e^{-\zeta x} \left[\, (t-x)x\, \right]^{-\frac{1}{2}} \cos\left(k \cos^{-1}\left(\frac{2x-t}{t}\right)\right) \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}v$$

$$= \frac{\sqrt{2}\varpi}{\pi\sqrt{\pi}} \int_0^\infty e^{-\beta v} \int_0^v e^{-\zeta x} \int_0^x (v-t)^{-1} t^{-\frac{3}{2}} [\,(v-x)(x-t)\,]^{-\frac{1}{2}} \quad \times$$

$$\sum_{k=0}^{\infty} k^2 e^{-\frac{2k^2\varpi^2}{t}} \cos\left(k \cos^{-1}\left(\frac{2x-v-t}{v-t}\right)\right) \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}v.$$

Combining two parts together gives (7.5). Further due to the convolution theorem,

$$\int_0^z \psi(x,t;\varpi)\,\mathrm{d}x \;=$$

$$\frac{4\varpi}{\pi^2} \int_0^z \int_0^x \frac{\varpi}{\sqrt{2\pi}} s^{-\frac{3}{2}} e^{-\frac{\varpi^2}{2s}} \left((t-z+x-s)^{-\frac{1}{2}} - (t-s)^{-\frac{1}{2}}\right) \psi_2(x-s,t)\,\mathrm{d}s\,\mathrm{d}x$$

and (7.6) follows immediately by differentiating with z,

$$\psi(z,t\,;\varpi) \;=\; \frac{2\varpi}{\pi^2} \int_0^z \psi_2(z-x,t)(t-x)^{-\frac{3}{2}} \int_0^x \frac{\varpi}{\sqrt{2\pi}} w^{-\frac{3}{2}} e^{-\frac{\varpi^2}{2w}}\,\mathrm{d}w\,\mathrm{d}x$$

$$= \frac{4\varpi}{\pi^2} \int_0^z \psi_2(z-x,t)(t-x)^{-\frac{3}{2}} \left(1 - \Phi\left[\frac{\varpi}{\sqrt{x}}\right]\right) \mathrm{d}x\,.$$

□

7.2.2 Convergence of Occupation Time to the Brownian Local Time

An important application on the convergence of $Z_t(\varpi)$ to the Brownian local time is discussed. To perform calculations, we scale the occupation measure by a factor ϖ^{-p} with p taking any non-negative values. To get a better understanding of the asymptotic behaviour of $\varpi^{-p} Z_t(\varpi)$ as $\varpi \to 0$, we take advantage of Lévy's continuity theorem, finding the limit of the corresponding characteristic function subject to a proper choice on p.

Lemma 7.2.4. *For all $\beta, \zeta > 0$ and $p > 0$, it holds that*

$$\lim_{\varpi \to 0} \mathcal{L}^{\beta,\zeta}\{\,\varpi^{-p} Z_t(\varpi)\,\} \;=\; \mathcal{L}^{\beta,\zeta}\left\{\,\lim_{\varpi\to 0}\,\varpi^{-p} Z_t(\varpi)\right\};$$

and in particular for $\boldsymbol{p = 1}$,

$$\mathcal{L}^{\beta,\zeta}\left\{\,\lim_{\varpi\to 0}\,\varpi^{-1} Z_t(\varpi)\right\} \;=\; \left(\beta + \sqrt{2\beta}\zeta\,\right)^{-1}. \tag{7.7}$$

Proof. Let us go back to the martingale process obtained in LEMMA 7.2.1. First we replace ζ with $\varpi^{-p}\zeta$. Restricted to the fact that $Y_t \geq \varpi > 0$ as a result of considering $\varpi \to 0^+$, we thus have

$$\lim_{\varpi\to 0} \mathcal{L}^{\beta,\zeta}\{\, \varpi^{-p} Z_t(\varpi)\,\} \;=\; \frac{1}{\beta} \;+\; \lim_{\varpi\to 0} C_3(\varpi),$$

and

$$C_3(\varpi) \;=\; -\,\frac{\sqrt{\beta + \frac{\zeta}{\varpi^p}}\left(\frac{1}{\beta} - \frac{1}{\beta+\frac{\zeta}{\varpi^p}}\right)e^{\varpi\sqrt{2\beta}}}{\sqrt{\beta + \frac{\zeta}{\varpi^p}} + \sqrt{\beta}\coth\left(\,\varpi\sqrt{2\left(\beta + \frac{\zeta}{\varpi^p}\right)}\,\right)}\,.$$

To evaluate the limit of C_3 when taking $\varpi \to 0$, we rewrite it into the form

$$\begin{aligned}
&e^{-\varpi\sqrt{2\beta}}C_3(\varpi) \;=\\
&\left(1 + \frac{\sqrt{\beta}}{\sqrt{\beta + \frac{\zeta}{\varpi^p}}}\right)\left(\frac{\frac{2}{\sqrt{\beta}}}{\sqrt{\beta + \frac{\zeta}{\varpi^p}} + \sqrt{\beta} - \left(\sqrt{\beta + \frac{\zeta}{\varpi^p}} - \sqrt{\beta}\right)e^{-2\varpi\sqrt{2\left(\beta+\frac{\zeta}{\varpi^p}\right)}}} \;-\; \frac{1}{\beta}\right).
\end{aligned}$$

First to notice is the first bracket that is equal to 1 as taking ϖ to zero with any $p > 0$. Considering the denominator in the other bracket, it reaches some limit by choosing $p = 1$ with the aid of L'Hôpital's Rule . As a result, the limiting denominator is calculated to

$$\begin{aligned}
&\lim_{\varpi\to 0}\;\left(\sqrt{\beta + \frac{\zeta}{\varpi^p}} + \sqrt{\beta}\right) - \left(\sqrt{\beta + \frac{\zeta}{\varpi^p}} - \sqrt{\beta}\right)e^{-2\varpi\sqrt{2\left(\beta+\frac{\zeta}{\varpi^p}\right)}}\\
=\;&\lim_{\varpi\to 0}\;\left\{\;\sqrt{\beta}\left(\;1 + e^{-2\varpi\sqrt{2\left(\beta+\frac{\zeta}{\varpi}\right)}}\;\right) \;+\right.\\
&\qquad\left. 2\sqrt{2}\left(1 - \frac{\sqrt{\beta}}{\sqrt{\beta + \frac{\zeta}{\varpi}}}\right)\left(\;\frac{2\varpi^2}{\zeta}\left(\beta + \frac{\zeta}{\varpi}\right)^2 - \varpi\left(\beta + \frac{\zeta}{\varpi}\right)\right)\,e^{-2\varpi\sqrt{2\left(\beta+\frac{\zeta}{\varpi}\right)}}\;\right\}\\
=\;&2\sqrt{\beta} + 2\sqrt{2}\zeta.
\end{aligned}$$

Then the result follows. □

The asymptotics provide an approximation to the Brownian local time at zero, whose representation coincides with Lévy's "Mesure du Voisinage". Thereby, in the sense of equality in distribution, we recover two well-known results concerning Lévy's local time.

Theorem 7.2.5. *Let L_t^B denote the Lebesgue measure of the time spent by Y at zero up to time t. By the continuity theorem, there is a limit relation holding for all $t > 0$ such that*

$$\varpi^{-1} Z_t(\varpi) \overset{\mathcal{D}}{\to} 2L_t^B, \qquad \text{as } \varpi \to 0 . \tag{7.8}$$

Furthermore, the following identity holds in law

$$\left\{ Y_t,\ L_t^B,\ t \geq 0 \right\} \overset{\text{law}}{=} \{ M_t - B_t,\ M_t,\ t \geq 0 \}, \tag{7.9}$$

where $\{M_t,\ t \geq 0\}$ defined by $M_t := \max_{0 \leq s \leq t} B_s$ is the running maximum before time t of a standard Brownian motion B_t.

Proof. The key to proving the convergence and to characterising the distribution of the corresponding limit process is inverting the Laplace transform in (7.7). We proceed with an integral representation of

$$\begin{aligned}
\mathcal{L}^{\beta,\zeta}\left\{ \lim_{\varpi \to 0} \varpi^{-1} Z_t(\varpi) \right\} &= -2 \int_0^\infty \frac{\partial}{\partial \beta}\left\{ e^{-\sqrt{2\beta}x} \right\} e^{-\zeta' x} \frac{\mathrm{d}x}{x} \\
&= -2 \int_0^\infty \frac{\partial}{\partial \beta}\left\{ \int_0^\infty e^{-\beta t} \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}} \,\mathrm{d}t \right\} e^{-\zeta' x} \frac{\mathrm{d}x}{x} \\
&= \int_0^\infty e^{-\beta t} \int_0^\infty e^{-\zeta' x} \frac{2}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned}$$

with $\zeta' = 2\zeta$. Obviously, the integrand is the density function of a running maximum of BM, and thus the other identity in (7.9) follows immediately by virtue of the reflection principle. □

Underlying the identity is a famous result known as "Lévy's Identity in Law". It provides a representation connecting the local time of a reflected BM with the running maximum of a standard BM [89, Chap 3]. The latter process is strictly increasing with upward jumps only occurring in a Poisson-like manner. Motivated by the concept of Poisson random measure, we shall present, equivalent to the one in (7.8), an alternate measure in terms of downcrossings.

7.3 Limit Theorem II : Local Time as Limit of Downcrossings

Approximating local times with Poisson-like variables is familiar to us. Recalling the construction in CHAPTER 5.4, we introduce a point process counting the number of times that an age process jumps down to zero. By virtue of the association between the age and the underlying Bessel/Brownian processes, the jump counting is equal in quantity to the number of completed excursions away from zero by the underlying path. In other words, it is equal to the number of times that the underlying process crosses down from ε to zero. Following a similar approach to that dealing with age processes, the objective of this section is to recast the "Lévy's Downcrossing Theorem".

On the same space that Y_t is defined, let $M_t(\varepsilon)$ be a process counting the number of downcrossings suffered by Y over a time interval $(0, t)$. A precise definition of this measure is presented by

$$M_t(\varepsilon) \;=\; \sum_{i=1}^{\infty} \mathbb{1}_{\{\tau_i^{\varepsilon} \,\le\, t\}},$$

with

$$\tau_i^{\varepsilon} \;\coloneqq\; \inf\big\{\, s > \tau_{i-1}^{\varepsilon} \,\big|\, Y_s = 0 \,\big\}, \qquad \tau_0 = 0.$$

On the basis of it, we define a scaled measure of the form

$$\varepsilon^{\kappa} M_t(\varepsilon), \qquad \varepsilon > 0.$$

It is worth to mention that ε^{κ} with κ taking non-negative values is a speed measure for characterising the convergence as ε approaches zero, and it corresponds to the (scaled) unit increase at each occurrence of downcrossing by the BM .

Denoted by $(Y_t, M_t(\varepsilon))$ the concerned Markov process, the infinitesimal generator acting on a non-negative C^2-function $f : \mathbb{R}^+ \times \mathbb{N}^0 \to \mathbb{R}^+$ has the form

$$\mathcal{A}_m f(y,m) \;=\; \frac{1}{2}\frac{\partial^2 f}{\partial y^2}, \qquad \text{with } \; f(0,m) \;=\; f(\,\varepsilon,\; m + \varepsilon^{\kappa}\,)\,. \tag{7.10}$$

To examine the limit of this counting measure, we find its characteristic function derived from a proper martingale process for $(Y_t, M_t(\varepsilon))$.

Lemma 7.3.1. *The limiting behaviour of $\varepsilon^\kappa M_t(\varepsilon)$ is characterised by the Laplace function given by*

$$\lim_{\varepsilon\to 0} \mathcal{L}^{\beta,\xi}\{\, \varepsilon^\kappa M_t(\varepsilon)\,\} \;=\; \mathcal{L}^{\beta,\xi}\Big\{ \lim_{\varepsilon\to 0} \varepsilon^\kappa M_t(\varepsilon) \Big\}\,.$$

For all $\beta,\xi > 0$, the limit is assured with $\boldsymbol{\kappa = 1}$ and then

$$\mathcal{L}^{\beta,\xi}\Big\{ \lim_{\varepsilon\to 0} \varepsilon M_t(\varepsilon) \Big\} \;=\; \left(\beta + \frac{\xi\sqrt{\beta}}{\sqrt{2}}\right)^{-1}. \tag{7.11}$$

Proof. The martingale of interest is implied by the solution to $\mathcal{A} f(y,m,w,t) = 0$, with f in the form

$$w + e^{-\beta t} e^{-\xi m} h(y).$$

Referring to LEMMA 7.2.1, a bounded and continuous solution should have the form

$$h(y) = \frac{1}{\beta} + Ce^{-y\sqrt{2\beta}}, \qquad y > 0\,.$$

The constant C is determined by the boundary condition: $h(0) = e^{-\xi\varepsilon^\kappa} h(\varepsilon)$, which generates

$$\frac{1}{\beta} + C \;=\; e^{-\xi\varepsilon^\kappa}\left(\frac{1}{\beta} + Ce^{-\varepsilon\sqrt{2\beta}}\right).$$

C is thus determined as

$$C \;=\; -\,\frac{1 - e^{-\xi\varepsilon}}{\beta\Big(1 - e^{-(\xi+\sqrt{2\beta})\varepsilon}\Big)}\,.$$

This proof is completed by taking ε to zero. □

Theorem 7.3.2. *The local time at zero of a reflected Brownian motion satisfies,*

$$\varepsilon M_t(\varepsilon) \overset{\mathcal{D}}{\to} L_t^B, \qquad as \quad \varepsilon \;\to\; 0\,. \tag{7.12}$$

Proof. Comparing the characteristic function in (7.11) with that in (7.7), we simply get

$$\mathbb{E}\Big[\exp\Big\{-\xi \lim_{\varepsilon\to 0} \varepsilon M_t(\varepsilon)\Big\}\Big] \;=\; \mathbb{E}\big[\exp\{-\zeta' L_t^B\}\big]$$

for every $\xi = \zeta'$. By the continuity theorem this implies the identity between $M_t(\varepsilon)$ and L_t^B. □

7.4 Limit Theorem III : A Central Limit Theorem for Local Time

It is a well-known fact that there are various ways of approximating a Brownian local time at zero, generally by limiting a sequence of processes that describe the behaviour in the neighbourhood of zero of Brownian trajectories. Among such processes, the occupation time and the quantity of downcrossings are being selected to construct limit relations between the local time and themselves. On the basis of the results achieved, the asymptotics of the (normalised) difference between the local time and the approximations is studied in this section.

7.4.1 A Scaled Difference about Local Time

In regard to the convergence of stochastic processes to the Brownian local time at zero, we have constructed two limit relations, summarised as below

$$\lim_{\varpi\to 0}\ (2\varpi)^{-1}Z_t(\varpi)\ \overset{\mathcal{D}}{=}\ L_t^B\ \overset{\mathcal{D}}{=}\ \lim_{\varepsilon\to 0}\ \varepsilon M_t(\varepsilon), \qquad \forall\, t\geq 0.$$

Following the identities, we define a stochastic process describing the difference by

$$V_t(\varepsilon,\varpi)\ =\ \varpi^{-q}\left(\frac{Z_t(\varpi)}{2\varpi}-\varepsilon M_t(\varepsilon)\right), \qquad t\geq 0,$$

with q a scaler taking non-negative values. In particular by allowing $\varepsilon\to 0$,

$$V_t(\varpi)\ =\ \lim_{\varepsilon\to 0}V_t(\varepsilon,\varpi)\ =\ \varpi^{-q}\left(\frac{Z_t(\varpi)}{2\varpi}-L_t^B\right).$$

We are concerned with the asymptotic results of $V_t(\varepsilon,\varpi)$ by taking ε and ϖ to zero, normalized by ϖ^q for measuring the speed that makes the convergence takes place. To characterise the difference process, we begin with a martingale process around $(Y_t, Z_t(\varpi), M_t(\varepsilon))$ whose generator is simply obtained by merging (7.1) with (7.10). See below for this result.

Lemma 7.4.1. *Let $\beta,\zeta\in\mathbb{R}^+$ and $h:\mathbb{R}^+\to\mathbb{R}^+$ be a non-negative bounded function in C^2. Then the process $f\big(Y_t, Z_t(\varpi), L_t^B\big)$ is a martingale of the following form*

$$\int_0^t e^{-\beta s}e^{-\zeta Z_s(\varpi)}e^{-\xi L_s^B}\,\mathrm{d}s\ +\ e^{-\beta t}e^{-\zeta Z_t(\varpi)}e^{-\xi L_t^B}h(Y_t), \tag{7.13}$$

where depending on the domain of Y_t, the function $h(y)$ has representations

$$h(y) \;=\; \begin{cases} \dfrac{1}{\beta+\zeta} \;+\; C_1(\varpi)\, e^{y\sqrt{2(\beta+\zeta)}} \;+\; C_2(\varpi)\, e^{-y\sqrt{2(\beta+\zeta)}}, & 0 \;\le\; y \;<\; \varpi\;; \\ \dfrac{1}{\beta} \;+\; C_3(\varpi) e^{-y\sqrt{2\beta}}, & y \;\ge\; \varpi\;. \end{cases} \tag{7.14}$$

C_1, C_2 and C_3 are defined via

$$C_1(\varpi) \;=\; \frac{\sqrt{\beta}\Big(\frac{1}{\beta} - \frac{1}{\beta+\zeta}\Big)\Big(\sqrt{2(\beta+\zeta)} + \xi\Big) \;-\; \frac{\xi}{\beta+\zeta}\big(\sqrt{\beta+\zeta} - \sqrt{\beta}\big)e^{-\varpi\sqrt{2(\beta+\zeta)}}}{d^+_\varpi \;-\; d^-_\varpi}\;;$$

$$C_2(\varpi) \;=\; \frac{\sqrt{\beta}\Big(\frac{1}{\beta} - \frac{1}{\beta+\zeta}\Big)\Big(\sqrt{2(\beta+\zeta)} - \xi\Big) \;-\; \frac{\xi}{\beta+\zeta}\big(\sqrt{\beta+\zeta} + \sqrt{\beta}\big)e^{\varpi\sqrt{2(\beta+\zeta)}}}{d^+_\varpi \;-\; d^-_\varpi}\;;$$

$$C_3(\varpi)\, e^{-\varpi\sqrt{2\beta}} \;=\; \frac{2}{\sqrt{\beta(\beta+\zeta)}}\,\frac{d^+_\varpi \;-\; \xi\sqrt{\beta}}{d^+_\varpi \;-\; d^-_\varpi} - \left(\frac{1}{\beta} + \frac{1}{\sqrt{\beta(\beta+\zeta)}}\right),$$

with

$$d^-_\varpi \;=\; \Big(\sqrt{\beta+\zeta} - \sqrt{\beta}\Big)\Big(\sqrt{2(\beta+\zeta)} - \xi\Big)e^{-\varpi\sqrt{2(\beta+\zeta)}}\;; \tag{7.15}$$

$$d^+_\varpi \;=\; \Big(\sqrt{\beta+\zeta} + \sqrt{\beta}\Big)\Big(\sqrt{2(\beta+\zeta)} + \xi\Big)e^{\varpi\sqrt{2(\beta+\zeta)}}\;. \tag{7.16}$$

Proof. According to what we have structured, the martingale in (7.13) is equivalent to

$$\lim_{\varepsilon\to 0}\left\{\int_0^t e^{-\beta s} e^{-\zeta Z_s(\varpi)} e^{-\xi\varepsilon M_s(\varepsilon)}\,\mathrm{d}s \;+\; e^{-\beta t} e^{-\zeta Z_t(\varpi)} e^{-\xi\varepsilon M_t(\varepsilon)} h_\varepsilon(Y_t)\right\}.$$

This will be achieved by two procedures: finding the representation of $h_\varepsilon(Y_t)$ and then taking the limit of ε to zero.

Due to the complexity involved in calculations, this proof only includes several key results and leaves the detailed steps to **Appendix I**. We have found that depending on the domain of Y_t, the function $h_\varepsilon(Y_t)$ admits representations

$$h_\varepsilon(y) \;=\; \begin{cases} \dfrac{1}{\theta} \;+\; C_1(\varepsilon,\varpi)\, e^{y\sqrt{2\theta}} \;+\; C_2(\varepsilon,\varpi)\, e^{-y\sqrt{2\theta}}, & 0 \;\le\; y \;<\; \varpi\;; \\ \dfrac{1}{\beta} \;+\; C_3(\varepsilon,\varpi)\, e^{-y\sqrt{2\beta}}, & y \;\ge\; \varpi\,, \end{cases} \tag{7.17}$$

and with $\theta := \beta + \zeta$

$$C_1(\varepsilon,\varpi) \;=\; -\left(\frac{1}{\sqrt{\beta}} - \frac{1}{\sqrt{\theta}}\right)\frac{e^{-\varpi\sqrt{2\theta}}}{\sqrt{\theta}}\,\frac{\left(1-e^{-\xi\varepsilon}\right)\sqrt{\beta} \;-\; d^{+}_{\varepsilon,\varpi}}{d^{-}_{\varepsilon,\varpi} \;+\; d^{+}_{\varepsilon,\varpi}}\;;$$

$$C_2(\varepsilon,\varpi) \;=\; -\left(\frac{1}{\sqrt{\beta}} + \frac{1}{\sqrt{\theta}}\right)\frac{e^{\varpi\sqrt{2\theta}}}{\sqrt{\theta}}\,\frac{\left(1-e^{-\xi\varepsilon}\right)\sqrt{\beta} \;+\; d^{-}_{\varepsilon,\varpi}}{d^{-}_{\varepsilon,\varpi} \;+\; d^{+}_{\varepsilon,\varpi}}\;;$$

$$e^{-\varpi\sqrt{2\beta}}\,C_3(\varepsilon,\varpi) \;=\; -\,\frac{2}{\sqrt{\beta\theta}}\,\frac{\left(1-e^{-\xi\varepsilon}\right)\theta\sqrt{\beta} \;-\; d^{+}_{\varepsilon,\varpi}}{d^{-}_{\varepsilon,\varpi} \;+\; d^{+}_{\varepsilon,\varpi}} - \frac{1}{\beta}\left(1 + \frac{\sqrt{\beta}}{\sqrt{\theta}}\right),$$

in which

$$d^{-}_{\varepsilon,\varpi} \;=\; \left(\sqrt{\theta} - \sqrt{\beta}\right)\left(1 - e^{-\varepsilon\left(\xi - \sqrt{2\theta}\right)}\right)e^{-\varpi\sqrt{2\theta}}\;;$$

$$d^{+}_{\varepsilon,\varpi} \;=\; \left(\sqrt{\theta} + \sqrt{\beta}\right)\left(1 - e^{-\varepsilon\left(\xi + \sqrt{2\theta}\right)}\right)e^{\varpi\sqrt{2\theta}}\;.$$

We proceed with calculating the limit of $h_\varepsilon(y)$. Given the solutions of C_1, C_2 and C_3, we now take ε to zero and obtain

$$\begin{aligned} C_1(\varpi) \;&=\; \lim_{\varepsilon\to 0} C_1(\varepsilon,\varpi) \\ &=\; \frac{\sqrt{\beta}\left(\frac{1}{\beta} - \frac{1}{\theta}\right)\left(\sqrt{2\theta} + \xi\right) \;-\; \frac{\xi}{\theta}\left(\sqrt{\theta} - \sqrt{\beta}\right)e^{-\varpi\sqrt{2\theta}}}{\left(\sqrt{\theta} + \sqrt{\beta}\right)\left(\sqrt{2\theta} + \xi\right)e^{\varpi\sqrt{2\theta}} \;-\; \left(\sqrt{\theta} - \sqrt{\beta}\right)\left(\sqrt{2\theta} - \xi\right)e^{-\varpi\sqrt{2\theta}}}\;; \end{aligned}$$

$$\begin{aligned} C_2(\varpi) \;&=\; \lim_{\varepsilon\to 0} C_2(\varepsilon,\varpi) \\ &=\; \frac{\sqrt{\beta}\left(\frac{1}{\beta} - \frac{1}{\theta}\right)\left(\sqrt{2\theta} - \xi\right) \;-\; \frac{\xi}{\theta}\left(\sqrt{\theta} + \sqrt{\beta}\right)e^{\varpi\sqrt{2\theta}}}{\left(\sqrt{\theta} + \sqrt{\beta}\right)\left(\sqrt{2\theta} + \xi\right)e^{\varpi\sqrt{2\theta}} \;-\; \left(\sqrt{\theta} - \sqrt{\beta}\right)\left(\sqrt{2\theta} - \xi\right)e^{-\varpi\sqrt{2\theta}}}\;; \end{aligned}$$

$$\begin{aligned} C_3(\varpi)\,e^{-\varpi\sqrt{2\beta}} \;&=\; \lim_{\varepsilon\to 0} C_3(\varepsilon,\varpi)\,e^{-\varpi\sqrt{2\beta}} \\ &=\; \frac{\frac{2}{\sqrt{\beta\theta}}\left(\sqrt{\theta} + \sqrt{\beta}\right)\left(\sqrt{2\theta} + \xi\right)e^{\varpi\sqrt{2\theta}} \;-\; \frac{2\xi}{\sqrt{\theta}}}{\left(\sqrt{\theta} + \sqrt{\beta}\right)\left(\sqrt{2\theta} + \xi\right)e^{\varpi\sqrt{2\theta}} \;-\; \left(\sqrt{\theta} - \sqrt{\beta}\right)\left(\sqrt{2\theta} - \xi\right)e^{-\varpi\sqrt{2\theta}}} \\ &\qquad -\; \frac{1}{\beta}\left(1 + \frac{\sqrt{\beta}}{\sqrt{\theta}}\right). \end{aligned}$$

For the sake of simplicity in presenting results, we introduce d^-_{ϖ} and d^+_{ϖ} as given by (7.15) and (7.16) respectively. After a bit simplification with the notations, the solutions are converted into the form presented. □

Given h a bounded function on all support of Y_t, due to the martingale property we have

$$\mathbb{E}\left[\int_0^\infty e^{-\beta s} e^{-\zeta Z_s(\varpi)} e^{-\xi L_s^B}\, \mathrm{d}s\right] = \frac{1}{\beta+\zeta} + C_1(\varpi) + C_2(\varpi).$$

In consideration of the case $\varpi \to 0$, however, as a result of the continuity constraint at the joint point ϖ, it is obvious to have

$$\frac{1}{\beta+\zeta} + C_1(\varpi)\, e^{\varpi\sqrt{2(\beta+\zeta)}} + C_2(\varpi)\, e^{-\varpi\sqrt{2(\beta+\zeta)}} = \frac{1}{\beta} + C_3(\varpi) e^{-\varpi\sqrt{2\beta}}.$$

While taking the limit of ϖ, the above equation still holds. That is to say,

$$\lim_{\varpi\to 0} \mathbb{E}\left[\int_0^\infty e^{-\beta s} e^{-\zeta Z_s(\varpi)} e^{-\xi L_s^B}\, \mathrm{d}s\right] = \frac{1}{\beta} + \lim_{\varpi\to 0} C_3(\varpi)\, e^{-\varpi\sqrt{2\beta}}.$$

7.4.2 Convergence of the Difference to a Brownian Motion

To decide a suitable normalizing factor that makes the limiting difference having a non-trivial distribution, we investigate the asymptotic behaviour through its characteristic function. The associated results are presented below.

Lemma 7.4.2. *We denote by $\mathcal{L}^{\beta,\vartheta}\{V_t(\varpi)\}$ the double Laplace transform of $V_t(\varpi)$. For all $\beta, \vartheta > 0$, it holds that*

$$\lim_{\varpi\to 0} \mathcal{L}^{\beta,\vartheta}\{V_t(\varpi)\} = \mathcal{L}^{\beta,\vartheta}\Big\{ \lim_{\varpi\to 0} V_t(\varpi) \Big\},$$

and in particular for $\boldsymbol{q} = \frac{1}{2}$,

$$\mathcal{L}^{\beta,\vartheta}\Big\{ \lim_{\varpi\to 0} V_t(\varpi) \Big\} = \left(\beta - \frac{\sqrt{\beta}}{\sqrt{2}}\vartheta^2\right)^{-1}. \tag{7.18}$$

Proof. Replacing $\zeta \to \frac{\vartheta}{2\varpi^{q+1}}$ and $\xi \to -\frac{\vartheta}{\varpi^{q}}$ produces,

$$\begin{aligned} h(\varpi,\vartheta) &= \frac{1}{\beta} + C_3(\varpi)\, e^{-\varpi\sqrt{2\beta}} \\ &= \frac{\frac{4}{\sqrt{2\beta}} d^{+}_{\varpi}(\vartheta;q) + \frac{4\vartheta}{\varpi^{q}}}{\sqrt{2\beta + \frac{\vartheta}{\varpi^{q+1}}}\left(d^{+}_{\varpi}(\vartheta;q) - d^{-}_{\varpi}(\vartheta;q)\right)} - \frac{1}{\sqrt{\beta\left(\beta + \frac{\vartheta}{2\varpi^{q+1}}\right)}}, \end{aligned}$$

and

$$\begin{aligned} d^{-}_{\varpi}(\vartheta;q) &= \left(\sqrt{2\beta + \frac{\vartheta}{\varpi^{q+1}}} - \sqrt{2\beta}\right)\left(\sqrt{2\beta + \frac{\vartheta}{\varpi^{q+1}}} + \frac{\vartheta}{\varpi^{q}}\right) e^{-\varpi\sqrt{2\beta + \frac{\vartheta}{\varpi^{q+1}}}}; \\ d^{+}_{\varpi}(\vartheta;q) &= \left(\sqrt{2\beta + \frac{\vartheta}{\varpi^{q+1}}} + \sqrt{2\beta}\right)\left(\sqrt{2\beta + \frac{\vartheta}{\varpi^{q+1}}} - \frac{\vartheta}{\varpi^{q}}\right) e^{\varpi\sqrt{2\beta + \frac{\vartheta}{\varpi^{q+1}}}}. \end{aligned}$$

The limit of $h(\varpi,\vartheta)$ as $\varpi \to 0$ has been calculated with $q = \frac{1}{2}$, so that

$$\lim_{\varpi \to 0} \frac{e^{-\varpi\sqrt{2\beta + \frac{\vartheta}{\sqrt{\varpi^3}}}}}{2\beta + \frac{\vartheta}{\sqrt{\varpi^3}}} \left\{ \frac{4\vartheta}{\sqrt{\varpi}} + \frac{4}{\sqrt{2\beta}} d^{+}_{\varpi}\left(\vartheta; \tfrac{1}{2}\right) \right\} = \frac{4}{\sqrt{2\beta}}, \tag{7.19}$$

and

$$\lim_{\varpi \to 0} \frac{e^{-\varpi\sqrt{2\beta + \frac{\vartheta}{\sqrt{\varpi^3}}}}}{\sqrt{2\beta + \frac{\vartheta}{\sqrt{\varpi^3}}}} \left\{ d^{+}_{\varpi}\left(\vartheta; \tfrac{1}{2}\right) - d^{-}_{\varpi}\left(\vartheta; \tfrac{1}{2}\right) \right\} = 2\sqrt{2\beta} - 2\vartheta^2. \tag{7.20}$$

These two results are obtained by tedious calculations, which has been excluded by this proof but relegated to **Appendix II**. □

Referring to the work presented in CHAPTER 5 within the context of Bessel age processes, the limiting difference between the local time at zero and its approximations is characterised converging in distribution to a Gaussian process. In the following, we show that this result is extended to the case of Brownian motion with no surprise.

Theorem 7.4.3. *By central limit theorem, it holds for all* $0 < \alpha < 1$ *that*

$$\varpi^{-\frac{1}{2}}\left(\frac{Z_t(\varpi)}{2\varpi} - L^B_t \right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma^2(t)\right), \qquad as \quad \varpi \to 0.$$

$\mathcal{N}$ *represents a normal distribution whose variance* $\sigma^2(t) = 2L^B_t$.

Proof. The result is obtained straightforwardly by inverting (7.18),

$$
\begin{aligned}
\frac{\sqrt{2}}{\sqrt{\beta}} \frac{1}{\sqrt{2\beta} - \vartheta^2} &= -2 \int_0^\infty \frac{\partial}{\partial \beta} \left\{ e^{-t\sqrt{2\beta}} \right\} e^{t\vartheta^2} \frac{\mathrm{d}t}{t} \\
&= -2 \int_0^\infty \frac{\partial}{\partial \beta} \left\{ \int_0^\infty e^{-\beta x} \frac{t}{\sqrt{2\pi x^3}} e^{-\frac{t^2}{2x}} \, \mathrm{d}x \right\} \left\{ \int_{-\infty}^\infty e^{-\vartheta y} \frac{1}{\sqrt{2\pi 2t}} e^{-\frac{y^2}{4t}} \, \mathrm{d}y \right\} \frac{\mathrm{d}t}{t} \\
&= \int_0^\infty e^{-\beta x} \int_{-\infty}^\infty e^{-\vartheta y} \int_0^\infty \frac{2}{\sqrt{2\pi x}} e^{-\frac{t^2}{2x}} \frac{1}{\sqrt{2\pi 2t}} e^{-\frac{y^2}{4t}} \, \mathrm{d}t \, \mathrm{d}y \, \mathrm{d}x.
\end{aligned}
$$

Therefore, the difference process can be realised as a driftless time-changed Brownian motion, replacing the real time by the local time L_t^B. □

This central limit theorem is a well-known result as a sequel to the Lévy's downcrossing theorem, early proved by Kasahara [63, 64] and later extended by Csáki et al. [25].

Appendix I : A Supplementary Proof for Theorem 7.4.1.

Proof. We begin with the infinitesimal generator acting on

$$\int_0^t e^{-\beta s} e^{-\zeta Z_s} e^{-\xi\varepsilon M_s(\varepsilon)}\,\mathrm{d}s \;+\; e^{-\beta t} e^{-\zeta Z_t} e^{-\xi\varepsilon M_t(\varepsilon)} h(Y_t)$$

with $h(y)$ a bounded Borel function. Denoted by $\mathcal{A}f(y,z,m,w,t)$ the generator , then

$$\mathcal{A}f(y,z,m,w,t) \;=\; \frac{\partial f}{\partial t} \;+\; \frac{1}{2}\frac{\partial^2 f}{\partial y^2} \;+\; \mathbb{1}_{\{y\,<\,\varpi\}}\frac{\partial f}{\partial z} \;+\; e^{-\beta t}e^{-\zeta z}e^{-\xi m}\frac{\partial f}{\partial w},$$

with a boundary condition $f(0,z,m,w,t) \;=\; f(\,\varepsilon,z,m+\varepsilon,w,t\,)$ and $0<\varepsilon<\varpi$. By substitution with

$$f(y,z,m,w,t) \;=\; w + e^{-\beta t}e^{-\zeta z}e^{-\xi m}h(y)\,,$$

we get a differential equation

$$h''(y) - 2\big(\,\beta + \zeta\mathbb{1}_{\{y\,<\,\varpi\}}\,\big)h(y) + 2 \;=\; 0\,. \tag{7.21}$$

Setting

$$h(y) \;=\; \begin{cases} h_1(y), & y \;<\; \varpi\,; \\ h_2(y), & y \;>\; \varpi\,, \end{cases}$$

recall that the bounded and continuous solutions to (7.21) are obtained before as

$$\begin{cases} h_1(y) \;=\; \dfrac{1}{\beta+\zeta} \;+\; C_1 e^{y\sqrt{2(\beta+\zeta)}} \;+\; C_2 e^{-y\sqrt{2(\beta+\zeta)}}, & 0 \;\le\; y \;<\; \varpi; \\ h_2(y) \;=\; \dfrac{1}{\beta} \;+\; C_3 e^{-y\sqrt{2\beta}}, & y \;\ge\; \varpi. \end{cases}$$

The values of $\{\,C_i,\; i=1,2,3\,\}$ are decided by fulfilling the conditions of continuity and boundedness. To be specific, we need

$$\text{I. } \lim_{y\to\infty} h_2(y) \ < \ \infty \, ;$$

$$\text{II. } h_1(\varpi) \ = \ h_2(\varpi) \, ;$$

$$\text{III. } h_1'(\varpi) \ = \ h_2'(\varpi) \, ;$$

$$\text{IV. } h_1(0) \ = \ e^{-\xi\varepsilon} h_1(\varepsilon) \, .$$

These produce a system of three equations

$$\begin{cases} \dfrac{1}{\beta+\zeta} \ + \ C_1 e^{\varpi\sqrt{2(\beta+\zeta)}} \ + \ C_2 e^{-\varpi\sqrt{2(\beta+\zeta)}} \ = \ \dfrac{1}{\beta} + C_3 e^{-\varpi\sqrt{2\beta}}; \\ C_1\sqrt{2(\beta+\zeta)} e^{\varpi\sqrt{2(\beta+\zeta)}} \ - \ C_2\sqrt{2(\beta+\zeta)} e^{-\varpi\sqrt{2(\beta+\zeta)}} \ = \ - C_3\sqrt{2\beta} e^{-\varpi\sqrt{2\beta}}; \\ \dfrac{1}{\beta+\zeta} + C_1 + C_2 \ = \ e^{-\xi\varepsilon}\left(\dfrac{1}{\beta+\zeta} \ + \ C_1 e^{\varepsilon\sqrt{2(\beta+\zeta)}} \ + \ C_2 e^{-\varepsilon\sqrt{2(\beta+\zeta)}} \right). \end{cases}$$

Rearranging terms gives,

$$\begin{aligned} 2C_1 \ &= \ \left(\frac{1}{\beta} - \frac{1}{\beta+\zeta}\right) e^{-\varpi\sqrt{2(\beta+\zeta)}} \ + \ C_3 e^{-\varpi\left(\sqrt{2(\beta+\zeta)}+\sqrt{2\beta}\right)} \left(1 - \frac{\sqrt{\beta}}{\sqrt{\beta+\zeta}}\right); \\ 2C_2 \ &= \ \left(\frac{1}{\beta} - \frac{1}{\beta+\zeta}\right) e^{\varpi\sqrt{2(\beta+\zeta)}} \ + \ C_3 e^{\varpi\left(\sqrt{2(\beta+\zeta)}-\sqrt{2\beta}\right)} \left(1 + \frac{\sqrt{\beta}}{\sqrt{\beta+\zeta}}\right); \\ C_1 E^-_{\varepsilon,\varpi} \ + \ C_2 E^+_{\varepsilon,\varpi} \ &= \ -\frac{1-e^{-\xi\varepsilon}}{\beta+\zeta}. \end{aligned}$$

Solving them in the system, we get solutions

$$C_1 \ = \ \frac{-\frac{\left(1-e^{-\xi\varepsilon}\right)}{\beta+\zeta}\left(1 - \frac{\sqrt{\beta}}{\sqrt{\beta+\zeta}}\right) e^{-\varpi\sqrt{2(\beta+\zeta)}} \ + \ \left(\frac{1}{\beta} - \frac{1}{\beta+\zeta}\right)\frac{\sqrt{\beta}}{\sqrt{\beta+\zeta}} E^+_{\varepsilon,\varpi}}{\left(1 - \frac{\sqrt{\beta}}{\sqrt{\beta+\zeta}}\right) e^{-\varpi\sqrt{2(\beta+\zeta)}} E^-_{\varepsilon,\varpi} \ + \ \left(1 + \frac{\sqrt{\beta}}{\sqrt{\beta+\zeta}}\right) e^{\varpi\sqrt{2(\beta+\zeta)}} E^+_{\varepsilon,\varpi}};$$

$$C_2 \ = \ \frac{-\frac{\left(1-e^{-\xi\varepsilon}\right)}{\beta+\zeta}\left(1 + \frac{\sqrt{\beta}}{\sqrt{\beta+\zeta}}\right) e^{\varpi\sqrt{2(\beta+\zeta)}} \ - \ \left(\frac{1}{\beta} - \frac{1}{\beta+\zeta}\right)\frac{\sqrt{\beta}}{\sqrt{\beta+\zeta}} E^-_{\varepsilon,\varpi}}{\left(1 - \frac{\sqrt{\beta}}{\sqrt{\beta+\zeta}}\right) e^{-\varpi\sqrt{2(\beta+\zeta)}} E^-_{\varepsilon,\varpi} \ + \ \left(1 + \frac{\sqrt{\beta}}{\sqrt{\beta+\zeta}}\right) e^{\varpi\sqrt{2(\beta+\zeta)}} E^+_{\varepsilon,\varpi}};$$

$$e^{-\varpi\sqrt{2\beta}} C_3 \ = \ \frac{-\frac{2\left(1-e^{-\xi\varepsilon}\right)}{\beta+\zeta} - \left(\frac{1}{\beta} - \frac{1}{\beta+\zeta}\right)\left(e^{-\varpi\sqrt{2(\beta+\zeta)}} E^-_{\varepsilon,\varpi} + e^{\varpi\sqrt{2(\beta+\zeta)}} E^+_{\varepsilon,\varpi}\right)}{\left(1 - \frac{\sqrt{\beta}}{\sqrt{\beta+\zeta}}\right) e^{-\varpi\sqrt{2(\beta+\zeta)}} E^-_{\varepsilon,\varpi} + \left(1 + \frac{\sqrt{\beta}}{\sqrt{\beta+\zeta}}\right) e^{\varpi\sqrt{2(\beta+\zeta)}} E^+_{\varepsilon,\varpi}},$$

where

$$E^-_{\varepsilon,\varpi} \ := \ \left(1 - e^{-\varepsilon\left(\xi \ - \ \sqrt{2(\beta+\zeta)}\right)} \right); \quad \text{and}$$

$$E^{+}_{\varepsilon,\varpi} \;:=\; \left(1 - e^{-\varepsilon\left(\xi \;+\; \sqrt{2(\beta+\zeta)}\right)}\right).$$

With a bit of simplification, the representation of $h(y)$ together with C_1, C_2 and C_3 yields $h_\varepsilon(y)$ in (7.17) □

Appendix II : A Supplementary Proof for Lemma 7.4.2 .

Proof. To find a proper result of $\lim_{\varpi\to 0} h(\varpi,\vartheta)$, we multiply both the nominator and the denominator with

$$e^{-\varpi\sqrt{2\beta+\frac{\vartheta}{\varpi^{q+1}}}}\left(2\beta+\frac{\vartheta}{\varpi^{q+1}}\right)^{-1},$$

so we get

$$h(\varpi,\vartheta) \;=\; \frac{Q_1(\varpi)}{Q_2(\varpi)} \;-\; \frac{1}{\sqrt{\beta\left(\beta+\frac{\vartheta}{2\varpi^{q+1}}\right)}}$$

and

$$Q_1(\varpi) \;=\; \frac{e^{-\varpi\sqrt{2\Theta(\varpi)}}}{2\Theta(\varpi)}\left\{\frac{4\vartheta}{\varpi^q} \;+\; \frac{4}{\sqrt{2\beta}}\left(\sqrt{2\Theta(\varpi)}+\sqrt{2\beta}\right)\left(\sqrt{2\Theta(\varpi)}-\frac{\vartheta}{\varpi^q}\right)e^{\varpi\sqrt{2\Theta(\varpi)}}\right\};$$

$$Q_2(\varpi) \;=\; \frac{e^{-\varpi\sqrt{2\Theta(\varpi)}}}{\sqrt{2\Theta(\varpi)}}\left\{\left(\sqrt{2\Theta(\varpi)}+\sqrt{2\beta}\right)\left(\sqrt{2\Theta(\varpi)}-\frac{\vartheta}{\varpi^q}\right)e^{\varpi\sqrt{2\Theta(\varpi)}} - \left(\sqrt{2\Theta(\varpi)}-\sqrt{2\beta}\right)\left(\sqrt{2\Theta(\varpi)}+\frac{\vartheta}{\varpi^q}\right)e^{-\varpi\sqrt{2\Theta(\varpi)}}\right\}.$$

First notice that for all $q > 0$,

$$\lim_{\varpi\to 0}\frac{1}{\sqrt{\beta\left(\beta+\frac{\vartheta}{2\varpi^{q+1}}\right)}} \;=\; 0,$$

so we put focus on the limits of $Q_1(\varpi)$ and $Q_2(\varpi)$. For simplicity in notations, let

$$2\Theta(\varpi) \coloneqq \left(2\beta+\frac{\vartheta}{\varpi^{q+1}}\right).$$

Considering the numerator, applying the L'Hôpital's Rule gives

$$\lim_{\varpi\to 0} Q_1(\varpi) \;=\; \lim_{\varpi\to 0}\left\{\frac{e^{-\varpi\sqrt{2\Theta(\varpi)}}}{\Theta(\varpi)}\frac{2\vartheta}{\varpi^q} \;+\; \frac{4}{\sqrt{2\beta}}\left(1+\frac{\sqrt{2\beta}}{\sqrt{2\Theta(\varpi)}}\right)\left(1-\frac{\vartheta}{\varpi^q\sqrt{2\Theta(\varpi)}}\right)\right\}$$
$$= \begin{cases} \dfrac{4}{\sqrt{2\beta}}, & 0 < q < 1\,; \\ \dfrac{4}{\sqrt{2\beta}}\left(1-\sqrt{\vartheta}\right), & q = 1\,; \\ -\infty, & q > 1\,. \end{cases}$$

Then for the denominator,

$$\lim_{\varpi\to 0} Q_2(\varpi) = \lim_{\varpi\to 0} \sqrt{2\Theta(\varpi)}\Bigg\{ \left(1+\frac{\sqrt{\beta}}{\sqrt{\Theta(\varpi)}}\right)\left(1-\frac{\vartheta}{\varpi^q\sqrt{2\Theta(\varpi)}}\right) - \left(1-\frac{\sqrt{\beta}}{\sqrt{\Theta(\varpi)}}\right)\left(1+\frac{\vartheta}{\varpi^q\sqrt{2\Theta(\varpi)}}\right)e^{-2\varpi\sqrt{2\Theta(\varpi)}} \Bigg\}. \tag{7.22}$$

It becomes indeterminate (i.e. neither 0 nor ∞) for $0 < q < 1$, in which case a limit is attainable with the help of L'Hôpital's Rule. (7.22) is then calculated to

$$\begin{aligned}
&\lim_{\varpi\to 0} \sqrt{2\beta}\left(1-\frac{\vartheta}{\varpi^q\sqrt{2\Theta(\varpi)}}\right) - 2\varpi\left(1+\frac{\sqrt{\beta}}{\sqrt{\Theta(\varpi)}}\right)\left(\frac{1-q}{1+q}\Theta(\varpi)-\beta\right) \\
&\quad + \left\{ \begin{array}{l} \sqrt{2\beta}\left(1+\frac{\vartheta}{\varpi^q\sqrt{2\Theta(\varpi)}}\right) - 2\varpi\left(1-\frac{\sqrt{\beta}}{\sqrt{\Theta(\varpi)}}\right)\left(\frac{1-q}{1+q}\Theta(\varpi)-\beta\right) \\ -4\Theta(\varpi)\left(1-\frac{\sqrt{\beta}}{\sqrt{\Theta(\varpi)}}\right)\left(1+\frac{\vartheta}{\varpi^q\sqrt{2\Theta(\varpi)}}\right)\left(\frac{\varpi(q-1)}{q+1}-\frac{4\beta\varpi^{q+2}}{(1+q)\vartheta}\right) \end{array} \right\} e^{-2\varpi\sqrt{2\Theta(\varpi)}} \\
&= 2\sqrt{2\beta} - \frac{1-q}{1+q}\vartheta \lim_{\varpi\to 0} \frac{1-e^{-2\varpi\sqrt{2\Theta(\varpi)}}\left(1-\frac{\sqrt{\beta}}{\sqrt{\Theta(\varpi)}}\right)\left(1+\frac{2\vartheta}{\varpi^q\sqrt{2\Theta(\varpi)}}\right)}{\varpi^q} \\
&= 2\sqrt{2\beta} - \frac{1-q}{1+q}\vartheta \cdot \lim_{\varpi\to 0} e^{-2\varpi\sqrt{2\Theta(\varpi)}} \\
&\quad \Bigg\{ \frac{\sqrt{\beta}}{\sqrt{2}}\frac{1+q}{q}\frac{\vartheta}{\varpi^{2q+1}(2\Theta(\varpi))^2}\left(1+\frac{2\vartheta}{\varpi^q\sqrt{2\Theta(\varpi)}}\right) - \left(1-\frac{\sqrt{\beta}}{\sqrt{\Theta(\varpi)}}\right)\frac{1-q}{q}\frac{\vartheta}{\varpi^{2q}\Theta(\varpi)} \\
&\quad - \left(1-\frac{\sqrt{\beta}}{\sqrt{\Theta(\varpi)}}\right)\left(\frac{1+q}{q}\frac{\vartheta}{2\varpi^{2q}\Theta(\varpi)}-\frac{2\varpi^{1-q}}{q}\right)\left(1+\frac{\vartheta}{2\varpi^{q+1}\Theta(\varpi)}+\frac{2\vartheta}{\varpi^q\sqrt{2\Theta(\varpi)}}\right)\Bigg\}.
\end{aligned}$$

If we further restrict q to $\left(\frac{1}{3},1\right)$, it produces

$$\begin{aligned}
&2\sqrt{2\beta} + \frac{1-q}{1+q}\vartheta \times \\
&\quad \lim_{\varpi\to 0} \frac{\frac{1-q}{q}\frac{2\vartheta}{\varpi^{2q}} + \left(1+\frac{\vartheta}{\varpi^{q+1}}+\frac{2\vartheta}{\varpi^q\sqrt{2\Theta(\varpi)}}\right)\left(\frac{1+q}{q}\frac{\vartheta}{\varpi^{2q}2\Theta(\varpi)}-\frac{2}{q\varpi^{q-1}}\right)}{\sqrt{2\Theta(\varpi)}} \\
&= 2\sqrt{2\beta} + \frac{1-q}{1+q}2\vartheta^2 \times \\
&\quad \lim_{\varpi\to 0}\left(\frac{q+2}{3q-1}\frac{\vartheta}{\varpi^{3q}2\Theta(\varpi)} - \frac{q}{3q-1}\frac{1}{\varpi^{2q-1}}\right)\left(\frac{1+q}{q}\frac{\vartheta}{\varpi^{1+q}2\Theta(\varpi)}-\frac{2}{q}\right) \\
&= 2\sqrt{2\beta} - 2\vartheta^2,
\end{aligned}$$

where q has been chosen as ½. This completes the proof of (7.20), as a result of which (7.19) follows immediately. □

References

[1] Andrews, G. E., Askey, R., and Roy, R. (2000). *Special functions*, volume 71. Cambridge University Press.

[2] Azéma, J. and Yor, M. (1989). Study of a remarkable martingale. *Strasbourg probability seminar*, 23:88–130.

[3] Bateman, H. (1954). *Tables of integral transforms [volumes I & II]*, volume 1. McGraw-Hill Book Company.

[4] Baurdoux, E. J., Pardo, J. C., Pérez, J. L., and Renaud, J.-F. (2016). Gerber–Shiu distribution at Parisian ruin for Lévy insurance risk processes. *Journal of Applied probability*, 53(2):572–584.

[5] Bertoin, J. (1996). *Lévy processes*, volume 121. Cambridge University Press.

[6] Bertoin, J. (1999). Subordinators: examples and applications. In *Lectures on probability theory and statistics*, pages 1–91. Springer.

[7] Bingham, N. (1971). Limit theorems for occupation times of Markov processes. *Probability Theory and Related Fields*, 17(1):1–22.

[8] Björk, T. (2011). An introduction to point processes from a martingale point of view. *lecture note, KTH*.

[9] Black, F. and Cox, J. C. (1976). Valuing corporate securities: Some effects of bond indenture provisions. *The Journal of Finance*, 31(2):351–367.

[10] Blumenthal, R. M. (2012). *Excursions of Markov processes*. Springer Science & Business Media.

[11] Blumenthal, R. M. and Getoor, R. K. (1964). Local times for Markov processes. *Probability Theory and Related Fields*, 3(1):50–74.

[12] Blumenthal, R. M. and Getoor, R. K. (2007). *Markov processes and potential theory*. Courier Corporation.

[13] Borodin, A. (1992). Distribution of integral functionals of the local time of a Bessel process. *Journal of Soviet Mathematics*, 61(1):1827–1840.

[14] Borodin, A. N. (1989). Brownian local time. *Russian Mathematical Surveys*, 44(2):1.

[15] Borodin, A. N. (2005). Distributions of Some Functionals of the Brownian Local Time. *Journal of Mathematical Sciences*, 128(1):2503–2510.

[16] Borodin, A. N. and Salminen, P. (1997). Handbook of Brownian Motion-Facts and Formulae. *Journal of the Royal Statistical Society-Series A Statistics in Society*, 160(3):596.

[17] Brémaud, P. (1981). *Point Processes and Queues : Martingale Dynamics*, volume 50. Springer.

[18] Byczkowski, T., Małecki, J., and Ryznar, M. (2013). Hitting times of Bessel processes. *Potential Analysis*, 38(3):753–786.

[19] Çetin, U. (2012). Filtered Azéma martingales. *Electronic Communications in Probability*, 17:1–13.

[20] Çetin, U., Jarrow, R., Protter, P., Yıldırım, Y., et al. (2004). Modeling credit risk with partial information. *The Annals of Applied Probability*, 14(3):1167–1178.

[21] Chesney, M., Jeanblanc-Picqué, M., and Yor, M. (1997). Brownian excursions and Parisian barrier options. *Advances in Applied Probability*, 29(1):165–184.

[22] Chung, K. L. and Durrett, R. (2008). Downcrossings and local time. In *Selected Works Of Kai Lai Chung*, pages 585–587. World Scientific.

[23] Collin-Dufresne, P., Goldstein, R., and Helwege, J. (2010). Is Credit Event Risk Priced ? Modeling Contagion Via the Updating of Beliefs. *SSRN Electronic Journal*.

[24] Costeniuc, M., Schnetzer, M., and Taschini, L. (2008). Entry and exit decision problem with implementation delay. *Journal of Applied Probability*, 45(4):1039–1059.

[25] Csáki, E., Földes, A., and Kasahara, Y. (1988). Around Yor's theorem on the Brownian sheet and local time. *Journal of Mathematics of Kyoto University*, 28(2):373–381.

[26] Czarna, I. and Palmowski, Z. (2011). Ruin probability with Parisian delay for a spectrally negative Lévy risk process. *Journal of Applied Probability*, 48(4):984–1002.

[27] Dassios, A. (1995). The distribution of the quantile of a Brownian motion with drift and the pricing of related path-dependent options. *The Annals of Applied Probability*, pages 389–398.

[28] Dassios, A. (1996). Sample quantiles of stochastic processes with stationary and independent increments. *The Annals of Applied Probability*, pages 1041–1043.

[29] Dassios, A. (2005). On the quantiles of Brownian motion and their hitting times. *Bernoulli*, 11(1):29–36.

[30] Dassios, A. and Lim, J. W. (2013). Parisian option pricing: a recursive solution for the density of the Parisian stopping time. *SIAM Journal on Financial Mathematics*, 4(1):599–615.

[31] Dassios, A. and Lim, J. W. (2018). Recursive formula for the double-barrier Parisian stopping time. *Journal of Applied Probability*, 55(1):282–301.

[32] Dassios, A., Qu, Y., and Zhao, H. (2018). Exact simulation for a class of tempered stable and related distributions. *ACM Transactions on Modeling and Computer Simulation (TOMACS)*, 28(3):1–21.

[33] Dassios, A. and Wu, S. (2008a). Parisian ruin with exponential claims. *LSE Research Online Documents on Economics*.

[34] Dassios, A. and Wu, S. (2008b). Two-sided Parisian option with single barrier. *Preprint, London School of Economics*.

[35] Dassios, A. and Wu, S. (2009). On barrier strategy dividends with Parisian implementation delay for classical surplus processes. *Insurance: Mathematics and Economics*, 45(2):195–202.

[36] Dassios, A. and Wu, S. (2010). Perturbed Brownian motion and its application to Parisian option pricing. *Finance and Stochastics*, 14(3):473–494.

[37] Dassios, A. and Wu, S. (2011a). Double-barrier Parisian options. *Journal of applied probability*, 48(1):1–20.

[38] Dassios, A. and Wu, S. (2011b). Semi-Markov model for excursions and occupation time of Markov processes. *LSE Research Online Documents on Economics*.

[39] Davis, M. H. (1984). Piecewise-deterministic Markov processes: A general class of non-diffusion stochastic models. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 353–388.

[40] Döring, L. and Kyprianou, A. E. (2016). Perpetual integrals for Lévy processes. *Journal of Theoretical Probability*, 29(3):1192–1198.

[41] Duffie, D. and Singleton, K. (1999). Modelling term structures of defaultable bonds. *Review of Financial Studies*, 12(4):687–720.

[42] Dynkin, E. B. (1961). Some limit theorems for sums of independent random variables with infinite mathematical expectations. In *Select. Transl. Math. Statist. and Probability*, volume 1, pages 171–189. Inst. Math. Statist. and Amer. Math. Soc., Providence, R.I.

[43] Embrechts, P. and Maejima, M. (2000). An introduction to the theory of self-similar stochastic processes. *International journal of modern physics B*, 14(12n13):1399–1420.

[44] Fristedt, B. and Taylor, S. (1983). Constructions of local time for a Markov process. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 62(1):73–112.

[45] Geman, H. and Yor, M. (1993). Bessel processes, Asian options, and perpetuities. *Mathematical finance*, 3(4):349–375.

[46] Getoor, R. and Sharpe, M. (1994). On the Arc-Sine Laws for Lévy Processes. *Journal of Applied Probability*, 31(1):76–89.

[47] Giesecke, K. (2001). *Default compensator, incomplete information, and the term structure of credit spreads*. Humboldt-Universität zu Berlin, Wirtschaftswissenschaftliche Fakultät.

[48] Göing-Jaeschke, A., Yor, M., et al. (2003). A survey and some generalizations of Bessel processes. *Bernoulli*, 9(2):313–349.

[49] Grandell, J. (2006). *Doubly stochastic Poisson processes*, volume 529. Springer.

[50] Guo, X., Jarrow, R. A., and Zeng, Y. (2009). Credit risk models with incomplete information. *Mathematics of operations research*, 34(2):320–332.

[51] Hamana, Y. and Matsumoto, H. (2012). The probability densities of the first hitting times of bessel processes. *Journal of Math-for-Industry*, 4(2):91–95.

[52] Hamana, Y. and Matsumoto, H. (2013). The probability distributions of the first hitting times of Bessel processes. *Transactions of the American Mathematical Society*, 365(10):5237–5257.

[53] Hübner, G. (2001). The analytic pricing of asymmetric defaultable swaps. *Journal of banking & finance*, 25(2):295–316.

[54] Itô, K. (1972). Poisson point processes attached to Markov processes. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971)*, volume 3, pages 225–239.

[55] Itô, K. and McKean, H. P. (2012). *Diffusion processes and their sample paths*. Springer Science & Business Media.

[56] Jacobsen, M. (2006). *Point process theory and applications: marked point and piecewise deterministic processes*. Springer Science & Business Media.

[57] Jarrow, R. A., Lando, D., and Turnbull, S. M. (1997). A Markov model for the term structure of credit risk spreads. *The review of financial studies*, 10(2):481–523.

[58] Jarrow, R. A. and Protter, P. (2006). Structural versus Reduced-Form Models: A New Information-Based Perspective. *The Credit Market Handbook: Advanced Modeling Issues, Hoboken, NJ [ua]*, pages 118–131.

[59] Jarrow, R. A., Protter, P., and Sezer, A. D. (2007). Information reduction via level crossings in a credit risk model. *Finance and Stochastics*, 11(2):195–212.

[60] Jarrow, R. A. and Turnbull, S. M. (1995). Pricing derivatives on financial securities subject to credit risk. *The journal of finance*, 50(1):53–85.

[61] Jeanblanc, M., Yor, M., and Chesney, M. (2009). *Mathematical methods for financial markets*. Springer Science & Business Media.

[62] Karatzsas, I. and Shreve, S. E. (1991). Brownian motion and stochastic calculus. *Graduate texts in Mathematics*, 113.

[63] Kasahara, Y. (1980). On Lévy's downcrossing theorem. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, 56(10):455–458.

[64] Kasahara, Y. (1984). Limit theorems for Lévy processes and Poisson point processes and their applications to Brownian excursions. *Journal of Mathematics of Kyoto University*, 24(3):521–538.

[65] Kasahara, Y. and Kotani, S. (1979). On limit processes for a class of additive functional of recurrent diffusion processes. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 49(2):133–153.

[66] Knight, F. B. (1988). Inverse local times, positive sojourns, and maxima for Brownian motion. In *Colloque Paul Lévy sur les Processus Stochastiques*, pages 233–247.

[67] Kusuoka, S. (2013). A Remark on default risk models. *Advances in Mathematical Economics*, 1:69.

[68] Kyprianou, A. E. (2014). *Fluctuations of Lévy processes with applications: Introductory Lectures*. Springer Science & Business Media.

[69] Lagerås, A. N. (2005). A renewal-process-type expression for the moments of inverse subordinators. *Journal of Applied Probability*, 42(4):1134–1144.

[70] Landriault, D., Renaud, J.-F., and Zhou, X. (2014). An insurance risk model with Parisian implementation delays. *Methodology and Computing in Applied Probability*, 16(3):583–607.

[71] Lawless, J. and Zhan, M. (1998). Analysis of interval-grouped recurrent-event data using piecewise constant rate functions. *Canadian Journal of Statistics*, 26(4):549–565.

[72] Lévy, P. (1940). On certain homogeneous stochastic processes. *Compositio mathematica*, 7:283–339.

[73] Lévy, P. and Loeve, M. (1965). *Stochastic processes and Brownian motion*. Gauthier-Villars Paris.

[74] Li, C. (2016). Bessel processes, stochastic volatility, and timer options. *Mathematical Finance*, 26(1):122–148.

[75] Linetsky, V. (2004). Lookback options and diffusion hitting times: A spectral expansion approach. *Finance and Stochastics*, 8(3):373–398.

[76] Linetsky, V. and Mendoza, R. (2010). Constant elasticity of variance (CEV) diffusion model. *Encyclopedia of Quantitative Finance*.

[77] Makarov, R. N. and Glew, D. (2010). Exact simulation of Bessel diffusions. *Monte Carlo Methods and Applications*, 16(3-4):283–306.

[78] Marcus, M. B. and Rosen, J. (2006). *Markov processes, Gaussian processes, and local times*, volume 100. Cambridge University Press.

[79] McKean, H. P. (2015). Brownian local times. In *Henry P. McKean Jr. Selecta*, pages 293–310. Springer.

[80] Merton, R. C. (1974). On the pricing of corporate debt: The risk structure of interest rates. *The Journal of finance*, 29(2):449–470.

[81] Nagasawa, M. (1964). Time reversions of Markov processes. *Nagoya Mathematical Journal*, 24:177–204.

[82] Nikeghbali, A. (2006). Some random times and martingales associated with $\mathrm{BES}(\delta)$ processes $(0 < \delta < 2)$. *ALEA Latin American Journal of Probability and Mathematical Statistics*, 2:67-89.

[83] Pitman, J. and Yor, M. (1981). Bessel processes and infinitely divisible laws. In *Stochastic integrals*, pages 285–370. Springer.

[84] Platen, E. (2006). Capital asset pricing for markets with intensity based jumps. In *Stochastic Finance*, pages 157–182. Springer.

[85] Salminen, P. and Yor, M. (2005). Properties of perpetual integral functionals of Brownian motion with drift. In *Annales de l'Institut Henri Poincare (B) Probability and Statistics*, volume 41, pages 335–347. Elsevier.

[86] Salminen, P., Yor, M., et al. (2005). Perpetual integral functionals as hitting and occupation times. *Electronic Journal of Probability*, 10:371–419.

[87] Williams, D. (1970). Decomposing the Brownian path. *Bulletin of the American Mathematical Society*, 76(4):871–873.

[88] Williams, D. (1974). Path Decomposition and Continuity of Local Time for One-Dimensional Diffusions, I. *Proceedings of the London Mathematical Society*, 3(4):738–768.

[89] Yen, J. and Yor, M. (2013). Local times and excursion theory for Brownian motion. *Lecture Notes in Mathematics*, 2088.

[90] Zolotarev, V. M. (1966). On representation of stable laws by integrals. *Selected translations in mathematical statistics and probability*, 6(1):84–5.