The Formalism of Non-Commutative Quantum Mechanics and its Extension to Many-Particle Systems

by

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Declaration

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Abstract

Non-commutative quantum mechanics is a generalisation of quantum mechanics which incorporates the notion of a fundamental shortest length scale by introducing non-commuting position coordinates. Various theories of quantum gravity indicate the existence of such a shortest length scale in nature. It has furthermore been realised that certain condensed matter systems allow effective descriptions in terms of non-commuting coordinates. As a result, non-commutative quantum mechanics has received increasing attention recently.

A consistent formulation and interpretation of non-commutative quantum mechanics, which unambiguously defines position measurement within the existing framework of quantum mechanics, was recently presented by Scholtz et al. This thesis builds on the latter formalism, extends it to many-particle systems and links it up with non-commutative quantum field theory via second quantisation. It is shown that interactions of particles, among themselves and with external potentials, are altered as a result of the fuzziness induced by non-commutativity. For potential scattering, generic increases are found for the differential and total scattering cross sections. Furthermore, the recovery of a scattering potential from scattering data is shown to involve a suppression of high energy contributions, disallowing divergent interaction forces. Likewise, the effective statistical interaction among fermions and bosons is modified, leading to an apparent violation of Pauli's exclusion principle and foretelling implications for thermodynamics at high densities.

Opsomming

Nie-kommutatiewe kwantummeganika is 'n veralgemening van kwantummeganika wat die idee van 'n fundamentele kortste lengteskaal invoer d.m.v. nie-kommuterende koördinate. Verskeie teorieë van kwantum-grawitasie dui op die bestaan van so 'n kortste lengteskaal in die natuur. Dit is verder uitgewys dat sekere gekondenseerde materie sisteme effektiewe beskrywings in terme van nie-kommuterende koordinate toelaat. Gevolglik het die veld van nie-kommutatiewe kwantummeganika onlangs toenemende aandag geniet.

'n Konsistente formulering en interpretasie van nie-kommutatiewe kwantummeganika, wat posisiemetings eenduidig binne bestaande kwantummeganika raamwerke defineer, is onlangs voorgestel deur Scholtz et al. Hierdie tesis brei uit op hierdie formalisme, veralgemeen dit tot veeldeeltjiesisteme en koppel dit aan nie-kommutatiewe kwantumveldeteorie d.m.v. tweede kwantisering. Daar word gewys dat interaksies tussen deeltjies en met eksterne potensiale verander word as gevolg van nie-kommutatiwiteit. Vir potensiale verstrooiïng verskyn generiese toenames vir die differensiële and totale verstroiïngskanvlak. Verder word gewys dat die herkonstruksie van 'n verstrooiïngspotensiaal vanaf verstrooiïngsdata 'n onderdrukking van hoë-energiebydrae behels, wat divergente interaksiekragte verbied. Soortgelyk word die effektiewe statistiese interaksie tussen fermione en bosone verander, wat ly tot 'n skynbare verbreking van Pauli se uitsluitingsbeginsel en dui op verdere gevolge vir termodinamika by hoë digthede.

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Introduction and Background

Quantum mechanics, and its extension, quantum field theory, are counted amongst the greatest scientific advances of the 20th century. This is not only due to the unprecedented success of quantum theory in explaining and predicting experimental results. Maybe more than anything, quantum theory represented a conceptual revolution, challenging the deterministic paradigm of classical physics. In quantum mechanics, a system is allowed to be in a superposition of different states simultaneously, and only upon the intervention of measurement is it forced to take on a definite state. Any prediction of experimental outcomes is therefore statistical in nature.

Despite its accomplishments, quantum mechanics appears to have one obvious limitation: It is formulated exclusively to describe point particles, regarded as entities without spatial extent or structure. Of course, any particle looks like a point from sufficiently far, and at low densities we would expect finite-size effects to play a minor role. However, at high densities and energies we would expect such effects to be significant.

Related to the notion of particle size is the concept of spatial resolution. Conventional quantum theory is formulated on continuous space-time, and this is the root of the divergences that plague quantum field theory. To be more precise, a function defined over continuous space has an infinite number of degrees of freedom. In quantum field theory, which is formulated in the language of functional integrals, this leads to ill-defined integration measures and divergences. To overcome this problem, various so-called renormalisation schemes have been launched, in which high energy contributions to the path integrals are left out by the introduction of some cut-off parameter. This renders otherwise infinite quantities expressible in terms of the cut-off parameter, and physical quantities may be sought for which the cut-off dependence vanishes [1]. A physical justification for the introduction of a high energy cut-off is given by Delamotte [2]: If we try to probe a particle of mass m beyond a length scale of the order of the Compton wavelength $\hbar/mc$, the energy involved is of order mc^2. This is enough energy to create new particles identical to the original one. The indistinguishability of these particles makes position measurement ambiguous, and effectively imposes a lower bound on the resolution of space.

Before renormalisation became popular, another alternative approach was proposed to tackle the divergence problem of field theories: In 1947, the idea of quantised spacetime was introduced in an article by Snyder [3]. His basic idea was to turn space and time coordinates into operators, and showing that the quantity $S^2 = c^2t^2 - \vec{x}^2$ can be Lorentz invariant even if the position operators have discrete spectra. More specifically, he proposed the commutation relations

$$[\hat{x}_i, \hat{x}_j] = \frac{ia^2}{\hbar}\epsilon_{ijk}\hat{L}_k, \qquad [\hat{t}, \hat{x}_i] = \frac{ia^2}{\hbar c}\hat{M}_i, \qquad i, j, k \in \{1, 2, 3\}, \tag{1}$$

where the $\hat{L}_i$'s and the $\hat{M}_i$'s are the generators of the Lorentz group, and ϵ_{ijk} is the completely anti-symmetric tensor. Here a plays the role of a fundamental unit of length, and for $a \to 0$ we recover our familiar commuting coordinates. For finite a, however, the eigenvalue spectrum of the position operators becomes discretised in integer steps of a. The effect of introducing such a shortest fundamental length scale is similar to the introduction of a high energy cut-off in the renormalisation approach. However, Snyder's idea never received much attention, whereas the renormalisation method quickly gained popularity. And, as it later turned out, non-commutative field theories are not necessarily renormalisable [4].

It was only much later, in the context of quantum gravity, that the idea of non-commuting spatial coordinates resurfaced: Theories of quantum gravity attempt to unify quantum mechanics with two other largely successful theories of the 20th century, the theories of special and general relativity. Whereas the first two are largely reconciled in the framework of quantum field theory, the incorporation of the latter remains among the greatest challenges in physics. The problems arise at high energies and short length scales, where both quantum theory and general relativity presumably play significant roles. The length scale in question is believed to be of the order of the Planck length $l_P = \sqrt{G\hbar/c^3}$, the fundamental unit of length which emerges by combining the fundamental constants of the constituting theories; Planck's constant, $\hbar$, of quantum mechanics, the speed of light, c, of special relativity and the gravitational constant, G, of general relativity.

A physical argument for the existence of a fundamental shortest length scale in nature was given by Doplicher, Fredenhagen and Roberts [5], and is quoted as the 'Principle of gravitational Stability against localization of events' by Doplicher [6]:

> *The gravitational field generated by the concentration of energy required by the Heisenberg Uncertainty Principle to localize an event in spacetime should not be so strong to hide the event itself to any distant observer - distant compared to the Planck scale.*

In [5, 6] it is argued that the above uncertainty principle should be incorporated through the introduction of non-commuting spacetime coordinates, although not necessarily governed by Snyder's commutation relations (1). Rather, it is argued that the commutators of the coordinates should depend on energy concentration and hence on the underlying metric of the background space.

In view of the 'principle of gravitational stability against localization of events', it is maybe not surprising that the concept of non-commuting coordinates has emerged independently from various theories of quantum gravity. For example, Seiberg and Witten showed that non-commutative spacetime may result from certain low energy limits of string theory [7]. Intuitively, the localisation of an extended string, as opposed to a point particle, is ambiguous. Arguments from string theory have been among the major motivations for the study of non-commutative theories [4].

Non-commutative quantum mechanics is not only of interest to the quantum gravity community; applications can also be found in condensed matter physics. A well known example is quantum Hall systems, in which charged particles move in a plane perpendicular to a magnetic field. The particles move on orbits of quantised radii and may be described in terms of a set of non-commuting guiding centre coordinates [8, 9]. Similarly, one can imagine that non-commutative quantum theory may provide an effective mathematical description of other physical systems which are experimentally realisable. For example, models of Graphene have been developed based on a non-commutative description [10]. Another exciting prospect is the use of non-commutative quantum mechanics to describe particles with structure, as attempted by Rohwer et al [11]. Bigatti and Susskind [12] have argued that non-commutative quantum mechanics is dual to a description of dipoles moving in a magnetic field. A review of the various topics in physics inspired by non-commutative geometry can be found in [13].

The above motivations have stimulated an increasing interest in the fields of non-commutative quantum mechanics, non-commutative field theory and non-commutative geometry over the last few decades. The starting point is usually to postulate commutation relations

$$[\hat{x}_i, \hat{x}_j] \;=\; i\theta_{i,j}, \tag{2}$$

with θ_{ij} being a constant antisymmetric matrix. In the context of quantum gravity formulated on four dimensional spacetime this implies a breaking of Poincaré invariance, if

$\theta_{i,j}$ are considered fundamental constants of nature that do not transform. It is, however, possible to restore a twisted Poincaré symmetry [4, 14]. In two dimensions one need not worry about such problems, because the coordinate commutation relation

$$[\hat{x}_i, \hat{x}_j] = i\epsilon_{i,j}\theta, \quad i, j \in \{1, 2\}, \quad \epsilon_{1,2} = -\epsilon_{2,1} = 1, \tag{3}$$

where θ is a scalar constant, can easily be shown to be invariant under rotation.

The Aim of This Thesis

In this thesis we will restrict our attention to non-commutative quantum mechanics in the plane. The origin of non-commutativity will not be of primary concern to us, and we will simply postulate the commutation relation (3), with $\sqrt{\theta}$ playing the role of a fundamental length scale. This was also the point of departure for Scholtz et al. in [15], where a consistent formulation and interpretation of single-particle non-commutative quantum mechanics was presented. We build on [15] to develop non-commutative scattering theory and to construct a non-commutative many-particle formalism. It is precisely in scattering situations and in many-particle systems that one might expect effects of non-commutativity to manifest themselves, especially at energies and densities corresponding to the fundamental length scale $\sqrt{\theta}$. One can therefore hope that the extended formalism presented here can shed new light on our understanding of interactions in the non-commutative plane.

Conventional quantum field theory may be derived from commutative quantum mechanics via many-body theory and second quantisation. In a similar way, we will show that non-commutative quantum field theory may be built from non-commutative quantum mechanics. This is in contrast to the historical development, where non-commutative field theories emerged as a generalisation of commutative field theory, without reference to any underlying theory of non-commutative quantum mechanics. This thesis may therefore serve as a bridge across an apparent gap in existing literature. Furthermore, the choice of star product, which is an ongoing debate in non-commutative field theory [16], is in our case determined by the underlying single-particle formalism.

We emphasise that by limiting the discussion to two spatial dimensions, this thesis cannot claim any relevance to quantum gravity in four-dimensional spacetime. However, we offer a generalisation of quantum mechanics which incorporates the notion of a shortest length scale and which can be understood within the usual interpretational framework of quantum mechanics. Hopefully, the simple structure of the two-dimensional theory may reveal generic effects of non-commuting coordinates which carry over to higher dimensions. The two-dimensional formulation should also have direct application to physically realisable systems such as quantum Hall systems, for which the guiding centre coordinates satisfy (3), with θ inversely proportional to the magnetic field strength.

The thesis is structured as follows: The first chapter is devoted to the single-particle formulation of non-commutative quantum mechanics. It is largely an elaboration on [15], and it paves the way for Chapter 2, which discusses scattering, and Chapter 3, where a many-particle formalism is constructed. Chapter 4 explores how non-commutativity manifests itself in many-particle systems by considering the simplest possible such system; that of only two particles. In particular we study the effective interactions between two bosons or fermions. To round of the thesis, a short concluding chapter is included which seeks to link up the previous chapters, provide some general conclusions and outline the prospects of future investigations. In the interest of fluency, most of the elaborate calculations have been referred to Appendix C.

Chapter 1
Single-Particle Formalism

The purpose of this chapter is to review the formalism of single-particle non-commutative quantum mechanics. Naturally, such a theory shares many of the features of conventional 'commutative' quantum mechanics, and hence it is appropriate to begin by briefly sketching the development and essential elements of the latter. The following account is inspired by Dirac [17], von Neumnann [18] and Bratteli [19].

In its infancy, quantum theory was a fragmented set of ideas introduced to explain specific experimental observations. Among these were Einstein's theory of light quanta and Planck's explanation of blackbody radiation. These ideas were taken further by people like de Broglie, who postulated wave-particle dualism as a general property of all matter, and by Bohr and Sommerfeld, who introduced empirical quantisation rules to account for the observed quantisation of atomic spectra.

Two rivalling mathematical frameworks were proposed to formalise the empirical Bohr-Sommerfeld quantisation rules: on the one hand the so called matrix formulation of Heisenberg, and on the other hand the wave mechanics developed by Schrödinger. In the latter approach the state of a system is described by a wave function $\psi(x_1, ..., x_n; t)$, where $\{x_i\}_{i=1}^n$ are particle position coordinates and t is time. For particles of mass m in a position dependent potential V, the dynamics is governed by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(x_1, ..., x_n; t) \quad = \quad \left(-\frac{\hbar^2}{2m}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} + V(x_1, ..., x_n)\right)\psi(x_1, ..., x_n; t). \qquad (1.1)$$

Here, and in the remainder of the thesis, $\hbar$ is Planck's constant divided by 2π, and i is the imaginary unit $\sqrt{-1}$. If the wavefunction ψ is an element of the space $L^2(\mathbb{R}^n)$ of square-integrable functions, it can be normalised, which naturally leads to the interpretation of the quantity

$$\rho(x_1, ..., x_n; t) \quad = \quad |\psi(x_1, ..., x_n; t)|^2\, dx_1...dx_n, \qquad (1.2)$$

as the probability for localising the particle in a volume element around the coordinates $x_1, ..., x_n$ at time t.

In Heisenberg's formalism position and momentum coordinates are replaced by the sets of matrices $\{X^i\}$ and $\{P^i\}$ respectively, satisfying the relations

$$\sum_k \left(X^i_{n,k} P^j_{k,m} - P^j_{n,k} X^i_{k,m} \right) \;=\; i\hbar\delta_{n,m}\delta_{i,j}, \tag{1.3}$$

where δ is the Kronecker delta (assuming the value 1 for identical arguments and 0 otherwise). The energy of a system is associated with a Hamiltonian operator, which conventionally takes the form

$$H \;=\; \sum_{i=1}^{n} \frac{(P^i)^2}{2m} + V(X^1, ..., X^n). \tag{1.4}$$

The time evolution of an arbitrary operator $A = A(X^1, ..., X^n, P^1, ..., P^n)$ (with no explicit time dependence) is determined by the Hamiltonian operator according to

$$\frac{\partial}{\partial t} A = \frac{i}{\hbar} \left(HA - AH \right). \tag{1.5}$$

The physical states of a system are represented by normalised vectors acted upon by the matrix operators. As in the Schrödinger formulation, the state vectors have a probabilistic interpretation: Given a system in the state ψ, the outcome of measuring an operator A is random, with the average value given by $\langle A \rangle = (\psi, A\psi)$. We note, however, that in the Schrödinger picture the states are time dependent, while in Heisenberg's formulation all the dynamics is contained in the matrix operators.

The first unified, axiomatic formulations of quantum mechanics are usually attributed to von Neumann [18] and Dirac [17], who phrased quantum mechanics in the mathematically rigorous language of abstract algebra. In this language, observable quantities, such as position and momentum, are associated with Hermitian operators, denoted observables, and states of a system are represented by unit vectors in a separable Hilbert space carrying a unitary representation of the algebra formed by the observables. We call this state space the quantum Hilbert space. In the case of a spinless particle moving in $\mathbb{R}^n$, the quantum mechanics would be governed by the so called Heisenberg algebra

$$\begin{aligned} [\hat{x}_i, \hat{x}_j] &= 0, \\ [\hat{p}_i, \hat{p}_j] &= 0, \\ [\hat{x}_i, \hat{p}_j] &= i\delta_{i,j}\hbar. \end{aligned} \tag{1.6}$$

Schrödinger's wave-mechanical formulation is an irreducible unitary representation of the algebra (1.6) on the separable Hilbert space $L^2(\mathbb{R}^n)$, with the position and momentum operators acting according to

$$\hat{x}_i\psi(x) = x_i\psi(x), \qquad \hat{p}_i\psi(x) = -i\hbar\frac{\partial\psi(x)}{\partial x_i}. \tag{1.7}$$

This is often called the position representation, and it is known, by the Stone-von Neumann theorem, to be unique up to unitary transformations [20].

The state space of Heisenberg's matrix formulation is isomorphic to the quantum Hilbert space $L^2(\mathbb{R}^n)$ of Schrödinger's wave mechanics, and the position and momentum matrices satisfy (1.6). The actions of the position and momentum matrices are easily mapped to (1.7), and constitute an equally valid formulation of quantum mechanics. Similarly, we could imagine other representations of the quantum Hilbert space giving rise to alternative, but related, formulations.

A more abstract form of the quantum Hilbert space is revealed by introducing pairs of canonical operators

$$a_i = \frac{1}{\sqrt{2\hbar}}\left(\hat{x}_i + i\hat{p}_i\right) \qquad a_i^\dagger = \frac{1}{\sqrt{2\hbar}}\left(\hat{x}_i - i\hat{p}_i\right), \tag{1.8}$$

satisfying $[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0$ and $[a_i, a_j^\dagger] = \delta_{i,j}$. This implies that the Heisenberg algebra (1.6) can be realised on the Fock space

$$\mathcal{F}^n = \otimes^n\mathcal{F} \quad = \quad \{|i_1, ..., i_n\rangle\}_{i_1,...,i_n=0}^{\infty}, \tag{1.9}$$

with $|i_1, ..., i_n\rangle = |i_1\rangle \otimes ... \otimes |i_n\rangle = \frac{1}{\sqrt{i_1!...,i_n!}}(a_1^\dagger)^{i_1}|0\rangle \otimes ... \otimes (a_n^\dagger)^{i_n}|0\rangle$ and $a_1|0, i_2, ..., i_n\rangle = a_2|i_1, 0, ..., i_n\rangle = ... = a_n|i_1, ..., i_{n-1}, 0\rangle = 0$. The elements of $\mathcal{F}^n$ therefore form a basis for the quantum Hilbert space, and a general state of the latter is expressible as a vector $|\psi\rangle = \sum_{i_1,...,i_n=0}^{\infty} c_{i_1,...,i_n}|i_1, ..., i_n\rangle$ with $c_{i_1,...,i_n} \in \mathbb{C}$.

Let us now restrict ourselves to two dimensions and consider the generalisation to non-commutative quantum mechanics. First we turn the position coordinates into operators and impose the commutation relation

$$[\hat{x}_i, \hat{x}_j] \quad = \quad i\epsilon_{i,j}\theta. \tag{1.10}$$

Here $\epsilon_{i,j}$ is the completely antisymmetric tensor with $\epsilon_{1,2} = 1$, and the non-commutative parameter, θ, is real with dimension of an area. Without loss of generality, we may take it to be positive. This modification is reflected at the level of the Heisenberg algebra by substituting the first line of (1.6) with (1.10). The formulation of non-commutative quantum mechanics then amounts to constructing a unitary representation of a modified Heisenberg algebra on a separable Hilbert space. In fact, since (1.10) does not change the rank of the Heisenberg algebra, the quantum Hilbert space of the non-commutative system must be identical to that of the commutative system on an abstract level. As we have seen, however, we are free to represent the quantum Hilbert space in whatever way we like, and we will proceed in the spirit of Scholtz et al. [15], where a consistent formulation and interpretation of non-commutative quantum mechanics is achieved by a representation of the quantum Hilbert space as the set of Hilbert-Schmidt operators. This approach, which is termed 'the operatorial approach' in [13], features a natural interpretation of position measurement in the non-commutative plane. The remainder of this chapter is largely based on [15] and serves as a point of departure for the development of scattering theory in Chapter 2 and the generalisation to many-particle systems in Chapter 3.

1.1 The Non-Commutative Classical Configuration Space

In the position representation (1.7) of commutative quantum mechanics in a plane, the classical configuration space $\mathbb{R}^2$ of position coordinates plays an important role as the field over which the wavefunctions of the quantum Hilbert space are defined. It is essential for this construction that the position coordinates commute, so that both coordinates may be specified simultaneously. In non-commutative quantum mechanics, however, (1.10) leads to an uncertainty relation for position measurement, analogous to Heisenberg's uncertainty relation for position and momentum:

$$\Delta x_1 \Delta x_2 \ \geq \ \frac{\theta}{2}. \tag{1.11}$$

Hence, one cannot specify the two position coordinates of a particle simultaneously. The purpose of this section is to construct the equivalent of a classical configuration space for non-commutative quantum mechanics. We will do so in a manner which leads to a particular representation of the quantum Hilbert space in section 1.2.

We begin by introducing a new pair of operators

$$b = \frac{1}{\sqrt{2\theta}}(\hat{x}_1 + i\hat{x}_2), \qquad b^\dagger = \frac{1}{\sqrt{2\theta}}(\hat{x}_1 - i\hat{x}_2). \tag{1.12}$$

It is easy to verify that $[b, b^\dagger] = 1$, and therefore the set $\{b, b^\dagger\}$, or alternatively $\{\hat{x}_1, \hat{x}_2\}$, form an irreducible set of operators on the boson Fock space $\mathcal{F} = \{|n\rangle\}_{n=0}^{\infty}$ with $|n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n |0\rangle$ and $b|0\rangle = 0$.

In terms of this Fock basis we may define a set of coherent states [21]:

$$\begin{aligned} |z\rangle &= e^{-\frac{1}{2}|z|^2} e^{zb^\dagger} |0\rangle \\ &= e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \, . \end{aligned} \tag{1.13}$$

z here is a dimensionless complex number, however, we may write it as $z = \frac{1}{\sqrt{2\theta}}(x_1 + ix_2)$, which leads to the natural interpretation of (x_2, x_2) as dimensionfull position coordinates. In fact, $|z\rangle$ is a minimum uncertainty state in position, satisfying (1.11) with equality, and $b|z\rangle = z|z\rangle$. Motivated by this, we define the non-commutative classical configuration space as

$$\mathcal{H}_c = \text{span}\,\{|n\rangle\}_{n=o}^{\infty} \, , \tag{1.14}$$

where the span is over the field of complex numbers. The notion of a point in this space is most closely emulated by the set of coherent states (1.13).

The number states $|n\rangle$ form an orthogonal basis, implying that the identity operator on $\mathcal{H}_c$ may be resolved as

$$\hat{I}_c = \sum_{n=0}^{\infty} |n\rangle \langle n| \, . \tag{1.15}$$

The coherent states, on the other hand, are not orthogonal. Rather, they constitute an overcomplete basis for $\mathcal{H}_c$, yet they do allow the following alternative resolution of the identity operator:

$$\hat{I}_c = \frac{1}{\pi} \int d^2 z \, |z\rangle \langle z| \, . \tag{1.16}$$

Here, and throughout the rest of the thesis, $d^2 z = dz d\bar{z}$. Coherent states will be used extensively in the remainder of this thesis, and a derivation of their properties mentioned above, including the resolution of the identity operator, is provided in Appendix A.

1.2 The Quantum Hilbert Space

Let us denote the set of operators acting in on the classical configuration space (1.14) by $\mathcal{S}$. As noted in section 1.1, the coordinate operators $\{\hat{x}_1, \hat{x}_2\}$ form an irreducible set on the classical configuration space, and hence any operator in $\mathcal{S}$ should be expressible in terms of these (i.e. $\hat{O}(\hat{x}_1, \hat{x}_2) \in \mathcal{S}$). But, since $\mathcal{H}_c$ has a Fock basis, any operator acting in on it can also be expressed in the form $\hat{O} = \sum_{i,j=0}^{\infty} c_{i,j} |i\rangle \langle j|$, and there exists a trivial map between operators of this form and the Fock space $\mathcal{F}^2 = \{|m,n\rangle\}_{m,n=0}^{\infty}$ defined in (1.9). We therefore conclude that $\mathcal{S}$ is isomorphic to $\mathcal{F}^2$, and as a result, it should be possible to represent states in the quantum Hilbert space by operators of the form $\psi(\hat{x}_1, \hat{x}_2) \in \mathcal{S}$. In [15] this is accomplished by equipping $\mathcal{S}$ with the trace inner product

$$\begin{aligned}(\phi(\hat{x}_1,\hat{x}_2),\psi(\hat{x}_1,\hat{x}_2)) &= \mathrm{tr}_c(\phi(\hat{x}_1,\hat{x}_2)^\dagger \psi(\hat{x}_1,\hat{x}_2)) \\ &= \sum_{n=0}^{\infty} \langle n|\, \phi(\hat{x}_1,\hat{x}_2)^\dagger \psi(\hat{x}_1,\hat{x}_2)\, |n\rangle \\ &= \frac{1}{\pi}\int d^2z\, \langle z|\, \phi(\hat{x}_1,\hat{x}_2)^\dagger \psi(\hat{x}_1,\hat{x}_2)\, |z\rangle\,, \end{aligned} \tag{1.17}$$

where the trace is taken over $\mathcal{H}_c$. The quantum Hilbert space $\mathcal{H}_q$ is then defined as

$$\mathcal{H}_q = \{\psi(\hat{x}_1,\hat{x}_2) : \psi(\hat{x}_1,\hat{x}_2) \in \mathcal{B}(\mathcal{H}_c), \big(\psi(\hat{x}_1,\hat{x}_2)^\dagger, \psi(\hat{x}_1,\hat{x}_2)\big) < \infty\}, \tag{1.18}$$

where $\mathcal{B}(\mathcal{H}_c) \subseteq \mathcal{S}$ is the set of bounded operators on $\mathcal{H}_c$. These are commonly known as the set of Hilbert-Schmidt operators, and the inner product (1.17) of a state with itself is the so called Hilbert-Schmidt norm.

In 'bra-ket' notation we will denote elements of the quantum Hilbert space by $|\psi) \equiv \psi(\hat{x}_1, \hat{x}_2)$, to distinguish them from elements of the classical configuration space denoted by $|\cdot\rangle$. We must also distinguish between Hermitian conjugation on $\mathcal{H}_c$ and $\mathcal{H}_q$. The symbol † will be reserved for the former and the symbol ‡ for the latter.

We define the quantum position and momentum operators to act on an element $\psi(\hat{x}_1, \hat{x}_2) \in \mathcal{H}_q$ as follows:

$$\begin{aligned}\hat{X}_i \psi(\hat{x}_1,\hat{x}_2) &= \hat{x}_i \psi(\hat{x}_1,\hat{x}_2), \\ \hat{P}_i \psi(\hat{x}_1,\hat{x}_2) &= \frac{\hbar}{\theta}\epsilon_{i,j}\,[\hat{x}_j, \psi(\hat{x}_1,\hat{x}_2)]\,. \end{aligned} \tag{1.19}$$

The position operator acts by left operator multiplication, whereas the momentum operator acts adjointly. The commutator takes the role of a spatial derivative with respect to the non-commutative coordinates. It is easily verified using (1.10) that (1.19) satisfies the following modified Heisenberg algebra:

$$\begin{aligned}[\hat{X}_i, \hat{X}_j] &= i\epsilon_{i,j}\theta, \\ [\hat{P}_i, \hat{P}_j] &= 0, \\ [\hat{X}_i, \hat{P}_j] &= i\delta_{i,j}\hbar. \end{aligned} \tag{1.20}$$

(1.19) is a unitary representation of (1.20) on (1.18) [15]. The non-commutative Schrödinger equation becomes

$$i\hbar\frac{\partial}{\partial t}\psi(\hat{x}_1, \hat{x}_2) = \left(\frac{1}{2m}\left(P_1^2 + P_2^2\right) + V(\hat{X}_1, \hat{X}_2)\right)\psi(\hat{x}_1, \hat{x}_2). \tag{1.21}$$

It is sometimes convenient to adopt an alternative notation, in which operators are labeled by their direction of action. This is adopted for example in [11], where one seeks to give an alternative interpretation to the right hand side of the quantum states. In this notation we first rename our position operators: $\hat{X}_i^L \equiv \hat{X}_i$. Secondly, we define a set of right-acting operators $\{\hat{X}_1^R, \hat{X}_2^R\}$, acting according to

$$\hat{X}_i^R\psi(\hat{x}_1, \hat{x}_2) = \psi(\hat{x}_1, \hat{x}_2)\hat{x}_i. \tag{1.22}$$

These satisfy $\left[\hat{X}_i^R, \hat{X}_j^R\right] = -i\epsilon_{i,j}\theta$. We will not attempt to attach any meaning to the right-acting operators here, but we note that they allow for an alternative expression of the momentum operators:

$$\hat{P}_i\psi(\hat{x}_1, \hat{x}_2) = \frac{\hbar}{\theta}\epsilon_{i,j}\left(\hat{X}_j^L - \hat{X}_j^R\right). \tag{1.23}$$

For our purposes it will often be most convenient to work with operators whose action involves left or right multiplication by b and $b^\dagger$ rather than $\hat{x}_1$ and $\hat{x}_2$. We therefore introduce the operators

$$\begin{aligned} B_L = \frac{1}{\sqrt{2\theta}}\left(\hat{X}_1^L + i\hat{X}_2^L\right), &\quad B_L^\ddagger = \frac{1}{\sqrt{2\theta}}\left(\hat{X}_1^L - i\hat{X}_2^L\right), \\ B_R = \frac{1}{\sqrt{2\theta}}\left(\hat{X}_1^R + i\hat{X}_2^R\right), &\quad B_R^\ddagger = \frac{1}{\sqrt{2\theta}}\left(\hat{X}_1^R - i\hat{X}_2^R\right), \\ \hat{P} = \hat{P}_1 + i\hat{P}_2, &\quad \hat{P}^\ddagger = \hat{P}_1 - i\hat{P}_2, \end{aligned} \tag{1.24}$$

with corresponding actions

$$\begin{aligned}
B_L\psi(\hat{x}_1,\hat{x}_2) &= b\psi(\hat{x}_1,\hat{x}_2),\\
B_L^\ddagger\psi(\hat{x}_1,\hat{x}_2) &= b^\dagger\psi(\hat{x}_1,\hat{x}_2),\\
B_R\psi(\hat{x}_1,\hat{x}_2) &= \psi(\hat{x}_1,\hat{x}_2)b,\\
B_R^\ddagger\psi(\hat{x}_1,\hat{x}_2) &= \psi(\hat{x}_1,\hat{x}_2)b^\dagger,\\
\hat{P}\psi(\hat{x}_1,\hat{x}_2) &= -i\hbar\sqrt{\frac{2}{\theta}}\left(B_L-B_R\right)\psi(\hat{x}_1,\hat{x}_2)\\
&= -i\hbar\sqrt{\frac{2}{\theta}}\left[b,\psi(\hat{x}_1,\hat{x}_2)\right],\\
\hat{P}^\ddagger\psi(\hat{x}_1,\hat{x}_2) &= i\hbar\sqrt{\frac{2}{\theta}}\left(B_L^\ddagger-B_R^\ddagger\right)\psi(\hat{x}_1,\hat{x}_2)\\
&= i\hbar\sqrt{\frac{2}{\theta}}\left[b^\dagger,\psi(\hat{x}_1,\hat{x}_2)\right].
\end{aligned} \tag{1.25}$$

It is straightforward to check that $\hat{P}\hat{P}^\ddagger=\hat{P}^\ddagger\hat{P}=\hat{P}_1^2+\hat{P}_2^2$ and $\left[B_L,B_L^\ddagger\right]=\left[B_R^\ddagger,B_R\right]=1$, $\left[\hat{P},\hat{P}^\ddagger\right]=\left[B_L,\hat{P}\right]=\left[B_R,\hat{P}\right]=\left[B_L^\ddagger,\hat{P}^\ddagger\right]=\left[B_R^\ddagger,\hat{P}^\ddagger\right]=0$ and $\left[B_L,\hat{P}^\ddagger\right]=\left[B_R,\hat{P}^\ddagger\right]=\left[B_L^\ddagger,\hat{P}\right]=\left[B_R^\ddagger,\hat{P}\right]=i\hbar\sqrt{\frac{2}{\theta}}$. In terms of these operators the Schrödinger equation (1.21) becomes

$$i\hbar\frac{\partial}{\partial t}\psi(b^\dagger,b) = \left(\frac{1}{2m}P^\ddagger P+V(B_L^\ddagger,B_L)\right)\psi(b^\dagger,b). \tag{1.26}$$

1.2.1 A Position Basis for the Quantum Hilbert Space

The quantum Hilbert space was defined in (1.18) as the set of bounded operators acting in on the classical configuration space. Operators acting in on the classical configuration space have the form $|\cdot\rangle\langle\cdot|$, hence a general element of the quantum Hilbert space, in 'bra-ket' notation, should take the form

$$|\psi) = \sum_{n,m=0}^{\infty} c_{n,m}\,|n,m)\,, \tag{1.27}$$

where $|n,m)=|n\rangle\langle m|$ and $c_{n,m}=\langle n|\,\psi(b^\dagger,b)\,|m\rangle\in\mathbb{C}$. The vectors $|\psi)$ are diagonal in this basis only if $[\psi(b^\dagger,b),b^\dagger b]=0$. However, according to a well-known property of operators on boson Fock space, any state $|\psi)\in\mathcal{H}_q$ has a diagonal form in terms of the states $|z) = |z\rangle\langle z|$ [21][1]:

$$|\psi) \;=\; \frac{1}{\pi}\int d^2z\,\left[e^{-\partial_{\bar{z}}\partial_z}\langle z|\,\psi(b^\dagger,b)\,|z\rangle\right]|z)\,. \tag{1.28}$$

It follows from the definition of the coherent state that,

$$B_L\,|z) \;=\; z\,|z)\,. \tag{1.29}$$

Furthermore,

$$|z) \;=\; T(z)\,|z=0)\,, \tag{1.30}$$

where $T(z) = e^{z(B_L^\ddagger - B_R^\ddagger) - \bar{z}(B_L - B_R)} = e^{\frac{i}{\hbar}(x_1\hat{P}_1 + x_2\hat{P}_2)}$ [11]. This looks exactly like the translation operator in commutative quantum mechanics. (1.28), (1.29) and (1.30) indicate that the states $|z)$ play a role analogous to position states in commutative quantum mechanics. Further justification for this is provided in [22]. From here onwards we will speak loosely of the states $|z)$ as position states, keeping in mind that this is merely an analogy.

The overlap of such a position state with a general state $|\psi) \in \mathcal{H}_q$ becomes

$$\begin{aligned}(z|\,\psi) \;&=\; tr_c\left(|z\rangle\langle z|\,\psi(b^\dagger,b)\right)\\ &=\; \langle z|\,\psi(b^\dagger,b)\,|z\rangle\\ &\equiv\; \psi(\bar{z},z).\end{aligned} \tag{1.31}$$

In the last step it is implied that the operator $\psi(b^\dagger,b)$ is normal ordered (i.e. with all $b^\dagger$ operators in $\hat{\psi}$ appearing to the left of the b operators), which is not a restriction, because any operator can be brought to this form by application of the commutation relation $[b,b^\dagger] = 1$. In particular, if $|\psi) = |w)$ is another position state, we obtain

$$(z|\,w) \;=\; e^{-|z-w|^2}. \tag{1.32}$$

Keeping in mind that the complex numbers z and w are normalised by a factor of $1/\sqrt{2\theta}$, we see that this Gaussian overlap has a width of order $\sqrt{\theta}$, which reflects the fact that particles may not be localised simultaneously in two directions beyond this length scale. In the commutative limit we recover a Dirac delta function, as we should for position

[1] See proof in C.1.1

states in commutative quantum mechanics.

Consider for a moment the full set of operators $\mathcal{S}$ acting in on the classical configuration space $\mathcal{H}_c$, of which $\mathcal{H}_q$ form a subset. Again we assume normal ordering. Now we define the map

$$\begin{aligned} M : \mathcal{S} &\rightarrow \mathbb{C}, \\ M(\hat{A}) &= A(\bar{z}, z), \end{aligned} \tag{1.33}$$

with $A(\bar{z}, z) = \langle z|\, \hat{A}\,|z\rangle$. M is a isomorphism [2], because it is onto, $M(\hat{I}) = 1$ and $M(\hat{A}\hat{B}) = A(\bar{z}, z) \star_z B(\bar{z}, z)$ with the star denoting the Voros product

$$\star_z = e^{\overleftarrow{\partial_z}\overrightarrow{\partial_{\bar{z}}}}, \tag{1.34}$$

where $\partial_z \equiv \frac{\partial}{\partial z}$ and $\partial_{\bar{z}} \equiv \frac{\partial}{\partial \bar{z}}$.

The operators acting on the quantum Hilbert space are expressible in terms of left or right acting operators on the classical configuration space. For a left acting operator, O_L, we have

$$\begin{aligned} (z|\, O_L\,|\psi) &= \langle z|\, O_L \psi(b^\dagger, b)\,|z\rangle \\ &= O_L(\bar{z}, z) \star_z \psi(\bar{z}, z). \end{aligned} \tag{1.35}$$

For a right acting operator, O_R,

$$\begin{aligned} (z|\, O_R\,|\psi) &= \langle z|\, \psi(b^\dagger, b) O_R\,|z\rangle \\ &= \psi(\bar{z}, z) \star_z O_R(\bar{z}, z). \end{aligned} \tag{1.36}$$

In particular, we have

$$\begin{aligned} (z|\, B_L\,|\psi) &= z \star_z \psi(\bar{z}, z), \\ (z|\, B_L^\ddagger\,|\psi) &= \bar{z} \star_z \psi(\bar{z}, z), \\ (z|\, B_R\,|\psi) &= \psi(\bar{z}, z) \star_z z, \\ (z|\, B_R^\ddagger\,|\psi) &= \psi(\bar{z}, z) \star_z \bar{z}. \end{aligned} \tag{1.37}$$

[2]See proof in C.1.2

These relations can be used to express the momentum operators in terms of derivatives:

$$(z|\hat{P}|\psi) = -i\hbar\sqrt{\frac{2}{\theta}}\frac{\partial}{\partial_{\bar{z}}}\psi(\bar{z},z),$$
$$(z|\hat{P}^{\ddagger}|\psi) = -i\hbar\sqrt{\frac{2}{\theta}}\frac{\partial}{\partial_{z}}\psi(\bar{z},z). \tag{1.38}$$

The Schrödinger equation (1.26) can be written in the form

$$i\hbar\frac{\partial}{\partial t}(z|\,\psi) = -\frac{\hbar^2}{m\theta}\frac{\partial^2}{\partial_z\partial_{\bar{z}}}(z|\,\psi) + V(\bar{z},z)\star_z(z|\,\psi)\,. \tag{1.39}$$

The map M allows us to formulate non-commutative quantum mechanics on a space of functions of scalar coordinates. This approach is common in non-commutative geometry [23] and non-commutative field theory [4], however it comes at the expense of modifying the operation of multiplication, which introduces an explicit non-locality in the theory, because knowledge of a function up to all its derivatives is required at every point.

As a final note of this section, consider the overlap of two arbitrary states in the quantum Hilbert space, $\phi(b^\dagger,b),\psi(b^\dagger,b)\in\mathcal{H}_q$. This is defined in (1.17) as a trace over $\mathcal{H}_c$, which in the $|z)$ basis becomes

$$\begin{aligned}(\phi|\,\psi) &= \frac{1}{\pi}\int d^2z\ \langle z|\,\phi(b^\dagger,b)^\dagger\psi(b^\dagger,b)\,|z\rangle\\ &= \frac{1}{\pi}\int d^2z\ \bar{\phi}(\bar{z},z)\star_z\psi(\bar{z},z)\\ &= \frac{1}{\pi}\int d^2z\ (\phi|\,z)\star_z(z|\,\psi)\,,\end{aligned} \tag{1.40}$$

as a result of the homomorphism 1.33. Since the states $\phi(b^\dagger,b)$ and $\psi(b^\dagger,b)$ were arbitrary, we conclude that the identity operator on the quantum Hilbert space may be expressed as

$$\hat{I}_q = \frac{1}{\pi}\int d^2z\ |z)\star_z(z|\,. \tag{1.41}$$

This will be very useful in the next section, where we discuss the meaning of position measurement in non-commutative quantum mechanics.

1.2.2 The Meaning of Position Measurement

The state of a quantum system may be encoded in the form of a density matrix $\hat{\rho}$. We distinguish between pure states, which may be expressed as $\hat{\rho}=|\psi)\,(\psi|,\ |\psi)\in\mathcal{H}_q$

and mixed states, of the form $\hat{\rho} = \sum_i c_i |\psi_i)(\psi_i|$, with $c_i \in \mathbb{R}^+$ and $\sum_i c_i = 1$ [3]. The probability of measuring a system in a state $|\alpha)$ given that it was prepared in the state $|\psi)$ would be $P(|\alpha)) = |(\alpha|\psi)|^2 = (\alpha|\hat{\rho}|\alpha)$. This may alternatively be expressed as $P(|\alpha)) = tr_q(|\alpha)(\alpha|\,\hat{\rho})$. We note, however, that if $|\alpha) = |z)$ the quantity $tr_q(|z)(z|\,\hat{\rho})$ does not measure the probability of finding the particle at position z, because $\hat{I}_q \neq \int d^2z\,|z)(z|$. To define non-commutative position measurement we need to introduce the concept of a Positive Operator Valued Measure (POVM):

Let us define the positive definite operator

$$\pi_z \;=\; \frac{1}{\pi}\,|z) \star_z (z|\,. \tag{1.42}$$

From (1.41) it is clear that $\sum_z \pi_z = 1$, and π_z fulfils the requirement of a positive POVM [25]: it decomposes the identity operator into positive definite terms, thereby allowing for a natural probabilistic interpretation. For a non-commutative particle prepared in a state $\hat{\rho}$ we therefore define position measurement by

$$P(z) \;=\; \mathrm{tr}_q(\pi_z \hat{\rho}). \tag{1.43}$$

If $\hat{\rho} = |\psi)(\psi|$ this is equivalent to $P(z) = \frac{1}{\pi}(\psi|\,z) \star_z (z|\,\psi)$. We note that this definition ensures that $\int d^2z\; P(z) \;=\; 1$.

The non-commutative position measurement is not projective, because $\pi_z^2 \neq \pi_z$, and, as a result, two consecutive position measurements need not yield the same answer. The cost of accommodating non-commutative position measurement is therefore to relax von Neumann's postulate of projective measurement, which states that by performing a measurement, the system is forced into the eigenstate of the observable corresponding to the realised outcome [18]. This relaxation is permissible within the familiar framework of quantum mechanics in terms of POVMs, however, it requires us to think carefully about what is meant by measuring a particle's position.

[25] refers to 'state discrimination' rather than 'measurement'. This captures the sense of a measurement as an either/or process; a detector either clicks or it does not when it detects a particle. Because position states in non-commutative quantum mechanics overlap, we cannot perfectly discriminate them, and all the position states contribute to the probability of a detector to click. The effect of this is that space becomes fuzzy; there

[3] An introduction to density matrices is given, for example, by Sakurai [24]

is a limit to the resolution of space and one cannot localise a particle precisely to a point z. One may only conclude that the particle most likely was within the peak of the Gaussian position state $|z)$. After measurement the system will be in a superposition of the different states that might have triggered the detection, and this explains why the measurement is non-projective. We conclude that the act of position measurement in non-commutative quantum mechanics perturbs the system.

1.2.3 The Non-Commutative Momentum Basis

In [15] it was demonstrated that the non-commutative analogue of a plane wave has the form

$$\psi(b^\dagger, b) \;=\; e^{\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(pb^\dagger+\bar{p}b)}. \tag{1.44}$$

Here p is the complex momentum $p = (p_1 + ip_2)$.

$$\psi(b^\dagger, b) \;=\; \frac{1}{\pi}\int d^2z e^{\frac{\theta}{4\hbar^2}|p|^2} e^{\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p\bar{z}+\bar{p}z)} \left|z\right\rangle \left\langle z\right|. \tag{1.45}$$

This provides a natural definition of a momentum ket in the quantum Hilbert space:

$$|p) \;\equiv\; \sqrt{\frac{\theta}{2\pi^3\hbar^2}}\int d^2z e^{\frac{\theta}{4\hbar^2}|p|^2} e^{\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p\bar{z}+\bar{p}z)} |z). \tag{1.46}$$

The prefactor is included for reasons that will become evident shortly. It is straightforward to show that

$$\begin{aligned} P\,|p) &= p\,|p), \\ P^\ddagger\,|p) &= \bar{p}\,|p), \end{aligned} \tag{1.47}$$

as desired. The overlap of a state $|z)$ with such a momentum state is

$$(z|\,p) \;=\; \sqrt{\frac{\theta}{2\pi\hbar^2}} e^{-\frac{\theta}{4\hbar^2}|p|^2} e^{\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p\bar{z}+\bar{p}z)}, \tag{1.48}$$

and the momentum representation of an arbitrary state $|\psi) \in \mathcal{H}_q$ is

$$
\begin{aligned}
(p|\,\psi) &= tr_c\left(\sqrt{\frac{\theta}{2\pi\hbar^2}}e^{-\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(pb^\dagger+\bar{p}b)}\psi(b^\dagger,b)\right)\\
&= \frac{1}{\pi}\int d^2z\;(p|z)\star_z(z|\psi)\\
&= \sqrt{\frac{\theta}{2\pi^3\hbar^2}}e^{\frac{\theta}{4\hbar^2}|p|^2}\int d^2z\;e^{-\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p\bar{z}+\bar{p}z)}\,(z|\,\psi)\,. \qquad (1.49)
\end{aligned}
$$

Inserting (1.48) in (1.49) one can show that that the momentum states form an orthogonal basis:

$$
(p|\,p') = \delta(p_1-p_1')\delta(p_2-p_2'). \qquad (1.50)
$$

It is also straightforward to prove from (1.41) and (1.48) that the identity operator on the quantum Hilbert space may be expressed as

$$
\hat{I}_q = \int d^2p\,|p)\,(p|\,, \qquad (1.51)
$$

where $d^2p = dpd\bar{p} = dp_1dp_2$.

(1.50) and (1.51) show that the momentum basis of non commutative quantum mechanics is identical to that of commutative quantum mechanics. The difference lies in the transformation between the position and momentum bases. In commutative quantum mechanics this is a Fourier transform. In non-commutative quantum mechanics it is a Fourier transform followed by multiplication by a factor of $\sqrt{\frac{2\theta}{\pi\hbar^2}}e^{\frac{\theta}{4\hbar^2}|p|^2}$, according to (1.49). In general, a function $f(\bar{z},z)$ in the non-commutative position space becomes $f(\bar{p},p) = \sqrt{\frac{2\theta}{\pi\hbar^2}}e^{\frac{\theta}{4\hbar^2}|p|^2}\tilde{f}(\frac{\sqrt{2\theta}}{\hbar}\bar{p},\frac{\sqrt{2\theta}}{\hbar}p)$ in momentum space, where $\tilde{f}(\frac{\bar{p}}{\hbar},\frac{p}{\hbar})$ is the normal Fourier transform of $f(\bar{z},z)$. If we did not know that the position space was non-commutative we might attempt to Fourier transform $\sqrt{\frac{2\theta}{\pi\hbar^2}}e^{\frac{\theta}{4\hbar^2}|p|^2}\tilde{f}(\frac{\sqrt{2\theta}}{\hbar}\bar{p},\frac{\sqrt{2\theta}}{\hbar}p)$ back to position space. However this is only possible if $\tilde{f}$ falls off sufficiently fast at high momenta. The correct inverse transform from the momentum to the position basis is

$$
f(\bar{z},z) = \sqrt{\frac{\theta}{2\pi\hbar^2}}\int d^2p e^{\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(\bar{p}z+p\bar{z})}\left(e^{-\frac{\theta}{4\hbar^2}|p|^2}f(\bar{p},p)\right). \qquad (1.52)
$$

We note that high energy contributions are suppressed. Since high momentum corresponds to short length scales, this effectively means that functions in the position representation must be smooth at the scale of $\sqrt{\theta}$. This is equivalent to requiring that the functions be of Schwartz class.

1.3 The Physical Implications of Non-Commuting Coordinates

The previous sections focused on the mathematical formulation of non-commutative quantum mechanics. This section looks at some generic physical implications of this formalism.

1.3.1 Breaking of Time Reversal Symmetry

In commutative quantum mechanics the time reversal operator is an antiunitary operator which acts on the quantum Hilbert space by complex conjugation [24]. In non-commutative quantum mechanics we define the time reversal operator Θ which acts by Hermitian conjugation [15]:

$$\Theta\psi(\hat{x}_1, \hat{x}_2) = \psi^\dagger(\hat{x}_1, \hat{x}_2). \tag{1.53}$$

From the above definition the following relations may be derived:

$$\begin{aligned} \Theta\hat{X}_i^L\Theta^{-1} &= \hat{X}_i^R, \\ \Theta\hat{X}_i^R\Theta^{-1} &= \hat{X}_i^L. \end{aligned} \tag{1.54}$$

As a result, the momentum operators, which act adjointly, change sign under time reversal, as they should:

$$\Theta\hat{P}_i\Theta^{-1} = -\hat{P}_i. \tag{1.55}$$

For a general Hamiltonian with a position dependent potential $V(B_L^\ddagger, B_L)$, $\Theta H\Theta^{-1} \neq H$, and hence time reversal symmetry is broken. The exception would be if $[V, \psi] = 0$ for all $\psi \in \mathcal{H}_q$, but Schur's lemma tells us that this is only possible if V is a multiple of the identity. Hence, time reversal symmetry is only present for constant potentials.

Breaking of time reversal symmetry has been explicitly shown for the non-commutative harmonic oscillator [15] and the non-commutative well [26], where in both cases states of opposite angular momenta are found to have different energies.

A possible way to restore time reversal symmetry is by adding a term to the Hamiltonian which acts from the right. This leads to speculations about possible interpretations of the right acting operators, and we will refrain from that discussion here.

1.3.2 Confinement

In section 1.2.3 it was shown that the momentum basis is unchanged by the introduction of non-commuting position coordinates, and, as a consequence, a free particle is unaffected by this modification [15]. The effects of non commutativity only manifest themselves once a particle is confined by some potential. First of all, as discussed in section 1.3.1, the presence of a non-constant potential must result in a breaking of time reversal symmetry. Secondly, there are several conceptual and practical problems concerned with confinement in a non-commutative space. As was established in section 1.2.3, there is a shortest length scale at which a wavefunction may vary, and therefore, one would suppose, a smallest area in which a particle may be confined. In addition, one cannot have sharp potential boundaries in a fuzzy space.

[26] generalises the concept of piece-wise constant potentials to non-commutative quantum mechanics by means of projection operators on the non-commutative quantum Hilbert space, however the method is only applicable to radially symmetric potentials. The definition of arbitrary piece-wise constant potentials in a non-commutative space is still an outstanding challenge.

Chapter 2
Scattering Theory

In this chapter we discuss scattering in the non-commutative plane, more specifically scattering of a particle by a potential. The scattering theory in two dimensions for commutative quantum mechanics is well known and is presented by Adhikari in [27].

Scattering theory in non-commutative quantum mechanics has also been studied previously. For example, the scattering from a non-commutative well has been studied numerically by Thom [28]. From a theoretical viewpoint, Demetrian and Kochan [29] have found non-commutative corrections to the scattering cross section to first order. The aim of this chapter is to develop non-commutative scattering theory in a more rigorous fashion, following the Lippmann-Schwinger approach and starting from the consistent interpretation of position measurement presented in the previous chapter.

2.1 Conserved Currents in Non-Commutative Quantum Mechanics

In 1.2.2 it was shown that for a system in a pure state $\hat{\rho} = |\psi)(\psi|$ with $|\psi) \in \mathcal{H}_q$, the probability of measuring the particle at position z is $P(z) = tr_q(\pi_z\hat{\rho}) = \frac{1}{\pi}(\psi|\, z) \star_z (z|\,\psi)$. If we view z as a free parameter we may therefore regard the function $P(z)$ as a probability density function. For a general dynamic system this probability distribution will change with time, however, the total probability must remain unity. Thus, there must exist a continuity equation for the conservation of probability. In conventional quantum mechanics this continuity equation is

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{j} \;=\; 0, \tag{2.1}$$

with $\rho = \bar{\psi}(x)\psi(x)$ and $\vec{j} = \frac{i\hbar}{2m}\left(\psi\vec{\nabla}\bar{\psi} - \bar{\psi}\vec{\nabla}\psi\right)$ [30]. Correspondingly, it was shown in [15] that the quantity $\hat{\rho} = \hat{\psi}^\dagger\hat{\psi}$ is conserved in non-commutative quantum mechanics:

$$\frac{\partial}{\partial t}\hat{\rho} \;=\; [\hat{x}_2, j_1] + [\hat{x}_1, j_2]. \tag{2.2}$$

This time the currents are defined by

$$\begin{aligned} j_1 &= \frac{\hbar}{2mi\theta^2}\left\{\hat{\psi}^\dagger[\hat{x}_2,\hat{\psi}] - [\hat{x}_2,\hat{\psi}^\dagger]\hat{\psi}\right\}, \\ j_2 &= \frac{\hbar}{2mi\theta^2}\left\{\hat{\psi}^\dagger[\hat{x}_1,\hat{\psi}] - [\hat{x}_1,\hat{\psi}^\dagger]\hat{\psi}\right\}. \end{aligned} \tag{2.3}$$

In C.2.1 we derive the equivalent of the latter continuity equation in the non-commutative position representation, i.e. for the quantity $\frac{1}{\pi}\langle z|\hat{\psi}^\dagger\hat{\psi}|z\rangle = \frac{1}{\pi}\bar{\psi}(z)\star_z\psi(z) = P(z)$, with $\psi(z) = (z|\psi)$. We find that the continuity equation

$$\frac{\partial}{\partial t}P(z) + \frac{\partial}{\partial z}j_z + \frac{\partial}{\partial \bar{z}}j_{\bar{z}} = 0, \tag{2.4}$$

is satisfied, with

$$\begin{aligned} j_z &= \frac{\hbar}{2i\pi m\theta}\left[\bar{\psi}(z)\star_z\left(\frac{\partial}{\partial\bar{z}}\psi(z)\right) - \left(\frac{\partial}{\partial\bar{z}}\bar{\psi}(z)\right)\star_z\psi(z)\right], \\ j_{\bar{z}} &= \frac{\hbar}{2i\pi m\theta}\left[\bar{\psi}(z)\star_z\left(\frac{\partial}{\partial z}\psi(z)\right) - \left(\frac{\partial}{\partial z}\bar{\psi}(z)\right)\star_z\psi(z)\right]. \end{aligned} \tag{2.5}$$

In scattering situations one often considers radially symmetric potentials, and it is most convenient to work with polar coordinates. We therefore introduce the radial coordinate r and angular coordinate ϕ, such that $z = re^{i\phi}$ and $\bar{z} = re^{-i\phi}$. Correspondingly,

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{e^{-i\phi}}{2}\left[\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\phi}\right], \\ \frac{\partial}{\partial\bar{z}} &= \frac{e^{i\phi}}{2}\left[\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\phi}\right], \end{aligned} \tag{2.6}$$

and it is shown in C.2.2 that the radial form of the continuity equation is

$$\frac{\partial}{\partial t}P(z) + \frac{1}{r}\frac{\partial}{\partial r}(rj_r) + \frac{1}{r}\frac{\partial}{\partial\phi}j_\phi = 0, \tag{2.7}$$

with

$$\begin{aligned} j_r &= \frac{1}{2}\left(\sqrt{\frac{\bar{z}}{z}}j_z + \sqrt{\frac{z}{\bar{z}}}j_{\bar{z}}\right), \\ j_\phi &= -\frac{i}{2}\left(\sqrt{\frac{\bar{z}}{z}}j_z - \sqrt{\frac{z}{\bar{z}}}j_{\bar{z}}\right). \end{aligned} \tag{2.8}$$

2.1.1 The Differential Scattering Cross Section

In a two-dimensional scattering situation, where we have an incoming current and an outgoing current, we define the differential cross section by

$$\frac{d\sigma}{d\phi} = \frac{|j_{out}|\,|r|d\phi}{|j_{in}|}. \tag{2.9}$$

This is a probability distribution for an incoming particle to be scattered in a certain angular direction, i.e.

$$\frac{d\sigma}{d\phi}d\phi = \frac{\#\, of\, particles\, scattered\, into\, d\phi\, per\, unit\, time}{\#\, of\, incident\, particles\, crossing\, unit\, "area"\, per\, unit\, time}. \tag{2.10}$$

(We say "area" to keep the analogy with three-dimensional scattering.) The total cross section is defined as

$$\sigma = \int_0^{2\pi} \frac{d\sigma}{d\phi} d\phi, \tag{2.11}$$

and could be thought of as the effective area of the scattering potential as seen by an incident particle.

To actually calculate the differential cross section we need to find the incoming and outgoing currents, which again requires knowledge of the full eigenfunction of the Hamiltonian describing the scattering process. To do this we will use the Lippmann-Schwinger approach.

2.2 The Lippmann-Schwinger Equation

Suppose a scattering process is described by the Hamiltonian $(\hat{H}_0+\hat{V})|\psi) = E|\psi)$, where $\hat{H}_0$ is the free Hamiltonian and $\hat{V}$ is the scattering potential. In the absence of a scattering potential the corresponding eigenfunctions would be those of a free particle, which we will denote by $|\phi)$, i.e. $\hat{H}_0|\phi) = E_0|\phi)$, with E_0 being the energy of the free particle. If $|\psi)$ is the eigenfunction of the full Hamiltonian, we have $\hat{H}|\psi) = (\hat{H}_0 + \hat{V})|\psi) = E|\psi)$. One can argue that, if the scattering process is elastic, the inclusion of the potential $\hat{V}$ should not alter the energy, i.e. $E = E_0$. This leads one to postulate the following implicit form of the full wavefunction:

$$|\psi) = |\phi) + \frac{1}{E - \hat{H}_0}\hat{V}|\psi). \tag{2.12}$$

If one does not worry for the moment about the singular nature of $1/(E - \hat{H}_0)$, it can be seen that multiplication of (2.12) with $E - \hat{H}_0$ returns the desired full-Hamiltonian eigenvalue equation.

To remedy the singularity problem in (2.12) it is customary to add a small imaginary part to the denominator of the operator $1/(E - \hat{H}_0)$, such that

$$|\psi^{\pm}) \; = \; |\phi) + \frac{1}{E - \hat{H}_0 \pm i\varepsilon}\hat{V}|\psi^{\pm}). \tag{2.13}$$

The latter is known as the Lippmann-Schwinger equation, and the + and - solutions correspond to a plane wave plus an outgoing or incoming circular wave respectively [24]. The physically interesting case is the one with a plus, corresponding to a incident plane wave plus an outgoing scattered wave.

The Lippmann-Schwinger equation is independent of the particular choice of basis, and it should therefore be valid in non-commutative quantum mechanics as in commutative quantum mechanics. We are interested in the non-commutative position representation, i.e. $(z|\psi^+)$:

$$(z|\psi^{+}) \; = \; (z|\phi) + (z|\frac{1}{E - \hat{H}_0 \pm i\varepsilon}\hat{V}|\psi^{+}). \tag{2.14}$$

After some tedious manipulation, as carried out in C.2.3, one obtains the more explicit form of (2.14),

$$(z|\psi^{+}) \; = \; \sqrt{\frac{\theta}{2\pi\hbar^2}}\left(e^{-\frac{\theta}{4\hbar^2}|q|^2+\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(q\bar{z}+\bar{q}z)} + \sqrt{i}\frac{e^{i\frac{\sqrt{2\theta}}{\hbar}|q||z|}}{\sqrt{\frac{\sqrt{2\theta}}{\hbar}|q||z|}}f(q,p)\right), \tag{2.15}$$

which evidently has the form of a incoming plane wave plus an outgoing circular wave. The amplitude of the latter is given by

$$f(q,p) \; = \; -\frac{m\theta}{\hbar^2}\sqrt{\frac{2}{\pi}}\sqrt{\frac{2\pi\hbar^2}{\theta}}\int d^2w\, e^{-\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p\bar{w}+\bar{p}w)}(w|\hat{V}|\psi^{+}). \tag{2.16}$$

This is valid in the limit where the point of observation is far outside the scattering potential, and (2.16) can be shown to coincide exactly with the result of [27] in the commutative limit.

We may now proceed to find an expression for the differential scattering cross-section in terms of the scattering amplitude (2.16). To do this we need to calculate the currents

j_z and $j_{\bar{z}}$ for both the incident and scattered particles. For the incident particles this is straightforward: Using (2.5) we have

$$\begin{aligned} j_z^{in} &= \frac{\hbar}{2i\pi m\theta}\frac{\theta}{2\pi\hbar^2}\left[e^{-\frac{\theta}{4\hbar^2}|q|^2-\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(q\bar{z}+\bar{q}z)}\star_z\left(\frac{\partial}{\partial\bar{z}}e^{-\frac{\theta}{4\hbar^2}|q|^2+\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(q\bar{z}+\bar{q}z)}\right)\right. \\ &\quad \left. -\left(\frac{\partial}{\partial\bar{z}}e^{-\frac{\theta}{4\hbar^2}|q|^2-\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(q\bar{z}+\bar{q}z)}\right)\star_z e^{-\frac{\theta}{4\hbar^2}|q|^2+\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(q\bar{z}+\bar{q}z)}\right] \\ &= \frac{1}{4i\pi^2 m\hbar}\left[e^{-\frac{\theta}{4\hbar^2}|q|^2-\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(q\bar{z}+\bar{q}z)}e^{\frac{\theta}{2\hbar^2}|q|^2}\left(\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}qe^{-\frac{\theta}{4\hbar^2}|q|^2+\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(q\bar{z}+\bar{q}z)}\right)\right. \\ &\quad \left. -\left(-\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}qe^{-\frac{\theta}{4\hbar^2}|q|^2-\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(q\bar{z}+\bar{q}z)}\right)e^{\frac{\theta}{2\hbar^2}|q|^2}e^{-\frac{1}{4\hbar^2}|q|^2+\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(q\bar{z}+\bar{q}z)}\right] \\ &= \frac{\sqrt{2\theta}q}{4\pi^2 m\hbar^2}, \end{aligned} \tag{2.17}$$

and similarly

$$j_{\bar{z}}^{in} = \frac{\sqrt{2\theta}\bar{q}}{4\pi^2 m\hbar^2}. \tag{2.18}$$

We may, without loss of generality, assume that the incoming current has only an x-component. Then

$$\begin{aligned} |j^{in}| &= |\frac{1}{2}\left(j_z^{in}+j_{\bar{z}}^{in}\right)| \\ &= \frac{\sqrt{2\theta}|q|}{4\pi^2 m\hbar^2}. \end{aligned} \tag{2.19}$$

The corresponding calculation for the outgoing scattered current is considerably more complicated and is provided in C.2.4. The final result is

$$\begin{aligned} j_r &= \frac{1}{4\pi^2 m\hbar|z|}e^{\frac{\theta}{2\hbar^2}|q|^2}|f(q,p)|^2, \\ j_\phi &= 0. \end{aligned} \tag{2.20}$$

Accordingly, by substituting (2.19) and (2.20) into (2.9), with $|r| = |z|\sqrt{2\theta}$, we obtain the differential cross section

$$\frac{d\sigma}{d\phi} = \frac{\hbar}{|q|}e^{\frac{\theta}{2\hbar^2}|q|^2}|f(q,p)|^2. \tag{2.21}$$

2.2.1 The Transfer Matrix

The Lippmann-Schwinger equation gives an implicit expression for the wavefunction of the full scattering Hamiltonian. This suggests that the full wavefunction could be obtained recursively. In C.2.5 it is shown that the scattering amplitude (2.16) may be expressed as

$$f(q,p) \quad = \quad -2m\pi^2\sqrt{\frac{2}{\pi}}e^{-\frac{\theta}{4\hbar^2}|p|^2}(p|\hat{V}|\psi^+). \tag{2.22}$$

Now, let us define the operator $\hat{T}$, such that

$$\hat{V}|\psi^+) \quad = \quad \hat{T}|q), \tag{2.23}$$

where $|q)$ is the incident plane wave. Then, acting on (2.13) from the left with $\hat{V}$ yields

$$\hat{T}|q) \quad = \quad \hat{V}|q) + \hat{V}\frac{1}{E-\hat{H}_0+i\varepsilon}\hat{T}|q), \tag{2.24}$$

and hence we have the following implicit definition of $\hat{T}$:

$$\hat{T} \quad = \quad \hat{V} + \hat{V}\frac{1}{E-\hat{H}_0+i\varepsilon}\hat{T}. \tag{2.25}$$

Now, let us define $\hat{G} = (E-\hat{H}_0+i\varepsilon)^{-1}$. This is known as the free propagator [1]. We then see that we may write

$$\hat{T} \quad = \quad \hat{V} + \hat{V}\hat{G}\hat{V} + \hat{V}\hat{G}\hat{V}\hat{G}\hat{V} + \dots\, . \tag{2.26}$$

Correspondingly, we may write the scattering amplitude as

$$\begin{aligned} f(q,p) \quad &= \quad -2m\pi^2\sqrt{\frac{2}{\pi}}e^{-\frac{\theta}{4\hbar^2}|p|^2}(p|\hat{T}|q) \\ &= \quad -2m\pi^2\sqrt{\frac{2}{\pi}}e^{-\frac{\theta}{4\hbar^2}|p|^2}\left\{(p|\hat{V}|q) + (p|\hat{V}\hat{G}\hat{V}|q) + (p|\hat{V}\hat{G}\hat{V}\hat{G}\hat{V}|q) + \dots\right\} \end{aligned} \tag{2.27}$$

The latter form of the scattering amplitude has a clear intuitive interpretation: The incident particle may be scattered several times while traversing the scattering potential, before emerging on the other side. This give rise to the series of terms in the scattering

[1] A subscript 0 is often included to distinguish it from the full propagator, however this is not an issue here since we do not discuss the full or 'dressed' propagator.

amplitude. A term $\hat{V}$ appear for each scattering event, and in between these events the particle propagates in a manner governed by the propagator $\hat{G}$.

The above interpretation is the same as in commutative quantum mechanics, and can be found in text books like [24]. In fact, since the commutative and non-commutative momentum states are identical, one might be lead to conclude that the term $(p|\hat{T}|q)$ in (2.27) is exactly the same as in commutative quantum mechanics. In that case, when inserting (2.27) in (2.21) one obtains the exact same differential cross section as for commutative scattering, because the exponential terms cancel out. However, the series of scattering terms $(p|\hat{V}\hat{G}...\hat{V}|q)$ is actually not the same as in commutative quantum mechanics, because $\hat{V}$ is now dependent on the non-commuting position coordinates. The consequence of this will be illustrated in the next section, where we focus on the contribution of the first term $(p|\hat{V}|q)$ in (2.27).

2.2.2 The Born Approximation

We consider scattering potentials with a finite range, and hence, far from the potential a particle will move freely. One would therefore expect the full wavefunction of the scattering potential to be merely a perturbation of that of the free particle. Therefore, as a first attempt, we will insert the incident free wavefunction on the right hand side of the Lippmann-Schwinger equation, i.e.

$$|\psi) = |p) + \frac{1}{E - \hat{H}_0}\hat{V}|p), \tag{2.28}$$

where $|p)$ is the usual momentum state of an incident particle. We note that this is identical to only including the first term in the expansion of the operator $\hat{T}$ discussed in the previous section. In this approximation, known as the Born approximation, the scattering amplitude becomes (remembering that we assumed elastic scattering, i.e. $|p| = |q|$),

$$\begin{aligned} f(q,p) &= -2m\pi^2\sqrt{\frac{2}{\pi}}e^{-\frac{\theta}{4\hbar^2}|p|^2}(p|\hat{V}|q) \\ &= -4m\pi^2\sqrt{\frac{2}{\pi}}\frac{\theta}{2\pi\hbar^2}e^{\frac{\theta}{4\hbar^2}(|q|^2-2q\bar{p})}\widetilde{V}(\frac{\sqrt{2\theta}(q-q)}{\hbar}). \end{aligned} \tag{2.29}$$

The intermediate steps are shown in C.2.6, and $\widetilde{V}(\frac{\sqrt{2\theta}(q-q)}{\hbar})$ is the normal Fourier transform of $V(\bar{z}, z)$. Inserting this in (2.21) we obtain, as a first approximation to the differential scattering cross section,

$$\frac{d\sigma}{d\phi} = 8\pi\left(\frac{m\theta}{\hbar^2}\right)^2\frac{\hbar}{|q|}e^{\frac{\theta}{2\hbar^2}|q-p|^2}\left|\widetilde{V}\left(\frac{\sqrt{2\theta}(q-p)}{\hbar}\right)\right|^2. \quad (2.30)$$

We see that there is a modification to the commutative case, namely the exponential term $e^{\frac{\theta}{2\hbar^2}|q-p|^2}$. This originates in the modified transformation from the position basis to the momentum basis, as discussed in 1.2.3.

We can draw the following conclusions from (2.30):

- Given a scattering potential $V(\bar{z}, z)$, the differential scattering cross section will be different from what it would be in the commutative case. For elastic scattering ($|q| = |p|$) the additional term $e^{\frac{\theta}{2\hbar^2}|q-p|^2}$ may be written as $e^{\frac{\theta}{\hbar^2}|q|^2(1-cos\alpha)}$, where α is the angle between the incident and scattered particle. This implies that for all directions, except for forward, the scattering amplitude is enhanced. One could interpret this as the scatterer appearing to be bigger as a result of its fuzzy nature.

- If we did not know that space is non-commutative we would take the differential cross section to be of the form $d\sigma/d\theta \propto |V_{eff}|^2$, where $\widetilde{V}_{eff} = \frac{1}{2\pi}e^{\frac{\theta}{4\hbar^2}|q-p|^2}\widetilde{V}$, i.e. without the extra exponential term. We might then attempt to recover the form of the scattering potential by taking the Fourier transform of $\widetilde{V}_{eff}$. What one should actually do is to Fourier transform $\frac{1}{2\pi}e^{-\frac{\theta}{4\hbar^2}|q-p|^2}\widetilde{V}_{eff}$. This will improve the convergence of the integral involved and might in some cases render what would be a diverging potential in commutative quantum mechanics finite.

- By Fourier transforming $\widetilde{V} = \frac{1}{2\pi}e^{-\frac{\theta}{4\hbar^2}|q-p|^2}\widetilde{V}_{eff}$ one finds that the non-commutative potential is a Gaussian convolution of the effective potential V_{eff}, i.e. $V(\bar{z}, z) = \frac{2}{\pi}\int d^2w\ e^{-2|w|^2}V_{eff}(\bar{z}+\bar{w}, z+w)$. It appears that, to first order, one could simulate non-commutativity by blurring the potentials in conventional quantum mechanics.

These conclusions should also extend to the higher order scattering contributions in (2.27). The second point might suggest a means to test our non-commutative theory, by analysing experimental scattering data in the two frameworks and comparing the results.

2.2.3 The Optical Theorem

To conclude this chapter, we look at the optical theorem in non-commutative quantum mechanics. The optical theorem is useful because it provides a means to obtain the total cross section from the differential cross section without having to integrate the latter. In commutative quantum mechanics in two dimensions, the optical theorem reads [27]:

$$\sigma_{tot} = \frac{\sqrt{8\pi}\hbar}{|q|}\mathrm{Im}\left(f(\alpha = 0)\right), \qquad (2.31)$$

where α is the scattering angle. We show in C.2.7 that the corresponding relationship in non-commutative quantum mechanics is

$$\sigma_{tot} = \frac{\sqrt{8\pi}\hbar}{|q|}e^{\frac{\theta}{4\hbar^2}|p|^2}\mathrm{Im}\left(f(\alpha = 0)\right). \qquad (2.32)$$

We see, therefore, that the total cross section in non-commutative quantum mechanics is modified in a similar way to the differential cross-section, if we take into account the exponential terms hidden in f. We also note that (2.31) and (2.32) coincide in the commutative limit.

Chapter 3
Many-Particle Formalism

In Chapter 1 we developed the formalism of single-particle non-commutative quantum mechanics and established a consistent interpretational framework, which was subsequently applied in Chapter 2 to develop the theory of potential scattering. In this chapter we venture into new territory by extending this formalism to many-particle systems, and eventually we will make contact with non-commutative field theory.

3.1 Bosons and Fermions

An important ingredient of quantum mechanics, which only enters at the level of many-particle systems, is the concept of indistinguishability of particles. In classical mechanics we speak of identical particles, however, we think of them as entities that may in principle be labelled (i.e. by a number on a billiard ball) to keep track of the individual particles' trajectories. In quantum mechanics the meaning of the word 'indistinguishable' is more literal: there is no way, even in principle, that two identical particles, such as two electrons, can be distinguished. If two identical particles are brought close to each other their wavefunctions would overlap and make it impossible to tell which is where. Even if the particles are far apart they may be entangled, as is illustrated by the famous Einstein-Podolsky-Rosen paradox [31]. The resolution is to not think of the particles as separate entities but rather consider the system as a whole and describe it by a single wavefunction.

As an extension of the notation of Chapter 1 we will write a N-particle wavefunction in the position representation as $\psi(z_1, ...z_N)$, where $\{z_1, ...z_N\}$ are complex, dimensionless particle position coordinates. Certain restrictions are imposed on such wavefunctions by the fact that particles are indistinguishable. Since we may not label the particles uniquely, the physical state should not change under arbitrary permutations of the particle labels. Swapping two particle labels around should at most multiply the wavefunction with a phase factor, and swapping them back should return the original wavefunction [32]. Mathematically,

$$\begin{aligned}\psi(z_1, ..., z_a, ..., z_b, ...z_N) &= \lambda\psi(z_1, ..., z_b, ..., z_a, ...z_N) \\ &= \lambda^2\psi(z_1, ..., z_a, ..., z_b, ...z_N), \end{aligned} \tag{3.1}$$

and we conclude that $\lambda^2 = 1$ and $\lambda = \pm 1$[1] Hence, there are two kinds of particles, namely bosons, having symmetric wavefunctions:

$$\psi(z_1, ..., z_a, ..., z_b, ...z_N) = +\psi(z_1, ..., z_b, ..., z_a, ...z_N); \tag{3.2}$$

and fermions, having antisymmetric wavefunctions:

$$\psi(z_1, ..., z_a, ..., z_b, ...z_N) = -\psi(z_1, ..., z_b, ..., z_a, ...z_N). \tag{3.3}$$

From (3.3) we immediately derive Pauli's exclusion principle: Two fermions may not be in the same state. (3.2), however, allows an arbitrary number of bosons to occupy the same state. Some of the most interesting features of quantum mechanics follows from this so called 'quantum statistics' of bosons and fermions.

It is appropriate to mention at this point that much work has been done on so called 'twisted statistics' in the context of non-commutative quantum mechanics, as a way to restore Poincaré invariance [14]. If one insists that the non-commutative parameter be an observer-independent constant of nature, such a twisting is required in three or more dimensions. In the non-commutative plane the coordinate commutation relation is already rotationally and translationally invariant, and twisting therefore becomes optional. We chose to not introduce twisting in this thesis, thereby allowing us to study other effects of non-commutativity separately from the effects caused by twisted statistics.

3.2 The N-Particle Classical Configuration Space

In Chapter 1 we defined the single-particle quantum Hilbert space $\mathcal{H}_q$ as the set of bounded Hilbert-Schmidt operators $\psi(b^\dagger, b)$, acting in on the classical configuration space $\mathcal{H}_c = span\,\{|n\rangle\}_{n=o}^{\infty}$. In commutative quantum mechanics, if the configuration space of a single particle is $\mathbb{R}^2$, the configuration space of a N-particle system would be $\mathbb{R}^{2N} = \otimes^N \mathbb{R}^2$. In complete analogy we therefore define the N-particle non-commutative configuration space as

$$\begin{aligned} \mathcal{H}_c^N &= \otimes^N \mathcal{H}_c \\ &= \text{span}\,\{|n_1, ..., n_N\rangle\}_{n_1,...,n_N=0}^{\infty}, \end{aligned} \tag{3.4}$$

[1] Strictly speaking, for a two-dimensional system the classical configuration space is not simply connected, and other phase factors are possible, leading to so called anyon statistics. This is a subtlety, however, and for the purpose of ths thesis we will only allow for pure bosonic or fermionic states.

where $|n_1, ..., n_N\rangle = |n_1\rangle \otimes ... \otimes |n_N\rangle$ and, as before, the span is over the complex field. In particular, this space contains the N-particle coherent states

$$|z_1, ..., z_N\rangle \quad = |z_1\rangle \otimes ... \otimes |z_N\rangle \,. \tag{3.5}$$

An irreducible set of operators acting in on $\mathcal{H}_c^N$ is formed by $\{b_1, b_1^\dagger, ..., b_N, b_N^\dagger\}$, with

$$\begin{aligned} b_i &= 1 \otimes ... \overbrace{\otimes b \otimes}^{i^{th} position} ... \otimes 1, \\ b_i^\dagger &= 1 \otimes ... \underbrace{\otimes b^\dagger \otimes}_{i^{th} position} ... \otimes 1, \end{aligned} \tag{3.6}$$

acting according to

$$\begin{aligned} b_i \, |z_1, ..., z_N\rangle &= z_i \, |z_1, ..., z_N\rangle \,, \\ \langle z_1, ..., z_N| \, b_i^\dagger &= \langle z_1, ..., z_N| \, \bar{z}_i. \end{aligned} \tag{3.7}$$

We may express the identity operator on $\mathcal{H}_c^N$ as

$$\hat{I}_c^N = \frac{1}{\pi^N} \int d^2 z_1 ... d^2 z_N \, |z_1, ..., z_N\rangle \langle z_1, ..., z_N| \tag{3.8}$$

A general operator acting in on the classical configuration space $\mathcal{H}_c^N$ is a function of all the b_i's and $b_i^\dagger$'s. A simple generalisation of (1.28) allows us to write any such operator as

$$\hat{O} \;=\; \frac{1}{\pi^N} \int d^2 z_1 ... d^2 z_N \left(e^{-\sum_{i=1}^N \partial_{\bar{z}_i} \partial_{z_i}} O(z_1, ..., z_N) \right) |z_1, ..., z_N\rangle \langle z_1, ..., z_N| \,, \tag{3.9}$$

i.e. diagonally in terms of $|z_1, ..., z_N\rangle \langle z_1, ..., z_N|$, and with the definition $O(z_1, ..., z_N) = \langle z_1, ..., z_N| \, \hat{O} \, |z_1, ..., z_N\rangle$ (implying again normal ordering).

3.3 The N-Particle Quantum Hilbert Space

As in the single-particle formulation of Chapter 1, the vectors of our N-particle quantum Hilbert space will be operators $\psi(b_1^\dagger, b_1, ..., b_N^\dagger, b_N)$ acting in on the N-particle classical configuration space $\mathcal{H}_c^N$. Evidently, any such operator is an element of the space $\mathcal{H}_q^N = \otimes^N \mathcal{H}_q$, however, physical states must be either bosonic or fermionic, and, accordingly, we impose the condition

$$\psi(...,b_i^\dagger,b_i,...,b_j^\dagger,b_j,...) \;=\; \pm\psi(...,b_j^\dagger,b_j,...,b_i^\dagger,b_i,...), \tag{3.10}$$

where the $+$ is for bosons and the $-$ is for fermions. This ensures that $\psi(z_1,...,z_N) = \langle z_1,...,z_N|\,\hat{\psi}\,|z_1,...,z_N\rangle$ satisfies either (3.2) or (3.3), and we will think of $\psi(z_1,...,z_N)$ as the non-commutative analogue of a many-particle wavefunction in position space. Let us formally define the N-particle non-commutative quantum Hilbert space for bosons and fermions as

$$\begin{aligned}\mathcal{H}_q^{N\pm} \;=\; & \Big\{\psi(b_1^\dagger,b_1,...,b_N^\dagger,b_N)\in\otimes^N\mathcal{H}_q : \\ & \psi(...,b_i^\dagger,b_i,...,b_j^\dagger,b_j,...) = \pm\psi(...,b_j^\dagger,b_j,...,b_i^\dagger,b_i,...)\forall i,j\in\{1,...,N\}\Big\}.\end{aligned} \tag{3.11}$$

The inner product is induced by the tensor product structure and the inner product on $\mathcal{H}_q$:

$$(\psi|\phi) = tr_{H_c^N}\Big\{\psi(b_1^\dagger,b_1,...,b_N^\dagger,b_N)^\dagger\phi(b_1^\dagger,b_1,...,b_N^\dagger,b_N)\Big\}. \tag{3.12}$$

3.3.1 A Position Basis for the N-Particle Quantum Hilbert Space

By defining the inner product (3.12) we are associating bras and kets with the elements of the N-particle quantum Hilbert space. As a natural generalisation of section 1.2.1, let us therefore define

$$\begin{aligned}|z_1,...,z_N) &= |z_1,...,z_N\rangle\langle z_1,...,z_N| \\ &= |z_1\rangle\langle z_1|\otimes...\otimes|z_N\rangle\langle z_N|.\end{aligned} \tag{3.13}$$

This allows us to write $\psi(z_1,...,z_N) = \langle z_1,...,z_N|\,\hat{\psi}\,|z_1,...,z_N\rangle = (z_1,...,z_N|\,\psi)$. [2] Using (3.9) we can then write arbitrary vectors in the quantum Hilbert space as

$$\begin{aligned}|\psi) &= \frac{1}{\pi^N}\int d^2z_1...d^2z_N\left(e^{-\sum_{i=1}^N\partial_{\bar{z}_i}\partial_{z_i}}\psi(z_1,...,z_N)\right)|z_1,...,z_N) \\ &= \frac{1}{\pi^N}\int d^2z_1...d^2z_N\,|z_1,...,z_N)\,e^{\sum_{i=1}^N\overleftarrow{\partial_{z_i}}\overrightarrow{\partial_{\bar{z}_i}}}\psi(z_1,...,z_N) \\ &= \frac{1}{\pi^N}\int d^2z_1...d^2z_N\,|z_1,...,z_N)\star_{z_1,...,z_N}(z_1,...,z_N|\,\psi),\end{aligned} \tag{3.14}$$

[2]This wavefunction obviously depend on the conjugate coordinates $\bar{z}_i$ as well, but for the purpose of shortening our expressions we will coomonly only write for example $\psi(z)$ when we mean $\psi(\bar{z},z)$.

where, in the second line, we have used integration by parts to change the direction of action of the ∂_{z_i}'s, and in the third line we have introduced the N-particle Voros star product

$$\star_{z_1,...,z_N} = e^{\sum_{i=1}^N \overleftarrow{\partial_{z_i}}\overrightarrow{\partial_{\bar{z}_i}}}. \tag{3.15}$$

We see from (3.14) that the N-particle identity operator may be written as

$$\hat{I}_q^N = \frac{1}{\pi^N}\int d^2z_1...d^2z_N\ |z_1,...,z_N)\star_{z_1,...,z_N}(z_1,...,z_N|\,. \tag{3.16}$$

The states $|z_1,...,z_N)$ are not symmetric or antisymmetric, and hence not physical. The identity operator (3.16) spans all of $\otimes^N\mathcal{H}_q$, while we would like to restrict ourselves to the bosonic or fermionic subspaces $\mathcal{H}_q^{N\pm}$. Let us therefore introduce the states

$$|z_1,...,z_N)_\pm = \frac{1}{\sqrt{N!}}\sum_{P\in S(N)}(\pm)^{\epsilon_P}|z_{P(1)},...,z_{P(N)}). \tag{3.17}$$

Here the sum is over all permutations of the N complex coordinates, i.e. P is an element of the permutation group $S(N)$, and $z_{P(i)}$ is the new i-th coordinate after permutation. ϵ_P is the parity of the permutation, i.e.

$$\epsilon_P = \begin{cases} 0 \text{ if P is an even permutation (even number of transpositions)} \\ 1 \text{ if P is an odd permutation (odd number of transpositions)} \end{cases}$$

With a + (3.17) becomes symmetric (i.e. bosonic), and with a - it becomes antisymmetric (i.e. fermionic). The overlap of two such states is (see C.3.1)

$$_\pm(z_1,...,z_N|\,w_1,...,w_N)_\pm = \sum_{P\in S(N)}(\pm)^{\epsilon_P}\prod_{i=1}^N e^{-|z_i-w_{P(i)}|^2}, \tag{3.18}$$

and in general, for a bosonic or fermionic state $|\psi)$ which is symmetric or antisymmetric according to (3.10),

$$_\pm(z_1,...,z_N|\psi) = \sqrt{N!}(z_1,...,z_N|\psi). \tag{3.19}$$

Note that this does not mean that $|z_1,...,z_N)_\pm = \sqrt{N!}|z_1,...,z_N)$! Because of (anti-)symmetry it is possible to write $_\pm(z_1,...,z_N|\psi) = \frac{1}{N!}\sum_{P\in S(N)}(\pm)^{\epsilon_P}{}_\pm(z_{P(1)},...,z_{P(N)}|\psi)$.

We may use this and (3.19) to rewrite (3.14):

$$
\begin{aligned}
|\psi) &= \frac{1}{\pi^N} \int d^2z_1 ... d^2z_N \, |z_1, ..., z_N) \star_{z_1,...,z_N} (z_1, ..., z_N| \, \psi) \\
&= \frac{1}{N!\pi^N} \sum_{P \in S(N)} (\pm)^{\epsilon_P} \int d^2z_1 ... d^2z_N \, |z_1, ..., z_N) \star_{z_1,...,z_N} (z_{P(1)}, ..., z_{P(N)}|\psi) \\
&= \frac{1}{N!\pi^N} \sum_{P \in S(N)} (\pm)^{\epsilon_P} \int d^2z_1 ... d^2z_N |z_{P(1)}, ..., z_{P(N)}) \star_{z_1,...,z_N} (z_1, ..., z_N|\psi) \\
&= \frac{1}{N!\pi^N} \int d^2z_1 ... d^2z_N \, |z_1, ..., z_N)_{\pm} \star_{z_1,...,z_N} {}_{\pm}(z_1, ..., z_N| \, \psi) \, . \qquad (3.20)
\end{aligned}
$$

From the second to the third line we have relabelled the integration variables, in order to shift the permutations from the bra to the ket. The final form makes it explicit that $|\psi)$ is diagonal in the bosonic or fermionic N-particle position states $|z_1, ..., z_N)_{\pm}$. The operators

$$
\hat{I}_q^{N\pm} = \frac{1}{N!\pi^N} \int d^2z_1 ... d^2z_N \, |z_1, ..., z_N)_{\pm} \star_{z_1,...,z_N} {}_{\pm}(z_1, ..., z_N| \, , \qquad (3.21)
$$

project states onto the symmetric or antisymmetric subspace of $\mathcal{H}_q^N$, and therefore play the roles of the identity operators on our quantum Hilbert spaces $\mathcal{H}_q^{N\pm}$.

3.3.2 N-Particle Position Measurement

In Chapter 1 position measurement in non-commutative quantum mechanics was defined in terms of the POVM $\pi_z = \frac{1}{\pi}|z) \star_z (z|$. The form of this POVM was derived from the identity operator on the single particle quantum Hilbert space. In a similar way, we may construct POVMs on a N-particle space from resolutions of the N-particle identity operator. However, the question arises whether one should use the full identity operator on $\mathcal{H}_q^N$ or the identity operator on the bosonic or fermionic subspaces? Suppose we construct a POVM from (3.16):

$$
\pi_{z_1,...,z_N} = \frac{1}{\pi^N} |z_1, ..., z_N) \star_{z_1,...,z_N} (z_1, ..., z_N|, \qquad (3.22)
$$

According to this definition, the probability of measuring N particles in the state $|\psi)$ at coordinates $z_1, ..., z_n$ is

$$
\begin{aligned}
P(z_1, ..., z_N) &= tr_{\mathcal{H}_c^N} \left(\pi_{z_1,...,z_N} |\psi)(\psi| \right) \\
&= \frac{1}{\pi^N} (\psi|z_1, ..., z_N) \star_{z_1,...,z_N} (z_1, ..., z_N|\psi). \qquad (3.23)
\end{aligned}
$$

Suppose instead we construct a POVM from (3.21):

$$\pi^{\pm}_{z_1,...,z_N} = \frac{1}{N!\pi^N} \left|z_1,...,z_N\right)_{\pm} \star_{z_1,...,z_N} {}_{\pm}\left(z_1,...,z_N\right|. \quad (3.24)$$

Then the probability of finding the N particles at coordinates $\{z_1,...,z_N\}$ is given by

$$\begin{aligned} P(z_1,...,z_N) &= tr_{\mathcal{H}_c^N}\left(\pi^{\pm}_{z_1,...,z_N}|\psi)(\psi|\right) \\ &= \frac{1}{N!\pi^N}(\psi|z_1,...,z_N)_{\pm} \star_{z_1,...,z_N} {}_{\pm}(z_1,...,z_N|\psi). \end{aligned} \quad (3.25)$$

However, if $|\psi)$ is a bosonic or fermionic state, then (3.23) and (3.25) are identical by insertion of (3.19). As a result, (3.22) and (3.24) measure the same quantity. The latter form (3.24) makes it explicit that $\pi^{\pm}_{z_1,...,z_N}$ does not distinguish which particle is where, but merely measures the probabilities of finding the N particles at the given coordinates, however, this is already implicit in the bosonic or fermionic nature of $|\psi)$. While the former version (3.22) may be applied to both fermionic and bosonic states, the latter version (3.24) sees either only bosons (+) or only fermions (-).

Rather than asking about the position of N particles it might be more natural to ask about the particle density at a position z. We do this by averaging out the probability of any of the N coordinates to be equal to z, i.e. we define the particle density operator for bosons or fermions:

$$\begin{aligned} \hat{\rho}^{\pm}_N(z) &= \int d^2z_1...d^2z_N \sum_{a=1}^{N} \delta(z-z_a)\pi^{\pm}_{z_1,...,z_N} \\ &= N\int d^2z_2...d^2z_N\ \pi^{\pm}_{z,z_2,...,z_N} \\ &= \frac{1}{(N-1)!\pi^N}\int d^2z_2...d^2z_N\ |z,z_2,...,z_N)_{\pm} \star_{z,z_2,...,z_N} {}_{\pm}(z,z_2,...,z_N|. \end{aligned} \quad (3.26)$$

We see that $\int d^2z\ \hat{\rho}^{\pm}_N(z) = N\hat{I}^{\pm}_N$, which confirms that there are N bosons or fermions in all of space.

3.4 Second Quantisation

In the previous section we regarded the particle number N as being fixed and proceeded to construct N-particle quantum states. In this section we will develop the formalism of 'second quantisation', which allows particles to be created and destroyed. Our development will follow closely that of Brown [33] for commutative quantum mechanics.

3.4.1 The Particle Fock Space

Let us start by enlarging our quantum Hilbert space to include states of arbitrary particle number:

$$\mathcal{H}_Q^{\pm} \;=\; \mathcal{H}_q^{0\pm} \oplus \mathcal{H}_q^{1\pm} \oplus \mathcal{H}_q^{2\pm} \oplus \mathcal{H}_q^{3\pm} \oplus \ldots . \tag{3.27}$$

Here $\mathcal{H}_q^{i\pm}$ is defined according to (3.11), and we have included the zero-particle space $\mathcal{H}_q^{0\pm}$. The latter contains only one state, the so called vacuum-state, which we will denote by $|\Omega)$. Given two states $|z_1, ..., z_N)_\pm, |w_1, ..., w_M)_\pm \in \mathcal{H}_Q^{\pm}$ the overlap is

$${}_\pm(z_1, ..., z_N|w_1, ..., w_M)_\pm \;=\; \delta_{N,M}\,{}_\pm(z_1, ..., z_N|w_1, ..., w_M)_\pm, \tag{3.28}$$

i.e. states of different particle number are orthogonal.

3.4.2 Particle Creation and Annihilation Operators

Let us define the following operator acting in on the enlarged quantum Hilbert space $\mathcal{H}_Q^{\pm}$:

$$\hat{\psi}^\ddagger(z) \;=\; \sum_{n=0}^{\infty} \frac{1}{N!\pi^N} \int d^2z_1 ... d^2z_N |z, z_1, ..., z_N)_\pm \star_{z_1,...,z_N} {}_\pm(z_1, ..., z_N|. \tag{3.29}$$

Note that the bra on the right is a N-particle state, while the ket on the left is a $(N+1)$-particle state. [3] The action of this operator on an arbitrary state $|z, ..., z_N)_\pm$ is

$$\hat{\psi}^\ddagger(z)|z_1, ..., z_N)_\pm \;=\; |z, z_1, ..., z_N)_\pm, \tag{3.30}$$

as is shown in C.3.2. It adds a new particle at position z to the state $|z, ..., z_N)_\pm$, and hence we will call it the particle creation operator. In fact, any ket $|z, ..., z_N)_\pm$ can be written in terms of creation operators acting in on the vacuum ket:

$$|z_1, ..., z_N)_\pm \;=\; \hat{\psi}^\ddagger(z_1)...\hat{\psi}^\ddagger(z_N)|\Omega). \tag{3.31}$$

[3] We have not included a label $\pm$ to indicate the bosonic or fermionic nature of the operator $\hat{\psi}^\ddagger(z)$. It will, however, always be clear from the context which one we mean.

From (3.31) we deduce the following commutation relation for the creation operators:

$$[\hat{\psi}^{\ddagger}(z), \hat{\psi}^{\ddagger}(w)]_{\mp} \;=\; 0, \tag{3.32}$$

because $|..., z_i, z_{i+1}, ...)_{\pm} = ...\hat{\psi}^{\ddagger}(z_a)\hat{\psi}^{\ddagger}(z_b)...|\Omega) = \pm|..., z_b, z_a, ...)_{\pm} = \pm...\hat{\psi}^{\ddagger}(z_b)\hat{\psi}^{\ddagger}(z_a)...|\Omega)$.

Let us next consider the Hermitian conjugate of (3.29), namely

$$\hat{\psi}(z) \;=\; \sum_{N=0}^{\infty} \frac{1}{N!\pi^N} \int d^2z_1...d^2z_N |z_1, ..., z_N)_{\pm} \star_{z_1,...,z_N} {}_{\pm}(z, z_1, ..., z_N|. \tag{3.33}$$

In this operator the ket to the left has one particle less than the bra on the right. By simple Hermitian conjugation of (3.31) we must have

$${}_{\pm}(z_1, ..., z_N| \;=\; (\Omega|\hat{\psi}(z_N)...\hat{\psi}(z_1), \tag{3.34}$$

and subsequently

$$[\hat{\psi}(z), \hat{\psi}(w)]_{\mp} \;=\; 0. \tag{3.35}$$

If we let $\hat{\psi}(z)$ act forward on a ket $|z, ..., z_N)_{\pm}$ we obtain

$$\hat{\psi}(z)|z_1, ..., z_N)_{\pm} \;=\; \sum_{a=1}^{N} (\pm)^{a+1} e^{-|z-z_a|^2} |z_1, ..., z_{a-1}, z_{a+1}, ..., z_N)_{\pm}, \tag{3.36}$$

as shown in C.3.3. We end up with a state of particle number $N-1$, and hence we call $\hat{\psi}(z)$ an annihilation operator. The operator $\hat{\psi}(z)$ removes the a-th particle of $|z, ..., z_N)_{\pm}$ with a probability dependent on the distance $|z - z_a|$. We cannot tell with certainty which particle has been removed, and therefore we must end up in a superposition of all the possible states that one obtains by removing one of the particles. In the commutative limit the Gaussian factor $e^{-|z-z_a|^2}$ would tend to a Dirac delta function, and a particle could only be annihilated if $z = z_a$.

It is straightforward to derive the commutation relation between an annihilation and a creation operator. This is shown in C.3.4, and the result is

$$[\hat{\psi}(z), \hat{\psi}^{\ddagger}(w)]_{\mp} \;=\; e^{-|z-w|^2}. \tag{3.37}$$

Compare this to the usual commutation relation $[\hat{\psi}(\vec{x}), \hat{\psi}^{\dagger}(\vec{x}')]_{\mp} = \delta(\vec{x}-\vec{x}')$ of conventional quantum mechanics [33].

3.4.3 Momentum Creation and Annihilation Operators

In section 1.2.3 we defined the momentum states $|p) = \sqrt{\frac{\theta}{2\pi^3\hbar^2}} \int d^2z e^{\frac{\theta}{4\hbar^2}|p|^2} e^{\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p\bar{z}+\bar{p}z)}|z)$. The fact that $|z) = \hat{\psi}^\dagger(z)|\Omega)$ suggest that we could write $|p) = \hat{a}_p^\ddagger|\Omega)$, with

$$\hat{a}_p^\ddagger = \sqrt{\frac{\theta}{2\pi^3\hbar^2}} \int d^2z e^{\frac{\theta}{4\hbar^2}|p|^2} e^{\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p\bar{z}+\bar{p}z)}\hat{\psi}^\ddagger(z), \tag{3.38}$$

and similarly we could introduce the Hermitian conjugate operator

$$\hat{a}_p = \sqrt{\frac{\theta}{2\pi^3\hbar^2}} \int d^2z e^{\frac{\theta}{4\hbar^2}|p|^2} e^{-\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p\bar{z}+\bar{p}z)}\hat{\psi}(z). \tag{3.39}$$

It is straightforward to check, using the commutation relations (3.32), (3.35) and (3.37), that the above pair of operators satisfy the commutation relations

$$\begin{aligned} [\hat{a}_p, \hat{a}_q]_\mp &= 0, \\ \left[\hat{a}_p^\ddagger, \hat{a}_q^\ddagger\right]_\mp &= 0, \\ \left[\hat{a}_p, \hat{a}_q^\ddagger\right]_\mp &= \delta^2(p-q). \end{aligned} \tag{3.40}$$

These operators therefore create and annihilate particles with momentum p and can be used to construct bosonic or fermionic N-particle states as follows:

$$|p_1, ..., p_N)_\pm = \hat{a}_{p_1}^\ddagger ... \hat{a}_{p_N}^\ddagger|\Omega). \tag{3.41}$$

Since $|z) = \int d^2p|p)(p|z)$ we derive the inverse transformation of (3.29) and (3.33) to be

$$\begin{aligned} \hat{\psi}(z) &= \sqrt{\frac{\theta}{2\pi\hbar^2}} \int d^2p e^{-\frac{\theta}{4\hbar^2}|p|^2} e^{-\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p\bar{z}+\bar{p}z)}\hat{a}_p, \\ \hat{\psi}^\dagger(z) &= \sqrt{\frac{\theta}{2\pi\hbar^2}} \int d^2p e^{-\frac{\theta}{4\hbar^2}|p|^2} e^{\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p\bar{z}+\bar{p}z)}\hat{a}_p^\ddagger. \end{aligned} \tag{3.42}$$

3.4.4 The Relationship Between Commutative and Non-Commutative Creation/Annihilation Operators

As discussed in Chapter 1, the momentum basis is the same in commutative and non-commutative quantum mechanics. Suppose we were working in the momentum basis and did not know that the position coordinates were non-commuting. We might then have proceeded as in commutative quantum mechanics and defined position creation and annihilation operators by a normal Fourier transform of (3.38) and (3.39):

$$\begin{aligned}\hat{\psi}_C(\vec{x}) &= \frac{1}{2\pi\hbar}\int d^2p e^{\frac{i}{\hbar}\vec{x}\cdot\vec{p}}\hat{a}_p,\\ \hat{\psi}_C^\dagger(r) &= \frac{1}{2\pi\hbar}\int d^2p e^{-\frac{i}{\hbar}\vec{x}\cdot\vec{p}}\hat{a}_p^\ddagger.\end{aligned} \tag{3.43}$$

These operators satisfy the commutation relations

$$\begin{aligned}[\hat{\psi}_C(\vec{x}), \hat{\psi}_C(\vec{x}')]_\mp &= 0,\\ [\hat{\psi}_C^\dagger(\vec{x}), \hat{\psi}_C^\dagger(\vec{x}')]_\mp &= 0,\\ [\hat{\psi}_C(\vec{x}), \hat{\psi}_C^\dagger(\vec{x}')]_\mp &= \delta(\vec{x}-\vec{x}').\end{aligned} \tag{3.44}$$

The inverse of the relations in (3.43) are obtained by an inverse Fourier transform and are

$$\begin{aligned}\hat{a}_p &= \frac{1}{2\pi\hbar}\int d^2p e^{-\frac{i}{\hbar}\vec{x}\cdot\vec{p}}\hat{\psi}_C(\vec{x}),\\ \hat{a}_p^\ddagger &= \frac{1}{2\pi\hbar}\int d^2p e^{\frac{i}{\hbar}\vec{x}\cdot\vec{p}}\hat{\psi}_C^\dagger(\vec{x}).\end{aligned} \tag{3.45}$$

Combining (3.42) and (3.45) we get

$$\begin{aligned}\hat{\psi}(z) &= \sqrt{\frac{2}{\pi\theta}}\int d^2r e^{-\frac{1}{\theta}|\vec{x}-\sqrt{2\theta}\vec{z}|^2}\hat{\psi}_C(\vec{x}),\\ \hat{\psi}^\dagger(z) &= \sqrt{\frac{2}{\pi\theta}}\int d^2r e^{-\frac{1}{\theta}|\vec{x}-\sqrt{2\theta}\vec{z}|^2}\hat{\psi}_C^\dagger(\vec{x}),\end{aligned} \tag{3.46}$$

where we have introduced the vectors $\vec{z} = (\mathrm{Re}(z), \mathrm{Im}(z))$. This provides us with a nice interpretation of the non-commutative creation operator $\hat{\psi}^\dagger(z)$: it is not creating a particle at the position z, but rather in a region around this point according to a sharply peaked Gaussian probability distribution. The non-commutative particle is in a superposition of commutative position states, in agreement with the interpretation of non-commutative position space as being fuzzy and the inability to localise a particle beyond a length scale of θ. This interpretation might be misleading, however, as the sharply defined position states of commutative quantum mechanics have no clear meaning if space is really non-commutative.

3.4.5 General Second Quantised Operators

In section 1.2.1 we showed that the non-commutative Schrödinger equation for a single particle can be expressed as

$$i\hbar\frac{\partial}{\partial t}(z|\psi) = -\frac{\hbar^2}{m\theta}\frac{\partial^2}{\partial_z\partial_{\bar{z}}}(z|\psi) + V(\bar{z},z)\star(z|\psi). \tag{3.47}$$

We now propose that the appropriate second quantised Hamiltonian operator for non-interacting particles in an external potential V is

$$\hat{H} = \frac{1}{\pi}\int d^2z\hat{\psi}^\dagger(z)\star_z\left(-\frac{\hbar^2}{m\theta}\frac{\partial^2}{\partial_z\partial_{\bar{z}}}\hat{\psi}(z)\right) + \hat{\psi}^\dagger(z)\star_z V(\bar{z},z)\star_z\hat{\psi}(z). \tag{3.48}$$

We will motivate the form of (3.48) by showing how it acts in on an N-particle wavefunction, and to do this we need to compute the commutator $[\hat{\psi}(w),\hat{H}]$. Using the identity $[AB,C] = A[B,C]_\mp \pm [A,C]_\mp B$ and the commutation relations (3.32), (3.35) and (3.37) of the position creation and annihilation operators we get

$$\begin{aligned}
[\hat{\psi}(w),\frac{1}{\pi}\int d^2z\hat{\psi}^\dagger(z)\star_z\left(\frac{\partial^2}{\partial_z\partial_{\bar{z}}}\hat{\psi}(z)\right)] &= \frac{1}{\pi}\int d^2z[\hat{\psi}(w),\hat{\psi}^\dagger(z)]_\mp\star_z\left(\frac{\partial^2}{\partial_z\partial_{\bar{z}}}\hat{\psi}(z)\right)\\
&= \frac{1}{\pi}\int d^2z\left(e^{-\partial_z\partial_{\bar{z}}}e^{-|z-w|^2}\right)\left(\frac{\partial^2}{\partial_z\partial_{\bar{z}}}\hat{\psi}(z)\right)\\
&= \int d^2z\delta(z-w)\left(\frac{\partial^2}{\partial_z\partial_{\bar{z}}}\hat{\psi}(z)\right)\\
&= \frac{\partial^2}{\partial_w\partial_{\bar{w}}}\hat{\psi}(w).
\end{aligned} \tag{3.49}$$

The property $e^{-\partial_z\partial_{\bar{z}}}e^{-|z-w|^2} = \pi\delta(z-w)$ is proved in B.2. Similarly, for the potential energy term,

$$\begin{aligned}
[\hat{\psi}(w),\frac{1}{\pi}\int d^2z\hat{\psi}^\dagger(z)\star_z V(\bar{z},z)\star_z\hat{\psi}(z)] &= \frac{1}{\pi}\int d^2z[\hat{\psi}(w),\hat{\psi}^\dagger(z)]_\mp\star_z V(\bar{z},z)\star_z\hat{\psi}(z)\\
&= \int d^2z\delta(z-w)V(\bar{z},z)\star_z\hat{\psi}(z)\\
&= V(\bar{w},w)\star_w\hat{\psi}(w).
\end{aligned} \tag{3.50}$$

Combining (3.49) and (3.50) we get

$$[\hat{\psi}(w),\hat{H}] = -\frac{\hbar^2}{m\theta}\frac{\partial^2}{\partial_w\partial_{\bar{w}}}\hat{\psi}(w) + V(\bar{w},w)\star_w\hat{\psi}(w). \tag{3.51}$$

We also note that $(\Omega|\hat{H} = 0$, because of the operator $\hat{\psi}^\ddagger$ being the leftmost operator in $\hat{H}$. As a result of this, and by repeated application of $[AB, C] = A[B, C] + [A, C]B$, we therefore have

$$\begin{aligned}
{}_\pm(z_1, ..., z_N|\hat{H}|\psi) &= (\Omega|\hat{\psi}(z_N)...\hat{\psi}(z_1), \hat{H}|\psi) \\
&= (\Omega|[\hat{\psi}(z_N)...\hat{\psi}(z_1), \hat{H}]|\psi) \\
&= (\Omega|\hat{\psi}(z_N)...[\hat{\psi}(z_1), \hat{H}]|\psi) \\
&\quad +(\Omega|\hat{\psi}(z_N)...[\hat{\psi}(z_2), \hat{H}]\hat{\psi}(z_1)|\psi) \\
&\quad +... + (\Omega|[\hat{\psi}(z_N), \hat{H}]\hat{\psi}(z_{N-1})...\hat{\psi}(z_1)|\psi) \\
&= \left\{\sum_{a=1}^{N}\left[-\frac{\hbar^2}{m\theta}\frac{\partial^2}{\partial_{z_a}\partial_{\bar{z}_a}} + V(\bar{z}_a, z_a)\star_{z_a}\right]\right\}{}_\pm(z_1, ..., z_N|\psi). \quad (3.52)
\end{aligned}$$

This is precisely what we would expect the Hamiltonian of a non-interacting many particle system to look like, with the total energy being the sum of the individual particles' energies.

For a two-particle interaction we can introduce the second quantised operator

$$\hat{W} = \frac{1}{2\pi^2}\int d^2zd^2w\hat{\psi}^\dagger(z)\hat{\psi}^\dagger(w) \star_{z,w} W(\bar{z}, z, \bar{w}, w) \star_{z,w} \hat{\psi}(w)\hat{\psi}(z). \quad (3.53)$$

As shown in C.3.5, this two-particle operator acts on a N-particle state according to

$${}_\pm(z_1, ..., z_N|\hat{W}|\psi) = \sum_{a=1}^{N}\sum_{b=a+1}^{N} W(\bar{z}_a, z_a, \bar{z}_b, z_b) \star_{z_a,z_b} {}_\pm(z_1, ..., z_N|\psi). \quad (3.54)$$

We would of course usually only consider interactions that depend on relative distance, in which case $W(\bar{z}, z, \bar{w}, w) = W(|z - w|)$.

As a last remark in this section, let us define the particle density operator

$$\begin{aligned}
\hat{\rho}(z) &= \frac{1}{\pi}\hat{\psi}^\ddagger(z) \star_z \hat{\psi}(z) \\
&= \sum_{N=0}^{\infty}\frac{1}{N!\pi^{N+1}}\int d^2z_1...d^2z_N\; |z, z_1, ..., z_N)_\pm \star_{z,z_1,...,z_N} {}_\pm(z, z_1, ..., z_N|. \quad (3.55)
\end{aligned}$$

We note that $\hat{\rho} = \sum_{N=0}^{\infty}\hat{\rho}_N^\pm$ where $\hat{\rho}_N^\pm$ is the N-particle density operator defined in 3.26. It is clear that that $|z_1, ..., z_N)_\pm$ is not an eigenstate of $\hat{\rho}$, so the action of measuring the particle density disturbs the system. This is in contrast to commutative quantum mechanics. We may, however, define the number operator

$$\hat{N} = \int d^2 z \hat{\rho}(z), \tag{3.56}$$

which returns the number of particles in a pure N-particle state, for any N, without altering the state, because $\hat{N} = \sum_{N=0}^{\infty} N \hat{I}_N^{\pm}$, with $\hat{I}_N^{\pm}$ the identity operator on the N-particle subspace.

3.4.6 Time Dependence

In order to describe the time evolution of a quantum system we need to amend the time-independent framework developed so far. We will introduce time dependence in accordance with the Heisenberg picture, where operators evolve in time according to

$$i\hbar \frac{\partial}{\partial t} \hat{O} = [\hat{O}, \hat{H}]. \tag{3.57}$$

In particular,

$$i\hbar \frac{\partial}{\partial t} \hat{\psi}(z,t) = [\hat{\psi}(z,t), \hat{H}]. \tag{3.58}$$

At any time t the equal-time commutation relations of the creation and annihilation operators are

$$\begin{aligned} [\hat{\psi}(z,t), \hat{\psi}(w,t)]_{\mp} &= 0, \\ [\hat{\psi}^{\ddagger}(z,t), \hat{\psi}^{\ddagger}(w,t)]_{\mp} &= 0, \\ [\hat{\psi}(z,t), \hat{\psi}^{\ddagger}(w,t)]_{\mp} &= e^{-|z-w|^2}. \end{aligned} \tag{3.59}$$

The analogous commutation relations for different time arguments are non-trivial and depend on the Hamiltonian governing their time evolution.

We may define a time dependent position state

$$_{\pm}(z_1, ..., z_N; t| = (\Omega| \hat{\psi}(z_N; t) ... \hat{\psi}(z_1; t). \tag{3.60}$$

The vacuum state is time independent, because $(\Omega|\hat{H} = 0$, and as a result, using (3.58),

$$i\hbar \frac{\partial}{\partial t} {}_{\pm}(z_1, ..., z_N; t| = {}_{\pm}(z_1, ..., z_N; t|\hat{H}. \tag{3.61}$$

Taking the inner product of both sides with a state $|\psi)$ gives the time dependent Schrödinger equation. For a time-independent Hamiltonian we get

$$ {}_{\pm}(z_1, ..., z_N; t| \;=\; \pm(z_1, ..., z_N; t_0|e^{-\frac{i(t-t_0)}{\hbar}\hat{H}}. \tag{3.62} $$

3.5 The Link With Non-Commutative Quantum Field Theory

With the second-quantisation formalism in place, it is straightforward to make a link with non-commutative quantum field theory. The aim of this section is merely to show that non-commutative quantum field theory can be derived from non-commutative quantum mechanics in the same way that commutative quantum field theory can be built from conventional quantum mechanics. The actual study and application of non-commutative field theory is beyond the scope of this thesis, and interested readers are referred to the vast amount of existing literature on this topic, for example [4]. Non-commutative field theory is closely linked to non-commutative geometry, which is an active field of research in mathematics. The standard reference for this theory is Connes [23].

3.5.1 Bosonic Fields

We define a bosonic field state by

$$ |\phi, t) \;=\; e^{\hat{D}_t[\phi]}|\Omega), \tag{3.63} $$

where

$$ \begin{aligned} \hat{D}_t[\phi] &= \frac{1}{\pi}\int d^2z \left\{\hat{\psi}^\ddagger(z,t) \star_z \phi(z) - \bar{\phi}(z) \star_z \hat{\psi}(z,t)\right\} \\ &= \int d^2p \left\{\hat{a}_p^\ddagger(t)\phi(p) - \bar{\phi}(p)\hat{a}_p(t)\right\}. \end{aligned} \tag{3.64} $$

$|\phi)$ has the form of a coherent state on the quantum Hilbert space and can be written as a superposition of states with different particle numbers. The fact that we cannot associate with $|\phi, t)$ a specific particle number leads to the interpretation of these states as fields. The square brackets around the argument in $\hat{D}_t[\phi]$ is used to emphasise the functional dependence of the operator on the whole function ϕ. The particular representation of ϕ is unimportant, and the position and momentum representations above are only two examples of representations of the operator $\hat{D}$. In fact, one can easily show that $\phi(z) = (z|\phi)$ and $\phi(p) = (p|\phi)$, i.e. the functions appearing in the operator $\hat{D}$ are the projections of the field operator $|\phi)$ onto single-particle states. One could therefore think of $e^{\hat{D}}$ as creating a many particle field state out of single particle wavefunctions when acting in on the vacuum state.

It is easy to check that $|\phi)$ is an eigenstate of the annihilation operator (in any representation). For example, in the position basis, $[\hat{\psi}(z,t), \hat{D}_t[\phi]] = \phi(z)$ and therefore, by the Baker-Campbell-Hausdorff formula, $e^{-\hat{D}_t[\phi]}\hat{\psi}(z,t)e^{\hat{D}_t[\phi]} = \hat{\psi}(z,t) + \phi(z)$. Consequently,

$$\begin{aligned} \hat{\psi}(z)|\phi,t) &= e^{\hat{D}_t[\phi]}e^{-\hat{D}_t[\phi]}\hat{\psi}(z,t)e^{\hat{D}_t[\phi]}|\Omega) \\ &= e^{\hat{D}_t[\phi]}\left\{\hat{\psi}(z,t) + \phi(z)\right\}|\Omega) \\ &= \phi(z)|\phi,t), \end{aligned} \tag{3.65}$$

and similarly, in the momentum representation (or any other representation for that matter)

$$\hat{a}_p(t)|\phi,t) = \phi(p)|\phi,t). \tag{3.66}$$

Since the annihilation operators return the field when acting in on a ket (and similarly the creation operator returns the conjugate field when acting backwards on a bra) the creation and annihilation operators are known as field operators in the context of quantum field theory.

The overlap between two field-states at equal time is

$$\begin{aligned} (\phi,t|\psi,t) &= \exp\left\{-\frac{1}{2\pi}\int d^2z\big(\bar{\phi}(z)\star_z\phi(z) + \bar{\psi}(z)\star_z\psi(z) - 2\bar{\phi}(z)\star_z\psi(z)\big)\right\} \\ &= \exp\left\{-\frac{1}{2}\int d^2p\big(\bar{\phi}(p)\phi(p) + \bar{\psi}(p)\psi(p) - 2\bar{\phi}(p)\psi(p)\big)\right\}. \end{aligned} \tag{3.67}$$

(The proof is a simple generalisation of the proof in Appendix A that $\langle z|w\rangle = e^{-\frac{1}{2}|z|^2-\frac{1}{2}|w|^2+\bar{z}w}$) This yields unity if $\psi = \phi$, which shows that the field states as defined in (3.63) are normalised. One can also show that $e^{\hat{D}_t[\phi]} = e^{\frac{it}{\hbar}\hat{H}}e^{\hat{D}_0[\phi]}e^{-\frac{it}{\hbar}\hat{H}}$ and thus

$$|\phi,t) = e^{-\frac{it}{\hbar}\hat{H}}|\phi,0). \tag{3.68}$$

As demonstrated in previous sections, the momentum representation of non-commutative quantum mechanics is equivalent to that of the commutative theory. By choosing to work in the momentum basis we can therefore proceed to construct a path integral formulation of quantum field theory exactly as in the commutative case. One does this by first noting that the identity operator on the quantum Hilbert space at a fixed time t can be expressed as a functional integral (see [33]):

$$\hat{I} = \int [d\phi][d\bar{\phi}]|\phi, t)(\phi, t|, \tag{3.69}$$

where the integration measure expressed in the momentum representation is

$$[d\phi][d\bar{\phi}] = \prod_{k=1}^{N} \frac{d\phi(p_k)d\bar{\phi}(p_k)dp}{2\pi i}. \tag{3.70}$$

By this it is implied that momentum space is discretised into N cells of size dp so that there are N degrees of freedom to integrate over. Finally the limit $N \to \infty$ is taken.

If one now wishes to calculate transition amplitudes from an initial field ϕ_i at time t_i to a final field ϕ_f at time t_f one may make use of the identity (3.69) to divide the transition into a sequence of smaller steps:

$$\begin{aligned}(\phi_f, t_f|\phi_i, t_i) &= \int \prod_{j=1}^{M-1} [d\phi_j][d\bar{\phi}_j](\phi_f, t_f|\phi_{M-1}, t_{M-1})(\phi_{M-1}, t_{M-1}|... \\ &\quad ...|\phi_2, t_2)(\phi_2, t_2|\phi_1, t_1)(\phi_1, t_1|\phi_i, t_i).\end{aligned} \tag{3.71}$$

If we make the time steps equal and of infinitesimal length $\epsilon = (t_f - t_i)/M$ (i.e. let $M \to \infty$) we may calculate the intermediate terms:

$$\begin{aligned}(\phi_{j+1}, t_{j+1}|\phi_j, t_j) &= (\phi_{j+1}, t_j|e^{-\frac{i\epsilon}{\hbar}\hat{H}}|\phi_j, t_j) \\ &\approx (\phi_{j+1}, t_j|1 - \frac{i\epsilon}{\hbar}\hat{H}|\phi_j, t_j) \\ &= (\phi_{j+1}, t_j|1 - \frac{i\epsilon}{\hbar}H(\phi_{j+1}, \phi_j)|\phi_j, t_j) \\ &= (\phi_{j+1}, t_j|\phi_j, t_j)e^{-\frac{i\epsilon}{\hbar}H(\phi_{j+1}, \phi_j)},\end{aligned} \tag{3.72}$$

where we have defined $H(\phi_{j+1}, \phi_j) = (\phi_{j+1}, t_j|\hat{H}|\phi_j, t_j)/(\phi_{j+1}, t_j|\phi_j, t_j)$. Substituting back into (3.71) and using (3.67) we obtain

$$\begin{aligned}(\phi_f, t_f|\phi_i, t_i) &= \lim_{M\to\infty} \int \prod_{j=1}^{M-1} [d\phi_j][d\bar{\phi}_j]e^{-\frac{1}{2}\int d^2p\left(\bar{\phi}_{j+1}\phi_{j+1} + \bar{\phi}_j\phi_j - 2\bar{\phi}_{j+1}\phi_j\right)} \\ &\quad \times e^{-\frac{i\epsilon}{\hbar}H(\phi_{j+1}, \phi_j)}.\end{aligned} \tag{3.73}$$

For convenience we have made the identifications $\phi_i = \phi_0$ and $\phi_f = \phi_M$ in the expression above. If one now realises that

$$\lim_{\epsilon\to 0}\bar{\phi}_{j+1}\frac{\phi_{j+1}-\phi_j}{\epsilon} = \bar{\phi}(t)\frac{\partial}{\partial t}\phi(t), \tag{3.74}$$

one may group the terms in such a way as to obtain, after taking the limit,

$$(\phi_f, t_f|\phi_i, t_i) = e^{\frac{1}{2}\int d^2p\left(\bar{\phi}_f(p)\phi_f(p)-\bar{\psi}_i(p)\psi_i(p)\right)}\int [d\phi][d\bar{\phi}]e^{\frac{i}{\hbar}\int_{t_i}^{t_f}dtL}. \tag{3.75}$$

Here L is the Lagrangian of the system,

$$L = i\int d^2p\bar{\phi}(p,t)\frac{\partial}{\partial t}\phi(p,t) - H(\bar{\phi}(p,t),\phi(p,t)), \tag{3.76}$$

and we have defined the measure

$$[d\phi][d\bar{\phi}] = \prod_{j=1}^{M-1}\prod_{k=1}^{N}\frac{d\bar{\phi}(p_k,t_j)d\phi(p_k,t_j)dp}{2\pi i}, \tag{3.77}$$

i.e. (3.70) $M-1$ times.

If we now want to express the path integral in terms of fields over the non-commutative position space we transform the fields according to (1.49):

$$\phi(p,t) = \sqrt{\frac{\theta}{2\pi^3\hbar^2}}e^{\frac{\theta}{4\hbar^2}|p|^2}\int d^2z e^{-\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p\bar{z}+\bar{p}z)}\phi(z,t). \tag{3.78}$$

The consequence is that a star product enters in the Lagrangian, more specifically, with the Hamiltonian written out in full:

$$\begin{aligned} L = {} & \frac{i}{\pi}\int d^2z\left\{\bar{\phi}(z,t)\star_z\frac{\partial}{\partial t}\phi(z,t) - \bar{\phi}(z,t)\star_z\left(-\frac{\hbar^2}{4m\theta}\frac{\partial^2}{\partial\bar{z}\partial z}\phi(z,t) + V(\bar{z},z)\star_z\phi(z,t)\right)\right\} \\ & +\frac{1}{\pi^2}\int d^2zd^2w\bar{\phi}(z,t)\bar{\phi}(w,t)\star_{z,w}W(\bar{z},z,\bar{w},w)\star_{z,w}\phi(w,t)\phi(z,t). \end{aligned} \tag{3.79}$$

In changing from the momentum to the position representation, a Jacobian function will enter in the integration measure (3.77). Since the transformation (3.78) and its complex conjugate are linear this should not pose any problem, and should just give rise to an irrelevant overall normalisation factor. The measure will be ill-defined, but this is the case anyway in conventional quantum field theory. One does not worry so much about this because the physically interesting quantities calculated in field theory are normalised such that these divergences cancel out. One might argue, however, that the space of functions that we integrate over in non-commutative field theory is a subspace of the full L^2 space of conventional field theory. More precisely, the functions are Schwartz-class and smooth on a length scale of order $\sqrt{\theta}$. This does not seem to be a concern in the existing literature on non-commutative field theories, possibly for the reason that the set of Schwartz-class functions is dense in L^2.

Without dwelling more on possible issues with the measure, we conclude that the essential difference between commutative and non-commutative quantum field theory is the replacement of the normal Cartesian product with a star product. This is in agreement with existing literature, although the symmetric Moyal product is usually considered instead of the Voros product resulting from our development [16].

3.5.2 Fermionic Fields

Just as in the bosonic case, we may define a fermionic field state

$$|\phi, t) = e^{\hat{D}_t[\phi]}|\Omega), \tag{3.80}$$

with

$$\begin{aligned} \hat{D}_t[\phi] &= \frac{1}{\pi}\int d^2z \left\{\hat{\psi}^\ddagger(z,t) \star_z \phi(z) - \bar{\phi}(z) \star_z \hat{\psi}(z,t)\right\} \\ &= \int d^2p \left\{\hat{a}_p^\ddagger(t)\phi(p) - \bar{\phi}(p)\hat{a}_p(t)\right\}. \end{aligned} \tag{3.81}$$

The only difference is that the field operators in $\hat{D}$ are now fermionic, and the functions ϕ and $\bar{\phi}$ are anti-commuting Grassmann variables[4] defined over the complex field. The states (3.80) are eigenstates of the fermionic annihilation operators, and since we are free to chose the momentum representation it is is possible to derive the fermionic path integral formalism in complete analogy to the commutative case. We may then proceed as in the previous section to transform the path integral into the non commutative position representation. In the end we obtain for an arbitrary transition amplitude,

$$(\phi_f, t_f|\phi_i, t_i) = e^{\frac{1}{2\pi}\int d^2z\left(\bar{\phi}_f(z)\star_z\phi_f(z)-\bar{\psi}_i(z)\star_z\psi_i(z)\right)} \int [d\phi][d\bar{\phi}]e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dtL}. \tag{3.82}$$

[4]A short introduction to Grassmann calculus in the context of quantum field theory is given by Brown [33].

with L identical to (3.79), only in terms of fermionic Grassmann functions ϕ. The functional integral is understood to be a Grassmannian functional integral and acts as a functional derivative.

Chapter 4
The Exchange Potential

At this stage we could have ventured into applications of the full machinery developed in the previous chapter to many particle systems. However, to gain an intuitive feeling for how non-commutative particles interact, it is instructive to consider a simpler system of only two particles.

Suppose we have two particles in the canonical ensemble, described by a density matrix

$$\hat{\rho} \;=\; \frac{1}{Z}e^{-\beta\hat{H}}, \tag{4.1}$$

where $\hat{H}$ is the free Hamiltonian and $\beta = 1/(kT)$ (k being the Boltzmann constant and T the temperature of the system [1]). $Z = \mathrm{tr}_Q\left(e^{-\beta\hat{H}}\right)$ is the partition function of the system and must be included for normalisation purposes.

We may now calculate the probability of measuring the particles at positions z_1 and z_2. This is done in C.4.1, and the result is

$$\begin{aligned} P(z_1, z_2) &= \mathrm{tr}_Q\left[\pi^{\pm}_{z_1,z_2}\hat{\rho}\right] \\ &= \frac{1}{Z\pi^2}\left(\frac{m\theta}{\beta\hbar^2}\right)^2\left\{1 \pm \frac{\beta\hbar^2}{\beta\hbar^2+2m\theta}e^{-\frac{2m\theta}{2m\theta+\beta\hbar^2}|z_2-z_1|^2}\right\}. \end{aligned} \tag{4.2}$$

(Note that we used the operators $\pi^{\pm}_{z_1,z_2}$, rather than π_{z_1,z_2}, since we want to measure either bosons or fermions (see 3.3.2).) In accordance with Pathria [34], we now define an effective potential between the two particles by

$$V_{eff}(z_1, z_2) \;=\; -\frac{1}{\beta}\ln(P(z_1, z_2)). \tag{4.3}$$

Substituting (4.2) into (4.3) we get

$$V_{eff}(|z_1 - z_2|) \;=\; -\frac{1}{\beta}\ln\left[1 \pm \frac{\beta\hbar^2}{\beta\hbar^2+2m\theta}e^{-\frac{2m\theta}{2m\theta+\beta\hbar^2}|z_2-z_1|^2}\right] + const. \tag{4.4}$$

The constant term absorbs the diverging partition function Z, but, since this term only shifts the potential by a global factor, it may be ignored. It may easily be verified that

[1] Temperature is maybe a misleading term for a system of two particles. What is meant is that the particles are in thermal equilibrium, for example by coupling to an external reservoir.

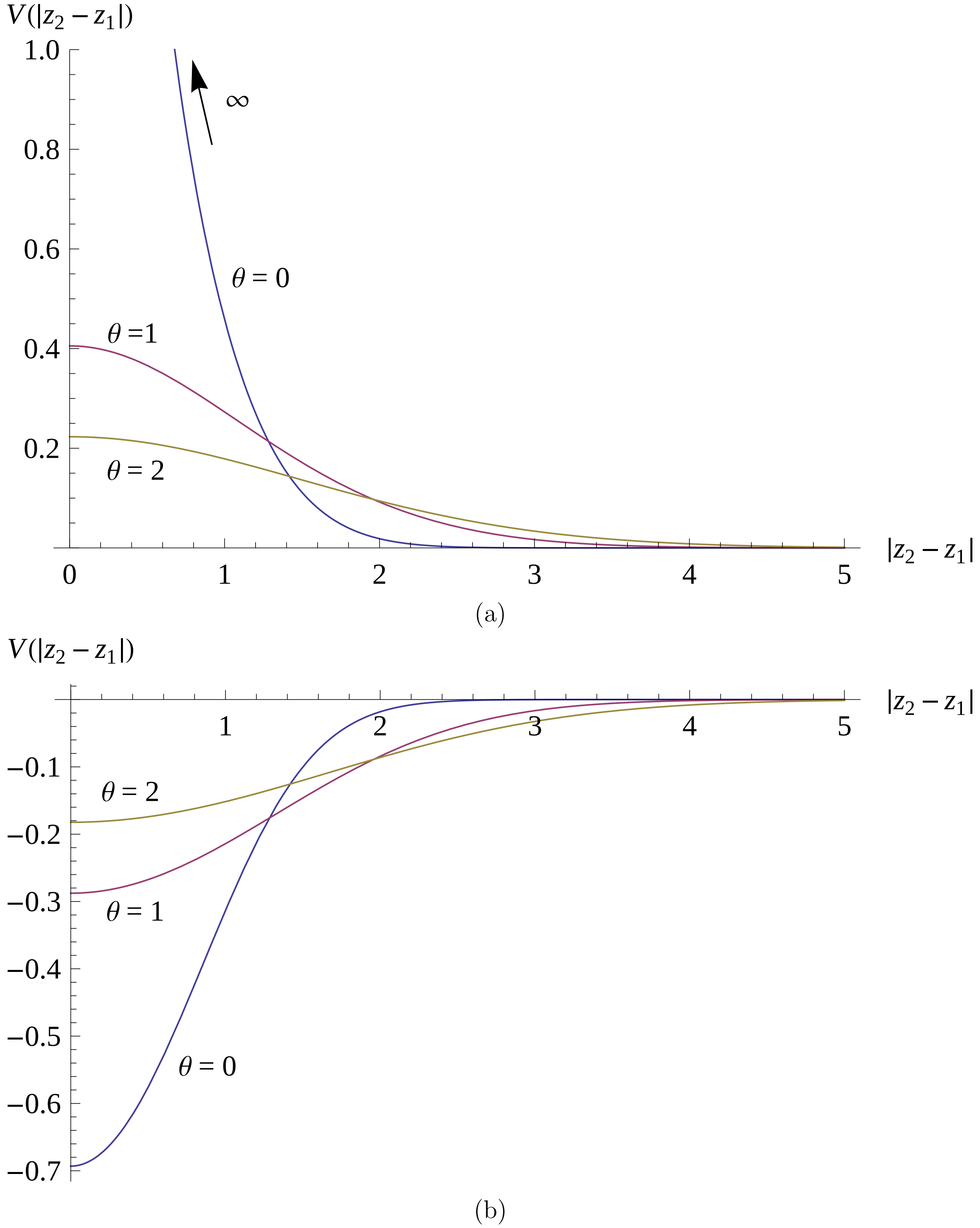
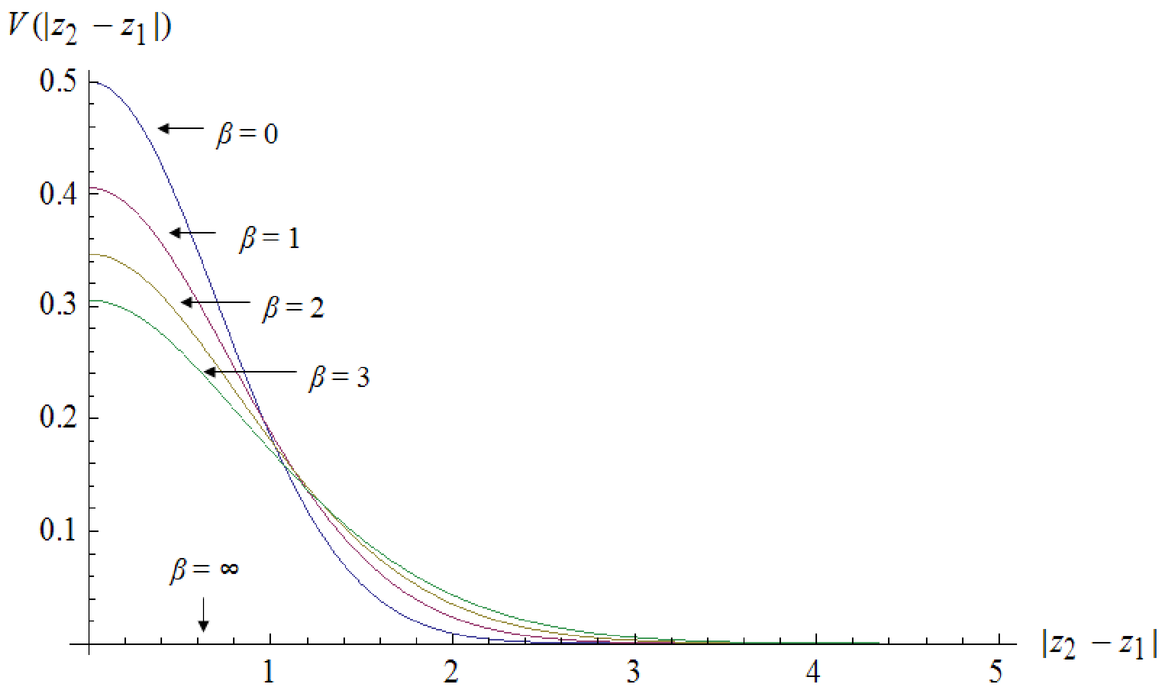
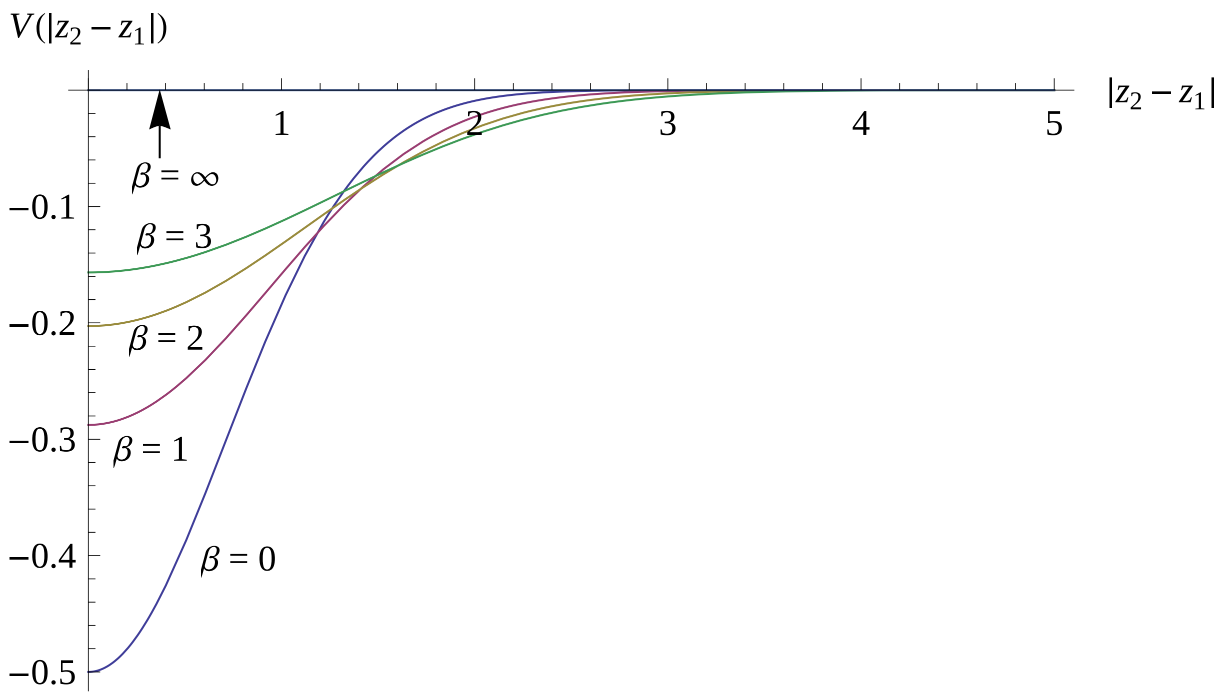


Figure 4.1: The exchange potential for two fermions (a) and two bosons (b) plotted for different values of θ for fixed β ($\beta = \hbar = m = 1$).



(a)



(b)

Figure 4.2: The exchange potential for two fermions (a) and two bosons (b) plotted for different values of β for fixed θ ($\theta = \hbar = m = 1$).

in the limit as $\theta \to 0$, after changing from the dimensionless coordinates z_1 and z_2 to the corresponding dimensionfull vectors $\vec{x}_1$ and $\vec{x}_2$, we retain the same form of the exchange potential as [34], namely

$$V_{eff}(\vec{x}_1 - \vec{x}_2) \;=\; -\frac{1}{\beta}\ln\left[1 \pm e^{-\frac{m}{\beta\hbar^2}(\vec{x}_2-\vec{x}_1)^2}\right]. \tag{4.5}$$

[34] defines the mean thermal wavelength $\lambda = \sqrt{\frac{2\pi\beta\hbar^2}{m}}$. The mean thermal wavelength can be interpreted as the length scale at which quantum effects become dominant. If the mean inter-particle distance in a gas is much larger than the mean thermal wavelength, we may regard the gas as a classical Maxwell-Boltzmann gas. However, as the mean particle separation approaches the mean thermal wavelength, the fermionic or bosonic nature of the particles start playing a role.

In non-commutative quantum mechanics we may define a θ-dependent mean thermal wavelength $\lambda_{NC} = \sqrt{\frac{2\pi(\beta\hbar^2+2m\theta)}{m}}$. In contrast to the commutative case, λ_{NC} is finite even for infinite temperature, i.e. $\lim_{\beta\to 0}\lambda_{NC} = 2\sqrt{\pi\theta}$. The behaviour of the exchange potential at high temperatures is further altered by the prefactor $\frac{\beta\hbar^2}{\beta\hbar^2+2m\theta}$ in front of the exponential term in 4.4. We conclude, therefore, that in non-commutative quantum mechanics, as opposed to normal quantum mechanics, the quantum effects do not necessarily become less important at higher temperatures. If the density is sufficiently high, the quantum effects will rather get more significant as the temperature increases.

The best way to get a feeling for the behaviour of the non-commutative exchange potential (4.4) is by plotting it, as has been done in Figures 4.1 and 4.2. We see that the potential indeed behaves differently from its commutative counterpart (4.5) on length scales of order θ. Most striking is the effect for the fermionic exchange potential, which diverges in commutative quantum mechanics but becomes finite at $|z_2 - z_1| = 0$ for $\theta > 0$. In the case of bosons the effect is not as dramatic, however, the non-commutative exchange potential is weakened at short length scales even for bosons.

In the commutative case (4.5) the temperature only has a scaling effect on the exchange potential. Although the temperature enters in a more complicated manner in (4.4), Figure 4.2 shows that the temperature plays a similar role for the non-commutative exchange potentials.

Particles experience forces, not potentials. Let us therefore define the exchange force

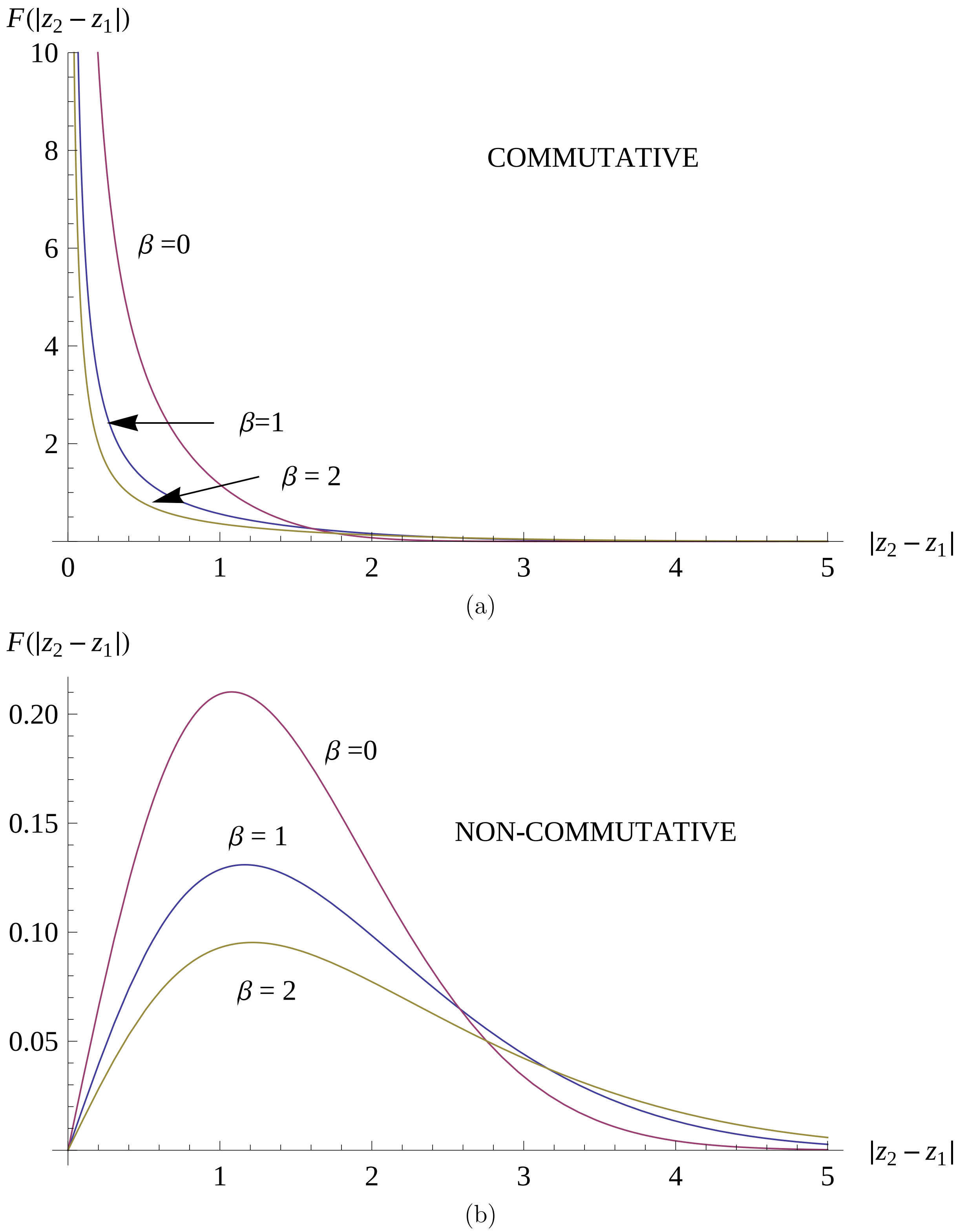


Figure 4.3: The exchange force for two fermions plotted for different values of β for the $\theta = 0$ (a) and $\theta = 1$ (b) ($\hbar = m = 1$).

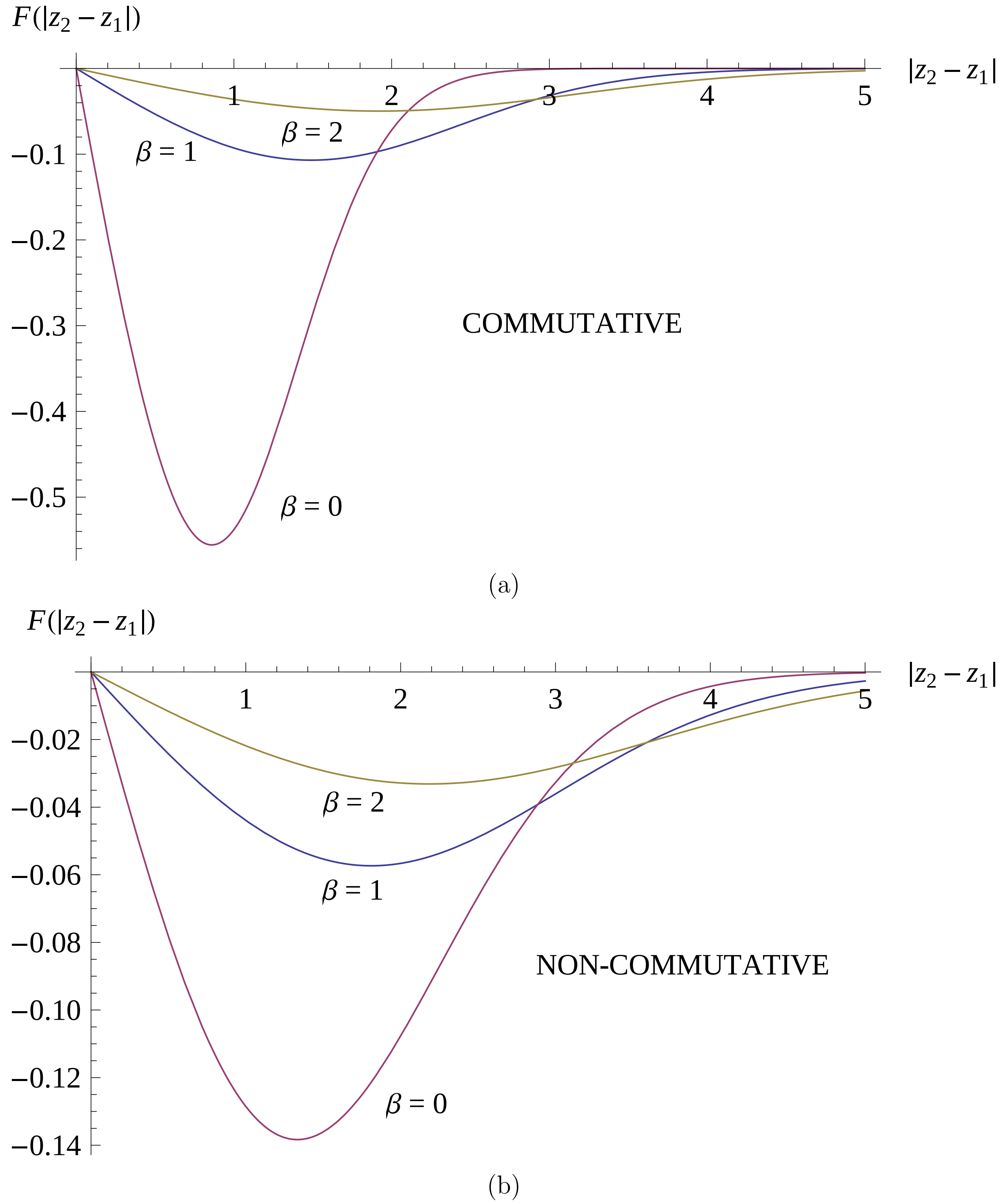


Figure 4.4: The exchange force for two bosons plotted for different values of β for the $\theta = 0$ (a) and $\theta = 1$ (b) ($\hbar = m = 1$).

between two particles as follows:

$$F_{eff} = -\frac{\partial}{\partial(|z_2 - z_1|)} V_{eff}(|z_2 - z_1|). \tag{4.6}$$

In Figures 4.3 and 4.4 the dependence of the exchange force on temperature is shown for the commutative and non-commutative cases. We see that the introduction of non-commuting coordinates makes the forces weaker. It also appears that the non-commutative exchange forces are less sensitive to temperature than in commutative quantum mechanics at short length scales. One could interpret this as the effect of θ becoming comparatively larger than the effect of β at short separations. For bosons we note that the separation at which the force attains its maximum strength is larger than in the commutative case. Even for fermions the force reaches a maximum, before decaying away as the separation approaches zero. Non-commutativity does not alter the sign of the forces, however; for bosons the exchange force is always attractive, and for fermions it is always repulsive.

One could take different views regarding the softening of the exchange potentials in non-commutative space: Firstly, one could argue that, because the distance between two particles may not be resolved beyond the length scale $\sqrt{\theta}$, it makes no sense to ask about the potential strength for smaller separations. Alternatively, since particles will experience each other as smeared out in space, we should not interpret the separation in the argument of V_{eff} as a sharply defined distance. Rather, one should interpret $V_{eff}(|z_2 - z_1|)$ as an average energy of two particles whose separation is only known up to a Gaussian distribution centered at $z_2 - z_1$ and with variance $\sqrt{\theta}$.

An alternative interpretation suggests itself if one realises that the modifications to the exchange potentials may be fully attributed to the POVMs for position measurement, $\pi^{\pm}_{z_1,z_2} = \frac{1}{2!\pi^2}|z_1, z_2)_{\pm} \star_{z_1,z_2} {}_{\pm}(z_1, z_2|$. At first glance, one might think that this operator reduces to the zero-operator if we set $z_1 = z_2$ in the fermionic case; after all it contains antisymmetric states. However, there is a star product present in $\pi^{\pm}_{z_1,z_2}$, and this star product must be allowed to act before the indices are set equal. This ensures that $\pi^{\pm}_{z,z} \neq 0$. One can see this easily if the star product is decomposed, for example as $e^{\overleftarrow{\partial_z}\overrightarrow{\partial_{\bar{z}}}} =$

$\frac{1}{\pi}\int d^2u\ e^{-|u|^2+u\overleftarrow{\partial_z}+\bar{u}\overrightarrow{\partial_{\bar{z}}}}$, as was done in C.4.1:

$$
\begin{aligned}
&|z_1,z_2)_\pm \star_{z_1,z_2}\ {}_\pm(z_1,z_2| \\
&= \frac{1}{2\pi^2}\int d^2ud^2v e^{-|u|^2-|v|^2}\Big\{|z_1)e^{u\overleftarrow{\partial_{z_1}}+\bar{u}\overrightarrow{\partial_{\bar{z}_1}}}(z_1|\otimes|z_2)e^{v\overleftarrow{\partial_{z_2}}+\bar{v}\overrightarrow{\partial_{\bar{z}_2}}}(z_2| \\
&+|z_2)e^{v\overleftarrow{\partial_{z_2}}+\bar{v}\overrightarrow{\partial_{\bar{z}_2}}}(z_2|\otimes|z_1)e^{u\overleftarrow{\partial_{z_1}}+\bar{u}\overrightarrow{\partial_{\bar{z}_1}}}(z_1|\pm|z_1)e^{u\overleftarrow{\partial_{z_1}}+\bar{v}\overrightarrow{\partial_{\bar{z}_2}}}(z_2|\otimes|z_2)e^{v\overleftarrow{\partial_{z_2}}+\bar{u}\overrightarrow{\partial_{\bar{z}_1}}}(z_1| \\
&\pm|z_2)e^{v\overleftarrow{\partial_{z_2}}+\bar{u}\overrightarrow{\partial_{\bar{z}_1}}}(z_1|\otimes|z_1)e^{u\overleftarrow{\partial_{z_1}}+\bar{v}\overrightarrow{\partial_{\bar{z}_2}}}(z_2|\Big\}. \qquad (4.7)
\end{aligned}
$$

For fermions, the last two terms of (4.7) will only cancel out the first two terms for $z_1 = z_2$ in the commutative limit; in this limit the Gaussian factors tend to Dirac delta functions, and the only contribution to the integrals is for $u = v = 0$.

Suggestions have been made, for example by Rohwer et al. [11], that decompositions of the star product, for instance like the example above, may be related to internal degrees of freedom of the particles. From this perspective, two non-commutative fermions can really sit on top of each other, and the Pauli principle is not violated, because the particles may be in different internal states. If we remove the integral in (4.7) and set $u = v$ the terms do cancel for fermions. One might say that the state has been completely specified by the 'internal degrees of freedom' u and v. Within this interpretation, the strong modifications of the exchange potentials in the high temperature limit could be explained by the fact that more and more internal degrees of freedom become accessible to the particles as the temperature increases.

If one merely regards u and v in (4.7) as auxiliary variables, instead of invoking extra physical degrees of freedom, one might say that $\pi^{\pm}_{z,z} \neq 0$ simply because of the nonlocality introduced by non-commutativity through the star product.

The exchange potential has previously been derived in the context of twisted statistics in [36], however, not based on the consistent interpretation of position measurement as presented in this thesis. The result obtained is similar in form to (4.4), and leads to the same apparent violation of Pauli's exclusion principle whenever twisting is present. However, since twisting in two dimensional non-commutative quantum mechanics is optional (i.e. the twisting parameter is arbitrary) the effect reported in [36] may be avoided by choosing not to twist. Our result is a more profound manifestation of the non-commutativity of space, which cannot be avoided, and which originates in the fact that position measurement does not provide complete knowledge of the state of a system. In higher dimensions, where the twisting parameter equals the non-commutative parameter by the requirement of Poincaré invariance, one would expect an additional distortion of the exchange potential due to the twisting, on top of the distortion reported here in two dimensions.

Lastly, let us remark that by setting $\hbar = m = \theta = 1$ in the plots in this section we have indeed chosen a very large non-commutative parameter, thereby exaggerating the effects one would expect in the context of quantum gravity, where θ presumably would assume a very small value. The energies required to observe violations of the Pauli principle would probably be unattainable in experiment. In any case, our non-relativistic approach would likely be invalid at very high energies, and a more careful analysis would be required.

Conclusion

To conclude this thesis, let us briefly summarise what has been achieved in the preceding chapters:

- We have derived the differential scattering cross section for scattering of particles by a potential in the non-commutative plane. It was found that this differential cross section is larger than predicted by commutative theory. We contribute this finding to the fact that interactions in a non-commutative space are not point interactions, since potentials and particles cannot be localised perfectly. This effectively increases the range of the potential and makes it appear larger. A lower limit is also imposed on the scale at which scattering potentials may fluctuate, as a direct result of the altered relationship between the position and momentum representations in non-commutative quantum mechanics.

- We have built a many-particle formalism of non-commutative quantum mechanics based on the interpretational framework developed in [15]. In particular we have phrased this formalism in the language of second quantisation. The fact that interactions are no longer point-interactions is captured by the modified commutation relation satisfied by the field operators:

$$[\hat{\psi}(z), \hat{\psi}^{\ddagger}(w)]_{\mp} = e^{-|z-w|^2}.$$

- We were able to make a link between non-commutative quantum mechanics and non-commutative field theory via the second quantised formalism. The step from commutative to non-commutative field theory essentially involves the substitution of the usual Cartesian product with a star product. This is in agreement with existing literature on non-commutative field theory, however, there is presently no consensus about the form of star product to be used. Besides the Voros product employed here, the slightly different Moyal product is commonly used, and there is an ongoing debate whether the two products result in the same physics or not [16]. Our development sees the Voros product emerging in a natural way, and provides an argument in favour of its selection.

- The effective interaction between particles due to particle statistics is modified in non-commutative quantum mechanics. We observed an effective increase in the mean thermal wavelength, indicating that quantum effects will play a significant role whenever particles are brought within a distance of order $\sqrt{\theta}$. Most striking is the effect on fermions, which in non-commutative quantum mechanics are allowed to sit on top of each other. This apparent violation of Pauli's exclusion principle is unavoidable and comes about without the introduction of twisted statistics. The effect has a more profound origin, stemming from the inherent non-locality of non-commutative quantum mechanics and the fact that a position measurement does not specify the state of a particle completely.

A general observation, which may summarise the first and last point above, is that non-commutativity appears to have a softening effect on interactions at the scale of $\sqrt{\theta}$. It appears that in a hypothetical non-commutative world singularities in space cannot exist.

As pointed out in the introduction, we have been strictly confined to two spatial dimensions, and the conclusions reached here do not necessarily extend to higher dimensions. An obvious suggestion for future research would therefore be to generalise this formalism to higher dimensions. In the event that the results from two dimensions survive, one could foresee implications in high energy physics and for the physics of dense astrophysical objects due to the alteration of forces at short length scales. The two-dimensional version of the formalism, as presented here, may find direct applications to systems such as the quantum Hall system, which allow for effective descriptions in terms of non-commutative coordinates.

This thesis has only considered a non-local description of non-commutative quantum mechanics. As previously mentioned, a local description of non-commutative quantum mechanics have recently been suggested, and attempts have been made to relate the new local degrees of freedom to particle structure or extent [11]. A generalisation of the many-particle formalism developed here to such a local description is a straightforward matter. Mathematically, the step from a non-local to a local description involves some decomposition of the star product, and this naturally extends to the many particle situation. Hence, should one wish to study many-particle systems in a local framework, the tools developed here are readily available.

Appendix A
Coherent States

This appendix serves as an introduction to the theory of coherent states, and should provide sufficient background for the purpose of this thesis. Interested readers are referred to Klauder and Skagerstam [21] for an extensive review of the theory of coherent states.

In 1.1 we defined the annihilation and creation operators $b = \frac{1}{\sqrt{2\theta}}(\hat{x}_1 + i\hat{x}_2)$ and $b^\dagger = \frac{1}{\sqrt{2\theta}}(\hat{x}_1 - i\hat{x}_2)$ on the classical configuration space $\mathcal{H}_c$, satisfying $[b, b^\dagger] = 1$. These operators act in on the number states of the Fock basis as follows:

$$\begin{aligned} b\,|n\rangle &= \sqrt{n}\,|n-1\rangle\,, \\ b^\dagger\,|n\rangle &= \sqrt{n+1}\,|n+1\rangle\,. \end{aligned} \tag{A.1}$$

It follows that

$$b^\dagger b\,|n\rangle = n\,|n\rangle\,. \tag{A.2}$$

Let us now define the coherent state

$$|z\rangle = e^{zb^\dagger - \bar{z}b}\,|0\rangle\,. \tag{A.3}$$

Using the Baker-Campbell-Hausdorff formula, one can show that

$$e^{-zb^\dagger + \bar{z}b} b e^{zb^\dagger - \bar{z}b} = b + z. \tag{A.4}$$

Keeping in mind that $b\,|0\rangle = 0$ according to (A.1) we therefore have

$$\begin{aligned} b\,|z\rangle &= be^{zb^\dagger - \bar{z}b}\,|0\rangle \\ &= e^{zb^\dagger - \bar{z}b}e^{-zb^\dagger + \bar{z}b}be^{zb^\dagger - \bar{z}b}\,|0\rangle \\ &= e^{zb^\dagger - \bar{z}b}(b+z)\,|0\rangle \\ &= z\,|z\rangle\,, \end{aligned} \tag{A.5}$$

i.e. the coherent state (A.3) is an eigenstate of the annihilation operator b. By complex conjugation we similarly obtain

$$\langle z|\, b^\dagger \;=\; \bar{z}\,\langle z|\,. \qquad \text{(A.6)}$$

Using the fact that $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ whenever $[A,B] = const$, one may write the coherent state (A.3) as

$$\begin{aligned} |z\rangle &= e^{zb^\dagger - \bar{z}b}\,|0\rangle \\ &= e^{zb^\dagger} e^{-\bar{z}b} e^{-\frac{1}{2}|z|^2}\,|0\rangle \\ &= e^{-\frac{1}{2}|z|^2} e^{zb^\dagger}\,|0\rangle \\ &= e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\,|n\rangle\,. \end{aligned} \qquad \text{(A.7)}$$

From the latter one deduces the action of the creation operator on a coherent state:

$$b^\dagger\,|z\rangle \;=\; \left(\frac{\partial}{\partial z} + \frac{\bar{z}}{2}\right)|z\rangle\,; \qquad \text{(A.8)}$$

and correspondingly,

$$\langle z|\,b \;=\; \left(\frac{\partial}{\partial \bar{z}} + \frac{z}{2}\right)\langle z|\,. \qquad \text{(A.9)}$$

One may also deduce from (A.7) the following overlaps:

$$\begin{aligned} \langle n|z\rangle &= e^{-\frac{1}{2}|z|^2} \frac{z^n}{\sqrt{n!}}, \\ \langle z|n\rangle &= e^{-\frac{1}{2}|z|^2} \frac{\bar{z}^n}{\sqrt{n!}}, \\ \langle z|w\rangle &= e^{-\frac{1}{2}|z|^2 - \frac{1}{2}|w|^2 + \bar{z}w}. \end{aligned} \qquad \text{(A.10)}$$

We know that the identity operator on $\mathcal{H}_c$ can be expressed in terms of the number states:

$$\hat{I}_c \;=\; \sum_{n=0}^{\infty} |n\rangle\,\langle n|\,. \qquad \text{(A.11)}$$

Let us now prove that the identity operator alternatively may be expressed in terms of the coherent states:

$$\begin{aligned}
\frac{1}{\pi}\int d^2\, z\, |z\rangle\langle z| &= \frac{1}{\pi}\int d^2 z\, e^{-|z|^2}\sum_{m,n=0}^{\infty}\frac{z^m\bar{z}^n}{\sqrt{m!n!}}|m\rangle\langle n| \\
&= \frac{1}{\pi}\sum_{m,n=0}^{\infty}\frac{1}{\sqrt{m!n!}}|m\rangle\langle n|\int r\,drd\phi\, e^{-r^2}r^{m+n}e^{i(m-n)\phi} \\
&= 2\sum_{m,n=0}^{\infty}\frac{\delta_{m,n}}{\sqrt{m!n!}}|m\rangle\langle n|\int dr\, e^{-r^2}r^{n+m+1} \\
&= 2\sum_{n=0}^{\infty}\frac{1}{n!}|n\rangle\langle n|\int dr\, e^{-r^2}r^{2n+1} \\
&= \sum_{n=0}^{\infty}|n\rangle\langle n|\,. \qquad \text{(A.12)}
\end{aligned}$$

Here we have expressed z in polar coordinates (i.e. $z = re^{i\phi}$) and used the fact that $2\int dr\, e^{-r^2}r^{2n+1} = \Gamma(n+1) = n!$ is the Gamma function. We therefore conclude that

$$\hat{I}_c = \frac{1}{\pi}\int d^2 z\ |z\rangle\langle z|\,. \qquad \text{(A.13)}$$

Using (A.10) and (A.13) one may express a number state in terms of coherent states:

$$|n\rangle = \frac{1}{\pi}\int d^z\ |z\rangle\, e^{-\frac{1}{2}|z|^2}\frac{\bar{z}^n}{\sqrt{n!}}. \qquad \text{(A.14)}$$

Let us lastly demonstrate that the coherent states satisfy the uncertainty relation (1.11) with equality:

$$\begin{aligned}
\langle z|\,\hat{x}_1\,|z\rangle &= \sqrt{\frac{\theta}{2}}\langle z|\left(b+b^\dagger\right)|z\rangle \\
&= \sqrt{\frac{\theta}{2}}(z+\bar{z}), \\
\langle z|\,(\hat{x}_1)^2\,|z\rangle &= \frac{\theta}{2}\langle z|\left(b+b^\dagger\right)^2|z\rangle \\
&= \frac{\theta}{2}\langle z|\left(b^2+(b^\dagger)^2+2b^\dagger b+1\right)|z\rangle \\
&= \frac{\theta}{2}(z^2+\bar{z}^2+2\bar{z}z+1). \qquad \text{(A.15)}
\end{aligned}$$

Therefore $\Delta x_1 = \sqrt{\langle z|\,(\hat{x}_1)^2\,|z\rangle - (\langle z|\,\hat{x}_1\,|z\rangle)^2} = \sqrt{\frac{\theta}{2}}$ and similarly $\Delta x_2 = \sqrt{\frac{\theta}{2}}$. It follows that $\Delta x_1\Delta x_2 = \frac{\theta}{2}$, and hence $|z\rangle$ is a minimum uncertainty state on $\mathcal{H}_c$.

Appendix B
Useful Mathematical Identities

B.1

Let z and k be complex variables, i.e. $z = z_x + iz_y$ and $k = k_x + ik_y$, with $z_x, z_y, k_x, k_y \in \mathbb{R}$. Then

$$\begin{aligned}
\frac{1}{4\pi^2}\int d^2k\, e^{\frac{i}{2}\left(k\bar{z}+\bar{k}z\right)} &= \frac{1}{4\pi^2}\int dk_x dk_y\, e^{\frac{i}{2}((k_x+ik_y)(z_x-iz_y)+(k_x-ik_y)(z_x+iz_y))} \\
&= \left(\frac{1}{2\pi}\int dk_x\, e^{ik_x z_x}\right)\left(\frac{1}{2\pi}\int dk_y\, e^{ik_y z_y}\right) \\
&= \delta(z_x)\delta(z_y), \qquad \text{(B.1)}
\end{aligned}$$

where δ denotes the Dirac delta function. In this thesis the shorthand notation $\delta(z) = \delta(z_x)\delta(z_y)$ is employed.

B.2

$$e^{-\partial_z\partial_{\bar{z}}}e^{-|z-w|^2} = \pi\delta(z-w). \qquad \text{(B.2)}$$

Proof:

$$\begin{aligned}
e^{-\partial_z\partial_{\bar{z}}}e^{-|z-w|^2} &= \frac{1}{4\pi}\int d^2v\, e^{-\frac{1}{4}|v|^2+\frac{i}{2}v\partial_z+\frac{i}{2}\bar{v}\partial_{\bar{z}}}e^{-|z-w|^2} \\
&= \frac{1}{4\pi}\int d^2v\, e^{-\frac{1}{4}|v|^2}e^{-(z+\frac{i}{2}v-w)(\bar{z}+\frac{i}{2}\bar{v}-\bar{w})} \\
&= \pi e^{-|z-w|^2}\frac{1}{4\pi^2}\int d^2v\, e^{\frac{i}{2}(v(\bar{w}-\bar{z})+\bar{v}(w-z))} \\
&= \pi e^{-|z-w|^2}\delta(z-w) \\
&= \pi\delta(z-w). \qquad \text{(B.3)}
\end{aligned}$$

Appendix C
Important Calculations

C.1 Calculations for Chapter 1

C.1.1 Diagonal Representation of Operators in Terms of Coherent States

Suppose we have a (normal ordered) operator $\hat{O}(b^\dagger, b)$. We may insert the identity operator on each side of this operator to obtain

$$\begin{aligned}\hat{O}(b^\dagger, b) &= \frac{1}{\pi^2}\int d^2z d^2w \; |z\rangle \langle z| \hat{O}(b^\dagger, b) |w\rangle \langle w| \\ &= \frac{1}{\pi^2}\int d^2z d^2w \; |z\rangle O(\bar{z}, w) \langle z|w\rangle \langle w| .\end{aligned}$$

($O(\bar{z}, w)$ here means that we have simply let the $b^\dagger$'s act to the left and the b's to the right such that they may be replaced by $\bar{z}$ and w respectively.) After making the change of variables $w = z + v$ and $\bar{w} = \bar{z} + \bar{v}$ and performing some manipulations, one obtains

$$\begin{aligned}\hat{O}(b^\dagger, b) &= \frac{1}{\pi^2}\int d^2z d^2v \; e^{-|v|^2} e^{zb^\dagger} |0\rangle O(\bar{z}, z+v) e^{-z(\bar{z}+\bar{v})} \langle 0| e^{(\bar{z}+\bar{v})b} \\ &= \frac{1}{\pi^2}\int d^2z d^2v \; e^{-|v|^2} e^{zb^\dagger} |0\rangle \left(e^{v\partial_z} O(\bar{z}, z)\right) \left(e^{\bar{v}\partial_{\bar{z}}} e^{-z\bar{z}} \langle 0| e^{\bar{z}b}\right) \\ &= \frac{1}{\pi}\int d^2z \; |z\rangle \langle z| \left(\frac{1}{\pi}\int d^2v \; e^{-|v|^2 + v\partial_z - \bar{v}\partial_{\bar{z}}} O(\bar{z}, z)\right) \\ &= \frac{1}{\pi}\int d^2z \; |z\rangle \langle z| \left(e^{-\partial_z\partial_{\bar{z}}} O(\bar{z}, z)\right) . \qquad \text{(C.1)}\end{aligned}$$

Between the second and third line integration by parts was used to make $\partial_{\bar{z}}$ act in on $O(\bar{z}, z)$. We can thus conclude that any operator on $\mathcal{H}_c$ has a diagonal representation in the coherent state basis.

C.1.2 Origin of the Star Product

We define the mapping

$$\begin{aligned}M : \mathcal{S} &\to \mathbb{C}, \\ M(\hat{A}) &= A(\bar{z}, z), \qquad \text{(C.2)}\end{aligned}$$

with $A(\bar{z}, z) = \langle z| \hat{A} |z\rangle$. We will now show that this is a homomorphism from $\mathcal{S}$ to $\mathbb{C}$ with the product $\star_z = e^{\overleftarrow{\partial_z}\overrightarrow{\partial_{\bar{z}}}}$:

$$\begin{aligned} M(\hat{A}\hat{B}) &= \langle z| \hat{A}\hat{B} |z\rangle \\ &= \frac{1}{\pi}\int d^2w \ \langle z| \hat{A} |w\rangle \langle w| \hat{B} |z\rangle \\ &= \frac{1}{\pi}\int d^2w \ |\langle z|w\rangle|^2 A(\bar{z}, w) B(\bar{w}, z). \end{aligned} \tag{C.3}$$

We now introduce a new variable $v = w - z$. Noting that $|\langle z|w\rangle|^2 = e^{-|z-w|^2} = e^{-|v|^2}$ we therefore have

$$\begin{aligned} M(\hat{A}\hat{B}) &= \frac{1}{\pi}\int d^2v \ e^{-|v|^2} A(\bar{z}, z+v) B(\bar{z}+\bar{v}, z) \\ &= A(\bar{z}, z)\left(\frac{1}{\pi}\int d^2v \ e^{-|v|^2 + v\overleftarrow{\partial_z} + \bar{v}\overrightarrow{\partial_{\bar{z}}}}\right) B(\bar{z}, z) \\ &= A(\bar{z}, z) e^{\overleftarrow{\partial_z}\overrightarrow{\partial_{\bar{z}}}} B(\bar{z}, z) \\ &= \langle z| \hat{A} |z\rangle \star_z \langle z| \hat{B} |z\rangle . \end{aligned} \tag{C.4}$$

Since M is clearly onto and $M(\hat{I}) = \langle z| \hat{I} |z\rangle = 1$, we conclude that M is an isomorphism.

C.2 Calculations for Chapter 2

C.2.1 Conserved Currents in the z-basis

We start by reminding of the Schrödinger equation in the non-commutative position basis:

$$i\hbar\frac{\partial}{\partial t}\psi(z) = -\frac{\hbar^2}{m\theta}\frac{\partial^2}{\partial_z\partial_{\bar{z}}}\psi(z) + V(\bar{z}, z) \star_z \psi(z). \tag{C.5}$$

The conjugate of this is (assuming that $V(\bar{z}, z)$ is real)

$$-i\hbar\frac{\partial}{\partial t}\bar{\psi}(z) = -\frac{\hbar^2}{m\theta}\frac{\partial^2}{\partial_z\partial_{\bar{z}}}\bar{\psi}(z) + \bar{\psi}(z) \star_z V(\bar{z}, z). \tag{C.6}$$

Using (C.5) and (C.6) one therefore has

$$
\begin{aligned}
\frac{\partial}{\partial t}P(z) &= \frac{1}{\pi}\left(\frac{\partial}{\partial t}\bar{\psi}(z)\right)\star_z \psi(z) + \frac{1}{\pi}\bar{\psi}(z)\star_z\left(\frac{\partial}{\partial t}\psi(z)\right) \\
&= \frac{\hbar}{i\pi m\theta}\left(\frac{\partial}{\partial z\partial\bar{z}}\bar{\psi}(z)\right)\star_z\psi(z) - \frac{1}{i\hbar\pi}\bar{\psi}(z)\star_z V(\bar{z},z)\star_z\psi(z) \\
&- \frac{\hbar}{i\pi m\theta}\bar{\psi}(z)\star_z\left(\frac{\partial}{\partial z\partial\bar{z}}\psi(z)\right) + \frac{1}{i\hbar\pi}\bar{\psi}(z)\star_z V(\bar{z},z)\star_z\psi(z) \\
&= \frac{\hbar}{i\pi m\theta}\left\{\left(\frac{\partial}{\partial z\partial\bar{z}}\bar{\psi}(z)\right)\star_z\psi(z) - \bar{\psi}(z)\star_z\left(\frac{\partial}{\partial z\partial\bar{z}}\psi(z)\right)\right\}. \qquad \text{(C.7)}
\end{aligned}
$$

However, we may use the product rule to write

$$
\begin{aligned}
\left(\frac{\partial}{\partial z\partial\bar{z}}\bar{\psi}(z)\right)\star_z\psi(z) &= \frac{1}{2}\left\{\frac{\partial}{\partial z}\left[\left(\frac{\partial}{\partial\bar{z}}\bar{\psi}(z)\right)\star_z\psi(z)\right] - \left(\frac{\partial}{\partial\bar{z}}\bar{\psi}(z)\right)\star_z\left(\frac{\partial}{\partial z}\psi(z)\right)\right. \\
&+ \left.\frac{\partial}{\partial\bar{z}}\left[\left(\frac{\partial}{\partial z}\bar{\psi}(z)\right)\star_z\psi(z)\right] - \left(\frac{\partial}{\partial z}\bar{\psi}(z)\right)\star_z\left(\frac{\partial}{\partial\bar{z}}\psi(z)\right)\right\}, \qquad \text{(C.8)}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{\partial}{\partial t}P(z) &= -\frac{\hbar}{2i\pi m\theta}\left\{\frac{\partial}{\partial z}\left[\bar{\psi}(z)\star_z\left(\frac{\partial}{\partial\bar{z}}\psi(z)\right) - \left(\frac{\partial}{\partial\bar{z}}\bar{\psi}(z)\right)\star_z\psi(z)\right]\right. \\
&- \left.\frac{\partial}{\partial\bar{z}}\left[\bar{\psi}(z)\star_z\left(\frac{\partial}{\partial z}\psi(z)\right) - \left(\frac{\partial}{\partial z}\bar{\psi}(z)\right)\star_z\psi(z)\right]\right\}, \qquad \text{(C.9)}
\end{aligned}
$$

and by defining the currents

$$
\begin{aligned}
j_z &= \frac{\hbar}{2i\pi m\theta}\left[\bar{\psi}(z)\star_z\left(\frac{\partial}{\partial\bar{z}}\psi(z)\right) - \left(\frac{\partial}{\partial\bar{z}}\bar{\psi}(z)\right)\star_z\psi(z)\right], \\
j_{\bar{z}} &= \frac{\hbar}{2i\pi m\theta}\left[\bar{\psi}(z)\star_z\left(\frac{\partial}{\partial z}\psi(z)\right) - \left(\frac{\partial}{\partial z}\bar{\psi}(z)\right)\star_z\psi(z)\right], \qquad \text{(C.10)}
\end{aligned}
$$

we obtain the continuity equation

$$
\frac{\partial}{\partial t}P(z) + \frac{\partial}{\partial z}j_z + \frac{\partial}{\partial\bar{z}}j_{\bar{z}} = 0. \qquad \text{(C.11)}
$$

C.2.2 Polar Currents

Using (2.6) and (2.5)

$$
\begin{aligned}
\frac{\partial}{\partial z} j_z + \frac{\partial}{\partial \bar{z}} j_{\bar{z}} &= \frac{e^{-i\phi}}{2}\left[\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \phi}\right] j_z + \frac{e^{i\phi}}{2}\left[\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \phi}\right] j_{\bar{z}} \\
&= \frac{\partial}{\partial r}\left[\frac{e^{-i\phi}}{2} j_z + \frac{e^{i\phi}}{2} j_{\bar{z}}\right] - \frac{ie^{-i\phi}}{2r}\frac{\partial}{\partial \phi} j_z + \frac{ie^{i\phi}}{2r}\frac{\partial}{\partial \phi} j_{\bar{z}} \\
&= \frac{\partial}{\partial r}\left[\frac{e^{-i\phi}}{2} j_z + \frac{e^{i\phi}}{2} j_{\bar{z}}\right] + \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{-ie^{-i\phi}}{2} j_z\right) - \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{-ie^{-i\phi}}{2}\right) j_z \\
&\quad + \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{ie^{i\phi}}{2} j_{\bar{z}}\right) - \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{ie^{i\phi}}{2}\right) j_{\bar{z}} \\
&= \frac{\partial}{\partial r}\left[\frac{e^{-i\phi}}{2} j_z + \frac{e^{i\phi}}{2} j_{\bar{z}}\right] + \frac{1}{r}\frac{\partial}{\partial \phi}\left[\frac{-ie^{-i\phi}}{2} j_z + \frac{ie^{i\phi}}{2} j_{\bar{z}}\right] + \frac{1}{r}\left[\frac{e^{-i\phi}}{2} j_z + \frac{e^{i\phi}}{2} j_{\bar{z}}\right] \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{e^{-i\phi}}{2} j_z + \frac{e^{i\phi}}{2} j_{\bar{z}}\right)\right] + \frac{1}{r}\frac{\partial}{\partial \phi}\left[\frac{-ie^{-i\phi}}{2} j_z + \frac{ie^{i\phi}}{2} j_{\bar{z}}\right]. \qquad \text{(C.12)}
\end{aligned}
$$

Since $e^{i\phi} = \sqrt{z/\bar{z}}$ and $r = \sqrt{z\bar{z}}$, we may define the radial and angular currents

$$
\begin{aligned}
j_r &= \frac{1}{2}\left(\sqrt{\frac{\bar{z}}{z}} j_z + \sqrt{\frac{z}{\bar{z}}} j_{\bar{z}}\right), \\
j_\phi &= -\frac{i}{2}\left(\sqrt{\frac{\bar{z}}{z}} j_z - \sqrt{\frac{z}{\bar{z}}} j_{\bar{z}}\right), \qquad \text{(C.13)}
\end{aligned}
$$

satisfying the radial continuity equation

$$
\frac{\partial}{\partial t} P(z) + \frac{1}{r}\frac{\partial}{\partial r}(r j_r) + \frac{1}{r}\frac{\partial}{\partial \phi} j_\phi = 0. \qquad \text{(C.14)}
$$

C.2.3 Explicit Form of the Lippmann-Schwinger Equation

Consider the second term of (2.14). By inserting the momentum representation of the identity operator we have

$$
\begin{aligned}
(z|\frac{1}{E - \hat{H}_0 + i\varepsilon}\hat{V}|\psi^+) &= \int d^2p\,(z|\frac{1}{E - \hat{H}_0 + i\varepsilon}|p)(p|\hat{V}|\psi^+) \\
&= \int d^2p\,(z|p)\frac{1}{E - \frac{p^2}{2m} + i\varepsilon}(p|\hat{V}|\psi^+). \qquad \text{(C.15)}
\end{aligned}
$$

E corresponds to the energy of the incoming particle, and we may write $E = q^2/2m$, with q the momentum of the incoming particle. If we also redefine $\varepsilon' = 2m\varepsilon$ we can write

$$\begin{aligned}
(z|\frac{1}{E - \hat{H}_0 + i\varepsilon}\hat{V}|\psi^+) &= 2m\int d^2p\,(z|p)\frac{1}{q^2 - p^2 + i\varepsilon'}(p|\hat{V}|\psi^+) \\
&= \frac{2m}{\pi}\int d^2pd^2w\,(z|p)\frac{1}{q^2 - p^2 + i\varepsilon'}(p|w)\star_w(w|\hat{V}|\psi^+) \\
&= \frac{m\theta}{\pi^2\hbar^2}\int d^2pd^2w\frac{e^{-\frac{\theta}{2\hbar^2}|p|^2 + \frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p(\bar{z}-\bar{w}) + \bar{p}(z-w))}}{q^2 - p^2 + i\varepsilon'}e^{\overleftarrow{\partial_w}\overleftarrow{\partial_{\bar{w}}}}(w|\hat{V}|\psi^+) \\
&= \frac{m\theta}{\pi^2\hbar^2}\int d^2pd^2w\frac{e^{\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p(\bar{z}-\bar{w}) + \bar{p}(z-w))}}{q^2 - p^2 + i\varepsilon'}(w|\hat{V}|\psi^+). \qquad \text{(C.16)}
\end{aligned}$$

In the last step we let the star product act backwards, and its action exactly cancels the Gaussian term in p. For convenience, let us temporarily introduce a new variable $r = \frac{\sqrt{2\theta}}{\hbar}(z - w)$. Furthermore let ϕ denote the angle between the momentum vector p and the relative position vector r. One can then show that the momentum integral evaluates as follows:

$$\begin{aligned}
\int d^2p\,\frac{e^{\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p(\bar{z}-\bar{w}) + \bar{p}(z-w))}}{q^2 - p^2 + i\varepsilon} &= \int d|p|d\phi\,|p|\frac{e^{i|p||r|\cos\phi}}{|q|^2 - |p|^2 + i\varepsilon} \\
&= 2\pi\int d|p|\,|p|\frac{1}{|q|^2 - |p|^2 + i\varepsilon}J_0(|p||r|) \\
&= -2\pi\frac{\pi i}{2}H_0^{(1)}(|q||r|) \\
&= -i\pi^2 H_0^{(1)}(|q||r|). \qquad \text{(C.17)}
\end{aligned}$$

J_n denotes Bessel functions of the first kind and $H_0^{(1)}$ is the first Hankel function. We thus have

$$(z|\frac{1}{E - \hat{H}_0 \pm i\varepsilon}\hat{V}|\psi^+) = -\frac{im\theta}{\hbar^2}\int d^2w\,H_0^{(1)}(\frac{\sqrt{2\theta}}{\hbar}|q||z - w|)(w|\hat{V}|\psi^+). \qquad \text{(C.18)}$$

We next note that

$$\lim_{|r|\to\infty} H_0^{(1)}(\frac{\sqrt{2\theta}}{\hbar}|q||z - w|) = \sqrt{\frac{2}{\pi\frac{\sqrt{2\theta}}{\hbar}|q||z - w|}}e^{i\frac{\sqrt{2\theta}}{\hbar}|q||z-w|}e^{-i\frac{\pi}{4}}. \qquad \text{(C.19)}$$

We also note that if $|z| \gg |w|$, i.e. the observation point is far away,

$$\begin{aligned} |z-w| &= \sqrt{(z-w)(\bar{z}-\bar{w})} \\ &= |z|\sqrt{1-\frac{w\bar{z}+\bar{w}z)}{|z|^2}-\frac{|w|^2}{|z|^2}} \\ &\approx |z|\sqrt{1-\frac{2|w||z|\cos\alpha}{|z|^2}} \\ &\approx |z|-|w|\cos\alpha, \end{aligned} \tag{C.20}$$

where α is the angle between z and w. If we assume that the scattering is elastic then the outgoing momentum p of a particle scattered in the direction of the observation point z has magnitude $|q|$ and direction α relative to w. Thus,

$$\begin{aligned} e^{i\frac{\sqrt{2\theta}}{\hbar}|q||z-w|} &\approx e^{i\frac{\sqrt{2\theta}}{\hbar}|q||z|}e^{-i\frac{\sqrt{2\theta}}{\hbar}|q||w|\cos\alpha} \\ &= e^{i\frac{\sqrt{2\theta}}{\hbar}|q||z|}e^{-\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p\bar{w}+\bar{p}w)}. \end{aligned} \tag{C.21}$$

Finally, we let $1/|z-w| \approx 1/|z|$. (Note that all the approximations we have made are the same as the approximations that are made in commutative scattering theory, see [24]). Putting everything back into (2.14) we finally obtain

$$(z|\psi^+) = \sqrt{\frac{\theta}{2\pi\hbar^2}}\left(e^{-\frac{\theta}{4\hbar^2}|q|^2+\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(q\bar{z}+\bar{q}z)} + \sqrt{i}\frac{e^{i\frac{\sqrt{2\theta}}{\hbar}|q||z|}}{\sqrt{\frac{\sqrt{2\theta}}{\hbar}|q||z|}}f(q,p)\right), \tag{C.22}$$

where

$$f(q,p) = -\frac{m\theta}{\hbar^2}\sqrt{\frac{2}{\pi}}\sqrt{\frac{2\pi\hbar^2}{\theta}}\int d^2w\, e^{-\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p\bar{w}+\bar{p}w)}(w|\hat{V}|\psi^+), \tag{C.23}$$

is the scattering amplitude for a particle to have outgoing momentum p given an incident momentum of q.

C.2.4 Outgoing Current

The exact form of the scattered outgoing wave is given by (C.18), where no approximations have yet been made. We also note that the Hankel functions can be expressed in terms of the Bessel and von Neumann functions as

$$\begin{aligned} H_n^{(1)}(x) &= J_n(x) + iN_n(x), \\ H_n^{(2)}(x) &= J_n(x) - iN_n(x), \end{aligned} \tag{C.24}$$

and hence $\bar{H}_n^{(1)}(x) = H_n^{(2)}(x)$. Using this together with (2.5) we may calculate the current j_z^{out}:

$$\begin{aligned} j_z^{out} &= \frac{\hbar}{2i\pi m\theta}\left[\left(\frac{im\theta}{\hbar^2}\int d^2w\, H_0^{(2)}(\frac{\sqrt{2\theta}}{\hbar}|q||z-w|)(\psi^+|\hat{V}|w)\right)\right. \\ &\quad \star_z \left(-\frac{im\theta}{\hbar^2}\frac{\partial}{\partial\bar{z}}\int d^2w'\, H_0^{(1)}(\frac{\sqrt{2\theta}}{\hbar}|q||z-w'|)(w'|\hat{V}|\psi^+)\right) \\ &- \left(\frac{im\theta}{\hbar^2}\frac{\partial}{\partial\bar{z}}\int d^2w\, H_0^{(2)}(\frac{\sqrt{2\theta}}{\hbar}|q||z-w|)(\psi^+|\hat{V}|w)\right) \\ &\quad \left.\star_z \left(-\frac{im\theta}{\hbar^2}\int d^2w'\, H_0^{(1)}(\frac{\sqrt{2\theta}}{\hbar}|q||z-w'|)(w'|\hat{V}|\psi^+)\right)\right] \\ &= \frac{\hbar}{2i\pi m\theta}\left(\frac{m\theta}{\hbar^2}\right)^2\int d^2w d^2w'\,(\psi^+|\hat{V}|w)(w'|\hat{V}|\psi^+) \\ &\quad \left\{H_0^{(2)}(\frac{\sqrt{2\theta}}{\hbar}|q||z-w|)\star_z\left(\frac{\partial}{\partial\bar{z}}H_0^{(1)}(\frac{\sqrt{2\theta}}{\hbar}|q||z-w'|)\right)\right. \\ &\quad \left.-\left(\frac{\partial}{\partial\bar{z}}H_0^{(2)}(\frac{\sqrt{2\theta}}{\hbar}|q||z-w|)\right)\star_z H_0^{(1)}(\frac{\sqrt{2\theta}}{\hbar}|q||z-w'|)\right\}. \end{aligned} \tag{C.25}$$

The terms in curly brackets may be evaluated by expanding the star product, writing $|z| = \sqrt{z\bar{z}}$ and noting that the following properties hold for Hankel functions (with $i = 1, 2$ and $a \in \mathbb{R}$):

$$\begin{aligned} \frac{\partial}{\partial z}\left[\left(-\frac{a}{2}\sqrt{\frac{\bar{z}}{z}}\right)^n H_n^i(a\sqrt{z\bar{z}})\right] &= \left(-\frac{a}{2}\sqrt{\frac{\bar{z}}{z}}\right)^{n+1} H_{n+1}^i(a\sqrt{z\bar{z}}), \\ \frac{\partial}{\partial \bar{z}}\left[\left(-\frac{a}{2}\sqrt{\frac{z}{\bar{z}}}\right)^n H_n^i(a\sqrt{z\bar{z}})\right] &= \left(-\frac{a}{2}\sqrt{\frac{z}{\bar{z}}}\right)^{n+1} H_{n+1}^i(a\sqrt{z\bar{z}}), \end{aligned} \tag{C.26}$$

(this may easily be verified using software such as in Mathematica) and

$$\lim_{x\to\infty} H_n^i(x) = \sqrt{\frac{\pm 2i}{\pi x}}e^{\pm ix \pm \frac{n}{2}\pi}, \tag{C.27}$$

with + for $i = 1$ and - for $i = 2$.

We now evaluate the terms in the curly brackets of (C.25), making the abbreviation $a = \frac{\sqrt{2\theta}}{\hbar}|q|$:

$$\begin{aligned}
&H_0^{(2)}(a|z-w|) \star_z \left(\frac{\partial}{\partial \bar{z}} H_0^{(1)}(a|z-w'|)\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\partial^n}{\partial z^n} H_0^{(2)}(a|z-w|)\right) \left(\frac{\partial^{n+1}}{\partial \bar{z}^{n+1}} H_0^{(1)}(a|z-w'|)\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{a}{2}\sqrt{\frac{\bar{z}-\bar{w}}{z-w}}\right)^n \left(-\frac{a}{2}\sqrt{\frac{z-w'}{\bar{z}-\bar{w'}}}\right)^{n+1} \\
&\times H_n^{(2)}(a|z-w|) H_{n+1}^{(1)}(a|z-w'|).
\end{aligned} \tag{C.28}$$

If we now take the limit $|z| \to \infty$, $|z| \gg |w|, |w'|$ and therefore $\left(\sqrt{\frac{\bar{z}-\bar{w}}{z-w}}\right)^n \left(\sqrt{\frac{z-w'}{\bar{z}-\bar{w'}}}\right)^{n+1} \approx \sqrt{\frac{z}{\bar{z}}}$. (C.28) then becomes

$$\begin{aligned}
\lim_{|z|\to\infty} : \quad & -\frac{a}{2}\sqrt{\frac{z}{\bar{z}}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{a}{2}\right)^{2n} \sqrt{\frac{-2i}{\pi a|z|}} e^{-ia|z-w|-\frac{n}{2}\pi} \sqrt{\frac{2i}{\pi a|z|}} e^{ia|z-w'|+\frac{n+1}{2}\pi} \\
&= -i\sqrt{\frac{z}{\bar{z}}} e^{\frac{a^2}{4}} \frac{1}{\pi|z|} e^{ia(|z-w'|-|z-w|)} \\
&\approx -i\sqrt{\frac{z}{\bar{z}}} e^{\frac{\theta}{2\hbar^2}|q|^2} \frac{1}{\pi|z|} e^{\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(q(\bar{w}-\bar{w}')+\bar{q}(w-w'))}.
\end{aligned} \tag{C.29}$$

In the last line we substituted back $a = \frac{\sqrt{2\theta}}{\hbar}|q|$ and used (C.20) to write $|q||z-w| \approx |q||z| - |q||w|cos\alpha = |q||z| - \frac{i}{2}(w\bar{q} + \bar{w}q)$ and similarly $|q||z-w'| \approx |q||z| - \frac{i}{2}(w'\bar{q} + \bar{w}'q)$. Similarly to (C.28) and (C.29) we obtain

$$\begin{aligned}
\lim_{|z|\to\infty} \quad & \left(\frac{\partial}{\partial \bar{z}} H_0^{(2)}(a|z-w'|)\right) \star_z H_0^{(1)}(a|z-w|) \\
&\approx i\sqrt{\frac{z}{\bar{z}}} e^{\frac{\theta}{2\hbar^2}|q|^2} \frac{1}{\pi|z|} e^{\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(q(\bar{w}-\bar{w}')+\bar{q}(w-w'))}.
\end{aligned} \tag{C.30}$$

Substituting everything back into (C.25) we have

$$\begin{aligned}
j_z^{out} &= \frac{\hbar}{2i\pi m\theta} \left(\frac{m\theta}{\hbar^2}\right)^2 \frac{-2i}{\pi|z|} \sqrt{\frac{z}{\bar{z}}} e^{\frac{\theta}{2\hbar^2}|q|^2} \left| \int d^2w \, (w|\hat{V}|\psi^+) e^{\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(q\bar{w}+\bar{q}w)} \right|^2 \\
&= \sqrt{\frac{z}{\bar{z}}} \frac{1}{4\pi^2 m\hbar|z|} e^{\frac{\theta}{2\hbar^2}|q|^2} |f(q,p)|^2,
\end{aligned} \tag{C.31}$$

where in the last line we have substituted in (2.16). Similarly,

$$j_{\bar{z}}^{out} = \sqrt{\frac{\bar{z}}{z}}\frac{1}{4\pi^2 m\hbar|z|}e^{\frac{\theta}{2\hbar^2}|q|^2}|f(q,p)|^2, \tag{C.32}$$

and hence, using (2.8), we obtain

$$\begin{aligned} j_r &= \frac{1}{4\pi^2 m\hbar|z|}e^{\frac{\theta}{2\hbar^2}|q|^2}|f(q,p)|^2, \\ j_\phi &= 0. \end{aligned} \tag{C.33}$$

C.2.5 Alternative Form of Scattering Amplitude

We note first that

$$\begin{aligned} \frac{1}{\pi}\int d^2w\, e^{-\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p\bar{w}+\bar{p}w)}(w|\hat{V}|\psi^+) &= e^{-\frac{\theta}{2\hbar^2}|p|^2}\frac{1}{\pi}\int d^2w\, e^{-\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p\bar{w}+\bar{p}w)}\star_w (w|\hat{V}|\psi^+) \\ &= e^{-\frac{\theta}{4\hbar^2}|p|^2}\frac{1}{\pi}\int d^2w\, e^{-\frac{\theta}{4\hbar^2}|p|^2-\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p\bar{w}+\bar{p}w)}\star_w (w|\hat{V}|\psi^+) \\ &= \sqrt{\frac{2\pi\hbar^2}{\theta}}e^{-\frac{\theta}{4\hbar^2}|p|^2}\frac{1}{\pi}\int d^2w\,(p|w)\star_w (w|\hat{V}|\psi^+) \\ &= \sqrt{\frac{2\pi\hbar^2}{\theta}}e^{-\frac{\theta}{4\hbar^2}|p|^2}(p|\hat{V}|\psi^+). \end{aligned} \tag{C.34}$$

Hence, from (2.16)

$$\begin{aligned} f(q,p) &= -\frac{m\pi\theta}{\hbar^2}\sqrt{\frac{2}{\pi}}\frac{2\pi\hbar^2}{\theta}e^{-\frac{\theta}{4\hbar^2}|p|^2}(p|\hat{V}|\psi^+) \\ &= -2m\pi^2\sqrt{\frac{2}{\pi}}e^{-\frac{\theta}{4\hbar^2}|p|^2}(p|\hat{V}|\psi^+). \end{aligned} \tag{C.35}$$

C.2.6 Scattering Amplitude in the Born Approximation

$$\begin{aligned}
f(q,p) &= -2m\pi^2\sqrt{\frac{2}{\pi}}e^{-\frac{\theta}{4\hbar^2}|p|^2}(p|\hat{V}|q) \\
&= -2m\pi\sqrt{\frac{2}{\pi}}e^{-\frac{\theta}{4\hbar^2}|p|^2}\int d^2w(p|w)\star_w (w|\hat{V}|q) \\
&= -2m\pi\sqrt{\frac{2}{\pi}}e^{-\frac{\theta}{4\hbar^2}|p|^2}\int d^2w(p|w)\star_w V(\bar{w},w)\star_w (w|q) \\
&= -2m\pi\sqrt{\frac{2}{\pi}}\frac{\theta}{2\pi\hbar^2}e^{-\frac{\theta}{4\hbar^2}|p|^2}\int d^2we^{-\frac{\theta}{4\hbar^2}|p|^2-\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p\bar{w})+\bar{p}w} \\
&\quad \star_w\left(V(\bar{w},w)\star_w e^{-\frac{\theta}{4\hbar^2}|q|^2+\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(q\bar{w})+\bar{q}w}\right) \\
&= -2m\pi\sqrt{\frac{2}{\pi}}\frac{\theta}{2\pi\hbar^2}e^{-\frac{\theta}{4\hbar^2}(2|p|^2+|q|^2-2|p|^2-2q(\bar{q}-\bar{p}))}\int d^2we^{\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}((q-p)\bar{w})+(\bar{q}-\bar{p})w}V(\bar{w},w) \\
&= -4m\pi^2\sqrt{\frac{2}{\pi}}\frac{\theta}{2\pi\hbar^2}e^{\frac{\theta}{4\hbar^2}(|q|^2-2q\bar{p})}\widetilde{V}(\frac{\sqrt{2\theta}(q-p)}{\hbar}), \qquad (C.36)
\end{aligned}$$

with $\widetilde{V}(\frac{\sqrt{2\theta}(q-p)}{\hbar})$ the normal Fourier transform of $V(\bar{z},z)$.

C.2.7 The Optical Theorem

Proof of the non-commutative optical theorem, following the steps of [24]:

$$\begin{aligned}
Im(q|\hat{T}|q) &= Im(q|\hat{V}|\psi^+) \\
&= Im\left[\left((\psi^+| - (\psi^+|\hat{V}\frac{1}{E-\hat{H}_0-i\varepsilon}\right)\hat{V}|\psi^+)\right] \\
&= -Im(\psi^+|\hat{V}\frac{1}{E-\hat{H}_0-i\varepsilon}\hat{V}|\psi^+), \qquad (C.37)
\end{aligned}$$

because the hermiticity of $\hat{V}$ implies that the first term vanishes.

We next use the relation

$$\frac{1}{E-\hat{H}_0-i\varepsilon} = Pr.\frac{1}{E-\hat{H}_0}+i\pi\delta(E-\hat{H}_0). \qquad (C.38)$$

We then get

$$\begin{aligned}
-Im(\psi^+|\hat{V}\frac{1}{E-\hat{H}_0-i\varepsilon}\hat{V}|\psi^+) &= -Im(\psi^+|\hat{V}Pr.\frac{1}{E-\hat{H}_0}\hat{V}|\psi^+) \\
&\quad -Im(\psi^+|\hat{V}i\pi\delta(E-\hat{H}_0)\hat{V}|\psi^+) \\
&= -\pi(\psi^+|\hat{V}\delta(E-\hat{H}_0)\hat{V}|\psi^+). \qquad \text{(C.39)}
\end{aligned}$$

Again the one term vanishes because of the hermiticity of $\hat{V}$ and $Pr.\frac{1}{E-\hat{H}_0}$. We thus end up with

$$\begin{aligned}
Im(q|\hat{T}|q) &= -\pi(\psi^+|\hat{V}\delta(E-\hat{H}_0)\hat{V}|\psi^+) \\
&= -\pi(q|\hat{T}^\dagger\delta(E-\hat{H}_0)\hat{T}|q) \\
&= -\pi\int d^2p\,(q|\hat{T}^\dagger\delta(E-\hat{H}_0)|p)(p|\hat{T}|q) \\
&= -\pi\int d^2p\,(q|\hat{T}^\dagger|p)(p|\hat{T}|q)\delta(E-\frac{p^2}{2m}) \\
&= -\pi\int d|p|d\phi\,|p||(p|\hat{T}|q)|^2\delta(E-\frac{|p|^2}{2m}) \\
&= -m\pi\int d\phi\,|(p|\hat{T}|q)|^2, \qquad \text{(C.40)}
\end{aligned}$$

where $|p|=|q|$. Therefore, using C.35 and $\hat{T}|q)=\hat{V}|\psi^+)$,

$$\begin{aligned}
Im\left(f(\phi=0)\right) &= -2m\pi^2\sqrt{\frac{2}{\pi}}e^{-\frac{\theta}{4\hbar^2}|q|^2}Im\left((q|\hat{T}|q)\right) \\
&= -2m\pi^2\sqrt{\frac{2}{\pi}}e^{-\frac{\theta}{4\hbar^2}|q|^2}\left(-m\pi\int d\phi\,|(p|\hat{T}|q)|^2\right) \\
&= 2m^2\pi^3\sqrt{\frac{2}{\pi}}e^{-\frac{\theta}{4\hbar^2}|q|^2}\int d\phi\left|\frac{-1}{2m\pi^2}\sqrt{\frac{\pi}{2}}e^{\frac{\theta}{4\hbar^2}|p|^2}f(q,p)\right|^2 \\
&= \frac{1}{\sqrt{8\pi}}e^{\frac{\theta}{4\hbar^2}|q|^2}\int d\phi|f(q,p)|^2 \\
&= \frac{1}{\sqrt{8\pi}}e^{\frac{\theta}{4\hbar^2}|q|^2}\int d\phi\left(\frac{|q|}{\hbar}e^{\frac{\theta}{2\hbar^2}|q|^2}\frac{d\sigma}{d\phi}\right) \\
&= \frac{1}{\sqrt{8\pi}}\left(\frac{|q|}{\hbar}\right)e^{-\frac{\theta}{4\hbar^2}|q|^2}\sigma_{tot}, \qquad \text{(C.41)}
\end{aligned}$$

i.e.

$$\sigma_{tot} = \frac{\sqrt{8\pi}\hbar}{|q|}e^{\frac{\theta}{4\hbar^2}|p|^2}Im\left(f(\phi=0)\right). \qquad \text{(C.42)}$$

C.3 Calculations for Chapter 3

C.3.1 Overlap of N-Particle Position States

Using the definition (3.17) we have

$$\begin{aligned} {}_\pm(z_1,...,z_N|\, w_1,...,w_N)_\pm &= \frac{1}{N!}\sum_{P,Q\in S(N)} (\pm)^{\epsilon_P+\epsilon_Q}\left(z_{P(1)},...,z_{P(N)}\middle|\, w_{Q(1)},...,w_{Q(N)}\right) \\ &= \frac{1}{N!}\sum_{P,Q\in S(N)} (\pm)^{\epsilon_P+\epsilon_Q}\prod_{i=1}^{N} e^{-|z_{P(i)}-w_{Q(i)}|^2} \\ &= \frac{1}{N!}\sum_{P,Q\in S(N)} (\pm)^{\epsilon_P+\epsilon_Q}\prod_{i=1}^{N} e^{-|z_i-w_{Q(P^{-1}(i))}|^2} \\ &= \frac{1}{N!}\sum_{P\in S(N)}\sum_{Q'\in S(N)} (\pm)^{\epsilon_{Q'}}\prod_{i=1}^{N} e^{-|z_i-w_{Q'(i)}|^2} \\ &= \sum_{Q'\in S(N)} (\pm)^{\epsilon_{Q'}}\prod_{i=1}^{N} e^{-|z_i-w_{Q'(i)}|^2}, \end{aligned} \tag{C.43}$$

where we have introduced $Q' = QP^{-1}$.

C.3.2 Action of the Creation Operator

$$\hat{\psi}^{\ddagger}(z)|z_1,...,z_N)_\pm = |z,z_1,...,z_N)_\pm. \tag{C.44}$$

Proof:

$$\begin{aligned} &\hat{\psi}^{\ddagger}(z)|z_1,...,z_N)_\pm \\ &= \sum_{N'=0}^{\infty}\frac{1}{N'!\pi^{N'}}\int d^2z_1'...d^2z_{N'}'|z,z_1',...,z_{N'}')_\pm \star_{z_1',...,z_{N'}'}\ {}_\pm(z_1',...,z_{N'}'|z_1,...,z_N)_\pm \\ &= \frac{1}{N!\pi^N}\int d^2z_1'...d^2z_N'|z,z_1',...,z_N')_\pm \star_{z_1',...,z_N'}\ {}_\pm(z_1',...,z_N'|z_1,...,z_N)_\pm \\ &= \frac{1}{N!\pi^N}\int d^2z_1'...d^2z_N'|z,z_1',...,z_N')_\pm e^{\sum_{i=1}^N \overleftarrow{\partial_{z'_i}}\overrightarrow{\partial_{\bar{z}_i'}}}\sum_P(\pm)^P\prod_{a=1}^N e^{-|z_a'-z_{P(a)}|^2} \\ &= \frac{1}{N!\pi^N}\int d^2z_1'...d^2z_N'|z,z_1',...,z_N')_\pm e^{\sum_{i=1}^N -\overrightarrow{\partial_{z'_i}}\overrightarrow{\partial_{\bar{z}_i'}}}\sum_P(\pm)^P\prod_{a=1}^N e^{-|z_a'-z_{P(a)}|^2} \end{aligned} \tag{C.45}$$

$$
\begin{aligned}
&= \frac{1}{N!\pi^N} \sum_P (\pm)^P \int d^2z'_1...d^2z'_N |z, z'_1, ..., z'_N)_\pm \prod_{a=1}^{N} \pi\delta(z'_a - z_{P(a)}) \\
&= \frac{1}{N!} \sum_P (\pm)^P |z, z_{P(1)}, ..., z_{P(N)})_\pm \\
&= \frac{1}{N!} \sum_P (\pm)^P (\pm)^P |z, z_1, ..., z_N)_\pm \\
&= |z, z_1, ..., z_N)_\pm .
\end{aligned} \tag{C.46}
$$

Here we have used integration by parts to make the derivative operators work forward, and B.2 has been used repeatedly to turn the Gaussian factors into Dirac deltas.

C.3.3 Action of the Annihilation Operator

$$
\hat{\psi}(z)|z_1, ..., z_N)_\pm = \sum_{a=1}^{N} (\pm)^{a+1} e^{-|z-z_a|^2} |z_1, ..., z_{a-1}, z_{a+1}, ..., z_N)_\pm . \tag{C.47}
$$

Proof:

$$
\begin{aligned}
&\hat{\psi}(z)|z_1, ..., z_N) \\
&= \sum_{N'=0}^{\infty} \frac{1}{N'!\pi^{N'}} \int d^2z'_1...d^2z'_{N'} |z'_1, ..., z'_{N'})_\pm e^{-\sum_{i=1}^{N'} \overrightarrow{\partial_{z'_i}} \overrightarrow{\partial_{\bar{z}'_i}}} {}_\pm(z, z'_1, ..., z'_{N'}|z_1, ..., z_N)_\pm \\
&= \frac{1}{(N-1)!\pi^{N-1}} \int d^2z'_1...d^2z'_{N-1} |z'_1, ..., z'_{N-1}) \\
&\times e^{-\sum_{i=1}^{N-1} \overrightarrow{\partial_{z'_i}} \overrightarrow{\partial_{\bar{z}'_i}}} {}_\pm(z, z'_1, ..., z'_{N-1}|z_1, ..., z_N)_\pm \\
&= \frac{1}{(N-1)!\pi^{N-1}} \int d^2z'_1...d^2z'_{N-1} |z'_1, ..., z'_{N-1})_\pm \\
&\times e^{-\sum_{i=1}^{N-1} \overrightarrow{\partial_{z'_i}} \overrightarrow{\partial_{\bar{z}'_i}}} \sum_P (\pm)^P e^{-|z-z_{P(N)}|^2} \prod_{a=1}^{N-1} e^{-|z'_a - z_{P(a)}|^2} \\
&= \frac{1}{(N-1)!\pi^{N-1}} \sum_P (\pm)^P \int d^2z'_1...d^2z'_{N-1} |z'_1, ..., z'_{N-1})_\pm \\
&e^{-|z-z_{P(N)}|^2} \prod_{a=1}^{N-1} \pi\delta(z'_a - z_{P(a)}) \\
&= \frac{1}{(N-1)!} \sum_P (\pm)^P e^{-|z-z_{P(N)}|^2} |z_{P(1)}, ..., z_{P(N-1)})_\pm \\
&= \sum_{a=1}^{N} (\pm)^{a+1} e^{-|z-z_a|^2} |z_1, ..., z_{a-1}, z_{a+1}, ..., z_N)_\pm .
\end{aligned} \tag{C.48}
$$

C.3.4 Commutation Relation Between Non-Commutative Position Annihilation and Creation Operators

$$[\hat{\psi}(z), \hat{\psi}^{\ddagger}(w)]_{\mp} = e^{-|z-w|^2}. \tag{C.49}$$

Proof:

$$\begin{aligned}
\hat{\psi}(z)\hat{\psi}^{\ddagger}(w)|z_1, ..., z_N)_{\pm} &= \hat{\psi}(z)|w, z_1, ..., z_N)_{\pm} \\
&= \sum_{a=0}^{N} (\pm)^a e^{-|z-z_a|^2} |w, ..., z_{a-1}, z_{a+1}, ..., z_N)_{\pm} \\
&= e^{-|z-w|^2}|z_1, ..., z_N)_{\pm} \\
&\quad + \sum_{a=1}^{N} (\pm)^a e^{-|z-z_a|^2} |w, ..., z_{a-1}, z_{a+1}, ..., z_N)_{\pm},
\end{aligned} \tag{C.50}$$

where we used $z_0 = w$. Also,

$$\begin{aligned}
\hat{\psi}^{\ddagger}(w)\hat{\psi}(z)|z_1, ..., z_N)_{\pm} &= \hat{\psi}^{\ddagger}(w) \sum_{a=1}^{N} (\pm)^{a+1} e^{-|z-z_a|^2} |z_1, ..., z_{a-1}, z_{a+1}, ..., z_N)_{\pm} \\
&= \sum_{a=1}^{N} (\pm)^{a+1} e^{-|z-z_{P(N)}|^2} |w, ..., z_{a-1}, z_{a+1}, ..., z_N)_{\pm}.
\end{aligned} \tag{C.51}$$

Hence, $[\hat{\psi}(z), \hat{\psi}^{\dagger}(w)]_{\star\mp} = [\hat{\psi}(z), \hat{\psi}^{\dagger}(w)]_{\mp} = e^{-|z-w|^2}$.

C.3.5 Action of a Two-Particle Operator

Suppose we have a two-particle operator $\hat{W}$ as defined in (3.53). We wish to show how such an operator acts on an n-particle state. To do this we follow the same steps as in 3.4.5:

$$\begin{aligned}
[\hat{\psi}(z), \hat{W}] &= \frac{1}{2\pi^2} \int d^2u d^2v \; [\hat{\psi}(z), \hat{\psi}^{\dagger}(u)\hat{\psi}^{\dagger}(v)] \star_{u,v} W(\bar{u}, u, \bar{v}, v) \star_{u,v} \hat{\psi}(v)\hat{\psi}(u) \\
&= \frac{1}{2\pi^2} \int d^2u d^2v \; \left\{ [\hat{\psi}(z), \hat{\psi}^{\dagger}(u)]_{\mp} \hat{\psi}^{\dagger}(v) \pm \hat{\psi}^{\dagger}(u) [\hat{\psi}(z), \hat{\psi}^{\dagger}(v)]_{\mp} \right\} \\
&\quad \star_{u,v} W(\bar{u}, u, \bar{v}, v) \star_{u,v} \hat{\psi}(v)\hat{\psi}(u) \\
&= \frac{1}{2\pi^2} \int d^2u d^2v \; \left\{ e^{-|z-u|^2} \hat{\psi}^{\dagger}(v) \pm \hat{\psi}^{\dagger}(u) e^{-|z-v|^2} \right\} \\
&\quad \star_{u,v} W(\bar{u}, u, \bar{v}, v) \star_{u,v} \hat{\psi}(v)\hat{\psi}(u).
\end{aligned} \tag{C.52}$$

We may swap the integration labels u and v around for the second term and use the fact that $\hat{\psi}(v)\hat{\psi}(u) = \mp\hat{\psi}(u)\hat{\psi}(v)$ to obtain

$$\begin{aligned}
[\hat{\psi}(z), \hat{W}] &= \frac{1}{\pi^2}\int d^2u d^2v\; e^{-|z-u|^2}\hat{\psi}^\dagger(v) \star_{u,v} W(\bar{u}, u, \bar{v}, v) \star_{u,v} \hat{\psi}(v)\hat{\psi}(u) \\
&= \frac{1}{\pi^2}\int d^2u d^2v\; \left(e^{-\partial_u\partial_{\bar{u}}}e^{-|z-u|^2}\right)\hat{\psi}^\dagger(v) \star_{v} W(\bar{u}, u, \bar{v}, v) \star_{u,v} \hat{\psi}(v)\hat{\psi}(u) \\
&= \frac{1}{\pi}\int d^2v\; \hat{\psi}^\dagger(v) \star_{v} W(\bar{z}, z, \bar{v}, v) \star_{z,v} \hat{\psi}(v)\hat{\psi}(z) \\
&= \hat{V}_W(z)\hat{\psi}(z), \qquad\qquad (C.53)
\end{aligned}$$

where we have introduced the operator $\hat{V}_W(z) = \frac{1}{\pi}\int d^2v\;\; \hat{\psi}^\dagger(v) \star_v W(\bar{z}, z, \bar{v}, v) \star_{z,v} \hat{\psi}(v)$ which acts as a single-particle operator.

Next we note that

$$\begin{aligned}
{}_\pm(z_1, ..., z_N|\hat{W}|\psi) &= (\Omega|\hat{\psi}(z_N)...\hat{\psi}(z_1), \hat{W}|\psi) \\
&= (\Omega|[\hat{\psi}(z_N)...\hat{\psi}(z_1), \hat{W}]|\psi) \\
&= (\Omega|\hat{\psi}(z_N)...[\hat{\psi}(z_1), \hat{W}]|\psi) \\
&\quad +(\Omega|\hat{\psi}(z_N)...[\hat{\psi}(z_2), \hat{W}]\hat{\psi}(z_1)|\psi) \\
&\quad +... + (\Omega|[\hat{\psi}(z_N), \hat{W}]\hat{\psi}(z_{N-1})...\hat{\psi}(z_1)|\psi). \qquad (C.54)
\end{aligned}$$

For each of these terms we have

$$(\Omega|\hat{\psi}(z_N)...[\hat{\psi}(z_i), \hat{W}]\hat{\psi}(z_{i-1})...|\psi) = (\Omega|\hat{\psi}(z_N)...\hat{V}_W(z_i)\hat{\psi}(z_i)\hat{\psi}(z_{i-1})...|\psi) \quad (C.55)$$

Furthermore,

$$\begin{aligned}
(\Omega|\hat{\psi}(z_N)...\hat{V}_W(z_i)\hat{\psi}(z_i)...|\psi) &= (\Omega|[\hat{\psi}(z_N)...\hat{V}_W(z_i)]\hat{\psi}(z_i)...|\psi) \\
&= \sum_{a=i+1}^{N} W(\bar{z}_i, z_i, \bar{w}_a, w_a) \star_{z_i,z_a} {}_\pm(z_1, ..., z_N|\psi), \quad (C.56)
\end{aligned}$$

as was shown in (3.52). Combining (C.54), (C.55) and (C.56) we finally obtain

$${}_\pm(z_1, ..., z_N|\hat{W}|\psi) = \sum_{a=1}^{N}\sum_{b=a+1}^{N} W(\bar{z}_a, z_a, \bar{z}_b, z_b) \star_{z_a,z_b} {}_\pm(z_1, ..., z_N|\psi). \qquad (C.57)$$

C.4 Calculations for Chapter 4

C.4.1 The Exchange Potential

We start from the expression

$$\begin{aligned} P(z_1, z_2) &= tr_Q \left[\pi^{\pm}_{z_1,z_2}\hat{\rho}\right] \\ &= \frac{1}{\pi^2} tr_Q[|z_1, z_2)_{\pm} \star_{z_1,z_2} {}_{\pm}(z_1, z_2|\hat{\rho}], \end{aligned} \tag{C.58}$$

with $\hat{H}$ denoting the two-particle free Hamiltonian. To proceed, we will exploit the fact that the star product may be decomposed as follows:

$$e^{\overleftarrow{\partial_z}\overrightarrow{\partial_{\bar{z}}}} = \frac{1}{\pi}\int d^2u \; e^{-|u|^2+u\overleftarrow{\partial_z}+\bar{u}\overrightarrow{\partial_{\bar{z}}}}. \tag{C.59}$$

It is also a well known fact that $e^{u\partial_z}$ and $e^{\bar{u}\partial_{\bar{z}}}$ are translation operators, i.e.

$$\begin{aligned} e^{u\partial_z} f(\bar{z}, z) &= f(\bar{z}, z+u), \\ e^{\bar{u}\partial_{\bar{z}}} f(\bar{z}, z) &= (\bar{z}+\bar{u}, z), \end{aligned} \tag{C.60}$$

for an arbitrary function f. Having split up the star product, we may express (C.58) in terms of an overlap. We must, however, be careful to ensure that the derivatives with respect to z_1 and z_2 only act on the the ket $|z_1, z_2)_{\pm}$, whereas the derivatives with respect to $\bar{z}_1$ and $\bar{z}_2$ only see the bra ${}_{\pm}(z_1, z_2|$. To ensure this we will temporarily let $|z_1, z_2)_{\pm} \to |w_1, w_2)_{\pm}$ and then let $w_1 \to z_1$ and $w_2 \to z_2$ at the end of the calculation.

As a first step we have

$$\begin{aligned} P(z_1, z_2) &= \frac{1}{\pi^2} tr_Q[|z_1, z_2)_{\pm} \star_{z_1,z_2} {}_{\pm}(z_1, z_2|\hat{\rho}] \\ &= \lim_{w_1\to z_1, w_2\to z_2} \frac{1}{\pi^4}\int d^2u d^2v \; e^{-|u|^2+u\partial_{w_1}+\bar{u}\partial_{\bar{z}_1}} e^{-|v|^2+v\partial_{w_2}+\bar{v}\partial_{\bar{z}_2}} \\ &\quad \times {}_{\pm}(z_1, z_2|\hat{\rho}|w_1, w_2)_{\pm}. \end{aligned} \tag{C.61}$$

For non-interacting particles the Hamiltonian has the form of a sum of two single-particle Hamiltonians, i.e. $\hat{H} = \hat{H}_1 \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{H}_2$. Therefore $e^{-\beta\hat{H}} = e^{-\beta\hat{H}_1} \otimes e^{-\beta\hat{H}_2}$, and if $\hat{H}_1$ and $\hat{H}_2$ are free Hamiltonians, the overlap in C.61 becomes

$$\begin{aligned} {}_{\pm}(w_1, w_2|e^{-\beta\hat{H}}|z_1, z_2)_{\pm} &= (w_1|e^{-\beta\hat{H}}|z_1)(w_2|e^{-\beta\hat{H}}|z_2) \pm (w_1|e^{-\beta\hat{H}}|z_2)(w_2|e^{-\beta\hat{H}}|z_1) \\ &= \Gamma^2\left\{e^{-\Gamma(|z_1-w_1|^2+|z_2-w_2|^2)} \pm e^{-\Gamma(|z_1-w_2|^2+|z_2-w_1|^2)}\right\}, \end{aligned} \tag{C.62}$$

where $\Gamma = \frac{m\theta}{m\theta+\beta\hbar^2}$, and we have used the result

$$\begin{aligned}(w|e^{-\beta\hat{H}}|z) &= \int d^2p\ (w|e^{-\beta\hat{H}}|p)(p|z) \\ &= \frac{\theta}{2\pi\hbar^2}\int d^2p\ e^{-(\frac{\beta}{2m}+\frac{\theta}{2\hbar^2})|p|^2+\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}(p(\bar{w}-\bar{z})+\bar{p}(w-z))} \\ &= \Gamma e^{-\Gamma|z-w|^2}, \end{aligned} \tag{C.63}$$

Acting in on (C.62) with the operators $e^{u\partial_{w_1}}$, $e^{v\partial_{w_2}}$, $e^{\bar{u}\partial_{\bar{z}_1}}$ and $e^{\bar{u}\partial_{\bar{z}_1}}$ simply translates the respective function arguments. We therefore have

$$\begin{aligned}&P(z_1,z_2) \\ &= \lim_{w_1\to z_1, w_2\to z_2}\frac{\Gamma^2}{\pi^4 Z}\int d^2u d^2v\ e^{-|u|^2-|v|^2}\left\{e^{-\Gamma((z_1-w_1-u)(\bar{z}_1+\bar{u}-\bar{w}_1)+(z_2-w_2-u)(\bar{z}_2+\bar{v}-\bar{w}_2))}\right. \\ &\left.\pm\ e^{-\Gamma((z_1-w_2-v)(\bar{z}_1+\bar{u}-\bar{w}_2)+(z_2-w_1-u)(\bar{z}_2+\bar{v}-\bar{w}_1))}\right\} \\ &= \frac{\Gamma^2}{\pi^4 Z}\int d^2u d^2v\ \left\{e^{-(1-\Gamma)\left(|u|^2+|v|^2\right)}\right. \\ &\left.\pm e^{-\left(|u|^2+|v|^2\right)+\Gamma\left(-2|z_1-z_2|^2+u(\bar{z}_2-\bar{z}_1+\bar{v})+\bar{u}(z_2-z_1+v)-v(z_2-z_1)-\bar{v}(z_2-z_1)\right)}\right\} \\ &= \frac{\Gamma^2}{\pi^3 Z}\int d^2v\ \left\{\frac{e^{-(1-\Gamma)|v|^2}}{1-\Gamma}\pm e^{-(2\Gamma-\Gamma^2)|z_2-z_1|^2}e^{-(1-\Gamma^2)|v|^2+(\Gamma^2-\Gamma)v(\bar{z}_2-\bar{z}_1)+(\Gamma^2-\Gamma)\bar{v}(z_2-z_1)}\right\} \\ &= \frac{\Gamma^2}{\pi^2 Z(1-\Gamma)^2}\pm\frac{\Gamma^2}{\pi^2 Z(1-\Gamma^2)}e^{-\left(2\Gamma-\Gamma^2-\frac{(\Gamma^2-\Gamma)^2}{1-\Gamma^2}\right)|z_2-z_1|^2}. \end{aligned} \tag{C.64}$$

Upon insertion of $\Gamma = \frac{m\theta}{m\theta+\beta\hbar^2}$ this simplifies to

$$P(z_1,z_2) = \frac{1}{Z\pi^2}\left(\frac{m\theta}{\beta\hbar^2}\right)^2\left\{1\pm\frac{\beta\hbar^2}{\beta\hbar^2+2m\theta}e^{-\frac{2m\theta}{2m\theta+\beta\hbar^2}|z_2-z_1|^2}\right\}. \tag{C.65}$$

We now define the exchange potential analogous to [34]:

$$\begin{aligned}V_{eff}(z_1,z_2) &= -\frac{1}{\beta}ln(P(z_1,z_2)) \\ &= -\frac{1}{\beta}ln\left[1\pm\frac{\beta\hbar^2}{\beta\hbar^2+2m\theta}e^{-\frac{2m\theta}{2m\theta+\beta\hbar^2}|z_2-z_1|^2}\right]+const. \end{aligned} \tag{C.66}$$

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