A Categorical Study of Compactness via Closure

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We have the familiar Kuratowski-Mrówka theorem in topology, where compactness is characterised by a closure and a projection-map (X is compact iff $p : X \times Y \to Y$ is a closed mapping, for any space Y, i.e. $p(\overline{A}) = \overline{p(A)}$, $\forall A \subseteq X \times Y$). Using this as our starting point, we generalise compactness to a categorical setting. We then generalise even further to "asymmetric" compactness. Then we discuss a functional approach to compactness, where we do not explicitly mention closure operators. All this provides economical proofs as well as applications in different areas of mathematics.

Ons ken die bekende Kuratowski-Mrówka stelling in topologie, waar kompaktheid deur afsluiting en 'n projeksie gekarakteriseer word (X is kompak as en slegs as $p : X \times Y \to Y$ 'n geslote afbeelding is, vir enige ruimte Y, m.a.w $p(\overline{A}) = \overline{p(A)}$, $\forall A \subseteq X \times Y$). Deur bogenoemde as ons beginpunt te gebruik, veralgemeen ons kompaktheid tot 'n kategoriële idee. Ons veralgemmen dan selfs verder tot "nie-simmetriese kompaktheid". Dan bespreek ons 'n funksionele benadering tot kompaktheid waar ons nie eksplisiet die idee van afsluiting noem nie. Deur al hierdie werk te bespreek kry ons ekonomiese bewyse asook heelwat toepassings in verskillende gebiede in wiskunde.

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Introduction

In the field of category theory numerous other mathematical fields can be generalized. Our focus is on the generalization of topolgical properties. We want to discuss compactness in a category. We know the notion from topology - "every open cover has a finite subcover". This could also be equivalently characterised by means of closure - A topological space is compact iff the projection $p : X \times Y \to Y$ is a closed mapping, for any space Y, the Kuratowski-Mrówka characterisation. This developed with Kuratowski proving in [15] that such projections are closed mappings for compact metric spaces, while Mrówka showed (cf. [17]) that compact topological spaces are characterised in this way. Originally, this characterisation was considered for Hausdorff spaces. We, however, are going to consider the case where no restrictions are put on the topological space (see for example [14]).

This characterisation opens the way for a categorical notion of compactness, and we are especially interested since we are going to work with categorical closure operators and (since category theorists are more interested in definitions and characterisations given in terms of morphisms rather than objects themselves) we are going to consider projection-morphisms. This leads to a strong generalisation of compactness and perfectness-ideas from topology, with the proofs of many topological theorems becoming much more economical and, in some instances, quite trivial. And we are working in a category after all, which means we can also apply all this theory to other branches of mathematics - so, in a certain sense, one general categorical proof/result in this context could lead to numerous results in different categories, yielding many useful applications.

A categorical study of compactness has been developed over the years, with Manes discussing Compact Hausdorff objects in 1974 by using a category of "sets with structure" (cf. [16]), while in 1987 Herrlich, Salicrup and Strecker discussed categorical compactness in [13] by considering a pair of factorization systems (without using closure operators)

In [3] (1990) Castellini generalised the work of Herrlich, Salicrup and Strecker by using the notion of closure operators equivalent to the notion we will be considering.

As mentioned before, many applications in different branches of mathematics are possible - in 1994 Dikranjan and Uspensjki (cf. [9]) established a result in the category of topological groups (proven in a non-categorical way), which is actually a result of the categorical Tychonoff-theorem. If you work in the category of locales (the dual of the category of frames) the Kuratowski-Mrówka characterisation of compact objects still holds - it was shown in [18] (Pultr an Tozzi) and [20], with and without choice respectively. And compact morphisms in the category of locales have been characterised in [20] and [21] by Vermeulen.

In 1996 all of this development was brought together in [5] with Clementino, Giuli and Tholen discussing categorical compactness via closure operators (in a category equipped with a proper factorization system), and providing some beautiful generalised topological results, like Tychonoff's and Frolik's Theorem. This article lead to Holgate's article on "Asymmetric compactness" in 2008 (cf. [14]), where two different closure operators are used to obtain a generalised notion of categorical compactness. This also uses examples from topology as motivation, and shows that topological properties such as Countably Compact and Lindelöf are captured by this notion. Compact Morphisms (generalised proper/perfect maps) are also discussed by working in the Comma Category, and the conclusion is made that, since the class of Countably Compact maps is larger than the class of Quasi-perfect maps, Countably Compact maps is probably the "better behaved" class to study.

In 2004 Clementino, Giuli and Tholen provided a more functional approach in [6] by assuming they have a class of morphisms $\mathcal{F}$ in their category, satisfying certain axioms. We think of this class as being the closed (i.e. closure preserving) morphisms (or closed maps if you think of the topological analogy). This class is then used to define categorical compactness in an even more generalised way without mentioning closure operators (This approach was first outlined by Tholen in [19](in 1999). A rather counter-intuitive example of such a $\mathcal{F}$ is given - open maps in a topological space, and generalised versions of Tychonoff's and Frolik's theorems are again provided, amongst others. Exponentiability is also discussed.

In Chapter 1 we discuss the Kuratowski-Mrówka characterisation for a topological space - X is compact iff $p : X \times Y \to Y$ is a closed mapping, for any

space Y. The usual closure in topology will be referred to as the Kuratowski-closure or "k-closure", however, we do still use the "$\overline{A}$"-notation to denote the k-closure of A. Different closures will be defined (the σ-closure and the θ-closure (cf. [10])), and similar characterisations are obtained (to the Kuratowski-Mrówka characterisation), which are also proven in [5]. It is then noted that we are actually only interested in the inlcusion $p(\overline{A}) \supseteq \overline{p(A)}$, since the projection-mapping is continuous. We then discuss examples where two different closures are used, in a similar inclusion, to obtain certain characterisations (this being the motivation for discussing asymmetric compactness in a category, mentioned in the second paragraph above). Characterisations for Countably Compact as well as Lindelöf, which are discussed in [14], will be discussed.

In Chapter 2 we investigate the categorical approach. Firstly $(\mathcal{E}, \mathcal{M})$ factorization systems are discussed, as they were introduced by Freyd and Kelly in 1972 (cf. [11]). $\mathcal{E}$ and $\mathcal{M}$ are two classes of morphisms which form a factorization system for a category $\mathcal{C}$ if they satisfy certain properties, and each morphism $f : X \to Y$ in $\mathcal{C}$ can be factorized $f = m \circ e$, with $e \in \mathcal{E}$ and $m \in \mathcal{M}$. And we mention that we are interested in factorization systems where $\mathcal{E}$ is a class of epimorphisms and $\mathcal{M}$ is a class of monomorphisms, which is called a "proper factorization system". All the $\mathcal{M}$-morphisms with codomain X are called the subobjects of X, generalised inclusion-maps. We also define an order on subobjects, and we discuss the image/pre-image adjunction.
We then define closure operators as they were introduced by Dikranjan and Giuli in 1986 (cf. [8]). We mention what is meant by a "C-closed subobject", where C refers to a closure operator C. We then proceed to define closure-preserving morphisms, and we show properties of closure operators as well as closure-preserving morphisms, following the strategies of [5] very closely.
We then discuss categorical compactness by using the Kuratowski-Mrówka characterisation (we follow the approach of [5]), and mention three topological theorems which we generalise to our categorical context - we only state them, since we want to prove them in the asymmetric case

Our attention then shifts onto the asymmetric case where we use different closure operators to define asymmetric compactness, as motivated by our topological examples. We show how all the properties and results of the "symmetric" case pass to the asymmetric case. We follow the strategies of

[14] very closely in proving those three generalised topological theorems -
"Closed Subspace of a Compact Space is Compact", "Compact Subspace of
a Hausdorff space is closed", and "Image of a Compact space is Compact"
We also consider compact morphisms (generalised proper/perfect maps) by
working in the Comma Category, and discuss countably compact maps in
topology.

In Chapter 3 we consider a functional approach to compactness, following [6]
very closely. In the first section we assume we have a class $\mathcal{F}$ of morphisms
(which we think of as being the closed morphisms) with certain properties.
We define $\mathcal{F}$-dense morphisms, $\mathcal{F}$-proper morphisms, and $\mathcal{F}$-separated mor-
phisms.

We then define $\mathcal{F}$-Hausdorff and $\mathcal{F}$-Compactness, and generalise this ap-
proach to capture the notion of Asymmetric Compactness - we assume we
have three classes of morphisms satisfying certain properties (using the clo-
sure operator approach of [14] as a guideline).

We end off by proving those three theorems, already proven in Chapter 2, in
this new context.

Notations and conventions

Firstly, we mention that our topological work mostly follows notations and
definitions from [10]. However, when we consider a compact topological
space, we do not require for it to be Hausdorff. In [10], compactness is de-
fined for Haudorff topological spaces, while compactness without Hausdorff
is called "quasi compact".

Throughout this thesis, the complement of a subset, say $A \subseteq X$ is denoted
by $X - A$ (X "minus" A). Also, we denote the class of neighbourhoods of a
point x by $\mathcal{U}_x$. We usually use the letters U and V to denote such neighbour-
hoods, and we use the letter N to denote the neighbourhood of a tuple, say
(x, y). We use "$\subseteq$" and "$\supseteq$" to denote inclusions. When a set is properly
contained in another, it will be made clear.

We denote the usual closure of A by $\overline{A}$, and (this will again be mentioned)
we call this closure the "Kuratowski-closure". We also use the notation A°,

which denotes the interior of A.

Categorical terminology follows [1], although it should be mentioned that we use the term "factorization system" while in [1] "factorization structure" is used. Also, we denote the category of sets and functions by $\mathcal{SET}$ and the category of topological spaces and continuous functions by $\mathcal{TOP}$. When we refer to a unique morphism we will denote it by "!", for example if we say that d is unique, we write $\exists! d$. By $Mor(\mathcal{C})$ we denote the class of morphisms of the category $\mathcal{C}$.

Chapter 2 follows the proof-strategies of [5] and [14], with one difference being that we use the notation of $C_1 C_2$-compact when we are working with asymmetric compactness, while in [14] $\alpha\beta$-compact is used. We also mention that we are going to use the notion of δ_X - the diagonal morphism of an object X (see [5]). This is of course the same idea as the diagonal $\Delta_X = \{(x,x)|x \in X\}$ where X is a set. Chapter 3 follows the notations and terminology of [6] very closely.

Contents

Chapter 1

Compactness in topology

We discuss the Kuratowski-Mrówka characterisation of compactness in topology ([17] and [15]). We then use different closures to obtain similar characterisations. Eventually we discuss examples where two different closures are used to obtain certain characterisations.

1.1 The Kuratowski-Mrówka theorem

1.1.1 The Kuratowski-closure and closed mappings

We define what we mean by a "closed" mapping, and then show this property is equivalent to being "closure-preserving", in the Kuratowski-case (in fact, whenever the closure is idempotent).

Definition 1.1.1 : *Let X and Y be topological spaces, and $f : X \to Y$ a continuous function. Then f is called a* **closed mapping** *if and only if:*
A closed in $X \Rightarrow f(A)$ closed in Y, $\forall A \subseteq X$.

Next we discuss a characterisation of closed mappings in terms of the Kuratowski-closure.

Lemma 1.1.1 : *For topological spaces X and Y, we have: $f : X \to Y$ is a closed mapping if and ony if $\overline{f(A)} = f(\overline{A})$, $\forall A \subseteq X$.*

Proof:

$\Rightarrow$: Let $A \subseteq X$. Assume f is a closed mapping. Let $y \in f(\overline{A})$.

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Then $y = f(x)$, for some $x \in \overline{A}$, meaning $\forall U \in \mathcal{U}_x, U \cap A \neq \emptyset$.

Now, let V be any neighbourhood of y. Then, by using the continuity of f, we conclude that $f^{-1}(V)$ is a neighbourhood of x, hence $f^{-1}(V) \cap A \neq \emptyset$. So, $\exists\, x' \in f^{-1}(V) \cap A$. We have $f(x') \in V \cap f(A)$, which means $y \in \overline{f(A)}$.

For the other inclusion, we use the fact that f is a closed mapping to conclude that $f(\overline{A})$ is closed in Y. And, since $f(A) \subseteq f(\overline{A})$, we have $\overline{f(A)} \subseteq f(\overline{A})$.

$\Leftarrow$: Let $A \subseteq X$ be closed, then $f(A) = f(\overline{A}) = \overline{f(A)}$, giving us the desired result. $\qquad\square$

1.1.2 The theorem

We proceed to state and prove the famous Kuratowski-Mrówka theorem for topological spaces. This was done in more restricted settings by Kuratowski and Mrówka in 1931 and 1959 respectively (cf. [15] and [17]).

Let X and Y be topological spaces, and $p : X \times Y \to Y$ the projection mapping. The Kuratowski-Mrówka theorem says the following:

Proposition 1.1.1 : *X is compact $\Leftrightarrow$ p is a closed mapping, for all Y.*

Proof:

$\Rightarrow$: Assume X is compact. And let $A \subseteq X \times Y$ be any subset. Let $y_0 \in Y$ but $y_0 \notin p(\overline{A})$. Then $\forall x \in X$, $(x, y_0) \notin \overline{A}$, meaning $\forall x \in X, \exists N_x \in \mathcal{U}_{(x,y_0)}$, with $N_x \cap A = \emptyset$.

Now, $\forall x \in X$, $\exists$ open A_x, B_x such that $A_x \times B_x \subseteq N_x$, where $A_x \in \mathcal{U}_x$ and $B_x \in \mathcal{U}_{y_0}$. And, clearly, $(A_x \times B_x) \cap A = \emptyset, \forall x \in X$.

The sets A_x cover X, i.e.

$$X = \bigcup_{x \in X} A_x.$$

Now, we have assumed X is compact, so $\exists A_{x_1}, \ldots, A_{x_t}$ such that

$$X = \bigcup_{j=1}^{t} A_{x_j}.$$

For $j = 1, \ldots, t$ consider the corresponding $A_{x_j} \times B_{x_j}$. The claim is that

$$\left(\bigcap_{j=1}^{t} B_{x_j} \right) \cap p(A) = \emptyset.$$

Assume

$$\left(\bigcap_{j=1}^{t} B_{x_j} \right) \cap p(A) \neq \emptyset.$$

We conclude the following:

$$\exists y \in \left(\bigcap_{j=1}^{t} B_{x_j} \right) \cap p(A),$$

meaning

$$y \in B_{x_j}, j = 1, \ldots, t \; ; \; y \in p(A).$$

So, $\exists \, x' \in X$ with $(x', y) \in A$. Now, since

$$X = \bigcup_{j=1}^{t} A_{x_j},$$

x' has to lie in some A_{x_i} where $i \in \{1, \ldots, t\}$, and since $y \in B_{x_j}$ for $j = 1, \ldots, t$, it means $(x', y) \in (A_{x_i} \times B_{x_i}) \cap A$, i.e. $(A_{x_i} \times B_{x_i}) \cap A \neq \emptyset$, which is a contradiction, since $(A_x \times B_x) \cap A = \emptyset, \forall x \in X$. Hence

$$\left(\bigcap_{j=1}^{t} B_{x_j} \right) \cap p(A) = \emptyset.$$

And hence we have $y_0 \notin \overline{p(A)}$ (clearly y_0 is in the intersection of those sets B_{x_j}, and that intersection will be open since it is a finite intersection)

And we have $\overline{p(A)} \subseteq p(\overline{A})$, meaning $\overline{p(A)} = p(\overline{A})$ (The inclusion $p(\overline{A}) \subseteq \overline{p(A)}$

follows from the continuity of the projection map p), and p is a closed mapping.

$\Leftarrow$: Assume p is a closed mapping. Let τ_X denote the topology on X. Let $\{F_i\}_{i \in I}$ be a family of closed sets in X with the finite intersection property. Let $Y := X \cup \{y_0\}$, where $y_0 \notin X$. Let the basic neighbourhoods of y_0 be sets of the form $y_0 \cup \{F_{i_1} \cap \ldots \cap F_{i_k}\} \cup F$, where $F \subseteq X$, and k some natural number. Note that $\{F_{i_1} \cap \ldots \cap F_{i_k}\}$ will never be empty since we have assumed thet $\{F_i\}_{i \in I}$ has the finite intersection property. For any other element of Y we let elements of $\mathcal{P}(X)$ (the power-set of X) be their neighbourhoods.

Now, let $A := \overline{\{(x,x) : x \in X\}} = \overline{\Delta_X}$.

Since A is closed in $X \times Y$, by assumption $p(A)$ will be closed in Y.

Now, $p(\Delta_X) = X \subseteq p(A) \Rightarrow \overline{X} \subseteq p(A)$, since $p(A)$ is closed. And $y_0 \in \overline{X}$ since it is not possible for any neighbourood of y_0 to have empty intersection with X. In fact, $\overline{X} = Y$. So, $\exists x_0 \in X$, with $(x_0, y_0) \in A$. Let U be any neighbourhood of x_0. Then $(U \times (\{y_0\} \cup F_i)) \cap \Delta_X \neq \emptyset$, $\forall i \in I$, i.e. $U \cap F_i \neq \emptyset$, $\forall i \in I$ but then $x_0 \in \overline{F_i} = F_i$, $\forall i \in I$. That means

$$x_0 \in \bigcap_{i \in I} F_i.$$

Hence

$$\bigcap_{i \in I} F_i \neq \emptyset,$$

and X is compact. $\qquad\square$

1.2 Sequential compactness

1.2.1 The Sequential-closure

We are now going to take a look at a different closure operator. Let X be a topological space, and $A \subseteq X$. We define the *sequential closure* of A, A^σ, as follows:

$x \in A^\sigma \Leftrightarrow \exists$ a sequence $(x_n) \subseteq A$, with $(x_n) \to x$.

First we are going to look at certain properties of the sequential closure.

Clearly, $\forall A \subseteq X$, we have $A \subseteq A^\sigma$ (For any $x \in A$ just consider the constant sequence $(x, x, x, \ldots)$. We are going to look at three more properties:

Proposition 1.2.1 : *Let X and Y be topological spaces. Then the following hold, for any $A, B \subseteq X$:*

(1) $A \subseteq B \Rightarrow A^\sigma \subseteq B^\sigma$

(2) If a mapping $f : X \to Y$ is continuous, then $f(A^\sigma) \subseteq (f(A))^\sigma$

(3) $(A \cup B)^\sigma = A^\sigma \cup B^\sigma$

Proof of (1): Assume $A \subseteq B$, and let $x \in A^\sigma$. Thus there is a sequence $(x_n) \subseteq A$, with $(x_n) \to x$. But (x_n) also lies in B, since $A \subseteq B$, so $x \in B^\sigma$.

Proof of (2): Let $f : X \to Y$ be continuous. Let $y \in f(A^\sigma)$, which menas $y = f(x)$, for some $x \in A^\sigma$.

So, there is a sequence (x_n) in A converging to x. And since f is continuous, we know $(f(x_n)) \to f(x) = y$. And clearly $(f(x_n)) \subseteq f(A)$, yielding $y \in (f(A))^\sigma$.

Proof of (3): Let $x \in A^\sigma \cup B^\sigma$. So, there is a sequence in A converging to x or there is a sequence in B converging to x. Hence there is a sequence in $A \cup B$ converging to x, yielding $x \in (A \cup B)^\sigma$.

Now, let $x \in (A \cup B)^\sigma$, meaning $\exists \, (x_n) \subseteq A \cup B$, with $(x_n) \to x$.

This means there must at least exist some subsequence of (x_n) in A or in B. And we know this subsequence will also converge to x. So there will be a sequence in A converging to x, or a sequence in B converging to x.
And we conclude that $x \in A^\sigma \cup B^\sigma$. $\qquad \square$

1.2.2   The theorem

A topological space X is called *sequentially compact* if every sequence in X has a convergent subsequence. Now we can prove a simlar result to the

Kuratowski-Mrowka theorem. This result is also mentioned in [5] although a fully written out proof is not given. We proceed to give a possible proof.

Let X and Y be topological spaces and $p : X \times Y \to Y$ the projection mapping.

Proposition 1.2.2 : *X is sequentially compact $\Leftrightarrow (p(A))^\sigma \subseteq p(A^\sigma)$, for all $A \subseteq X \times Y$.*

Proof:

$\Rightarrow$: Assume X is sequentially compact.

Let $y \in (p(A))^\sigma$, i.e. $\exists (y_n) \subseteq p(A)$, with $(y_n) \to y$.

Take any sequence $(x_n) \subseteq X$ with $((x_n, y_n)) \subseteq A$. Now, X is sequentially compact, meaning (x_n) has some convergent subsequence (x_{n_k}). Let $x \in X$ denote the limit of (x_{n_k}), i.e. $(x_{n_k}) \to x$.

Now, for each x_{n_k} consider the corresponding y_{n_k}, and then consider the sequence $((x_{n_k}, y_{n_k}))$. We know $(y_n) \to y$. Any subsequence of (y_n) will also converge to y. So, in fact, $((x_{n_k}, y_{n_k})) \to (x, y)$. And, since $((x_{n_k}, y_{n_k})) \subseteq A$, we have that $y \in p(A^\sigma)$, hence $(p(A))^\sigma \subseteq p(A^\sigma)$.

$\Leftarrow$: Let X be any topological space. Let $Y := \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Let the topology on Y be $\tau_Y := \mathcal{B} \cup \{\emptyset\}$,
where $\mathcal{B} := \{B_n : n \in \mathbb{N}\}$, and $B_n := \{\frac{1}{k} : k \geq n\}$.

Let (x_n) be any sequence in X, and let $A := \{(x_n, \frac{1}{n}) : n \in \mathbb{N}\} \subseteq X \times Y$.

We know $(\frac{1}{n}) \to 0$, so $0 \in (p(A))^\sigma$. Now, $0 \in p(A^\sigma)$, so $\exists x \in X$ such that $(x, 0)$ is the limit of some sequence in A. All of the non-zero elements of Y are of the form $\frac{1}{n}$. So, the Y-components of this sequence (for which $(x, 0)$ is the limit) will be of the form $\frac{1}{n_k}$, where n_k is a natural number. We know $(\frac{1}{n_k}) \to 0$. That means (n_k) must have an increasing subsequence.

Moreover, the X-components of the elements of A are actually indexed by the Y-components, which means that the sequence in A converging to $(x, 0)$

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will look like this: $((x_{n_{k_i}}, \frac{1}{n_{k_1}}))$, where $(x_{n_{k_1}})$ is a subsequence of (x_{n_k}) (and clearly also of x_n). And since $(x_n) \subseteq X$ was arbitrary, X is sequentially compact. $\square$

1.3 H-closedness

1.3.1 The Theta-closure

The θ-closure will be the third closure-operator we'll be discussing. For a topological space X, and $A \subseteq X$, define A^θ in the following way:

$$x \in A^\theta \Leftrightarrow \forall U \in \mathcal{U}_x, \overline{U} \cap A \neq \emptyset.$$

Similar to what we did in the case of the sequential closure, we are going to look at a few properties of the theta-closure.

Again, it is clear that $A \subseteq A^\theta$, $\forall A \subseteq X$, since $\overline{A} \subseteq A^\theta$.

Proposition 1.3.1 *: Let X be a topological space. Then the following hold, for any $A, B \subseteq X$:*

(1) $A \subseteq B \Rightarrow A^\theta \subseteq B^\theta$

(2) If a mapping $f : X \to Y$ is continuous, then $f(A^\theta) \subseteq (f(A))^\theta$

(3) $(A \cup B)^\theta = A^\theta \cup B^\theta$

Proof of (1): Assume $A \subseteq B$ and let $x \in A^\theta$. This means for all neighbourhoods U of x, we have $\overline{U} \cap A \neq \emptyset$. But since $A \subseteq B$, we have $x \in B^\theta$ (because $\overline{U} \cap A \subseteq \overline{U} \cap B, \forall \, U \in \mathcal{U}_x$).

Proof of (2): Let $f : X \to Y$ be continuous.
Assume $y \in f(A^\theta)$, i.e. $y = f(x)$, for some $x \in A^\theta$. Hence $\forall U \in \mathcal{U}_x$, and we have $\overline{U} \cap A \neq \emptyset$.

Now, let V be any neighbourhood of y. By using the continuity of f, we conclude that $f^{-1}(V)$ is a neighbourhood of x, i.e. $\overline{f^{-1}(V)} \cap A \neq \emptyset$. Hence $f^{-1}(\overline{V}) \cap A \neq \emptyset$, since $\overline{f^{-1}(V)} \subseteq f^{-1}(\overline{V})$. Hence $\exists$ some $x' \in f^{-1}(\overline{V}) \cap A \neq \emptyset$,

so $f(x') \in \overline{V} \cap f(A) \neq \emptyset$, meaning $\overline{V} \cap f(A) \neq \emptyset$, and we have $y \in (f(A))^\theta$.

Proof of (3): Let $x \in A^\theta \cup B^\theta$, i.e. $\forall U \in \mathcal{U}_x, \overline{U} \cap A \neq \emptyset$ or $\forall U \in \mathcal{U}_x, \overline{U} \cap B \neq \emptyset$. Hence $\forall U \in \mathcal{U}_x, (\overline{U} \cap (A)) \cup (\overline{U} \cap (B)) \neq \emptyset$, i.e. $\overline{U} \cap (A \cup B) \neq \emptyset, \forall U \in \mathcal{U}_x$, which is equivalent to $x \in (A \cup B)^\theta$.

Assume now $x \in (A \cup B)^\theta$, i.e. $\forall U \in \mathcal{U}_x, \overline{U} \cap (A \cup B) \neq \emptyset$, meaning $\forall U \in \mathcal{U}_x, (\overline{U} \cap A) \cup (\overline{U} \cap B) \neq \emptyset$.

Now, assume $\exists$ neighbourhoods U_1 and U_2 of x, such that $\overline{U}_1 \cap A = \emptyset$, and $\overline{U}_2 \cap B \neq \emptyset$, and also $\overline{U}_1 \cap B = \emptyset$, while $\overline{U}_2 \cap A \neq \emptyset$.

Consider $U_1 \cap U_2$, which will also be a neighbourhood of x.
Then, $(\overline{U}_1 \cap \overline{U}_2) \cap (A \cup B) = \emptyset$, meaning $\overline{U_1 \cap U_2} \cap (A \cup B) = \emptyset$, which is a contradiction.

So we conclude that $\forall U \in \mathcal{U}_x, \overline{U} \cap A \neq \emptyset$ or $\forall U \in \mathcal{U}_x, \overline{U} \cap B \neq \emptyset$, and hence we have $x \in A^\theta \cup B^\theta$. $\square$

1.3.2 The theorem

The property "H-closed" was originally defined for Hausdorff-spaces, and in the following way: A Hausdorff space X is called H-closed if X is a closed subspace of every Hausdorff space in which it is contained - closed in the Kuratowski-sense. We consider an alternative (equivalent) characterisation of H-closed (proof of the equivalence can be found in [10]), and we do not require the space to be Hausdorff.
A topological space X is *H-closed* if and only if the following holds:

$$X = \bigcup_{i \in I} W_i \Rightarrow \exists W_{i_1}, \dots, W_{i_t}, X = \bigcup_{i=1}^{t} \overline{W_{i_j}}.$$

With the W_i's open subsets of X.
H-closed could also be called "Theta-compact".

The following proposition is also discussed in [5] but it is given with two other equivalent properties and is proven accordingly. We proceed to prove it directly.

Let X be a topological space and p the projection-map as before (again, for any space Y).

Proposition 1.3.2 X *is H-closed* $\Leftrightarrow (p(A))^\theta \subseteq p(A^\theta)$ *for any* $A \subseteq X \times Y$.

Proof:

$\Rightarrow$: Assume X is H-closed. Let $A \subseteq X \times Y$ be any subset, and let $y_0 \in Y$ but $y_0 \notin p(A^\theta)$, i.e. $\forall x \in X, (x, y_0) \notin A^\theta$, meaning $\forall x \in X, \exists N_x \in \mathcal{U}_{(x, y_0)}$ with $\overline{N_x} \cap A = \emptyset$. Now, $\forall x \in X, \exists$ open $A_x \in \mathcal{U}_x$ and $\exists$ open $B_x \in \mathcal{U}_{y_0}$, with $A_x \times B_x \subseteq N_x$. Hence $\overline{A_x \times B_x} \subseteq \overline{N_x}$, i.e. $\overline{A_x} \times \overline{B_x} \subseteq \overline{N_x}$, since $\overline{A_x \times B_x} = \overline{A_x} \times \overline{B_x}$. And, clealy, $\forall x \in X, (\overline{A_x} \times \overline{B_x}) \cap A = \emptyset$. Now, the sets A_x form a cover of X, i.e.

$$X = \bigcup_{x \in X} A_x.$$

Hence there are sets $A_{x_1}, \ldots, A_{x_t}$, with

$$X = \bigcup_{j=1}^{t} \overline{A_{x_j}}.$$

For $j = 1, \ldots, t$ consider $\overline{A_{x_j}} \times \overline{B_{x_j}}$. The claim is

$$\overline{(\bigcap_{j=1}^{t} B_{x_j})} \cap p(A) = \emptyset.$$

Now, since

$$\overline{\bigcap_{j=1}^{t} B_{x_j}} \subseteq \bigcap_{j=1}^{t} \overline{B_{x_j}},$$

it suffices to show

$$(\bigcap_{j=1}^{t} \overline{B_{x_j}}) \cap p(A) = \emptyset.$$

Assume

$$(\bigcap_{j=1}^{t} \overline{B_{x_j}}) \cap p(A) \neq \emptyset.$$

Hence there exists $y \in Y$, with $y \in \overline{B_{x_j}}$, for $j = 1, \ldots, t$, and $y \in p(A)$, so $\exists x' \in X$, with $(x', y) \in A$. Since

$$X = \bigcup_{j=1}^{t} \overline{A_{x_j}},$$

we know x' has to lie in some $\overline{A_{x_i}}$, where $i \in \{1, \ldots, t\}$. And, using the fact that $y \in \overline{B_{x_j}}$ for $j = 1, \ldots, t$, we can conclude that $(x', y) \in \overline{A_{x_i}} \times \overline{B_{x_i}}$, meaning $(\overline{A_{x_i}} \times \overline{B_{x_i}}) \cap A \neq \emptyset$. This is a contradiction, since $(\overline{A_x} \times \overline{B_x}) \cap A = \emptyset$, $\forall x \in X$. Hence

$$\left(\bigcap_{j=1}^{t} \overline{B_{x_j}}\right) \cap p(A) = \emptyset.$$

Hence

$$\overline{\left(\bigcap_{j=1}^{t} B_{x_j}\right)} \cap p(A) = \emptyset.$$

The intersection of these sets B_{x_j} will be open, since it is a finite intersection. And y_0 is contained in the intersection, hence $y_0 \notin (p(A))^{\theta}$. Hence $(p(A))^{\theta} \subseteq p(A^{\theta})$

$\Leftarrow$: Assume $p(A^{\theta}) \subseteq (p(A))^{\theta}, \forall A \subseteq X \times Y$. Let X be any topological space. Assume X is not H-closed. So

$$X = \bigcup_{1 \in I} W_i,$$

(some open cover) but for each $W_{i_1}, \ldots, W_{i_t}$ (finite collection),

$$X \neq \bigcup_{j=1}^{t} \overline{W_{i_j}}.$$

Thus,

$$\bigcap_{j=1}^{t} (X - \overline{W_{i_j}}) \neq \emptyset.$$

Define Y in the following way: $Y := X \cup \{y_0\}$, where $y_0 \notin X$. Let the

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topology on Y be generated by the discrete topology on X, and the basic open neighbourhoods of y_0 be sets of the form $\{y_0\} \cup ((X - \overline{W_{i_1}}) \cap \ldots \cap (X - \overline{W_{i_t}}))$, where the intersection on the right is never empty. Let $A := \Delta_X = \{(x,x) : x \in \underline{X}\} \subseteq X \times Y$. Now, take any arbitrary neighbourhood of y_0, and consider $\overline{\{y_0\} \cup ((X - \overline{W_{i_1}}) \cap \ldots \cap (X - \overline{W_{i_t}}))} \cap \underline{p(A)}$. Clearly this intesection can never be empty, since $\overline{((X - \overline{W_{i_1}}) \cap \ldots \cap (X - \overline{W_{i_t}}))}$ lies in X, and $p(A) = X$. So, $y_0 \in (p(A))^\theta$. Which means $y_0 \in p(A^\theta)$, using our assumption. That means $\exists x_0 \in X$, with $(x_0, y_0) \in A^\theta$, hence $\forall N \in \mathcal{U}_{(x_0, y_0)}$, $\overline{N} \cap A \neq \emptyset$. Let U be any neighbourhood of x_0, then

$$\overline{(U \times (\{y_0\} \cup (X - \overline{W_i})))} \cap A \neq \emptyset, \forall i \in I.$$

Hence

$$\overline{U} \cap \overline{(X - \overline{W_i}))} \neq \emptyset, \forall i \in I.$$

Since the topology on Y is discrete (apart from the neighbourhoods of y_0), we have that

$$\overline{(X - \overline{W_i})} = (X - \overline{W_i}), \forall i \in I.$$

So, $\forall U \in \mathcal{U}_{x_0}, \forall i \in I$, we have

$$\overline{U} \cap (X - \overline{W_i}) \neq \emptyset.$$

Assume $\exists U' \in \mathcal{U}_{x_0}$ with $U' \cap (X - \overline{W_i}) = \emptyset$, hence $U' \subseteq \overline{W_i}$. So, $\overline{U'} \subseteq \overline{W_i}$, yielding $\overline{U'} \cap (X - \overline{W_i}) = \emptyset$, a contradiction. So, $\forall U \in \mathcal{U}_{x_0}$, we have $U \cap (X - \overline{W_i}) \neq \emptyset, \forall i \in I$, i.e. $x_0 \in \overline{(X - \overline{W_i})}, \forall i \in I$ (in X). So, $\forall i \in I, x_0 \notin X - \overline{X - \overline{W_i}} = (\overline{W_i})^\circ$. Hence $x_0 \notin (W_i)^\circ = W_i, \forall i \in I$, meaning $x_0 \in X - W_i, \forall i \in I$. Hence

$$x_0 \in \bigcap_{i \in I} (X - W_i) = X - \bigcup_{i \in I} W_i.$$

But we have assumed that

$$X = \bigcup_{i \in I} W_i,$$

i.e.

$$X - \bigcup_{i \in I} W_i = \emptyset.$$

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So we have found a contradiction, and hence X is H-closed. $\qquad\qquad\square$

1.4   Asymmetric compactness in topology

We have now seen three different forms of compactness and characterisations of them, using three different closure operators.

Eventually, when we get to the categorical approach, we want to generalise further to where we use two different closure operators to characterise a certain compactness.

We will now look at two examples which motivate our interest into studying such a categorical generalisation.

1.4.1   kσ-compactness

We now use both the Kuratowski-closure and the Sequential closure to obtain a characterisation similar to the three which were discussed in the three sections above. We have seen that we are working with a "closure-preserving"-expression, i.e. an equality like the following must hold: $p(\overline{A}) = \overline{p(A)}$, for example. In fact the inclusion $\subseteq$ already holds because of the continuity of the projection-mapping p, and hence we are actually interested in the inclusion $\supseteq$.

Keeping this in mind, we investigate the case where we have an inclusion of the following form:

$p(\overline{A}) \supseteq (p(A))^\sigma.$

We mention that the following proposition is proven in [14], and we provide a similar proof.

Let X and Y again be topological spaces, and $p : X \times Y \rightarrow Y$ be the projection map.

Proposition 1.4.1 *: Every sequence in X has a cluster point*
$\Leftrightarrow (p(A))^\sigma \subseteq p(\overline{A})$, $\forall A \subseteq X \times Y$.

Proof:

$\Rightarrow$: Let $A \subseteq X \times Y$ be any subset. Assume every sequence in X has a cluster point, and let $y \in (p(A))^\sigma$.

Let (y_n) be the sequence in $p(A)$, converging to y. Consider the sequence $(x_n) \subseteq X$, with $((x_n, y_n)) \subseteq A$. Note that (x_n) does exist since $(y_n) \subseteq p(A)$.

Let x be the cluster point of (x_n). Let N be any neighbourhood of the tuple (x, y). We know there will exist $U \in \mathcal{U}_x$ and $V \in \mathcal{U}_y$, with $U \times V \subseteq N$.

Now, for V, there will exist some $m \in \mathbb{N}$ such that $y_k \in V$, $\forall k \geq m$.
And we know $\forall n \in \mathbb{N}$, $\exists l \geq n$, such that $x_l \in U$. So, in particular, if we consider $n = m$, we conclude that there has to exist some $k' \geq m$, with $(x_{k'}, y_{k'}) \in U \times V$. And, since $(x_{k'}, y_{k'}) \in A$, we know $(U \times V) \cap A \neq \emptyset$, hence $N \cap A \neq \emptyset$. And we have $y \in p(\overline{A})$.

$\Leftarrow$: Let X be any topological space. Define Y as we did in Proposition 1.2.2:

$Y := \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$, with the topology on Y again being $\mathcal{B} \cup \{\emptyset\}$.

Let (x_n) be any sequence in X and, as before, let A be the set $\{(x_n, \frac{1}{n}) : n \in \mathbb{N}\}$.

We know $0 \in (p(A))^\sigma$, hence $0 \in p(\overline{A})$. So there exists some $x \in X$, with $(x, 0) \in \overline{A}$.

Let U be any neighbourhood of x, and let $m \in \mathbb{N}$. Choose B_m as the neighbourhood of 0, and consider $U \times B_m$, which will be a neighbourhood of $(x, 0)$. We know $(U \times B_m) \cap A \neq \emptyset$. And we know the elements of A are of the form $(x_n, \frac{1}{n})$.

Now, by using the definition of B_m and the fact that $(U \times B_m) \cap A \neq \emptyset$, we know there has to exist some $k \in \mathbb{N}$, with $k \geq m$, and $x_k \in U$. Hence x is the cluster point of (x_n). $\qquad\square$

This above proposition is equivalent to the property "countably compact",

i.e, any countable open cover has a finite subcover. In our context, we also call it "kσ-compact", referring to the Kuratowski(k)-closure and the sequential(σ)-closure.

1.4.2 kγ-compactness

For our next example we define a new closure, the "γ-closure". In order to do that we need to recall what is meant by a "G_δ-set".

Definition 1.4.1 *Let X be a topological space, and $G \subseteq X$. G is called a G_δ-**set** if $G = \bigcap \{O_i | i \in \mathbb{N}\}$, where all the O_1's are open sets in X, i.e. G is a countable intersection of open sets in X.*

Let X again be a topological space, and $M \subseteq X$. We define $\gamma(M)$ as follows:

$x \in \gamma(M) \Leftrightarrow G \cap M \neq \emptyset$, $\forall$ G_δ-sets containing x.

We also mention what it means for a topological space to be Lindelöf: every countable Fopen cover has a finite subcover. We are going to use an alternative characterisation of Lindelöf in the following proposition, namely that a space X is Lindelöf if and only if the intersection of any family of closed sets in X will be non-empty, if this family has the countable intersection property. By the countable intersection property we mean that any countable intersection of sets from this family is non-empty.

One can also find a very similar proof of the following proposition in [14].

Proposition 1.4.2 $\gamma(p(M)) \subseteq p(\overline{M})$, $\forall M \subseteq X \times Y$ *(with projection $p : X \times Y \to Y$), for any space $Y \Leftrightarrow X$ is Lindelöf.*

Proof:

$\Rightarrow$: Let $\mathcal{F}$ be any family of closed sets in X with the countable intersection property.

Define Y as follows:

$Y := X \cup \{\infty\}$, and let the subbase for its topology be:
$\mathcal{S} := \mathcal{P}(X) \cup \{F \cup \{\infty\} | F \in \mathcal{F}\}$.

Clearly $\Delta_X = \{(x,x)|x \in X\} \subseteq X \times Y$.

Now, $\gamma(X) = \gamma(p(\Delta_X)) \subseteq p(\overline{\Delta_X})$, by assumption.

We use the fact that $\mathcal{F}$ has the countable intersection property to conclude that $\infty \in \gamma(X)$ - any countable intersection of open neighbourhoods (which won't be empty) of ∞ will meet X.

That means $\exists x' \in X$ such that $(x', \infty) \in \overline{\Delta_X}$.

Let $U \in \mathcal{U}_{x'}$. Then, we have $(\forall F \in \mathcal{F})$, $U \times (F \cup \{\infty\}) \cap \Delta_X \neq \emptyset$, i.e. $U \cap F \neq \emptyset$. Hence $x' \in F$, $\forall F \in \mathcal{F}$, and X is Lindelöf.

$\Leftarrow$: Let X be Lindelöf and $M \subseteq X \times Y$, with Y any space. Assume $y \notin p(\overline{M})$, then $\forall x \in X$, $(x,y) \notin \overline{M}$. That means, $\forall x \in X$, $\exists N_x \in \mathcal{U}_{(x,y)}$, with $N_x \cap M = \emptyset$.

So, $\forall x \in X$, $\exists$ open $U_x \in \mathcal{U}_x$ and open $V_x \in \mathcal{U}_y$ with $U_x \times V_x \cap M = \emptyset$.

$\{U_x | x \in X\}$ is an open cover for X, and will thus have a countable sub-cover $\{U_{x_i} | i \in \mathbb{N}\}$. Define $V := \bigcap\{V_{x_i} | i \in \mathbb{N}\}$.

Now, clearly V is a G_δ-set containing y. Assume $V \cap p(M) \neq \emptyset$. That means $\exists y_0 \in Y$ such that $y_0 \in V$ and $y_0 \in p(M)$. The latter yields an $x_0 \in X$ such that $(x_0, y_0) \in M$.

And, since the U_{x_i}'s form a cover for X and $y_0 \in V_{x_i}, \forall i \in \mathbb{N}$, we conclude that $\exists j \in \mathbb{N}$ such that $(x_0, y_0) \in U_{x_j} \times V_{x_j}$, which is a contradiction, since $U_x \times V_x \cap M = \emptyset$, $\forall x \in X$.

Hence $V \cap p(M) = \emptyset$, and $y \notin \gamma(p(M))$. $\qquad\square$

Chapter 2

Categorical compactness

This chapter considers the question: "What is a compact object in a category?" We have seen the characterisation - by means of closure - of different compactness notions in topology.

We define what we mean by a *closure operator* in a category, and then proceed to define categorical compactness by using our topological knowledge.

2.1 Factorization systems

The modern notion of factorization systems was introduced by Freyd and Kelly in 1972 (cf. [11]). A good revision could also be found in [2]. In this section we mention definitions, properties and results following work done in [11], [2], [5] and [6].

We also mention that throughout the rest of the thesis we assume our category has finite products and pullbacks.

2.1.1 Prefactorization and factorization systems

Consider the following square:

where $v \circ e = m \circ u$.

Definition 2.1.1 : *We say e **is orthogonal to** m (or $e \perp m$) if, for any square $v \circ e = m \circ u$ $\exists !d : B \to C$, with $d \circ e = u$ and $m \circ d = v$.*

As an example, consider the category **SET**. Let e be surjective and m injective.

Now, since e is surjective, we know that any $b \in B$ will be of the form $e(a)$, for some $a \in A$. We want do define d. Note that such a d will be unique (due to the fact that e is an epimorphism).

For any $b \in B$, let $d(b) := u(a)$, where $e(a) = b$. So, by definition the square commutes. We have to check if d is indeed well-defined.

Let $e(a_1) = e(a_2) = b \in B$, then $v(e(a_1)) = v(e(a_2)) \Leftrightarrow m(u(a_1)) = m(u(a_2))$. But m is injective which means that $u(a_1) = u(a_2)$, and d is well-defined.

Consider a category $\mathcal{C}$. Let $\mathcal{F}$ be a class of morphisms in $\mathcal{C}$. Then we define the following:

$\mathcal{F}^{\uparrow} := \{g \in Mor(\mathcal{C}) \mid g \perp f , \forall f \in \mathcal{F}\}$ and $\mathcal{F}^{\downarrow} := \{g \in Mor(\mathcal{C}) \mid f \perp g , \forall f \in \mathcal{F}\}$

Definition 2.1.2 *Let $\mathcal{E}$ and $\mathcal{M}$ be two classes of morphisms in $\mathcal{C}$. We say the pair $(\mathcal{E}, \mathcal{M})$ is a **prefactorization system** for $\mathcal{C}$ if $\mathcal{E}^{\downarrow} = \mathcal{M}$ and $\mathcal{M}^{\uparrow} = \mathcal{E}$.*

The orthogonality-example we have seen previously is an axample of a prefactorization system, with $\mathcal{E} = \{f \in Mor(\textbf{SET}) \mid f$ surjective $\}$, and $\mathcal{M} = \{f \in Mor(\textbf{SET}) \mid f$ injective $\}$.

Because of our context, we will especially be interested in prefactorization systems where (as in the **SET**-example) $\mathcal{E}$ is a class of epimorphisms and $\mathcal{M}$ a class of monomorphisms. This is called *proper*.

Prefactorization systems have certain properties:

(F1) $\mathcal{E} \cap \mathcal{M} = Iso(\mathcal{C})$.

(F2) Both $\mathcal{E}$ and $\mathcal{M}$ are closed under composition.

(F3) $\mathcal{M}$ is left-cancellable (i.e. if $m \circ n \in \mathcal{M}$, we have that $n \in \mathcal{M}$). And dually $\mathcal{E}$ is right cancellable.

(F4) $\mathcal{M}$ is stable under pullback.

We'll take a look at a proof for (F2) (Further proofs can be found in [1]):

Assume $m_1 : K \to D$ and $m_2 : C \to K$ are in $\mathcal{M}$. We want to show the composition $m_1 \circ m_2$ is again contained in $\mathcal{M}$. Now let e be any morphism contained in $\mathcal{E}$, and let the following square be commutative:

Create a new commutative square out of the one above:

Since $e \in \mathcal{E}$, $\exists! d_1 : B \to K$, with $d_1 \circ e = m_2 \circ u$ and $m_1 \circ d_1 = v$. Consider now a third commutative square:

$$
\begin{array}{ccc}
A & \xrightarrow{\ e\ } & B \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle d_1} \\
C & \xrightarrow{\ m_2\ } & K
\end{array}
$$

Again, we find a diagonal: $\exists! d : B \to C$, with $d \circ e = u$ and $m_2 \circ d = d_1$. Putting everything together now yields the following:

$$
\begin{array}{ccc}
A & \xrightarrow{\ e\ } & B \\
\downarrow{\scriptstyle u} & \nearrow{\scriptstyle d} & \downarrow{\scriptstyle v} \\
C & \xrightarrow{\ m_1 \circ m_2\ } & D
\end{array}
$$

with both the upper and lower triangles commute.

We want to show that d is also the unique diagonal for the square above. Let $d' : B \to C$ also be a diagonal. That means that $m_2 \circ d'$ is a diagonal for the square $v \circ e = m_1 \circ m_2 \circ u$, so $m_2 \circ d' = d_1$, since d_1 is unique. We thus have that d' is a diagonal for the square $d_1 \circ e = m_2 \circ u$, meaning that $d' = d$.

The proof for $\mathcal{E}$ is dual.

For our purposes we are also interested in situations where $\mathcal{E}$ is stable under pullback, which is called a *stable* prefactorization system. Later on this will become crucial when we want to prove certain propositions.

Next we discuss factorization systems.

Definition 2.1.3 : *A prefactorization system $(\mathcal{E}, \mathcal{M})$ is called a* **factoriza-**

tion system *for $\mathcal{C}$ if each morphism f in $\mathcal{C}$ can be factorized: $f = m \circ e$, with $e \in \mathcal{E}$ and $m \in \mathcal{M}$.*

If we again visit the example in **SET**, we'll see that each function on a set can be factorized in this way. Let X and Y be sets and $f : X \to Y$ a function. Then f can be factorized through its image:

$$X \xrightarrow{\quad f \quad} Y$$

With e (the restriction of 1 to its image) clearly a surjective map, and m being the inclusion-map, clearly injective.

In the category of topological spaces and continuous maps, having $\mathcal{E}$ and $\mathcal{M}$ the surjective and injective maps is not enough. To ensure the unique diagonal is continuous, we need $\mathcal{M}$ to be embeddings (initial injective functions).

2.1.2   Subobjects

$\mathcal{C}$ will denote an arbitrary category from here on in.

Let $(\mathcal{E}, \mathcal{M})$ be our factorization system for $\mathcal{C}$. Let $X \in Obj(C)$.

We define the subobjects of X as follows:

Definition 2.1.4 $Sub(X) := \{ m \in \mathcal{M} \mid Cod(m) = X \ \}.$

These $\mathcal{M}$-morphisms could be seen as "inclusion-maps" (cf. [7]). So, when we talk about

$$M \xrightarrow{\quad m \quad} X \ ,$$

we can think of it as similar to $M \subseteq X$ in **SET**. We'll frequently denote

the domain of subobject m by M.

Now we are going to define an order relation on subobjects.
Let m and n be subobjects of X. Then

$m \leq n \Leftrightarrow \exists l : M \to N$ such that

$$
\begin{array}{ccc}
M & \xrightarrow{\quad m \quad} & X \\
& \searrow{\scriptstyle l} \quad \nearrow{\scriptstyle n} & \\
& N &
\end{array}
$$

commutes.

This order is transitive and reflexive. And, since $m \leq n$ and $n \leq m \Rightarrow m \cong n$ (by $m \cong n$ we mean there is an isomorphism $h : M \to N$ with $m \circ h = m$) we'll consider $\cong$ as "equal" and view the subobjects of an object in a category as a partially ordered class.

This might be a good time to briefly mention that $(\mathcal{E}, \mathcal{M})$-factorizations unique (up to isomorphism, of course). Let $f : X \to Y$ be a morphism in $\mathcal{C}$, with $f = m \circ e = m' \circ e'$ two factorizations of f, with $e, e' \in \mathcal{E}$ and $m, m' \in \mathcal{M}$. Then, by using orthogonality, we get $m = m'$. And, since we are working with a proper factorization system, we have $e = e'$.

2.1.3 Image and pre-image

We want to generalise the notion of image and pre-image. We have as starting point our idea of what this means when we work with sets.

Consider $\mathcal{C}$ $(\mathcal{E}, \mathcal{M})$ as before. Let X and Y be objects of $\mathcal{C}$, and $f : X \to Y$ a morphism. For $m \in Sub(X)$, we define $f(m)$ (the image of m under f) by considering the following commuting diagram:

$$M \xrightarrow{\; m \;} X \xrightarrow{\; f \;} Y$$

(diagram: $M \xrightarrow{m} X \xrightarrow{f} Y$, with $e : M \to f(M)$ and $f(m) : f(M) \to Y$)

where $f(m) \circ e$ is the $(\mathcal{E}, \mathcal{M})$-factorization of $f \circ m$. This is very intuitive as we can clearly see its analogy to the set-theoretic idea.

Let $n \in Sub(Y)$. We define $f^{-1}(n)$ (the pre-image of n under f) by considering the following pullback square:

$$
\begin{array}{ccc}
f^{-1}(N) & \xrightarrow{\; f^{-1}(n) \;} & X \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
N & \xrightarrow{\; n \;} & Y
\end{array}
$$

Since $\mathcal{M}$ is stable under pullback, $f^{-1}(m)$ will also be in $\mathcal{M}$, and hence a subobject of X. If we use the order defined earlier, we find a close relationship between f and f^{-1}.

First we mention waht a Galois connection is: let A and B be partially ordered sets, with $f : A \to B$ and $g : B \to A$ order-preserving functions. Then f and g form a Galois connection with f being the left adjoint of g if $f(x) \leq y \Leftrightarrow x \leq g(y)$, $\forall \, x \in A$ and $\forall \, y \in B$.

Proposition 2.1.1 : *Let $f : X \to Y$ in $\mathcal{C}$, then $f : Sub(X) \to Sub(Y)$ and $f^{-1} : Sub(Y) \to Sub(X)$ form a Galois connection, with f being the left adjoint of f^{-1}.*

Proof: We want to prove (a) both f and f^{-1} are order-preserving and (b) $f(m) \leq n \Leftrightarrow m \leq f^{-1}(n)$, $\forall \, m \in Sub(X)$ and $\forall \, n \in Sub(Y)$.

Proof of (a): Let $m_1 : M_1 \to X$ and $m_2 : M_2 \to X$ be subobjects of X, with $m_1 \leq m_2$. So, $\exists l : M_1 \to M_2$ such that $m_1 = m_2 \circ l$.

Now, let $f : X \to Y$ be a morphism. If we factorize $f \circ m_1$ and $f \circ m_2$ to obtain $f(m_1)$ and $f(m_2)$ respectively, we get the following commuting square:

$$
\begin{array}{ccc}
M_1 & \xrightarrow{\ e_1\ } & f(M_1) \\
\downarrow{\scriptstyle e_2 \circ l} & & \downarrow{\scriptstyle f(m_1)} \\
f(M_2) & \xrightarrow[f(m_2)]{} & Y
\end{array}
$$

with e_1 and e_2 being the morphisms in $\mathcal{E}$ also obtained when we found $f(m_1)$ and $f(m_2)$ respectively.

Now, $e_1 \in \mathcal{E}$ and $f(m_2) \in \mathcal{M} \Rightarrow \exists! d : f(M_1) \to f(M_2)$, such that $f(m_2) \circ d = f(m_1)$ and $d \circ e_1 = e_2 \circ l$. So we have $f(m_1) \leq f(m_2)$.

Let $n_1 : N_1 \to Y$ and $n_2 : N_2 \to Y$ be subobjects of Y, with $n_1 \leq n_2$. Again, we know $\exists k : N_1 \to N_2$, with $n_2 \circ k = n_1$. If we consider the pullback diagrams of $f^{-1}(n_1)$ and $f^{-1}(n_2)$ respectively, we find the following situation:

$$
\begin{array}{ccc}
f^{-1}(N_1) & & \\
 & \searrow{\scriptstyle f^{-1}(n_1)} & \\
 f^{-1}(N_2) & \xrightarrow[f^{-1}(n_2)]{} & X \\
{\scriptstyle k \circ g_1}\searrow \quad \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle f} \\
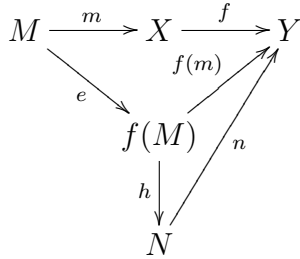N_2 & \xrightarrow[n_2]{} & Y
\end{array}
$$

with $g_1 : f^{-1}(N_1) \to N_1$ being part of the pullback-diagram of $f^{-1}(n_1)$.

The outer square clearly commutes. So $\exists! h : f^{-1}(N_1) \to f^{-1}(N_2)$, with $f^{-1}(n_2) \circ h = f^{-1}(n_1)$ and $k \circ g_1 = g_2 \circ h$.

So, we have $f^{-1}(n_1) \leq f^{-1}(n_2)$.

Proof of (b) :

$\Rightarrow$: Let $m \in Sub(X)$, $n \in Sub(Y)$ and $f : X \to Y$ a morphism. Assume $f(m) \leq n$, which yields the following diagram:

$$M \xrightarrow{\ m\ } X \xrightarrow{\ f\ } Y$$

with morphisms e, $f(m)$, $f(M)$, n, h, and N as drawn.

We can now consider the following diagram, where we use the definition of $f^{-1}(n)$:

$$
\begin{array}{c}
M \\
f^{-1}(N) \xrightarrow{f^{-1}(n)} X \\
h \circ e \quad g \qquad f \\
N \xrightarrow{\ n\ } Y
\end{array}
$$
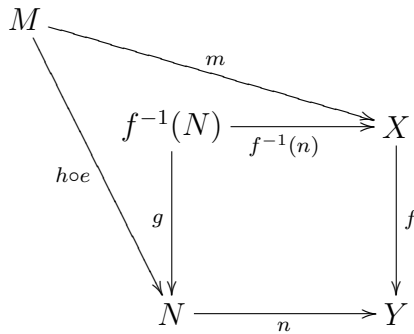
The outer square clearly commutes, meaning that we find a morphism $k : M \to f^{-1}(N)$ such that $g \circ k = h \circ e$ and $f^{-1}(n) \circ k = m \Rightarrow m \leq f^{-1}(n)$.

$\Leftarrow$: Assume now that $m \leq f^{-1}(n)$.

If we use the morphism we find from this inequality $(l : M \to f^{-1}(N))$, and the definition of $f^{-1}(n)$, we find the following commuting square:

$$
\begin{array}{ccc}
M & \xrightarrow{\ e\ } & f(M) \\
\downarrow{\scriptstyle g \circ l} & & \downarrow{\scriptstyle f(m)} \\
N & \xrightarrow{\ n\ } & Y
\end{array}
$$

and due to the orthogonality of e and n we find a unique morphism $d :$ $f(M) \to N$ such that $g \circ l = d \circ e$ and $n \circ d = f(m)$.

Hence we have $f(m) \leq n$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The following properties also hold, for $f : X \to Y$ and $g : Y \to X$ in $\mathcal{C}$:

(a) For $m \in Sub(X)$, and $n \in Sub(M)$, we have $m \circ n = m(n)$.

(b) For $m \in Sub(X)$, $(g \circ f)(m) = g(f(m))$,

(c) $g \in \mathcal{M} \Rightarrow g^{-1}(g(m)) = m$, $\forall m \in Sub(Y)$

(d) $f \in \mathcal{E} \Rightarrow f(f^{-1}(m)) = m$, $\forall m \in Sub(Y)$.

For (a), we use properties (F2) and (F3) of factorization systems. To prove (b), we use the fact that $(\mathcal{E}, \mathcal{M})$-factorizations are essentially unique. For (c) and (d) we use (a) above, and the fact that $(\mathcal{E}, \mathcal{M})$ is stable (any pullback of a morphism in $\mathcal{E}$ is again in $\mathcal{E}$).

2.2 Categorical closure operators

The first notion of categorical closure operators (as we use it) was formally introduced in [8], while [7] and [4] are good sources for studying closure operators. [3] also discussed the subject and contains useful examples. We use definitions and strategies similar to [7] and [5].

In Chapter 1 we looked at three different closures in topology. We now have the knowledge to start defining what we mean by a "closure operator" in a category.

Definition 2.2.1 : *A **closure operator** C of $\mathcal{C}$ is given by a family of maps:*
$C = (c_X)_{X \in Obj(\mathcal{C})}$, *where* $c_X : Sub(X) \to SubX$, *such that the following properties hold for every* $X \in Obj(\mathcal{C})$:

(C1) $m \le c_X(m)$, $\forall m \in Sub(X)$ (Extension)

(C2) $m_1 \le m_2$ in $Sub(X) \Rightarrow c_X(m_1) \le c_X(m_2)$ (Monotonicity)

(C3) $f(c_X(m)) \le c_Y(f(m))$, $\forall f : X \to Y$ and $\forall m \in Sub(X)$ (Continuity)

Because of (C1), we have the following commutative diagram:

The morphism j_m is also in $\mathcal{M}$ by (F3).

Proposition 2.2.1 : *(C2) and (C3) are jointly equivalent to (C4) (This is proven in [7]) :*

(C4) $(f(m) \le n \Rightarrow f(c_X(m)) \le c_Y(n))$, $\forall f : X \to Y$, $\forall m \in Sub(X)$ and $\forall n \in Sub(Y)$.

Proof :

$\Rightarrow$: Assume $f(m) \leq n$. Then (C2) gives $c_Y(f(m)) \leq c_Y(n)$. And by using (C3) we have that $f(c_X(m)) \leq c_Y(n)$.

$\Leftarrow$: Let f be $1_X : X \to X$, and let m_1 and m_2 be subobjects of X, with $m_1 \leq m_2$.

So $f(m_1) = 1_X(m_1) \leq m_2$.
By using (C4) we conclude that $1_X(c_X(m_1)) \leq c_X(m_2)$, and we have (C2).

If we consider $n = f(m)$, we have, by (C4):
$f(m) \leq f(m) \Rightarrow f(c_X(m)) \leq c_Y(f(m))$, which is (C3). $\qquad\qquad$ $\square$

Due to the adjunction between f and f^{-1}, we obtain an equivalent formulation of (C3):

(C3') $c_X(f^{-1}(n)) \leq f^{-1}(c_Y(n))$, $\forall n \in Sub(Y)$.

Next we mention a few definitions (similar to [5]) linked to closure operators. We keep the intuitive analogy from topology in the back of our mind - the analogy using the Kuratowski-closure being the more familiar one of course.

Definition 2.2.2 : *A subobject m of X is called* **C-closed** *if $m \cong c_X(m)$.*

The diagram of the above situation will be as we have seen:

$$M \xrightarrow{\ \ m\ \ } X$$

with j_m being an isomorphism.

In topology a subspace $M \subseteq X$ is closed when $M = \overline{M}$ - it's clearly intuitive how a C-closed subobject is defined.

Definition 2.2.3 : *A subobject m of X is called* **C-dense** *if $c_X(m) \cong 1_X$.*

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We again consider the triangle above. The subobject m being C-dense means that $c_X(m)$ is an isomorphism. In topology a subspace $M \subseteq X$ is dense in X when $\overline{M} = X$. Again, the analogy can be seen.

Definition 2.2.4 : *A closure operator C is called* **idempotent** *if $c_X(c_X(m)) \cong c_X(m)$, $\forall\, m \in SubX$, $\forall X \in Obj(\mathcal{C})$ i.e. the C-closure of m is C-dense.*

This definition is quite clearly what it means for a closure to be idempotent in topology. The Kuratowski-closure is of course idempotent, while the sequential closure is not necessarily.

Definition 2.2.5 : *A closure operator C is called* **weakly hereditary** *if $c_{c_X(M)}(j_m) \cong 1_{c_X(M)}$, $\forall X \in Obj(\mathcal{C})$ i.e. m is C-dense in its C-closure.*

The Kuratowski-closure is an example of a closure operator which is weakly hereditary. Clearly, a subspace $M \subseteq X$ is dense in $\overline{M}$.

The term "weakly hereditary" implies that this is a weaker form of "hereditary". This is indeed the case.

Consider the following commuting square:

where m, n are subobjects of X and $X \in Obj(\mathcal{C})$.
A closure operator C is called **hereditary** if $c_N(j) \cong n^{-1}(c_X(m))$.

Again, the Kuratowski-closure is hereditary. Let $M \subseteq N$ be subspaces of X. For a moment, let's denote the Kuratowski-closure by k. Then, $k_N(M)$ $= k_X(M) \cap N$. This is an alternative way of considering the above definition in topology.

The following does hold:
Hereditary $\Rightarrow$ Weakly hereditary.

From now on (because of the partial order we obtain amongst subobjects) we'll write $m = n$ instead of $m \cong n$.

Lemma 2.2.1 : *Let $f : X \to Y$ be a morphism in $\mathcal{C}$, and $n \in Sub(Y)$, then if n is C-closed, so is $f^{-1}(n)$.*

Proof: Assume that n is C-closed. By (C3') we have $c_X(f^{-1}(n)) \leq f^{-1}(c_Y(n))$. But $f^{-1}(n) = f^{-1}(c_Y(n))$, which means that $c_X(f^{-1}(n)) \leq f^{-1}(n)$, hence $c_X(f^{-1}(n)) = f^{-1}(n)$ and $f^{-1}(n)$ is C-closed. $\qquad\square$

We have discussed this Lemma to show how we generalise the topological property of continuous functions, where the pre-image maps closed sets onto closed sets.

It will also be important in applications later on, where we will work with pullbacks of closed subobjects.

2.3   Closure-preserving morphisms

Let X and Y be objects of $\mathcal{C}$, with closure operator C. And let $f : X \to Y$ be a morphism.

Definition 2.3.1 *f is called* **C-preserving** *if and only if*
$f(c_X(m)) = c_Y(f(m))$, $\forall m \in Sub(X)$.

A morphism f being C-preserving is equivalent to saying that f maps C-closed subobjects to C-closed subobjects *if C is idempotent* but not in general.

Closure-preserving morphisms have certain properties, which are discussed in [5]:

(CP1) Every isomorphism in $\mathcal{C}$ is C-preserving.

(CP2) Let f and g be C-preserving, then $f \circ g$ is also C-preserving.

(CP3) If $g \circ f$ are C-preserving, then f is C-preserving if $g \in \mathcal{M}$, and g

will be C-preserving if $f \in \mathcal{E}$, with $\mathcal{E}$ being stable under pullback along $\mathcal{M}$-morphisms.

(CP4) If C is weakly hereditary, then every C-closed subobject is C-preserving.

(CP5) If C is idempotent and if every C-closed subobject is also C-preserving, then C is weakly hereditary.

(CP6) Every C-preserving morphism in $\mathcal{M}$ is a C-closed subobject.

(CP7) If C is hereditary and $\mathcal{E}$ is stable under pullback along $\mathcal{M}$-morphisms, then every pullback af a C-preserving map along an $\mathcal{M}$-morphism is C-preserving.

From now on we'll replace the term C-preserving by C-closed, (which must not be confused with morphisms mapping closed subobjects onto closed subobjects). The term C-closed is commonly used when referring to C-preserving in the literature.

We also mention that we'll denote a closure operator by C for the rest of this chapter.

2.4   Compactness

As mentioned before, the categorical study of compactness has been developed in [3], [13] and [16]. Everything is brought together in [5], where fundamental properties of compact objects are discussed, amongst other work. We discuss categorical compactness similarly to [5].

Using the Kuratowski-Mrówka characterisation of compactness discussed in Chapter 1, we now generalise the situation.

Definition 2.4.1 *: An object X of $\mathcal{C}$ is called* **C-compact** *if the product projection $p_Y : X \times Y \to Y$ is C-closed, $\forall\ Y \in Obj(\mathcal{C})$.*

We will also be using the defintion of a C-Hausdorff object in $\mathcal{C}$. We use the characterisation in topology to obtain:

Definition 2.4.2 : $X \in Obj(\mathcal{C})$ *is called* **C-Hausdorff** *if and only if the diagonal morphism* $\delta_X : X \to X \times X$ *is C-closed.*

Some theorems about topological compactness can now be discussed. We are going to state the theorems but delay the proofs for now since we will soon be discussing categorical asymmetric compactness where the results still hold and proofs will be given in that context. Proofs can be found in [5].

Proposition 2.4.1 :
(1) For a morphism $f : X \to Y$ *in* $\mathcal{E}$*, with* $\mathcal{E}$ *stable under pullback, if* X *is C-compact, so is* Y.

(2) If X *in* $\mathcal{C}$ *is C-compact and* $m : M \to X$ *is C-closed, with* C *weakly hereditary, then* M *is C-compact.*

(3) If X *is C-compact and* Y *is C-Hausdorff, then every morphism* $f : X \to Y$ *is C-closed.*

2.5   Asymmetric compactness

This section follows the same route as [14] where the Kuratowski-Mrówka characterisation is mentioned and then the situation is generalised to asymmetric compactness, and eventually compact morphisms (generalised proper/perfect maps) are discussed.

2.5.1   The generalisation from the symmetric case

We have seen the example in Chapter 1 where countable compactness is characterised by using both the Kuratowski- and σ-closure. This motivates our approach to try and generalise this to a categorical setting. The following definitions can be found in [14].

Let C_1 and C_2 be closure operators for our category $\mathcal{C}$, with $(\mathcal{E}, \mathcal{M})$ a proper stable factorization system, which will be the type of factorization system we consider from now on.

Definition 2.5.1 : *A morphism $f : X \rightarrow Y$ in $\mathcal{C}$ is called $C_1 C_2$-**closed** if and only if $f(c_1(m)) \geq c_2(f(m))$, $\forall m \in Sub(X)$.*

Note that the subscript (indicating the codomain of the subobject) is ommitted. We to this for the sake of simplicity, and since we'll always make it clear in which context we are working.

This definition comes from the fact that a C-closed morphism $f : X \rightarrow Y$ satisfies the equality $f(c(m)) = c(f(m))$, $\forall m \in Sub(X)$ and since f is C-continuous, we are actually only interested in the inequality $f(c(m)) \geq c(f(m))$ (note the analogy from our topological examples).

$C_1 C_2$-closed morphisms map C_1-closed subobjects to C_2-closed subobject and, if C_1 is idempotent, the converse is also true.

When we write $C_1 \leq C_2$, for closure operators C_1 and C_2, we mean $c_1(m) \leq c_2(m)$, $\forall m \in \mathcal{M}$. We just extend the subobject ordering pointwise to closure operators.

And we write C_1-closed for $C_1 C_1$-closed (when $C_1 = C_2$), which just then reduces to our symmetric case.

Our next observation is that all C_1-closed and C_2-closed morphisms, respectively, are also $C_1 C_2$-closed if $C_1 \geq C_2$.

The following results follow from the definition of a $C_1 C_2$-closed morphism and seem trivial and not even worth proving. However, they become important in applications later on. They are mentioned in [14] but not proven. We'll briefly prove them.

Proposition 2.5.1 : *Let C_1, C_2 and C_3 be closure operators on $\mathcal{C}$ with $C_1 \geq C_2$, and let $f : X \rightarrow Y$ be a morphism in $\mathcal{C}$, then:*

(1) f is $C_3 C_1$-closed $\Rightarrow$ f is $C_3 C_2$-closed.

(2) f is $C_2 C_3$-closed $\Rightarrow$ f is $C_1 C_3$-closed.

Proof of (1): Let f be as above, and let $m \in Sub(X)$. Then:

$$f(c_3(m)) \geq c_1(f(m)) \geq c_2(f(m)).$$

Proof of (2): Again, consider f and m. We have $f(c_1(m)) \geq f(c_2(m)) \geq c_3(f(m))$. □

Note that, from the above proposition it follows that if f is C_2C_1-closed, it will also be both C_1-closed and C_2-closed. We are , however unlikely to consider such f. All the examples we consider (in topology) are those where $C_1 \geq C_2$.

In fact, since we are always interested in how a certain class of morphisms behaves w.r.t. the isomorphisms in a category, we look at the following (which is mentioned in [14] without proof):

Proposition 2.5.2 *All the isomorphisms in $\mathcal{C}$ are C_1C_2-closed if and only if $C_1 \geq C_2$.*

Proof:

$\Rightarrow$: We use the fact that the identity-morphisms are isomorphisms.

$\Leftarrow$: All isomorphisms are C_2-closed. □

We are also going to look at proofs for the following results (which are taken directly from [14]):

Proposition 2.5.3 *: Let C_1, C_2 and C_3 be closure operators on $\mathcal{C}$, $f : X \to Y$ and $g : Y \to Z$ morphisms in $\mathcal{C}$, then:*

(1) f C_1C_2-closed and g C_2C_3-closed $\Rightarrow$ $g \circ f$ C_1C_3-closed.

(2) $g \circ f$ C_1C_2-closed and $g \in \mathcal{M}$ $\Rightarrow$ f C_1C_2-closed.

(3) $g \circ f$ C_1C_2-closed and $f \in \mathcal{E}$ $\Rightarrow$ g C_1C_2-closed.

Proof of (1): Let $m \in Sub(X)$, and assume f is C_1C_2-closed and g C_2C_3-closed. Then:

$$(g \circ f)(c_1(m)) = g(f(c_1(m))) \geq g(c_2(f(m))) \geq c_3(g(f(m))) = c_3((g \circ f)(m)).$$

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Proof of (2): Let $m \in Sub(X)$ and assume $g \circ f$ is C_1C_2-closed and $g \in \mathcal{M}$. Then:

$$f(c_1(m)) = g^{-1}(g(f(c_1(m)))) \geq g^{-1}(c_2(g(f(m)))) \geq c_2(g^{-1}(g(f(m)))) = c_2(f(m)).$$

Proof of (3): Let $m \in Sub(Y)$ and assume $g \circ f$ is C_1C_2-closed and $f \in \mathcal{E}$. Then:

$$g(c_1(m)) = g(f(f^{-1}(c_1(m)))) \geq g(f(c_1(f^{-1}(m)))) \geq c_2(g(f(f^{-1}(m)))) = c_2(g(m)). \qquad \square$$

We also have the following result about closed subobjects and closed morphisms in $\mathcal{M}$, also proven in [14]:

Proposition 2.5.4 : Let $m : M \to X$ be in $\mathcal{M}$. Then:

(1) m a C_1C_2-closed morphism $\Rightarrow m$ a C_2-closed subobject.

(2) Let C_2 be weakly hereditary and $C_1 \geq C_2$. Then m a C_2-closed subobject $\Rightarrow m$ a C_1C_2-closed morphism.

(3) If C_1 is idempotent, then:
(m a C_1-closed subobject $\Rightarrow m$ a C_1-closed morphism, $\forall m \in \mathcal{M}$) $\Leftrightarrow C_1$ weakly hereditary.

Proof of (1): Let m be a C_1C_2-closed morphism.

Now, $m = m(1_M) = m(c_1(1_M))$.

Then, by using the C_1C_2-closedness of m, we have $m(c_1(1_M)) \geq c_2(m(1_M)) = c_2(m)$.

And hence $c_2(m) = m$.

Proof of (2): Let $n \in Sub(M)$.

Now, $m(n) = m \circ n \leq m$. So $c_2(m \circ n) \leq c_2(m)$, by the fact that C_2 is monotone.

By assuming that m is a C_2-closed subobject, we have $c_2(m) = m$, and hence we have the morphism j in the following triangle:

$$
\begin{array}{ccc}
 & \bullet & \\
g \nearrow & \downarrow j & \searrow c_2(m \circ n) \\
N \xrightarrow{n} & M \xrightarrow{m} & X
\end{array}
$$

The morphism g is C_2-dense because C_2 is weakly hereditary, and $j \leq c_2(n)$.

And hence we have:

$$m(c_1(n)) \geq m(c_2(n)) \geq m(j) = c_2(m \circ n) = c_2(m(n)).$$

And m is a $C_1 C_2$-closed morphism.

Proof of (3) $\Rightarrow$: Assume C_1 is idempotent, thus $c_1(n) = c_1(c_1(n))$, $\forall n \in \mathcal{M}$.

So $c_1(n)$ is a C_1-closed morphism.

Consider the following commuting triangle resulting from considering the C_1-closure of n:

$$
\begin{array}{ccc}
N & \xrightarrow{\quad n \quad} & \bullet \\
 & \searrow_{jm} \qquad \nearrow_{c_1(n)} & \\
 & c_1(N) &
\end{array}
$$

Then:

$$c_1(n) \circ c_1(j) = c_1(n)(c_1(j)) \geq c_1(c_1(n)(j)) = c_1(c_1(n) \circ j) = c_1(n).$$

And hence $c_1(j)$ is an isomorphism by the fact that $c_1(n)$ is a monomorphism, and C_1 is weakly hereditary.

$\Leftarrow$: Apply (2) above, with $C_1 = C_2$. $\qquad\qquad\qquad\qquad\qquad\square$

Now we discuss asymmetric compactness (cf. [14]).

Definition 2.5.2 : $X \in Obj(\mathcal{C})$ *is called C_1C_2-***compact** *if and only if the projection $p_Y : X \times Y \to Y$ is C_1C_2-closed, $\forall Y \in Obj(\mathcal{C})$.*

Similarly to closed morphisms, we will write C_1-compact when we talk about C_1C_1-compact. If we apply the results of Proposition 2.5.1 to projection-mappings, we have the following (cf. [14]):

Proposition 2.5.5 : *Let C_1, C_2 and C_3 be closure operators on $\mathcal{C}$, with $C_1 \geq C_2$, then:*

(1) X is C_3C_1-compact $\Rightarrow$ X is C_3C_2-compact.

(2) X is C_2C_3-compact $\Rightarrow$ X is C_1C_3-compact.

The above results become quite useful in applications.

Due to Proposition 2.5.3(1) we have (cf. [14]):

Proposition 2.5.6 *For closure operators C_1, C_2 and C_3 on $\mathcal{C}$, we have: X C_1C_2-compact and Y C_2C_3-compact $\Rightarrow$ $X \times Y$ C_1C_3-compact.*

Now we return to those theorems stated in the previous section, regarding "symmetric" compactness. We generalise them to our asymmetric context. We also keep in mind our analogy from the category of topological spaces and continuous functions. All of the following propositions use the same proof-strategies as [14], which in turn are similar to those found in [5]. We have filled in a number of details.

2.5.2   Image of Compact is Compact

Firstly, we are going to discuss a property of compactness which we know well from topology: The continuous image of a compact space is compact. We especially remember the case in topology where we work with the Kuratowksi-closure ("familiar" compactness). Now we discuss it in the generalised form, yielding even more results for free, as such.

Proposition 2.5.7 : *Let X be C_1C_2-compact, then:*
$f : X \to Y$ *is in* $\mathcal{E} \Rightarrow Y$ *is C_1C_2-compact.*

Proof: Let $Z \in Obj(\mathcal{C})$, and consider the following square:

$$
\begin{array}{ccc}
X \times Z & \xrightarrow{\ f \times 1_Z\ } & Y \times Z \\
{\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle q_1} \\
X & \xrightarrow{\quad f \quad} & Y
\end{array}
$$

with $f \in \mathcal{E}$, and p_1, q_1 projections. Now, let $A \in Obj(\mathcal{C})$, with $u : A \to Y \times Z$ and $v : A \to X$ morphisms such that $q_1 \circ u = f \circ v$. And let $q_2 : Y \times Z \to Z$ be the second projection.

Define $d : A \to X \times Z$ as follows: $d = \langle v, q_2 \circ u \rangle$.

It is easy to see that $f \times 1_Z \circ d = u$ and $p_1 \circ d = v$. And, by using the product-property, we have that d is unique. So, the square above is a pull-back. So, since $(\mathcal{E}, \mathcal{M})$ is stable, we have $f \times 1_Z \in \mathcal{E}$. Next, consider the following commutative triangle:

$$
\begin{array}{ccc}
X \times Z & \xrightarrow{\ f \times 1_Z\ } & Y \times Z \\
 & {\scriptstyle p_2}\searrow \quad \swarrow {\scriptstyle q_2} & \\
 & Z &
\end{array}
$$

Since X is C_1C_2-compact, the projection p_2 is C_1C_2-closed. We then use the fact that the triangle commutes and $f \in \mathcal{E}$ to conclude (from Proposition 2.5.3(3)) that the projection q_2 is C_1C_2-closed and hence Y is C_1C_2-compact. $\qquad\square$

2.5.3 Closed subspace of Compact Space is Compact

Our next compactness-property to be discussed is: Closed subspaces of compact spaces are compact. We again recall this result using our topological

knowledge. And, as with the previous theorem, we think of the example where the Kuratowski-closure is used.

Proposition 2.5.8 : *Let X be C_1C_2-compact, and C_1 weakly hereditary. Then:*
$m : M \to X$ *a C_1-closed subobject $\Rightarrow$ M is C_1C_2-compact.*

Proof: Let $Y \in Obj(\mathcal{C})$ and consider the following square:

$$
\begin{array}{ccc}
M \times Y & \xrightarrow{\ m \times 1_Y\ } & X \times Y \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle q_1} \\
M & \xrightarrow{\quad m \quad} & X
\end{array}
$$

where m is C_1-closed and X C_1C_2-compact, with C_1 weakly hereditary.

We'll again prove that the diagram is a pullback. As before, let $B \in Obj(\mathcal{C})$, with $u : B \to X \times Y$ and $v : B \to M$ morphisms such that $q_1 \circ u = m \circ v$, and let $q_2 : X \times Y \to Y$ be the second projection.

Define $d : B \to M \times Y$ as follows: $d := \langle v, q_2 \circ u \rangle$

Using the projections q_1 and q_2, it's easy to see that $m \times 1_Y \circ d = u$ and $p_1 \circ d = v$. And d will be unique due to the product-property.

Now, $m \times 1_Y$ will thus be C_1-closed. And since C_1 is weakly hereditary, we use Proposition 2.5.4(2) to conclude that $m \times 1_Y$ is a C_1-closed morphism. In the following commuting triangle the projection q_2 is C_1C_2-closed due to the C_1C_2-compactness of X:

$$
\begin{array}{ccc}
M \times Y & \xrightarrow{\ m \times 1_Y\ } & X \times Y \\
& {\scriptstyle p_2}\searrow \quad \swarrow {\scriptstyle q_2} & \\
& Y &
\end{array}
$$

By Proposition 2.5.3(1) we have that $q_2 \circ (m \times 1_Y) = p_2$ is C_1C_2-closed.

And hence M is C_1C_2-compact. □

2.5.4   Compact subspace of Hausdorff space is closed

In topology we have that compact subspaces of Hausdorff spaces are closed (again, the word "closed" here refers to "closed" in the Kuratowski-closure sense). We generalise this to our categorical asymmetric context as our third familiar result.

Proposition 2.5.9 *Let X be C_1C_2-compact and Y C_1-Hausdorff. Then any $m : X \to Y$ in $\mathcal{M}$ is a C_2-closed subobject.*

Proof: Let $m : X \to Y$ be in $\mathcal{M}$. Again consider a square:

$$
\begin{array}{ccc}
X & \xrightarrow{\langle 1_X, m \rangle} & X \times Y \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m \times 1_Y} \\
Y & \xrightarrow[\delta_Y]{} & Y \times Y
\end{array}
$$

This square is a pullback as well: let $B \in Obj(\mathcal{C})$ and $u : B \to X \times Y$ and $v : B \to Y$ such that $(m \times 1_Y) \circ u = \delta_Y \circ v$, and let q_1 and q_2 be the projections from $X \times Y$ to X and Y respectively.

Define $d : B \to X$ as follows: $d := q_1 \circ v$. Again, by using the projections q_1 and q_2, one can easily show that $\langle 1_X, m \rangle \circ d = u$ and $m \circ d = v$. And, as before, the product-property ensures the uniqueness of d.

The diagonal morphism δ_Y is a C_1-closed subobject, so $\langle 1_X, m \rangle$ is also a C_1-closed subobject. Consider the commuting triangle:

$$
\begin{array}{ccc}
X & \xrightarrow{\langle 1_X, m \rangle} & X \times Y \\
& {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle p_2} & \\
& Y &
\end{array}
$$

The projection p_2 is C_1C_2-closed by the C_1C_2-compactness of X.

$$m = p_2 \circ \langle 1_X, m \rangle = p_2(\langle 1_X, m \rangle) = p_2(c_1(\langle 1_X, m \rangle)) \geq c_2(p_2(\langle 1_X, m \rangle)) = c_2(m).$$

And hence m is C_2-closed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.5.5   More Topological Examples

For a moment we turn our attention back to topology.

We have seen examples in Chapter 1. Now we mention a few more (cf. [14]), and we see how our categorical knowledge becomes useful.

We are now working in the category of topological spaces and continuous functions, with our $(\mathcal{E}, \mathcal{M})$-factorization system being (Surjective, Embedding).

Let X be a topological space and $m : M \to X$ an embedding. We view M as a subset of X. Note we use notation similar to our categorical context.

By $k(M)$ (as in the last example in Chapter 1) we mean the (Kuratowski)-closure in the usual sense, while $\sigma(M)$, $\theta(M)$ and $\gamma(M)$ are as mentioned before.

We also define $z(M) := \{x \in X | C \cap M \neq \emptyset$ for each co-zero set C containing $x\}$. C is called a co-zero set if $C = X - f^{-1}(0)$, for some continuous function $f : X \to \mathbb{R}$.

We have the following ordering: $\sigma \leq k \leq \theta \leq z$, and $\gamma \leq k$ but σ and γ can't be compared.

• We have seen that $k\sigma$-compact is countably compact. Note that we can say $k\sigma$-compact $\Rightarrow \theta\sigma$-compact $\Rightarrow z\sigma$-compact, by Proposition 2.5.5.

• We have also seen that θ-compact is H-closed. By Proposition 2.5.5, we have θ-compact $\Rightarrow \theta k$-compact. And, in fact, the converse is also true (cf. [14]).

Proofs for the following can be found in [14].

- z-compact $=$ $z\theta$-compact $=$ zk-compact $=$ Functionally compact.

- Also, $z\sigma$-compact $\Rightarrow$ Pseudocompact. It is not known whether or not the converse is true.

We have discussed those three familiar compactness-theorems which we know from topology. The usual notion of compactness in topology is normally what we would think of when seeing those results.

In fact, we get a lot more, for example we can now conclude that the image of Lindelöf is Lindelöf, or that a θ-closed subspace of an H-closed topological space is also H-closed.

2.6   Compact Morphisms

2.6.1   Working in the Comma Category

Since morphisms in a category can be viewed as generalised objects, we are interested in investigating how a "compact morphism" will look. We again use the same strategy as [5] and [14], and look at an example which is also discussed therein. Let X be a fixed object in our category $\mathcal{C}$, and consider $\mathcal{C}/X$, the comma category over X.

Our factorization system $(\mathcal{E}, \mathcal{M})$ transfers to $\mathcal{C}/X$ - consider the following diagram:

We have (A, f) and (B, g) objects in $\mathcal{C}/X$, and h the morphism between them. The factorization of h in $\mathcal{C}/X$ will then be the $(\mathcal{E}, \mathcal{M})$-factorization of h in $\mathcal{C}$, which will again be over X.

The product of (A, f) and (B, g) in $\mathcal{C}/X$ is given by the following pullback

square in $\mathcal{C}$:

$$
\begin{array}{ccc}
A \times_X B & \xrightarrow{\ p_1\ } & A \\
\downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle f} \\
B & \xrightarrow{\ g\ } & X
\end{array}
$$

with p_1 and p_2 the projections. The object $f : A \to X$ in $\mathcal{C}/X$ will be C_1C_2-compact if p_2 is C_1C_2-closed for any object $g : B \to X$ in $\mathcal{C}/X$, i.e. any pullback of f in $\mathcal{C}$ is C_1C_2-closed. We call such f *stably C_1C_2-closed.*

Proposition 2.6.1 : $f : A \to X$ (considered as an object in $\mathcal{C}/X$) is C_1C_2-compact iff f is stably C_1C_2-closed in $\mathcal{C}$.

We can now view a compact object in $\mathcal{C}$ from a different perspective. We use the fact that $\mathcal{C} \cong \mathcal{C}/1$, with 1 being the terminal object in $\mathcal{C}$.

Proposition 2.6.2 *An object X of $\mathcal{C}$ is C_1C_2-compact iff $!_X : X \to 1$ is C_1C_2-compact in $\mathcal{C}/1$ iff $!_X : X \to 1$ is stably C_1C_2-closed in $\mathcal{C}$.*

We now conclude that C_1C_2-compact morphisms are closed and have compact fibres (like proper/perfect maps in the symmetric topological case).

Proposition 2.6.3 *If $f : A \to X$ is C_1C_2-compact in $\mathcal{C}/X$, then f is C_1C_2-closed in $\mathcal{C}$ and any fibre of f is C_1C_2-compact.*

Proof: Assume f is C_1C_2-compact in $\mathcal{C}/X$. So, f is stably C_1C_2-closed in $\mathcal{C}$ and since f is the pullback of itself along the identity-morphism 1_X, f is C_1C_2-closed.

Consider the following pullback-square:

$$
\begin{array}{ccc}
F & \xrightarrow{\ q\ } & A \\
\downarrow{\scriptstyle !_F} & & \downarrow{\scriptstyle f} \\
1 & \xrightarrow{\ p\ } & X
\end{array}
$$

F is a fibre of f (we think of $p : 1 \to X$ as a "point" of X, since the terminal object in $\mathcal{SET}$ is the singleton).

We use the composability of pullback-diagrams to conclude that, since f is stably closed, $!_F$ is not only C_1C_2-closed but in fact stably C_1C_2-closed. And by Proposition 2.6.2. we have that F is C_1C_2-compact. $\qquad\square$

2.6.2 A Topological example

We are going to discuss $k\sigma$-maps in topology (cf. [14]). By Proposition 2.6.3 $k\sigma$-maps are $k\sigma$-closed with $k\sigma$-compact fibres (i.e. countably compact fibres).

Conversely, any continuous function $f : A \to X$ which is $k\sigma$-closed with countably compact fibres is $k\sigma$-compact in $\mathcal{TOP}/X$. Let $g : B \to X$ be continuous and consider the following pullback:

$$
\begin{array}{ccc}
A \times_X B & \xrightarrow{\ p_2\ } & B \\
\Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle g} \\
A & \xrightarrow[\ f\ ]{} & X
\end{array}
$$

Where $A \times_X B = \{(a, b) \in A \times B \mid f(a) = g(b)\}$. We have to show that the projection p_2 is $k\sigma$-closed. Let $M \subseteq A \times_X B$ ba a nonempty subset. Assume $b \in \sigma(p(M))$, so there is a sequence (b_n) in $p(M)$ with $(b_n \to b)$. There is also a sequence (a_n) in A such that $(a_n, b_n) \in M$.

For $n \in \mathbb{N}$ let $A_n := \{a_m \mid m \geq n\}$. Then, by using the $k\sigma$-closedness of f, we have that $\sigma(f(A_n)) \subseteq f(\overline{A_n})$.

Now, $f(A_n) = \{f(a_m) \mid m \geq n\} = \{g(b_m) \mid m \geq n\}$. And, since g is continuous, $g((b_n)) \to g(b)$. So, $g(b) \in \sigma(f(A_n)) \subseteq f(\overline{A_n})$, $\forall n \in \mathbb{N}$.

$f^{-1}(g(b))$ is countably compact, by assumption. So, since $\overline{A_n} \cap f^{-1}(g(b)) \neq \emptyset \ \forall n \in \mathbb{N}$, we have that $\bigcap_{n \in \mathbb{N}} (\overline{A_n} \cap f^{-1}(g(b))) \neq \emptyset$.

Let $a \in \bigcap_{n \in \mathbb{N}} \overline{A_n}$ with $f(a) = g(b)$. So, $(a, b) \in A \times_X B$.

Let $U \subseteq A$, $V \subseteq B$ be open with $a \in U$ and $b \in V$. Now, $\exists n' \in \mathbb{N}$ such that $b_m \in V$, $\forall m \geq n'$, and $U \cap A_{n'} \neq \emptyset$. So, we have $m \geq n'$ with $(a_m, b_m) \in (U \times V) \cap M$, and $b \in p_2(\overline{M})$.

Note that the class of quasi-perfect maps (maps which are closed and have countably compact fibres (cf. [12])) is contained in the class of $k\sigma$-compact maps, yielding the conclusion the these maps might be the better class to study.

Chapter 3

Categorical Compactness without explicitly referring to closure

This chapter (especially the symmetric case) closely follows the work done in [6], which was preceeded by [19], where the ideas were already mentioned.

We have now seen how the notion of a compact object in a category can be characterised by a closure operator (or two, in the asymmetric case).
The theory of closure operators became essential in our quest to study closed morphisms and compactness, where the latter makes good use of the former.

In this chapter we are going to travel a slightly different route: we are going to start off by assuming we have a class of morphisms $\mathcal{F}$ with certain properties. We think of these as being closed morphisms but we do not explicitly state what they are, so this is indeed a generalisation of our previous work. We also still hold our topological knowledge in the back of our mind as an analogy.

We will discuss this briefly for the symmetric (usual) compactness, and then move to the asymmetric case where we will show how this new approach generalises what we have done thus far. The following definitions and results can be found in [6].

3.1 The symmetric case

3.1.1 The distinguished class $\mathcal{F}$

We again consider our category $\mathcal{C}$, with $(\mathcal{E}, \mathcal{M})$ our proper stable factorization system (in [6] the stable-condition is not assumed). We then assume we have a class of morphisms, $\mathcal{F}$, with the following properties:

(F1) All the isomorphisms are contained in $\mathcal{F}$, and $\mathcal{F}$ is closed under composition;

(F2) $\mathcal{F} \cap \mathcal{M}$ is stable under pullback;

(F3) For $g \circ f$ in $\mathcal{F}$, we have that if $f \in \mathcal{E}$ then g in $\mathcal{F}$.

If we think of the morphisms in $\mathcal{F}$ as being the C-closed morphisms, for some closure operator C in $\mathcal{C}$, we see how (F1)-(F3) coincides with what we have done previously:

- All the isomorphisms are C-closed

- The composition of C-closed morphisms is again C-closed.

- A C-closed morphism in $\mathcal{M}$ is also a C-closed subobject (also vice versa if C is weakly hereditary, which holds in the topological case). C-closed subobjects are stable under pullback.

- If $g \circ f$ is C-closed and $f \in \mathcal{E}$, then g is C-closed.

We'll often call a morphism in $\mathcal{F}$ "$\mathcal{F}$-closed".

We now discuss a few definitions of concepts we have seen previously.

Definition 3.1.1 *A morphism $f : A \to B$ in $\mathcal{C}$ is called $\mathcal{F}$-**dense** if for any factorization*

$$
\begin{array}{ccc}
A & \xrightarrow{\quad f \quad} & B \ , \\
& h \searrow \quad \nearrow n & \\
& C &
\end{array}
$$

with an $\mathcal{F}$-closed subobject n, we necessarily have that n is an isomorphism.

3.1.2 Proper Maps, Compact Objects and Separation

In topology the pullback of a closed morphism need not be closed (cf. [6]). We are thus interested in those closed morphisms which *are* stable under pullback.

Definition 3.1.2 *We call a morphism f $\mathcal{F}$-**proper** if in any pullback-diagram,*

$$
\begin{array}{ccc}
X' & \xrightarrow{\quad f' \quad} & Y' \\
\downarrow g & & \downarrow h \\
X & \xrightarrow{\quad f \quad} & Y
\end{array}
$$

We have that f' is in $\mathcal{F}$.

In this case we say f belongs *stably* to $\mathcal{F}$ but we'll mostly use the term $\mathcal{F}$-proper. We'll denote the class of $\mathcal{F}$-proper morphisms by $\mathcal{F}^*$. If we choose g and h in the above pullback-diagram to be identity-morphisms, we conclude $\mathcal{F}^* \subseteq \mathcal{F}$.

We'll now discuss some important properties of the class $\mathcal{F}^*$ ([6]).

Proposition 3.1.1 *For the class of $\mathcal{F}$-proper morphisms, the following hold:*

(1) $\mathcal{F}^*$ contains $\mathcal{F} \cap \mathcal{M}$, and is closed under composition.

(2) $\mathcal{F}^*$ is the largest pullback-stable subclass of $\mathcal{F}$.

(3) If $g \circ f$ is in $\mathcal{F}^*$, with g monic, then f is in $\mathcal{F}^*$.

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(4) If $g \circ f$ is in $\mathcal{F}^*$, with $f \in \mathcal{E}^*$ then g is in $\mathcal{F}^*$ ($\mathcal{E}^*$ is all morphism which are stably in $\mathcal{E}$ - note that in our case $\mathcal{E} = \mathcal{E}^*$, since $(\mathcal{E}, \mathcal{M})$ is stable).

Proof of (1): $\mathcal{F} \cap \mathcal{M}$ is stable under pullback, and a pullback of $g \circ f$ can be obtained from adjacent pullbacks of f and g.

Proof of (2): We also use the composability of adjacent pullback-diagrams.

Proof of (3): We use the fact that for any *any* pullback-stable class $\mathcal{M}$ we have that if $n \circ m \in \mathcal{M}$, with n monic, then $m \in \mathcal{M}$.

Proof of (4):Consider the following diagram:

$$
\begin{array}{ccccc}
A & \xrightarrow{\ f'\ } & B & \xrightarrow{\ g'\ } & C \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle l} \\
X & \xrightarrow[\ f\ ]{} & Y & \xrightarrow[\ g\ ]{} & Z
\end{array}
$$

Where the square on the right is an arbitrary pullback of g, and the one on the left is the pullback of f along k. The outer rectangle will be a pullback of $g \circ f$.

Since $g \circ f \in \mathcal{F}^*$, we conclude that $g' \circ f' \in \mathcal{F}$, while $f \in \mathcal{E}^*$ means f' will be in $\mathcal{E}$. And by property (F3) we conclude that $g' \in \mathcal{F}$. So g is proper, i.e. contained in $\mathcal{F}^*$. $\qquad\square$

We now turn our attention to compact objects. The following definition is related to Proposition 2.6.2.

Definition 3.1.3 *An object X of $\mathcal{C}$ is called $\mathcal{F}$-**compact** if the unique morphism $!_X : X \to 1$ to the terminal object is $\mathcal{F}$-proper.*

Now, pullbacks of $!_X$ look like this:

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{\;\;p_2\;\;} & Y \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle !_Y} \\
X & \xrightarrow{\;\;!_X\;\;} & 1
\end{array}
$$

We see that to say $!_X$ is $\mathcal{F}$-proper means exactly that p_2 is in $\mathcal{F}$, or "$\mathcal{F}$-closed". Again this coincides nicely with our previous work in Chapter 2 where we worked with closure operators.

We have seen the categorical generalisation of a Hausdorff space. We make use of the diagonal morphism. We now discuss our approach to define a Hausdorff object without referring to closure operators.

Let $f : X \rightarrow Y$ be a morphism in $\mathcal{C}$. We then consider the following morphism: $\delta_f : X \rightarrow X \times_Y X$, with $X \times_Y X$ belonging to the following pullback-diagram:

$$
\begin{array}{ccc}
X \times_Y X & \xrightarrow{\;\;f_2\;\;} & X \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} \\
X & \xrightarrow{\;\;f\;\;} & Y
\end{array}
$$

Now, we use the identity-morphism on X to obtain the following:

We use the product-property to conclude that δ_f is an equaliser for f_1 and f_2 (cf. [6]). And, since $\mathcal{E}$ is a class of epimorphisms, we have that $\delta_f \in \mathcal{M}$.

In this separation-section, we want to consider those morphisms f with $\delta_f \in \mathcal{F}^*$. But, since $\delta_f \in \mathcal{M}$, and we have $\mathcal{F} \cap \mathcal{M}$ stable under pullback, it suffices to require that $\delta_f \in \mathcal{F}$.

Definition 3.1.4 *We call a morphism f $\mathcal{F}$-separated if $\delta_f \in \mathcal{F}$.*

We look at an example from topology (cf. [6]):

Let X be a topological space. Then $X \times_Y X = \{(x_1, x_2) | f(x_1) = f(x_2)\}$, with $f : X \to Y$.
Now, to say $\Delta_X = \{(x, x) | x \in X\}$ is closed in $X \times_Y X$, means $(X \times_Y X) - \Delta_X$ is open.
This means $\forall x_1, x_2 \in X$, with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$, $\exists$ open neighbourhoods $U_1 \in \mathcal{U}_{x_1}$ and $U_2 \in \mathcal{U}_{x_2}$, with $U_1 \times U_2 \subseteq (X \times_Y X) - \Delta_X$.

So $U_1 \cap U_2 \neq \emptyset$.

Hence, in topology, a morphism $f : X \to Y$ is $\mathcal{F}$-separated if and only if each pair of distinct points in any fibre (the pre-image of a point) of f could be separated by disjoint open neighbourhoods.

Next we take a look at what we mean by an $\mathcal{F}$-separated object.

Definition 3.1.5 *An object X of $\mathcal{C}$ is called $\mathcal{F}$-separated or $\mathcal{F}$-Hausdorff if the unique morphism $!_X : X \to 1$ to the terminal object is $\mathcal{F}$-separated.*

The above definition simply means that $\langle 1_X, 1_X \rangle = \delta_X : X \to X \times X$ must be in $\mathcal{F}$, i.e. (using our topological analogy) "the diagonal is closed".

3.1.3 An example

As already stated, we think of the class $\mathcal{F}$ as being the closed morphisms. And we have definitions like $\mathcal{F}$-*closed*. We are going to look at an example from topology which is rather counter-intuitive (to show that there are examples where closures/closure-preserving maps are not used):

Consider the category of topological spaces and continuous functions, $\mathcal{TOP}$, with the $(\mathcal{E}, \mathcal{M})$-factorization system being (Epimorphisms, Embeddings).

Take the class $\mathcal{F}$ to be all the open maps, i.e. maps mapping open subsets onto open subsets (this is mentioned in [6] - we are going to verify that axioms F1-F3 are satisfied).

Every homeomorphism is an open map. And the composition of open is again open (Let $f : X \to Y$ and $g : Y \to Z$ be open maps. Let O be any open set in X. Then, clearly, $g(f(O))$ will be open in Z). We have thus shown that property (F1) holds.

Consider an open embedding. The pre-image of this will also be an open embedding, i.e. $\mathcal{F} \cap \mathcal{M}$ is stable under pullback and hence (F2) is satisfied.

Let $g \circ f$ be an open map, with $f : X \to Y$ and $g : Y \to Z$ and f a surjective continuous function. Let U be any open set in Y. Then $f^{-1}(U)$ will be open in X. Then $g(U) = g(f(f^{-1}(U)))$ will be open in Z, and (F3) holds.

We are ready to look at how we'll deal with asymmetric compactness.

3.2 The Asymmetric Case

3.2.1 Introductory

Firstly, assuming we only have the one class of morphisms $\mathcal{F}$ might not suffice. Say we think of $\mathcal{F}$ as being the C_1C_2-closed morphisms. This is not necessarily closed under composition, i.e. the first part of (F1) fails. We proceed to use our knowledge from [5] and [14] to obtain the generalisation of the asymmetric case.

We make use of the following:

Lemma 3.2.1 *Assume* $f : B \to C$, $f_1 : A \to B$ *and* $f_2 : C \to D$ *are* C_1C_2-*closed,* C_1-*closed and* C_2-*closed respectively, then* $f_2 \circ f \circ f_1$ *is also* C_1C_2-*closed.*

Proof: Let $m : M \to A$ be in $\mathcal{M}$.
Then $f_2 \circ f \circ f_1(c_1(m)) = f_2(f(f_1(c_1(m)))) = f_2(f(c_1(f_1(m)))) \geq f_2(c_2(f(f_1(m)))) = c_2(f_2(f(f_1(m))))$, yielding the desired result. $\qquad\square$

By $\mathcal{F}_2 \circ \mathcal{F} \circ \mathcal{F}_1$ we denote the class of morphisms of the form $f_2 \circ f \circ f_1$, where $f \in \mathcal{F}$, $f_1 \in \mathcal{F}_1$ and $f_2 \in \mathcal{F}_2$.

3.2.2 The distinguished classes

Assume $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are classes of morphisms in $\mathcal{C}$ with the following properties:

(A1) $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy (F1)-(F3) (on p55), $\mathcal{F}$ contains all the isomorphisms, and $\mathcal{F}_2 \circ \mathcal{F} \circ \mathcal{F}_1 \subseteq \mathcal{F}$;

(A2) $\mathcal{F} \cap \mathcal{M}$ is stable under pullback;

(A3) For $g \circ f$ in $\mathcal{F}$ and $f \in \mathcal{E}$, we have $g \in \mathcal{F}$.

We think of $\mathcal{F}$ as being the C_1C_2-closed morphisms, with $\mathcal{F}_1$ and $\mathcal{F}_2$ being the C_1-closed and C_2-closed morphisms respectively - Lemma 3.2.1 leads us to require the third part of (A1).

Keeping this closure operator approach in mind, note that if $\mathcal{F} = \mathcal{F}_1 = \mathcal{F}_2$, we are reduced to our symmetric case.

We can prove the following (generalisation of Proposition 2.5.1.):

Proposition 3.2.1 $\mathcal{F}_1 \subset \mathcal{F}$ and $\mathcal{F}_2 \subset \mathcal{F}$.

Proof: The isomorphisms in $\mathcal{C}$ are contained in all three the above classes of morphisms, hence the identity-morphisms are as well.

So, assume $f_1 : X \to Y$ is in $\mathcal{F}_1$. Pick $f_2 \in \mathcal{F}_2$ and $f \in \mathcal{F}$ to be 1_Y.

Then, clearly, $f_1 \in \mathcal{F}$. Similarly, any $f_2 \in \mathcal{F}_2$ is in $\mathcal{F}$. $\qquad\square$

Note that we have only used the fact that the isomorphisms in $\mathcal{C}$ are contained in $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$. In the asymmetric closure-operator analogy, saying that the isomorphisms are contained in $\mathcal{F}$ (the $C_1 C_2$-closed morphisms) is equivalent to saying $C_1 \geq C_2$, as we have seen before.

And if $C_1 \geq C_2$, then morphisms which are C_1-closed and C_2-closed, respectively, will also be $C_1 C_2$-closed (Proposition 2.5.1.).

We mention that the definition of an $\mathcal{F}$-proper morphism and that of an $\mathcal{F}$-compact object are both carried over to our "new" class $\mathcal{F}$.

Also, we offer the following lemma, where we discuss an important property also being inherited by the proper morphisms in $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively.

Lemma 3.2.2 : *For our classes,* $\mathcal{F}$, $\mathcal{F}_1$ *and* $\mathcal{F}_2$, *the following holds:*
$\mathcal{F}_2{}^* \circ \mathcal{F}^* \circ \mathcal{F}_1{}^* \subseteq \mathcal{F}^*$

Proof: Consider the following diagram:

$$A \xrightarrow{\ f_1\ } B \xrightarrow{\ f\ } C \xrightarrow{\ f_2\ } D$$

With f, f_1 and f_2 contained in $\mathcal{F}^*$, $\mathcal{F}_1{}^*$ and $\mathcal{F}_2{}^*$ respectively.

Now consider adjacent pullback-diagrams:

The entire diagram will thus also be a pullback, and since pullbacks are essentially unique, we conclude that any pullback of $f_2 \circ f \circ f_1$ must essentially be made up of adjacent pullbacks, as above.

And since $f \in \mathcal{F}^*$, $f_1 \in \mathcal{F}_1^*$ and $f_2 \in \mathcal{F}_2^*$, we have that $f' \in \mathcal{F}$, $f_1' \in \mathcal{F}_1$ and $f_2' \in \mathcal{F}_2$.

By using the last part of property (A1), we conclude that $f_2' \circ f' \circ f_1' \in \mathcal{F}$, which completes our proof. $\qquad\qquad\square$

We will now again look at those three compactness-theorems which we proved in Chapter 2. We are going to prove them again, using our new approach.

3.2.3   Image of Compact is Compact

We briefly recall what Proposition 2.5.7 says:

Assume $f : X \to Y$ is in $\mathcal{E}$ and X is C_1C_2-compact, then Y is C_1C_2-compact.

We translate this to our new language:

Proposition 3.2.2 *Let $f : X \to Y$ be in $\mathcal{E}$. If X is $\mathcal{F}$-compact, so is Y.*

Proof: Consider the following commuting diagram:

$!_X$ is in $\mathcal{F}^*$, since X is $\mathcal{F}$-compact. But $f \in \mathcal{E} = \mathcal{E}^*$, and hence $!_Y \in \mathcal{F}^*$ (by Proposition 3.1.1.(4)), i.e. Y is $\mathcal{F}$-compact. $\square$

3.2.4 Closed subspaces of Compact spaces are Compact

Recall Proposition 2.5.8.: Let X be C_1C_2-compact, and C_1 be weakly hereditary, then if $m : M \to X$ is a C_1-closed subobject, M is C_1C_2-compact.

We mention that if C_1 is weakly hereditary, all C_1-closed subobjects are also C_1-closed morphisms (the converse is always true). So, in the following Proposition, when we assume $m \in \mathcal{F}_1 \cap \mathcal{M}$, we actually only consider the case where m is an $\mathcal{F}$-closed morphism. We do not have the notion of an "$\mathcal{F}$-closed subobject".

Proposition 3.2.3 *Let X be $\mathcal{F}$-compact. If $m : M \to X$ is in $\mathcal{F}_1 \cap \mathcal{M}$, then M is $\mathcal{F}$-compact.*

Proof: Consider the following commuting diagram:

$$
\begin{array}{ccc}
M & \xrightarrow{\ m\ } & X \\
& \searrow{\scriptstyle !_M} \quad \swarrow{\scriptstyle !_X} & \\
& 1 &
\end{array}
$$

Since m is in $\mathcal{F}_1 \cap \mathcal{M}$, we have that $m \in \mathcal{F}_1^*$. By assumption, $!_X \in \mathcal{F}^*$. We then have the following:

$!_M = 1_1 \circ !_X \circ m \in \mathcal{F}^*$, where the identity-morphism 1_1 is clearly in $\mathcal{F}_2^*$. $\square$

3.2.5 Compact subspace of Hausdorff is Closed

Recall Proposition 2.5.9 : Let X be C_1C_2-compact, and Y C_1-Hausdorff. Then any $m : X \to Y$ in $\mathcal{M}$ is a C_2-closed subobject.

We are going to now prove this for any morphism, and not just for a subobject - we'll have a few remarks after the proof,

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Proposition 3.2.4 *: Let X be $\mathcal{F}$-compact and Y $\mathcal{F}_1$-Hausdorff. If $f : X \to Y$ is in $\mathcal{M}$ then $f \in \mathcal{F}$.*

Proof: Consider the following commuting triangle:

$$X \xrightarrow{\langle 1_X, f \rangle} X \times Y$$

(triangle with $f : X \to Y$ and $p_2 : X \times Y \to Y$)

The projection p_2 is $\mathcal{F}$-closed, since X is $\mathcal{F}$-compact. Since Y is $\mathcal{F}_1$-Hausdorff, we know the diagonal $\delta_Y : Y \to Y \times Y$ is $\mathcal{F}_1$-closed. Thus $f^{-1}(\delta_Y) = \langle 1_X, f \rangle$ is $\mathcal{F}_1$-closed.

So, we have $f = 1_Y \circ p_2 \circ \langle 1_X, f \rangle \in \mathcal{F}$. $\qquad\qquad\qquad\qquad$ $\square$

Note we haven't assumed that f is in $\mathcal{M}$ (i.e. a subobject of Y) and, as mentioned before, we do not have the notion of a "closed subobject". And we do not prove that $f \in \mathcal{F}_2$, since this is quite generalised. But if we apply this Proposition to our asymmetric closure operator counterpart, while assuming $f \in \mathcal{M}$, Proposition 2.5.9. is obtained as a consequence since if f is a $C_1 C_2$-closed morphism in $\mathcal{M}$, it is also a C_2-closed subobject.

We have now seen how the categorical theory of compactness has developed by starting with the intuitive topolgical ideas. Category theory gives us the possibility of generalising the topological work, and opens the way to apply the categorical theory to other fields of mathematics. And, for example - as we have mentioned in Chapter 2 - the class of $k\sigma$-maps looks like an interesting and useful class of morphisms to study.

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