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Title: Generalized Hyperfocused Arcs in $PG(2, p)$

Summary:

Let $PG(2, q)$ be the projective plane over F_q, the finite field with q elements. A $k-$arc in $PG(2, q)$ is a set of k points with no 3 on a line. A line containing 1 or 2 points of a $k-$arc is said to be a tangent or secant to the $k-$arc, respectively.

A *blocking set* of a family of lines $\mathcal{F}$ is a point-set $\mathcal{B} \subset PG(2, q)$ having non-empty intersection with each line in $\mathcal{F}$. If this is the case, we also say that the lines in $\mathcal{F}$ are *blocked* by $\mathcal{B}$.

A *generalized hyperfocused arc* $\mathcal{H}$ in $PG(2, q)$ is a k-arc with the property that the $k(k-1)/2$ secants can be blocked by a set $\mathcal{B}$ of $k-1$ points not belonging to the arc. Points of the arc $\mathcal{H}$ will be called *white points* and points of the blocking set $\mathcal{B}$ *black*. In case $k > 1$, since every secant to the arc contains a unique black point, the $k-1$ black points induce a factorization, i.e. a partition into matchings, of the white k-arc and k is forced to be even. For $k = 2$, we only have a trivial example: $\mathcal{B}$ consists of a unique point out of $\mathcal{H}$ on the line through the two points of $\mathcal{H}$.

An non trivial example of generalized hyperfocused arc is any 4-arc of white points with its three black diagonal points and **our main result is that this is the only non trivial example, provided q is an odd prime**.

For q even, there are many examples with all black points on a line; in this case $\mathcal{H}$ is simply called a *hyperfocused arc*. As a consequence of the main result of [3], hyperfocused arcs only exist if q is even. When q is even, a nice result is that generalized hyperfocused arcs contained in a conic are hyperfocused [1]; moreover it is known that there exist examples of generalized hyperfocused arcs which are not hyperfocused [6]. However, although much more is known about hyperfocused arcs, there are still many open problems concerning them [1, 5, 6].

The study of these arcs is motivated by a relevant application to cryptography in connection with constructions of efficient secret sharing schemes [7, 8]. Interestingly, our problem is also related to the (strong) cylinder conjecture [2].

References:

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