

UNIVERSITEIT • STELLENBOSCH • UNIVERSITY

On the analysis of refinable functions with
respect to mask factorisation, regularity and
corresponding subdivision convergence

by

Wouter de Vos de Wet



*Dissertation presented at Stellenbosch University
for the degree of*

Doctor of Philosophy in Mathematics

Promoter: Prof. J.M. de Villiers

December 2007

Declaration

I, the undersigned, hereby declare that the work contained in this dissertation is my own original work and that I have not previously in its entirety or in part submitted it at any university for a degree.

Signature:

W.d.V. de Wet

Date:

Summary

We study refinable functions where the dilation factor is not always assumed to be 2. In our investigation, the role of convolutions and refinable step functions is emphasized as a framework for understanding various previously published results. Of particular importance is a class of polynomial factors, which was first introduced for dilation factor 2 by Berg and Plonka and which we generalise to general integer dilation factors.

We obtain results on the existence of refinable functions corresponding to certain reduced masks which generalise similar results for dilation factor 2, where our proofs do not rely on Fourier methods as those in the existing literature do.

We also consider subdivision for general integer dilation factors. In this regard, we extend previous results of De Villiers on refinable function existence and subdivision convergence in the case of positive masks from dilation factor 2 to general integer dilation factors. We also obtain results on the preservation of subdivision convergence, as well as on the convergence rate of the subdivision algorithm, when generalised Berg-Plonka polynomial factors are added to the mask symbol.

We obtain sufficient conditions for the occurrence of polynomial sections in refinable functions and construct families of related refinable functions.

We also obtain results on the regularity of a refinable function in terms of the mask symbol factorisation. In this regard, we obtain much more general sufficient conditions than those previously published, while for dilation factor 2, we obtain a characterisation of refinable functions with a given number of continuous derivatives.

We also study the phenomenon of subsequence convergence in subdivision, which explains some of the behaviour that we observed in non-convergent subdivision processes during numerical experimentation. Here we are able to establish different sets of sufficient conditions for this to occur, with some results similar to standard subdivision convergence, e.g. that the limit function is refinable. These results provide generalisations of the corresponding results for subdivision, since subsequence convergence is a generalisation of subdivision convergence. The nature of this phenomenon is such that the standard subdivision algorithm can be extended in a trivial manner to allow it to work in instances where it previously failed.

Lastly, we show how, for masks of length 3, explicit formulas for refinable functions can be used to calculate the exact values of the refinable function at rational points.

Various examples with accompanying figures are given throughout the text to illustrate our results.

Samevatting

Ons bestudeer verfynbare funksies waar die skaalfaktor nie noodwendig 2 is nie. In ons ondersoek word die rol van konvolusies en verfynbare trapfunksies beklemtoon as 'n raamwerk om verskeie vorige resultate te verstaan. Van besondere belang is 'n klas polinoomfaktore wat deur Berg en Plonka bekendgestel is vir die skaalfaktor 2 en wat ons na algemene heeltallige skaalfaktore uitbrei.

Ons verkry resultate aangaande die bestaan van verfynbare funksies wat ooreenstem met sekere verminderde maskers, wat soortgelyke resultate vir skaalfaktor 2 veralgemeen, waar ons bewyse nie soos die voriges op Fourier-metodes staatmaak nie.

Ons beskou ook subdivisie vir algemene heeltallige skaalfaktore. In hierdie verband veralgemeen ons vorige resultate van De Villiers aangaande die bestaan van verfynbare funksies en subdivisie-konvergensie vir positiewe maskers van skaalfaktor 2 na 'n algemene heeltallige skaalfaktor. Ons verkry ook resultate oor die behoud van subdivisie-konvergensie, asook oor die konvergensie-tempo van subdivisie, wanneer veralgemeende Berg-Plonka faktore by die maskersimbool gevoeg word.

Ons verkry ook voldoende voorwaardes vir die voorkoms van polinoomstukke in verfynbare funksies en konstrueer families van verwante verfynbare funksies.

Verder verkry ons resultate oor die gladheid van 'n verfynbare funksie in terme van die maskersimbool-faktorisasie. In hierdie verband verkry ons baie meer algemene voldoende voorwaardes as wat te vore gepubliseer is, terwyl ons vir skaalfaktor 2 'n karakterisering verkry van verfynbare funksies met 'n gegewe aantal kontinue afgeleides.

Ons bestudeer ook die verskynsel van subry-konvergensie in subdivisie, wat sommige van die gedrag verklaar wat ons tydens numeriese eksperimentasie in nie-konvergente subdivisie-prosesse waargeneem het. Ons bepaal verskillende stelle voldoende voorwaardes waarvoor subry-konvergensie voorkom en verkry sommige resultate soortgelyk aan gewone subdivisie-konvergensie, bv. dat die limietfunksie verfynbaar is. Hierdie resultate veralgemeen die ooreenstemmende resultate vir subdivisie, aangesien subry-konvergensie 'n veralgemening van subdivisie-konvergensie is. Die aard van hierdie verskynsel laat ons toe om die gewone subdivisie-algoritme op 'n triviale manier aan te pas sodat dit werk vir gevalle waar dit te vore nie gewerk het nie.

Ten slotte wys ons, vir maskers met lengte 3, hoe eksplisiete formules vir verfynbare funksies gebruik kan word om die presiese waardes van die funksie by rasionale punte te bereken.

Verskeie voorbeelde met gepaardgaande grafika word deurgaans gegee om ons resultate toe te lig.

Acknowledgements

I would like to thank my promoter, Prof. Johan de Villiers, for his guidance and support throughout the duration of this study. He has provided me with many valuable insights and references and of course suggested most of the research topics considered in this work. The fact that this manuscript is presented for a Ph.D., rather than for an M.Sc., is largely due to his persistence and enthusiasm. Also, his meticulous proof reading of draft versions of this manuscript has been extremely helpful.

I also thank Prof. Thomas Sauer of the University of Giessen for alerting me to the presence of and providing me with references for various papers on multivariate subdivision, as well as for his prompt and helpful responses to various questions on subdivision which I sent to him via email.

Thanks is also extended to Rinske van der Bijl for reading a draft version of this manuscript and for pointing out various errors and unclarities in those versions.

I would also like to acknowledge here the generous financial support of the Harry Crossley foundation for the last two years of my study, as well as the financial support provided to me by the University of Stellenbosch. The department of Mathematical Sciences has also been very kind to me in many diverse ways, for which I thank them.

I also thank my family for their support and encouragement during the course of these studies.

Lastly and most importantly, I thank the Lord Jesus Christ, the risen Ruler and only Saviour for all mankind, for His amazing grace in my life.

Contents

Summary	ii
Samevatting	iii
Acknowledgements	iv
Contents	v
List of Figures	vii
List of Abbreviations	viii
List of Symbols	ix
1 Introduction	1
1.1 Notation	2
1.2 Polynomial operators	3
1.3 Preliminary results	5
1.3.1 Subdivision	10
1.3.2 The cascade algorithm	13
1.4 Our contribution	14
1.4.1 Remarks on non-integer dilation factors	15
2 A step function approach to the analysis of refinable functions	16
2.1 Convolutions and inverse convolutions	16
2.2 Step functions and special mask symbol factors	18
2.3 Equivalency of reduced masks	24
2.4 Existence and subdivision convergence for positive masks	29
2.5 Preservation of subdivision convergence	35
2.6 Polynomial sections in refinable functions	40
2.6.1 The operator L_m	41
2.6.2 Constant sections : “Table Mountlets”	42
2.6.3 Polynomial sections of any degree	46
3 Regularity analysis in terms of the mask symbol factorisation	51
3.1 Sufficient conditions for regularity	51
3.2 Necessary conditions for regularity: dilation factor 2	54

<i>Contents</i>	vi
4 Subsequence convergence in subdivision	61
4.1 Applications	74
4.2 Nested sets of refinement masks	77
5 A note on explicit formulas for masks of length 3	79
Bibliography	84

List of Figures

2.1	Derivation of 2-GBP factors up to level 3	21
2.2	Graphs of $\sigma_{\mathbb{C}}$ and $\sigma_{\mathbb{R}}$	24
2.3	The shifted De Rham function ϕ^{DR} of Example 2.28.	35
2.4	The interesting refinable function ψ of Example 2.29.	36
2.5	The plots of $m\phi_m$, $m = 1, 2, 3, 4, 5$ for Example 2.37.	44
2.6	The plots of $m\phi_m$, $m = 1, 2, 3, 4, 5$ for Example 2.38.	45
2.7	Linear polynomial sections for dilation factor 3	49
2.8	A family of refinable functions with growing linear sections	50
3.1	Plots of (a) $\tilde{\phi}$; and (b) $\tilde{\phi}'$ in Example 3.11.	54
3.2	Illustration of inverse convolution by a step function	57
4.1	Plots of (a) $S_a^4\delta$; and (b) $(S_a^8\delta)_{2.+e}$, $e = 0, 1$, in Example 4.1.	61
4.2	Plots of (a) $S_a^4\delta$; and (b) $(S_a^8\delta)_{3.+e}$, $e = 0, 1, 2$ in Example 4.9.	69
4.3	Illustration of 4-subsequence convergence for a complex-valued mask.	73
4.4	The complex-valued refinable function $\phi_{\mathbb{C}}$ of Example 4.11.	74
4.5	Using subsequence convergence for decorative effects	76
4.6	Effect of translation on the appearance of the decoration	76
4.7	Influence of the parameter β on the appearance of the decoration	77
5.1	Illustration of the explicit formula for a mask of length 3.	83

List of Abbreviations

GBP	generalised Berg-Plonka
LLS	Lee-Lawton-Shen
<i>p.p.</i>	<i>presque partout</i> (almost everywhere)
SCC	subsequence convergence constants

List of Symbols

Note. Whenever an entry in the list contains the words “the corresponding”, the correspondence is with the entry immediately preceding it, with multiple symbols corresponding in the specific order in which they are specified.

α	real dilation factor
p	integer dilator factor
$\phi, \tilde{\phi}, \psi$	refinable functions
A, \tilde{A}, B	the corresponding refinement mask symbols
a, \tilde{a}, b	the corresponding refinement masks
$c^{(r)}, \tilde{c}^{(r)}, d^{(r)}$	the r 'th iterates of the corresponding subdivision schemes
σ	refinable step function
P	LLS or GBP factor
$P_{\mathbb{C}}, P_{\mathbb{R}}$	non-trivial 2-GBP factors
$\sigma_{\mathbb{C}}, \sigma_{\mathbb{R}}$	the corresponding refinable step functions
ϕ^{DR}	shifted De Rham function
A_m	refinement mask symbol in a family parametrised by m
ϕ_m	the corresponding refinable function
i	the imaginary unit $\sqrt{-1}$
δ	the Kronecker delta sequence
χ	the characteristic function of the interval $[0, 1)$
E_m	elementary polynomial of degree m
N_l	the cardinal B -spline of order $l \in \mathbb{N}$
$a^{l,p}$	the p -refinement mask corresponding to N_l
$T_{a,p}, T_a$	cascade operator with mask a (and dilation factor p)
$S_{a,p}, S_a$	subdivision operator with mask a (and dilation factor p)
π_l^I	the set of functions which coincide with a polynomial of degree l on the interval I
$M(\mathbb{R})$	the set of complex-valued functions defined on \mathbb{R}
$M_0(\mathbb{R})$	the set of functions in $M(\mathbb{R})$ with compact support
$M_0^+(\mathbb{R})$	the set $\{f \in M_0(\mathbb{R}) : f(x) = 0, x < 0\}$
$M(\mathbb{Z})$	the set of complex-valued bi-infinite sequences
$M_0(\mathbb{Z})$	the set of sequences in $M(\mathbb{Z})$ with finite support
$M_0^+(\mathbb{Z})$	the set $\{a \in M_0(\mathbb{Z}) : \min \{j : a_j \neq 0\} = 0\}$

Chapter 1

Introduction

In this thesis, we consider refinable functions and some related concepts. A non-trivial function ϕ is called α -refinable with mask a if it satisfies the refinement equation

$$\phi = \sum_{j \in \mathbb{Z}} a_j \phi(\alpha \cdot -j), \quad (1.1)$$

for some real or complex sequence a and some constant $\alpha \in (1, \infty)$.

Refinable functions play an important role in the theory of wavelets, which in turn are used in various fields of application, e.g. signal processing and image processing (see e.g. [11; 17; 46] and references therein). Refinable functions also occur in the study of computer aided geometric design (CAGD), specifically in connection with subdivision processes and splines (see e.g. [10; 24; 34] and references therein).

Although these are the most well-known fields of use of refinable functions, they have also arisen in some other contexts. In [23] some results are given on the refinement equation

$$\phi = \frac{\alpha}{4} \phi(\alpha \cdot) + \frac{\alpha}{2} \phi(\alpha \cdot -1) + \frac{\alpha}{4} \phi(\alpha \cdot -2), \quad \alpha \in (1, \infty), \quad (1.2)$$

which arises in the context of spatially chaotic structures in amorphous glassy fluids in physics (see [23] and references therein for details). Overviews of results and open problems related to (1.2) can be found in the survey papers [1; 25], which also contain additional references for it.

Certain infinite Bernoulli convolutions also satisfy a refinement equation of the type (1.1) or a generalisation thereof (see [8; 18], the survey paper [38] and references therein).

Some of the questions one could ask regarding (1.1) are the following:

1. For what combinations of α and a does a function (or distribution) ϕ exist that satisfies (1.1) and in what class (e.g. $L^1(\mathbb{R})$, $C(\mathbb{R})$, $L^\infty(\mathbb{R})$) is ϕ ?
2. How regular (smooth) is ϕ ? For instance, how many continuous derivatives does ϕ possess? In CAGD it is desirable in many applications to have a high order of smoothness, so that the design will appear smooth.
3. If we do not have a simple closed formula for ϕ , what algorithms can be used to approximate it?

4. How large is the support of ϕ ? In both wavelet and CAGD applications, it is often desirable to have as small a support of ϕ as possible (subject to given constraints), because this improves the locality of the corresponding operators. For instance, in the design of surfaces, one usually wants the effect of a small perturbation to be sufficiently localised.

Although these questions have been intensively studied, especially since the appearance of the classical papers by Daubechies & Lagarias [18; 19], many open questions still remain. In this work, we provide some further partial answers by giving extensions of previously published results and some new results that address these issues.

Classes of the dilation factor

The value of the dilation factor α plays a critical role in the analysis of (1.1). One of the reasons why this is important, is that it affects the frequency localisation in a wavelet scheme, as explained in [12] and [17: Chapter 10].

The case $\alpha = 2$ has been especially intensively studied in the last two decades. The case $\alpha = p \in \mathbb{N}$, $p \geq 2$, which we refer to as the (general) integer dilation case, has also received some attention (see e.g. [2; 5; 6; 27; 29; 32; 41; 42] and references therein) and is also covered by more general papers on multivariate subdivision and/or wavelets (e.g. [13; 16; 14; 31; 28] and references therein). Some papers that deal with general $\alpha \in (1, \infty)$ include [8; 1; 15; 18; 22; 23; 36] and the references therein.

Throughout this work, the class of the dilation factor will be prominent, as many results are only derived for certain classes of the dilation factor. Further remarks on the differences between the integer and non-integer case appear in Section 1.4.1.

1.1 Notation

Before proceeding, we shall introduce various pieces of notation that we will find useful.

We start with the following note on our use of the placeholder notation: whenever we employ the placeholder symbol \cdot in an equation in the context of functions, e.g. as in (1.1), we mean that the equation holds for all real values of the placeholder argument, even when working with functions in $L^1(\mathbb{R})$. (We only consider functions that are defined everywhere on the real line.) Specifically, we are interested in functions ϕ which satisfy (1.1) for all real x , and not just “almost everywhere”.

Throughout this work, δ shall denote the Kronecker delta sequence given by

$$\delta_j = \begin{cases} 1 & \text{if } j = 0; \\ 0 & \text{if } j \in \mathbb{Z} \setminus \{0\}, \end{cases} \quad (1.3)$$

while χ shall denote the characteristic function of $[0, 1)$.

We write \sum_j for $\sum_{j \in \mathbb{Z}}$ and \sup_j for $\sup_{j \in \mathbb{Z}}$. We let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ denote the set of non-negative integers. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the largest integer $\leq x$ and $\lceil x \rceil$ denotes the smallest integer $\geq x$.

For $m \in \mathbb{N}$, \mathbb{Z}_m denotes the set $\{0, \dots, m-1\}$. We shall use the facts that

$$\mathbb{Z}_{mn} = \{jm + l : j \in \mathbb{Z}_n, l \in \mathbb{Z}_m\} \quad \text{and} \quad \mathbb{Z} = \{jm + l : j \in \mathbb{Z}, l \in \mathbb{Z}_m\},$$

which allows us to partition sums into appropriate double summations and vice versa.

For $j, m \in \mathbb{Z}$, $j \bmod m$ denotes the remainder in \mathbb{Z}_m when j is divided by m , that is

$$j \bmod m = j - m \left\lfloor \frac{j}{m} \right\rfloor, \quad j, m \in \mathbb{Z}.$$

We let the set of functions from \mathbb{R} to \mathbb{C} be denoted by $M(\mathbb{R})$. The support of a function f is the closure of the set $\{x \in \mathbb{R} : f(x) \neq 0\}$ and is denoted by $\text{supp}(f)$. The set of functions in $M(\mathbb{R})$ with compact support is denoted by $M_0(\mathbb{R})$, while $M_+(\mathbb{R})$ denotes the set of functions in $M(\mathbb{R})$ that vanish left of the origin. We set $M_0^+(\mathbb{R}) = M_0(\mathbb{R}) \cap M_+(\mathbb{R})$, $C_+^m(\mathbb{R}) = C^m(\mathbb{R}) \cap M_+(\mathbb{R})$ and $C_+(\mathbb{R}) = C(\mathbb{R}) \cap M_+(\mathbb{R})$. The set of functions that are piecewise continuous on the real line is denoted by $C^{-1}(\mathbb{R})$.

$M(\mathbb{Z})$ denotes the set of bi-infinite complex sequences. For any $a \in M(\mathbb{Z})$ we define $\text{supp}(a) = \{j \in \mathbb{Z} : a_j \neq 0\}$, called the support of a . The sequences in $M(\mathbb{Z})$ of finite support is denoted by $M_0(\mathbb{Z})$. For $a \in M_0(\mathbb{Z}) \setminus \{0\}$, we define $\downarrow a \downarrow = \min\{j \in \mathbb{Z} : a_j \neq 0\}$ and $\uparrow a \uparrow = \max\{j \in \mathbb{Z} : a_j \neq 0\}$, which we respectively call the lower and upper support bounds of a . We let $M_0^+(\mathbb{Z})$ denote the set $\{a \in M_0(\mathbb{Z}) \setminus \{0\} : \downarrow a \downarrow = 0\}$. Δ denotes the backward difference operator defined by $(\Delta c)_j = c_j - c_{j-1}$, $j \in \mathbb{Z}$, $c \in M(\mathbb{Z})$ and by $\Delta^\infty(\mathbb{Z})$ we mean the subspace $\{c \in M(\mathbb{Z}) : \Delta c \in l^\infty(\mathbb{Z})\}$ of $M(\mathbb{Z})$.

$C_u(\mathbb{R})$ denotes the Banach space of bounded functions on \mathbb{R} with respect to the norm $\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$. We shall also use the notation $\|\cdot\|_\infty$ for the norm of $l^\infty(\mathbb{Z})$ —the meaning will be clear from the context.

1.2 Polynomial operators

We shall sometimes exploit the one-to-one correspondence between Laurent polynomials and compactly supported sequences in our proofs. To this end, we define the following operators.

Definition 1.1. For a sequence $p = (p_j : j \in \mathbb{Z}) \in M_0(\mathbb{Z})$, define the Laurent polynomial $\text{Lpol}(p)$ by

$$(\text{Lpol}(p))(z) = \sum_j p_j z^j, \quad z \in \mathbb{C} \setminus \{0\}. \quad (1.4)$$

Definition 1.2. For a Laurent polynomial P defined by $P(z) = \sum_j p_j z^j$, $z \in \mathbb{C} \setminus \{0\}$, define the sequence $[P] \in M_0(\mathbb{Z})$ by

$$[P]_j = p_j, \quad j \in \mathbb{Z}, \quad (1.5)$$

so that $[P]_j$ equals the coefficient of z^j in P .

Observe that $\text{Lpol}(p)$ is actually a polynomial if $p \in M_0^+(\mathbb{Z})$, while it holds, for non-zero Laurent polynomials P and Q , that

$$\uparrow[PQ]\uparrow = \uparrow[P]\uparrow + \uparrow[Q]\uparrow \quad \text{and} \quad \downarrow[PQ]\downarrow = \downarrow[P]\downarrow + \downarrow[Q]\downarrow. \quad (1.6)$$

We will also make frequent use of the following operator and its properties.

Definition 1.3. For a Laurent polynomial P and $m \in \mathbb{N}$, define $P^{(m)}$ to be the Laurent polynomial given by

$$P^{(m)}(z) = P(z^m), \quad z \in \mathbb{C} \setminus \{0\}.$$

Note that $P^{(1)} = P$. Also, provided $P \neq 0$, we have

$$\uparrow[P^{(m)}]\uparrow = m \uparrow[P]\uparrow \quad \text{and} \quad \downarrow[P^{(m)}]\downarrow = m \downarrow[P]\downarrow. \quad (1.7)$$

The next lemma states some further properties of this operator.

Lemma 1.4. Suppose P and Q are Laurent polynomials and $m, n \in \mathbb{N}$. Then the following identities hold:

$$(P^{(m)})^{(n)} = (P^{(n)})^{(m)} = P^{(mn)} \quad (1.8a)$$

$$(PQ)^{(m)} = P^{(m)}Q^{(m)} \quad (1.8b)$$

$$[PQ^{(m)}]_j = \sum_k [P]_{j-km} [Q]_k, \quad j \in \mathbb{Z}. \quad (1.8c)$$

Proof. For $z \in \mathbb{C} \setminus \{0\}$ we have

$$(P^{(m)})^{(n)}(z) = P^{(m)}(z^n) = P^{(mn)}(z) = P^{(n)}(z^m) = (P^{(n)})^{(m)}(z)$$

and

$$(PQ)^{(m)}(z) = (PQ)(z^m) = P(z^m)Q(z^m) = (P^{(m)}Q^{(m)})(z).$$

Noting that, for $k \in \mathbb{Z}$, $l \in \mathbb{Z}_m$, the identity $[Q^{(m)}]_{km+l} = \delta_l [Q]_k$ holds, we also have

$$[PQ^{(m)}]_j = \sum_k [P]_{j-k} [Q^{(m)}]_k = \sum_k \sum_{l=0}^{m-1} [P]_{j-km-l} [Q^{(m)}]_{km+l} = \sum_k [P]_{j-km} [Q]_k, \quad j \in \mathbb{Z}.$$

□

1.3 Preliminary results

Definition 1.5. We say that (A, ϕ) is an α -refinement pair if, for some constant $\alpha \in (1, \infty)$, sequence $a \in M_0(\mathbb{Z})$ and function $\phi \in L^1(\mathbb{R}) \setminus \{0\}$, the refinement equation (1.1) is satisfied and $A = \frac{1}{\alpha} \text{Lpol}(a)$.

We call α the dilation factor, a the (refinement) mask and A the (refinement) mask symbol. If (A, ϕ) is an α -refinement pair, the function ϕ is said to be α -refinable, a is called the corresponding mask and A the corresponding mask symbol, while we also sometimes say that ϕ corresponds to a or A .

Note 1.6. Throughout this work, we will use the conventions $A = \frac{1}{\alpha} \text{Lpol}(a)$, $\tilde{A} = \frac{1}{\alpha} \text{Lpol}(\tilde{a})$ and $B = \frac{1}{\alpha} \text{Lpol}(b)$. In other cases, the relationship (if any) between lowercase and uppercase Roman alphabetic symbols will be stated explicitly and must not be assumed.

Definition 1.7. A mask $a \in M_0(\mathbb{Z})$ is called *non-negative* if $a_j \geq 0$, $j \in \mathbb{Z}$, while a is said to be *positive* if it satisfies the condition

$$a_j > 0, \quad j \in \{\downarrow a \downarrow, \dots, \uparrow a \uparrow\}. \quad (1.9)$$

Equation (1.1) is also called a two-scale difference equation (e.g. in [18]) or dilation equation (e.g. in [36]), while some authors call α the scale factor.

From [18: Theorem 2.1, Corollary 2.2 & Theorem 3.1] we have the following necessary conditions for the existence of an α -refinement pair.

Theorem 1.8. *Suppose (A, ϕ) is an α -refinement pair. Then the following assertions hold true:*

- (a) *If (A, ψ) is an α -refinement pair, then $\psi = K\phi$ for some real constant K .*
- (b) *$A(1) = \alpha^m$ for some $m \in \mathbb{Z}_+$.*
- (c) *If, in (b), we have $m \geq 1$, then there exists a function $\psi \in L^1(\mathbb{R})$ such that $(\alpha^{-m}A, \psi)$ is an α -refinement pair, where, with a proper choice of scale, we have $\frac{d^m}{dx^m}\psi = \phi$ p.p.*
- (d) *ϕ is finitely supported, with*

$$\phi(x) = 0, \quad x \notin \left[\frac{\downarrow a \downarrow}{\alpha - 1}, \frac{\uparrow a \uparrow}{\alpha - 1} \right]. \quad (1.10)$$

By virtue of points (b) and (c) above, we shall henceforth assume, without essential loss of generality, that $A(1) = 1$, i.e. $\sum_j a_j = \alpha$. In this case we have, again from [18], that $\int_{-\infty}^{\infty} \phi(x) dx \neq 0$. We call a refinable function ϕ a normalised refinable function if $\int_{-\infty}^{\infty} \phi(x) dx = 1$. In this case we call (A, ϕ) a normalised α -refinement pair.

It can easily be checked that if (1.1) holds, then it follows with $\psi = \phi(\cdot + \frac{\downarrow a \downarrow}{\alpha - 1})$ and $b_j = a_{j + \downarrow a \downarrow}$, $j \in \mathbb{Z}$, that $\psi = \sum_{j=0}^{\uparrow a \uparrow - \downarrow a \downarrow} b_j \psi(\alpha \cdot - j)$. Thus we shall henceforth assume that

$\downarrow a \downarrow = 0$, $\uparrow a \uparrow = N$ with $N \in \mathbb{N}$, so that the mask symbol A is a polynomial of degree N with $A(0) \neq 0$.

Our next result shows that, for a given dilation factor, a function can be refinable with at most one mask.

Lemma 1.9. *If both (A, ϕ) and (B, ϕ) are α -refinement pairs, then $A = B$.*

Proof. Since $\phi \neq 0$, whereas (1.10) and the assumption $\downarrow a \downarrow = 0$ yields $\phi(x) = 0$, $x < 0$, the real number $I = \inf \{x \in \mathbb{R} : \phi(x) \neq 0\}$ exists and satisfies $I \geq 0$. Now suppose that $I > 0$ and choose $\varepsilon = \min \{1, (\alpha - 1)I\}$. Then $\varepsilon > 0$ and from the definition of I there exists a real number $x_\varepsilon \in [I, I + \varepsilon)$ such that $\phi(x_\varepsilon) \neq 0$. Since $j \in \mathbb{N}$ implies $x_\varepsilon - j < I$, while $\downarrow a \downarrow = 0$, also implying $a_0 \neq 0$, we now find from (1.1) that

$$\phi\left(\frac{x_\varepsilon}{\alpha}\right) = a_0 \phi(x_\varepsilon) \neq 0,$$

which yields a contradiction, since $\frac{x_\varepsilon}{\alpha} < I$. Thus we conclude $I = 0$.

Since both (A, ϕ) and (B, ϕ) are α -refinement pairs, we obtain

$$\sum_j (a_j - b_j) \phi(\alpha \cdot -j) = \sum_j a_j \phi(\alpha \cdot -j) - \sum_j b_j \phi(\alpha \cdot -j) = \phi - \phi = 0.$$

Now suppose $A \neq B$ and let $k = \min \{j \in \mathbb{Z}_+ : a_j \neq b_j\}$. Then $x < \frac{k+1}{\alpha}$ implies that $\phi(\alpha x - j) = 0$ for $j \geq k + 1$, so that we obtain

$$(a_k - b_k) \phi(\alpha x - k) = 0, \quad x < \frac{k+1}{\alpha}.$$

Since $a_k \neq b_k$, this means that $I \geq 1$, which contradicts $I = 0$. Thus we conclude that $A = B$. \square

The following result, which shows that stretching a function by an integer factor preserves refinability, will prove useful later.

Theorem 1.10. *Suppose (A, ϕ) is an α -refinement pair.*

- (a) *Then for $q \in \mathbb{N}$, $(A^{(q)}, \frac{1}{q} \phi(\frac{\cdot}{q}))$ is an α -refinement pair.*
- (b) *Conversely, if there exists a polynomial B and $q \in \mathbb{N}$ such that $A = B^{(q)}$, then $(B, q\phi(q \cdot))$ is an α -refinement pair.*

Proof. To prove (a), let $A^{(q)} = \alpha [A^{(q)}]$ and observe that $a_{qj+i}^{(q)} = \delta_i a_j$, $j \in \mathbb{Z}$, $i \in \mathbb{Z}_q$.

Let $\psi = \frac{1}{q}\phi\left(\frac{\cdot}{q}\right)$, so that $\phi = q\psi(q\cdot)$. Then we obtain

$$\begin{aligned}\psi &= \frac{1}{q} \sum_j a_j \phi\left(\frac{\alpha}{q} \cdot -j\right) = \sum_j a_j \psi(\alpha \cdot -qj) = \sum_j \sum_{i=0}^{q-1} \delta_i a_j \psi(\alpha \cdot -qj - i) \\ &= \sum_j \sum_{i=0}^{q-1} a_{qj+i}^{(q)} \psi(\alpha \cdot -qj - i) = \sum_j a_j^{(q)} \psi(\alpha \cdot -j),\end{aligned}$$

so that $(A^{(q)}, \psi) = \left(A^{(q)}, \frac{1}{q}\phi\left(\frac{\cdot}{q}\right)\right)$ is an α -refinement pair.

To prove the converse statement (b), assume the existence of B and q as stated. Let $\psi = q\phi(q\cdot)$, so that $\phi = \frac{1}{q}\psi\left(\frac{\cdot}{q}\right)$. Since $a_{qj+i} = \delta_i b_j$ for $j \in \mathbb{Z}$ and $i \in \mathbb{Z}_q$, we get

$$\begin{aligned}\psi &= q \sum_j a_j \phi(\alpha q \cdot -j) = q \sum_j \sum_{i=0}^{q-1} a_{qj+i} \phi(\alpha q \cdot -qj - i) \\ &= \sum_j b_j q \phi(q(\alpha \cdot -j)) = \sum_j b_j \psi(\alpha \cdot -j).\end{aligned}$$

Hence $(B, q\phi(q\cdot))$ is an α -refinement pair. \square

Remark 1.11. Note that $\|f\|_1 = \|Kf(K\cdot)\|_1$ for any function $f \in L^1(\mathbb{R})$ and any constant $K \in \mathbb{C} \setminus \{0\}$. Specifically, in Theorem 1.10, if (A, ϕ) is normalised, then so is $\left(A^{(q)}, \frac{1}{q}\phi\left(\frac{\cdot}{q}\right)\right)$ and $(B, q\phi(q\cdot))$.

In the analysis of refinable functions with integer dilation factor, an important role is played by the so called *sum rules*

$$\sum_j a_{pj+l} = 1, \quad l \in \mathbb{Z}_p, \quad (1.11)$$

where $p \in \mathbb{Z}$, $p \geq 2$ is the dilation factor and a the mask. The next result shows a first important application of the sum rules. Our proof uses analogous methods to those used in [21], where only the case $p = 2$ was considered.

Lemma 1.12. *For $p \in \mathbb{Z}$, $p \geq 2$, if (A, ϕ) is a p -refinement pair such that ϕ is continuous and the mask a satisfies (1.11), then ϕ has the property*

$$\sum_j \phi(x - j) = \int_{-\infty}^{\infty} \phi(s) ds, \quad x \in \mathbb{R}. \quad (1.12)$$

Proof. By repeated application of (1.1) we obtain, for $r \in \mathbb{N}$,

$$\sum_j \phi\left(\frac{j}{p^r}\right) = \sum_j \sum_{k_1} a_{k_1} \phi\left(\frac{j}{p^{r-1}} - k_1\right)$$

$$\begin{aligned}
&= \sum_j \sum_{k_1} a_{k_1} \sum_{k_2} a_{k_2} \phi \left(\frac{j}{p^{r-2}} - pk_1 - k_2 \right) \\
&= \dots \\
&= \sum_j \sum_{k_1} a_{k_1} \sum_{k_2} a_{k_2} \dots \sum_{k_r} a_{k_r} \phi \left(j - p^r k_1 - p^{r-1} k_2 - \dots - k_r \right) \\
&= \sum_{k_1} a_{k_1} \sum_{k_2} a_{k_2} \dots \sum_{k_r} a_{k_r} \sum_j \phi \left(j - p^r k_1 - p^{r-1} k_2 - \dots - k_r \right) \\
&= \sum_{k_1} a_{k_1} \sum_{k_2} a_{k_2} \dots \sum_{k_r} a_{k_r} \left(\sum_j \phi(j) \right) \\
&= p^r \left(\sum_j \phi(j) \right),
\end{aligned}$$

after recalling also that our assumption $A(1) = 1$ is equivalent to the mask condition $\sum_j a_j = p$. We thus obtain

$$\int_{-\infty}^{\infty} \phi(s) ds = \lim_{r \rightarrow \infty} \frac{1}{p^r} \sum_j \phi \left(\frac{j}{p^r} \right) = \sum_j \phi(j). \quad (1.13)$$

By repeated use of (1.1) and (1.11), we also have, for $j \in \mathbb{Z}$ and $r \in \mathbb{Z}_+$,

$$\begin{aligned}
\sum_k \phi \left(\frac{j}{p^r} - k \right) &= \sum_k \sum_l a_l \phi \left(\frac{j}{p^{r-1}} - pk - l \right) \\
&= \sum_k \sum_l a_{l-pk} \phi \left(\frac{j}{p^{r-1}} - l \right) \\
&= \sum_l \left(\sum_k a_{l-pk} \right) \phi \left(\frac{j}{p^{r-1}} - l \right) \\
&= \sum_l \phi \left(\frac{j}{p^{r-1}} - l \right) \\
&= \dots \\
&= \sum_l \phi(j - l) = \sum_l \phi(l). \quad (1.14)
\end{aligned}$$

Since the set $\left\{ \frac{j}{p^r} : j \in \mathbb{Z}, r \in \mathbb{Z}_+ \right\}$ is dense in \mathbb{R} , while the continuity of ϕ implies that $\sum_j \phi(\cdot - j)$ is a continuous function, (1.13) and (1.14) now yield the required result (1.12). \square

Remark 1.13. In Lemma 1.12, if (A, ϕ) is a normalised α -refinement pair, we have

$$\sum_j \phi(x - j) = 1. \quad (1.15)$$

A function ϕ that satisfies (1.15) is said to form a *partition of unity*.

Another concept that occurs often in the analysis of refinable functions, is that of *stability*. A refinable function ϕ is said to be stable, or to have stable integer shifts, if there exist positive constants C_1 and C_2 such that

$$C_1 \|c\|_\infty \leq \left\| \sum_j c_j \phi(\cdot - j) \right\|_\infty \leq C_2 \|c\|_\infty, \quad c \in l^\infty(\mathbb{Z}). \quad (1.16)$$

Elementary Polynomials

The following class of polynomials will prove useful in various proofs. Define, for a given $m \in \mathbb{Z}_+$, the polynomial E_m by

$$E_m(z) = \frac{1}{m+1} \sum_{j=0}^m z^j = \frac{1}{m+1} \prod_{j=1}^m (z - e^{2j\pi i/(m+1)}), \quad z \in \mathbb{C}, \quad (1.17)$$

where the second equality follows from the fact that the first equality yields

$$E_m(z) = \frac{1 - z^{m+1}}{(m+1)(1-z)}, \quad z \in \mathbb{C} \setminus \{1\}, \quad m \in \mathbb{Z}_+. \quad (1.18)$$

Note that $E_m(1) = 1$, $m \in \mathbb{N}$ and that E_0 is the constant polynomial 1. It is also easy to verify that the following identities hold for $p \in \mathbb{N}$:

$$E_{m-1}^{(p)} E_{p-1} = E_{pm-1}, \quad m \in \mathbb{N}; \quad (1.19a)$$

$$\prod_{j=0}^{r-1} E_{p-1}^{(p^j)} = E_{p^r-1}, \quad r \in \mathbb{N}; \quad (1.19b)$$

$$E_{p^k-1} E_{p^{r-k}-1}^{(p^k)} = E_{p^r-1}, \quad k \in \{0, 1, \dots, r\}, \quad r \in \mathbb{N}. \quad (1.19c)$$

Cardinal B -splines

Well-known examples of refinable functions are provided by the so-called *cardinal B -splines*. Define the family $\{N_l : l \in \mathbb{N}\}$ recursively by

$$N_1 = \chi \quad \text{and} \quad N_{l+1} = N_l * N_1, \quad l \in \mathbb{N}. \quad (1.20)$$

Then N_l is the cardinal B -spline of order l . As indicated by the name, N_l is a spline function: for every $j \in \mathbb{Z}$, there is some polynomial P of degree at most $l-1$ so that N_l coincides with P on the interval $[j, j+1)$, while $N_l \in C^{l-2}(\mathbb{R})$. Furthermore it is known that, for any $p \in \mathbb{Z}$, $p \geq 2$ and $l \in \mathbb{N}$, $((E_{p-1})^l, N_l)$ is a p -refinement pair (see e.g. [32]). This result will be shown to be a special case of Theorem 2.1 in Chapter 2. Henceforth we let the p -refinement

mask corresponding to N_l be denoted by

$$a^{l,p} = p \left[(E_{p-1})^l \right], \quad l \in \mathbb{N}, \quad p \in \mathbb{Z}, \quad p \geq 2. \quad (1.21)$$

The following properties also hold for $l \in \mathbb{N}$ (see e.g. [11: Chapter 4]):

$$\text{supp}(N_l) = [0, l]; \quad (1.22a)$$

$$N_l(x) > 0, \quad x \in (0, l); \quad (1.22b)$$

$$\sum_j N_l(x - j) = 1, \quad x \in \mathbb{R}. \quad (1.22c)$$

Note in particular that N_2 is the hat function defined by

$$N_2(x) = \max\{0, 1 - |1 - x|\}, \quad x \in \mathbb{R}. \quad (1.23)$$

1.3.1 Subdivision

The classic monograph on subdivision for the case $p = 2$ is the one by Cavaretta, Dahmen & Micchelli [10]. Various extensions, especially to the multivariate case with general integer dilation matrix, have been studied (see e.g. [14; 30; 31; 40] and references therein). The subdivision operator that we proceed to define is the univariate subdivision operator with general integer dilation factor.

For a sequence $a \in M_0(\mathbb{Z})$, called the subdivision mask, and dilation factor $p \in \mathbb{Z}$, $p \geq 2$, we define the subdivision operator $S_{a,p} : M(\mathbb{Z}) \rightarrow M(\mathbb{Z})$ by

$$(S_{a,p}c)_j = \sum_i a_{j-pi} c_i, \quad j \in \mathbb{Z}. \quad (1.24)$$

For a given initial sequence $c \in M(\mathbb{Z})$, we then recursively define

$$c^{(0)} = c; \quad c^{(r)} = S_{a,p}c^{(r-1)}, \quad r \in \mathbb{N}.$$

We call this the subdivision scheme $(S_{a,p}, c)$.

For $p \in \mathbb{Z}$, $p \geq 2$, suppose $a, c \in M_0(\mathbb{Z})$ and let $A = \frac{1}{p} \text{Lpol}(a)$ and $C = \text{Lpol}(c)$. Note that by (1.8c), the definition (1.24) is equivalent to $\text{Lpol}(S_{a,p}c) = pAC^{(p)}$, so that, by repeated use of (1.8a) and (1.8b),

$$\text{Lpol}(c^{(r)}) = pA (\text{Lpol}(c^{(r-1)}))^{(p)} = \dots = p^r \left(\prod_{j=0}^{r-1} A^{(p^j)} \right) C^{(p^r)}, \quad r \in \mathbb{N}. \quad (1.25)$$

In the special case $c = \delta$, (1.25) becomes

$$\text{Lpol}(S_{a,p}^r \delta) = p^r \prod_{j=0}^{r-1} A\langle p^j \rangle, \quad r \in \mathbb{N}. \quad (1.26)$$

We say that the subdivision scheme $(S_{a,p}, c)$ converges (or that subdivision converges) if there exists a function $\Phi \in C(\mathbb{R}) \setminus \{0\}$, called the limit function of the subdivision scheme, such that

$$\sup_{j \in \mathbb{Z}} \left| \Phi\left(\frac{j}{p^r}\right) - c_j^{(r)} \right| \rightarrow 0, \quad r \rightarrow \infty. \quad (1.27)$$

The following necessary condition for subdivision convergence was proved for the case $p = 2$ in [10: Proposition 2.1] and was subsequently extended to the general multi-dimensional setting in [31: Proposition 1]. For our purposes, it is sufficient to state the result of [31] only in the one dimensional case, as follows.

Theorem 1.14. *For $p \in \mathbb{Z}$, $p \geq 2$ and $a \in M_0(\mathbb{Z})$, suppose there exists a sequence $c \in l^\infty(\mathbb{Z})$ such that the subdivision scheme $(S_{a,p}, c)$ converges to Φ . Then the sum rules (1.11) hold for the mask a .*

The next result shows that our assumption that $A(1) = 1$ is consistent with subdivision convergence and also gives a first indication of the usefulness of the polynomials defined by (1.17). It provides an equivalent formulation of the sum rules in terms of the mask symbol factorisation that is well-known in the case $p = 2$. We derived the proof given below independently and subsequently found similar proofs in the literature (see e.g. [22: Lemma 3.4]).

Theorem 1.15. *The sum rules (1.11) have the equivalent formulation*

$$A(1) = 1 \quad \text{and} \quad E_{p-1} \mid A. \quad (1.28)$$

Proof. Suppose first that (1.11) holds. Then

$$A(1) = \frac{1}{p} \sum_j a_j = \frac{1}{p} \sum_{l=0}^{p-1} \sum_j a_{pj+l} = \frac{1}{p} \sum_{l=0}^{p-1} 1 = 1.$$

Since

$$\sum_{n=0}^{p-1} e^{\frac{2\pi i l n}{p}} = \frac{e^{2\pi i l} - 1}{e^{\frac{2\pi i l}{p}} - 1} = 0, \quad l \in \{1, \dots, p-1\}, \quad (1.29)$$

we have for all $l \in \{1, \dots, p-1\}$,

$$\begin{aligned} A\left(e^{\frac{2\pi i l}{p}}\right) &= \frac{1}{p} \sum_j a_j e^{\frac{2\pi i l}{p} j} = \frac{1}{p} \sum_{n=0}^{p-1} \sum_j a_{jp+n} e^{2\pi i l j + \frac{2\pi i l}{p} n} \\ &= \frac{1}{p} \sum_{n=0}^{p-1} e^{\frac{2\pi i l}{p} n} \sum_j a_{jp+n} = \frac{1}{p} \sum_{n=0}^{p-1} e^{\frac{2\pi i l}{p} n} = 0. \end{aligned}$$

Thus the set $\Gamma = \left\{ e^{\frac{2\pi il}{p}} : l \in \{1, \dots, p-1\} \right\}$ is contained in the set of roots of A . But by (1.17), Γ is exactly the set of roots of the polynomial E_{p-1} . Thus $E_{p-1}|A$, thereby establishing (1.28).

Conversely, suppose that (1.28) is satisfied. Since $E_{p-1}(1) = 1$, this means $A = E_{p-1}B$ for some polynomial B with $B(1) = 1$. Thus

$$a_j = p [E_{p-1}B]_j = \sum_{l=0}^{p-1} [B]_{j-l}, \quad j \in \mathbb{Z}, \quad (1.30)$$

so that, for $l \in \mathbb{Z}_p$, we have

$$\sum_j a_{jp+l} = \sum_j \sum_{n=0}^{p-1} [B]_{jp+l-n} = \sum_j [B]_{j+l} = \sum_j [B]_j = B(1) = 1.$$

□

The link between the subdivision algorithm and refinable functions is borne out by the following result, which is given in [10: Theorem 2.1] for the case $p = 2$. It is extended to the general integer multi-dimensional case in [14: Section 3], but we once again use the formulation of [31: Proposition 2] restricted to the univariate case.

Theorem 1.16. *Suppose, for $p \in \mathbb{Z}$, $p \geq 2$, $a \in M_0(\mathbb{Z})$ and $c \in l^\infty(\mathbb{Z})$, that $(S_{a,p}, c)$ converges to a function Φ . Then there exists a unique compactly supported continuous function ϕ such that (A, ϕ) is a p -refinement pair and such that ϕ satisfies the partition of unity property (1.15). Furthermore,*

$$\Phi = \sum_j c_j \phi(\cdot - j). \quad (1.31)$$

Note 1.17. We see from Remark 1.13 and Theorem 1.16 that subdivision convergence is consistent with the normalisation $\int_{-\infty}^{\infty} \phi(s) ds = 1$.

The next result shows that to check subdivision convergence for all initial sequences in $l^\infty(\mathbb{Z})$, it is sufficient to consider the initial sequence δ . The result was first proved for $p = 2$ in [10: Proposition 2.2] and the extended multi-dimensional proof is given in [31: Lemma 4], which we once again only state in one-dimensional form.

Theorem 1.18. *For $p \in \mathbb{Z}$, $p \geq 2$ and $a \in M_0(\mathbb{Z})$, the subdivision scheme $(S_{a,p}, c)$ converges for all $c \in l^\infty(\mathbb{Z}) \setminus \{0\}$ if and only if $(S_{a,p}, \delta)$ converges.*

We generalise Theorem 1.18 and Theorem 1.16 in Lemma 4.6 and Theorem 4.7, where we consider subsequence convergence in subdivision, of which standard subdivision convergence is a special case.

It is also well-known (see e.g. [33: Lemma 5]) that a necessary condition for subdivision convergence in the case $p = 2$ is $\gcd(\{j \in \mathbb{Z} : a_j \neq 0\}) = 1$. In view of Theorem 1.10, this

does not pose a major restriction, since if $\gcd(\{j \in \mathbb{Z} : a_j \neq 0\}) = k$, then with the mask b defined by $b_j = a_{kj}$, $j \in \mathbb{Z}$, we have $\gcd(\{j \in \mathbb{Z} : b_j \neq 0\}) = 1$ and Theorem 1.10 yields that (A, ϕ) is a 2-refinement pair if and only if $(B, k\phi(k \cdot))$ is a 2-refinement pair.

1.3.2 The cascade algorithm

For a sequence $a \in M_0(\mathbb{Z})$ and dilation factor $\alpha \in (1, \infty)$, one defines the cascade operator $T_{a,\alpha} : M(\mathbb{R}) \rightarrow M(\mathbb{R})$ by

$$T_{a,\alpha}f = \sum_j a_j f(\alpha \cdot -j), \quad f \in M(\mathbb{R}). \quad (1.32)$$

Observe that the operator $T_{a,\alpha}$ is a linear operator that maps $C_u(\mathbb{R})$ into itself and that $T_{a,\alpha}$ is a bounded operator on $C_u(\mathbb{R})$ with operator norm $\|T_{a,\alpha}\|_\infty \leq \sum_j |a_j|$.

For a given initial function $g \in M(\mathbb{R})$ we let $f_0 = g$ and recursively define

$$f_r = T_{a,\alpha}f_{r-1}, \quad r \in \mathbb{N}.$$

This algorithm is called the cascade algorithm and denoted by $(T_{a,\alpha}, g)$.

A well-known relationship between the subdivision and cascade algorithms (see, e.g. the proof of Theorem 3.1 in [27]) is that for any $p \in \mathbb{Z}$, $p \geq 2$, $f \in M_0(\mathbb{R})$, $a \in M_0(\mathbb{Z})$ and sequence $c \in M(\mathbb{Z})$, we have

$$\sum_i c_i (T_{a,p}^r f)(x - i) = \sum_i (S_{a,p}^r c)_i f(p^r x - i), \quad x \in \mathbb{R}, \quad r \in \mathbb{N}. \quad (1.33)$$

Now if we take $f = N_2$ and $c = \delta$, we have from (1.23) and (1.33) that

$$(T_{a,p}^r N_2) \left(\frac{j}{p^r} \right) = (S_{a,p}^r \delta)_{j-1}, \quad j \in \mathbb{Z}, \quad r \in \mathbb{Z}_+. \quad (1.34)$$

Using (1.33), (1.34) and the fact that the set $\left\{ \frac{j}{p^r} : j \in \mathbb{Z}, r \in \mathbb{Z}_+ \right\}$ is dense in \mathbb{R} , it can now be shown, analogously to the proof of the case $p = 2$ in [10: Theorem 2.1], that the following result holds. We omit the proof.

Theorem 1.19. *For $a \in M_0(\mathbb{Z})$ and integers N, p with $N \geq p \geq 2$, the subdivision scheme $(S_{a,p}, c)$ converges for every $c \in l^\infty(\mathbb{Z}) \setminus \{0\}$ to $\Phi_c \in C(\mathbb{R})$ if and only if the cascade algorithm $(T_{a,p}, N_2)$ converges uniformly to $\phi \in C(\mathbb{R})$, where the function ϕ is such that (A, ϕ) is a p -refinement pair and ϕ and Φ_c are related by*

$$\Phi_c = \sum_j c_j \phi(\cdot - j), \quad c \in l^\infty(\mathbb{Z}).$$

1.4 Our contribution

Many results in the literature make use of Fourier (frequency domain) methods. For our own proofs, we prefer to use direct (time domain) methods. Also, many results in the literature give characterisations of certain properties of refinable functions in terms of conditions that are not necessarily easily checkable, for instance the spectral radii of certain matrices. It is our goal to state results in terms of conditions that can be easily checked, for instance the factorisation of the mask symbol.

We proceed to give a short overview of the main results derived in this work.

For the general case $\alpha \in (1, \infty)$, we derive a fundamental result that links multiplication of the mask symbols and convolution of the corresponding refinable functions. Special cases of this result occur frequently in the literature. A converse result, linking factorisation of the mask symbol and inverse convolutions, is obtained by using a result from operational calculus.

For the integer dilation case, we extend the definition of special polynomial factors, first considered for dilation factor 2 by Berg & Plonka [3], to the dilation factor $p \in \mathbb{Z}$, $p \geq 2$ and extend some of the results of [3] to this case. These polynomial factors correspond to a special case of p -refinable step functions, which play an important role throughout this work. We use these step functions to derive existence results for certain reduced masks and to derive sufficient conditions for the occurrence of constant and polynomial sections in refinable functions which do not necessarily have other polynomial sections anywhere in their supports.

Furthermore, we extend the positive existence and subdivision convergence results of De Villiers [20] from dilation factor 2 to the general integer dilation case and obtain much more general sufficient conditions for regularity in terms of the mask symbol factorisation than those in the existing literature that we are aware of. Our sufficient conditions for regularity are more general both with respect to the dilation factor (being stated for general integer dilation) and with respect to the form of factors considered.

For dilation factor 2, we establish necessary conditions for regularity in terms of the mask symbol factorisation when the refinable function is not assumed to be stable, which, to our knowledge, has been an open problem until now.

We study the phenomenon of subsequence convergence in subdivision, which, to our knowledge, has never been formally studied. Here we are able to establish different sets of sufficient conditions for this to occur, with some results similar to standard subdivision convergence, e.g. that the limit function is refinable. Since subsequence convergence is a proper generalisation of subdivision convergence, our results are thus generalisations of the corresponding results for subdivision convergence. The nature of this phenomenon is such that the standard subdivision algorithm can still be used for graphical purposes with an easy modification at the end of the process.

Lastly, for dilation factor 2 and mask sequences of length three, we note how explicit formulas for refinable functions can be used to calculate the exact value of a refinable function at any rational point.

1.4.1 Remarks on non-integer dilation factors

There are some notable differences between the cases of integer and non-integer dilation factors. As can be seen from Theorem 1.19, there is a strong relationship between the cascade and subdivision algorithms in the integer case.

The integer case also gives rise to the following eigenvalue problem [18: Section 5]. Suppose a continuous solution ϕ of (1.1) with dilation factor $p \in \mathbb{Z}$, $p \geq 2$ exists and let $N_0 = \lceil a \rceil / (p - 1)$. Define M to be the $N_0 \times N_0$ matrix with entries given by $M_{ij} = a_{pi-j}$ for $i, j = 1, \dots, N_0$ and define $\mathbf{v} \in \mathbb{R}^{N_0}$ by $\mathbf{v}_j = \phi(j)$, $j = 1, \dots, N_0$. Since (1.10) and the continuity of ϕ yields $\phi(j) = 0$ if $j \notin \{1, \dots, N_0\}$, evaluating (1.1) at the different argument values $1, \dots, N_0$ now yields $\mathbf{v} = M\mathbf{v}$. For non-integer dilation factors, this eigenvalue problem does not occur.

While the cascade operator is defined for any dilation factor, extending the subdivision algorithm to non-integer dilation factors is not so trivial. For a rational dilation factor $\frac{p}{q}$, one extension is studied by Rioul & Blu [39] in the context of rational filter banks in signal processing. However, their scheme does not lead to a single refinable limit function, but to a set of limit functions which together satisfy a refinement equation, although the individual functions are not solutions of (1.1). A good overview of this case is provided in [7], which provides further references.

The strong link between refinable functions and wavelets also breaks down in the non-integer case, as explained in [12: Section 3]. Thus, although it is shown in [9] that certain irrational dilation factors admit no orthonormal wavelets, this does not imply that no refinable functions exist for those dilation factors.

During the course of our studies, we tried various ways of extending our time domain methods for positive masks to rational dilation factors, without success. We also experimented with a novel approach to proving existence of a refinable function for a certain class of masks with the dilation factor in a bounded real interval, but could not get this approach to work either. It appears that in general the analysis for the non-integer dilation factor case is much more difficult than for the integer case when using direct methods.

Chapter 2

A step function approach to the analysis of refinable functions

In this chapter we consider especially the role of refinable step functions and their corresponding mask symbols in the analysis of refinable functions. When combined with the convolution results presented next, this provides a general framework in which various previously known results can be understood, as well as providing tools for the establishment of some other results, e.g. the occurrence of polynomial sections in refinable functions and the regularity results of the next chapter. We also present some results on subdivision.

2.1 Convolutions and inverse convolutions

The following result shows the link between polynomial multiplication of the mask symbols and convolution of the corresponding refinable functions. Special cases of this theorem appear often in the literature, e.g. in [10: Proposition 2.5], where $\alpha = 2$ and only masks for which subdivision is convergent are considered. Our theorem shows a much more fundamental and general result, depending only on refinability and allowing non-integer dilation factors. After deriving this result independently, we found a similar result in the recent paper [36], but because a more general setting is considered there, the result in [36] is not stated in terms of the polynomial multiplication of the mask symbols. Towards the end of our study, we also discovered a remark in this direction in [15: p. 375].

Theorem 2.1. *If (A, ϕ) and (B, ψ) are α -refinement pairs, then $(AB, \phi * \psi)$ is an α -refinement pair.*

Proof. Remembering that $a = \alpha[A]$ and $b = \alpha[B]$, we obtain, by using amongst others the refinability of ψ and ϕ , that

$$\begin{aligned}
 \sum_j \alpha[AB]_j (\phi * \psi)(\alpha \cdot -j) &= \sum_j \alpha \sum_k \frac{1}{\alpha} a_k \frac{1}{\alpha} b_{j-k} \int_{-\infty}^{\infty} \phi(s) \psi(\alpha \cdot -j - s) ds \\
 &= \frac{1}{\alpha} \sum_k a_k \sum_j b_{j-k} \int_{-\infty}^{\infty} \phi(s) \psi(\alpha \cdot -j - s) ds \\
 &= \frac{1}{\alpha} \sum_k a_k \sum_j b_j \int_{-\infty}^{\infty} \phi(s) \psi(\alpha \cdot -j - k - s) ds \\
 &= \frac{1}{\alpha} \sum_k a_k \int_{-\infty}^{\infty} \phi(s) \sum_j b_j \psi\left(\alpha \left(-\frac{k+s}{\alpha} - j\right)\right) ds
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\alpha} \sum_k a_k \int_{-\infty}^{\infty} \phi(s) \psi\left(\cdot - \frac{k+s}{\alpha}\right) ds \\
&= \sum_k a_k \int_{-\infty}^{\infty} \phi(\alpha s - k) \psi(\cdot - s) ds \\
&= \int_{-\infty}^{\infty} \sum_k a_k \phi(\alpha s - k) \psi(\cdot - s) ds \\
&= \int_{-\infty}^{\infty} \phi(s) \psi(\cdot - s) ds \\
&= \phi * \psi,
\end{aligned}$$

yielding the desired result. \square

Remark 2.2. Note that the recursive definition (1.20) of the cardinal B -splines provides a (very well-known) special case of Theorem 2.1, since (E_{p-1}, N_1) forms a p -refinement pair for $p \in \mathbb{Z}$, $p \geq 2$ (as can be easily verified directly). It then immediately follows inductively that $((E_{p-1})^l, N_l)$ is a p -refinement pair for all $l \in \mathbb{N}$.

To derive a converse result of Theorem 2.1, we need the following theorem. It was first proved for a class of continuous functions by Titchmarsh [43] and is given in the form we use by Mikusiński [35]. Recall that a function f is called locally Lebesgue integrable if the Lebesgue integral $\int_{\alpha}^{\beta} f(x) dx$ exists for every compact interval $[\alpha, \beta] \subset \mathbb{R}$.

Proposition 2.3. *If f and g are both locally Lebesgue integrable and both vanish left of the origin, then $f * g = 0$ p.p. implies that $f = 0$ p.p. or $g = 0$ p.p.*

Proof. Under the assumption that f, g vanish left of the origin, we have

$$(f * g)(x) = \int_0^x f(x-s) g(s) ds, \quad x \in \mathbb{R},$$

which is equivalent to the definition of convolution used by Mikusiński. For the rest of the proof, see [35: Part 6, Chapter 2]. \square

We can now derive the following inverse convolution result, which can be interpreted as a converse of Theorem 2.1. We shall rely on this result in Section 3.2.

Theorem 2.4. *If there exist polynomials A, B and functions ϕ, ψ vanishing left of the origin such that ϕ is continuous and compactly supported and (B, ψ) and $(AB, \phi * \psi)$ are α -refinement pairs, then (A, ϕ) is an α -refinement pair.*

Proof. By again using $a = \alpha [A]$, $b = \alpha [B]$, as well as the refinability of $\phi * \psi$ and ψ , we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(s) \psi(\cdot - s) ds &= \phi * \psi \\
&= \sum_j \alpha [AB]_j (\phi * \psi)(\alpha \cdot -j) \\
&= \sum_j \frac{1}{\alpha} \sum_k a_k b_{j-k} \int_{-\infty}^{\infty} \phi(s) \psi(\alpha \cdot -j - s) ds \\
&= \frac{1}{\alpha} \sum_k a_k \sum_j b_{j-k} \int_{-\infty}^{\infty} \phi(s) \psi(\alpha \cdot -j - s) ds \\
&= \frac{1}{\alpha} \sum_k a_k \sum_j b_j \int_{-\infty}^{\infty} \phi(s) \psi(\alpha \cdot -j - k - s) ds \\
&= \frac{1}{\alpha} \sum_k a_k \int_{-\infty}^{\infty} \phi(s) \sum_j b_j \psi\left(\alpha \left(\cdot - \frac{k+s}{\alpha}\right) - j\right) ds \\
&= \frac{1}{\alpha} \sum_k a_k \int_{-\infty}^{\infty} \phi(s) \psi\left(\cdot - \frac{k+s}{\alpha}\right) ds \\
&= \sum_k a_k \int_{-\infty}^{\infty} \phi(\alpha s - k) \psi(\cdot - s) ds \\
&= \int_{-\infty}^{\infty} \sum_k a_k \phi(\alpha s - k) \psi(\cdot - s) ds.
\end{aligned}$$

Thus

$$\int_{-\infty}^{\infty} \left[\sum_k a_k \phi(\alpha s - k) - \phi(s) \right] \psi(\cdot - s) ds = 0.$$

Note that both ψ and $f = \sum_k a_k \phi(\alpha \cdot -k) - \phi$ are compactly supported and hence locally Lebesgue integrable. Moreover, f vanishes left of the origin. Since $\psi \neq 0$, it follows from Proposition 2.3 that $f(x) = 0$ for almost all $x \in \mathbb{R}$. The continuity of ϕ now yields that $f(x) = 0$, $x \in \mathbb{R}$, which, together with the compact support of ϕ , as well as the fact that $\phi \neq 0$ by virtue of $\phi * \psi \neq 0$, shows that (A, ϕ) is an α -refinement pair. \square

2.2 Step functions and special mask symbol factors

One important application of Theorem 2.1 is that the smoothness of a p -refinable function can be increased by adding the factor E_{p-1} , which corresponds to the refinable function $N_1 = \chi$, to the mask symbol. If we can use a more general refinable step function of the form

$$\sigma = \sum_j r_j \chi(\cdot - j) \tag{2.1}$$

with $r \in M_0^+(\mathbb{Z})$, we can achieve the same increase in smoothness. In this section we consider such step functions. We start with some results from Lawton, Lee & Shen [32].

Definition 2.5. We say that a polynomial P is m -closed if P divides $P^{(m)}$.

Remark. Lawton *et al.* actually use another definition for m -closed polynomials and then prove that it is equivalent to the above. The above definition will be sufficient for our purposes.

We now have the following result from [32: Theorem 2.1].

Proposition 2.6. For $p \in \mathbb{Z}$, $p \geq 2$, a function σ of the form (2.1), with $r \in M_0^+(\mathbb{Z})$, is p -refinable if and only if the polynomial Q , defined by $Q(z) = \frac{1}{p}(z-1)\left(\sum_j r_j z^j\right)$, is p -closed. In this case the refinement mask symbol is given by $Q^{(p)}/Q$.

Definition 2.7. In view of the above proposition, for a given dilation factor $p \in \mathbb{Z}$, $p \geq 2$, we will call a polynomial P a p -LLS (Lawton-Lee-Shen) factor if $P = Q^{(p)}/Q$, where Q is a p -closed polynomial of the form $Q(z) = \frac{1}{p}(z-1)\left(\sum_j r_j z^j\right)$ for some $r \in M_0^+(\mathbb{Z})$.

Remark 2.8. In view of (1.18), we deduce that a polynomial P is a p -LLS factor if and only if P has the form

$$P = E_{p-1} \frac{R^{(p)}}{R}, \quad (2.2)$$

where $R = \text{Lpol}(r)$ for some $r \in M_0^+(\mathbb{Z})$.

Note. For $p \in \mathbb{Z}$, $p \geq 2$, the simplest example of a p -LLS factor is exactly $P = E_{p-1}$, being obtained by choosing $r = \delta$ in Definition 2.7, which yields $R(z) = 1$, $z \in \mathbb{C}$ in (2.2).

The following result will be important later.

Corollary 2.9. For $p \in \mathbb{Z}$, $p \geq 2$, suppose that (B, ψ) is a p -refinement pair and that the polynomial A has the form $A = PB$, where P is a p -LLS factor. Then (A, ϕ) is a p -refinement pair, where ϕ is given by

$$\phi(x) = \sum_j [R]_j \int_{x-j-1}^{x-j} \psi(s) ds, \quad x \in \mathbb{R}, \quad (2.3)$$

with R as in Remark 2.8.

Proof. By Proposition 2.6, (P, σ) is a p -refinement pair. Thus, by Theorem 2.1, $(A, \psi * \sigma)$ is a p -refinement pair. After noting that $[R]_j = r_j$, $j \in \mathbb{Z}$, we obtain, for $x \in \mathbb{R}$,

$$\begin{aligned} (\psi * \sigma)(x) &= \int_{-\infty}^{\infty} \psi(s) \sum_j r_j \chi(x-s-j) ds \\ &= \sum_j r_j \int_{-\infty}^{\infty} \psi(s) \chi(x-s-j) ds \\ &= \sum_j [R]_j \int_{x-j-1}^{x-j} \psi(s) ds, \end{aligned}$$

which yields the desired result. □

We present next a special case of LLS factors, which are generalisations of the polynomial factors considered for the case $p = 2$ by Berg & Plonka [3; 4].

Definition 2.10. For $p \in \mathbb{Z}$, $p \geq 2$, we say that a polynomial P is a p -GBP (generalised Berg-Plonka) factor if there is an integer $k \in \mathbb{Z}_+$ such that $P = P_k$ can be iteratively obtained as follows:

1. $P_0 = E_{p-1}$;
2. For $l = 1, 2, \dots, k$, P_l is obtained by replacing z by z^p in P_{l-1} or in a proper polynomial factor of degree at least 1 of P_{l-1} .

The importance of GBP factors for the case $p = 2$ is highlighted by the following result, which is given in [3: Theorem 3.3].

Theorem 2.11. *Suppose (A, ϕ) is a 2-refinement pair. Then A contains a 2-GBP factor.*

We say that P is a p -GBP factor of level k if k is the smallest integer such that $P = P_k$, where P_k can be obtained in the algorithm above. For instance, although in the case $p = 2$ one can derive $\frac{1}{2}(1 + z^4)$ by

$$\frac{1}{2}(1 + z) \rightarrow \frac{1}{2}(1 + z^2) = \frac{1}{2}(1 + iz)(1 - iz) \rightarrow \frac{1}{2}(1 + iz^2)(1 - iz) \rightarrow \frac{1}{2}(1 + iz^2)(1 - iz^2),$$

in which case $k = 3$ in the GBP algorithm, the factor $\frac{1}{2}(1 + z^4)$ is a 2-GBP factor of level 2, since its shortest possible derivation is

$$\frac{1}{2}(1 + z) \rightarrow \frac{1}{2}(1 + z^2) \rightarrow \frac{1}{2}(1 + z^4).$$

Example 2.12. In Definition 2.10, an important special case is obtained if we form P_l by replacing z by z^p in P_{l-1} for every $l = 1, \dots, k$, in which case it follows inductively that $P = E_{p-1}^{\langle p^k \rangle}$. If a p -GBP factor is not of this special form, we shall call it a non-trivial p -GBP factor.

We show some non-trivial 2-GBP factors in Figure 2.1, which depicts the derivation of all 2-GBP factors up to level 2 and the 2-GBP factors of level 3 with real coefficients. Although we do not show the calculations here, it is interesting to note that there are a further eighteen 2-GBP factors of level 3, yielding a total of twenty-six 2-GBP factors of level at most 3.

Remark 2.13. GBP factors have a useful equivalent formulation, which was noted in the proof of [3: Theorem 3.4] for the case $p = 2$. P is a p -GBP factor if and only if there is an integer $k \in \mathbb{Z}_+$ and polynomials $q_l, r_l : l \in \{0, \dots, k\}$ with $\deg(r_l) \geq 1$, $l \in \{0, \dots, k\}$ such

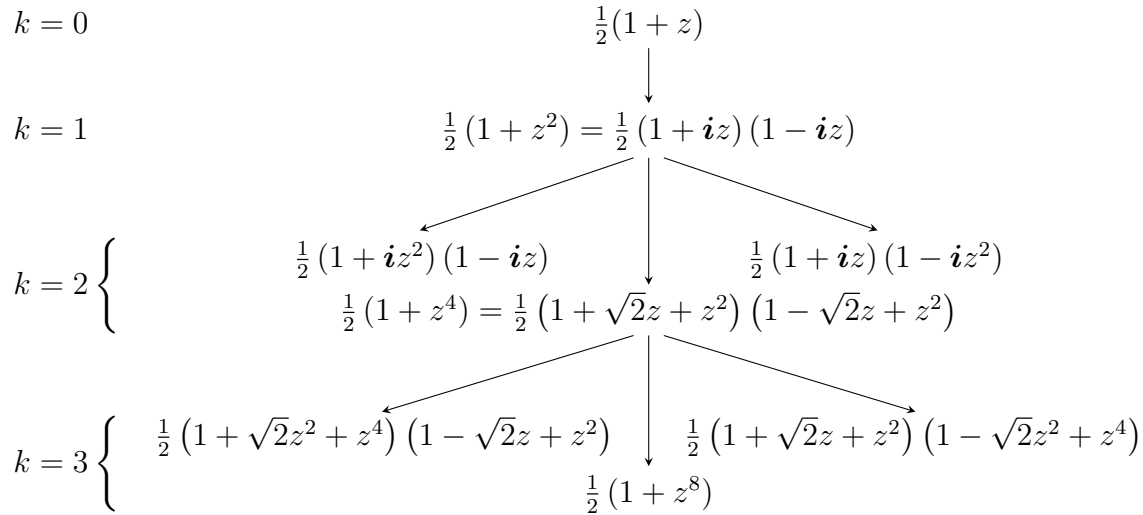


Figure 2.1: A graphic showing the derivation of the 2-GBP factors up to level 2 and those of level 3 with real coefficients. There are eighteen other 2-GBP factors of level 3.

that

$$q_0 r_0 = E_{p-1}, \quad (2.4a)$$

$$q_l r_l = q_{l-1} r_{l-1}^{(p)}, \quad l = 1, 2, \dots, k, \quad (2.4b)$$

and

$$q_k r_k = P. \quad (2.4c)$$

To see the equivalence of this definition, note that $q_l r_l = P_l$ for $l = 0, 1, \dots, k$, with r_{l-1} representing the polynomial factor of P_{l-1} in which z is replaced by z^p for $l = 1, 2, \dots, k$. Also note that, since $E_{p-1}(1) = 1$ and

$$q_l(1) r_l(1) = q_{l-1}(1) r_{l-1}^{(p)}(1) = q_{l-1}(1) r_{l-1}(1), \quad l = 1, 2, \dots, k,$$

it follows inductively that $P_l(1) = 1$ for $l = 0, 1, \dots, k$. Thus without loss of generality we can always choose the q_l, r_l such that

$$q_l(1) = r_l(1) = 1, \quad l = 0, 1, \dots, k. \quad (2.5)$$

The next lemma establishes some useful properties of GBP factors, which we shall employ in the proofs of various subsequent results on subdivision convergence and subsequence convergence in subdivision.

Lemma 2.14. For $p \in \mathbb{Z}$, $p \geq 2$, suppose that P is a p -GBP factor of level k . Then P is a p -LLS factor such that $R(1) = 1$, with R as in Remark 2.8. Furthermore, the function W , given by $W = E_{p^k-1}/R$, is a polynomial satisfying $W(1) = 1$.

Proof. By rewriting (2.4b) as $q_l = q_{l-1} \frac{r_{l-1}^{(p)}}{r_l}$, we obtain from (2.4a)–(2.4c) that

$$P = q_{k-1} r_{k-1}^{(p)} = q_{k-2} r_{k-2}^{(p)} \frac{r_{k-1}^{(p)}}{r_{k-1}} = \cdots = q_0 r_0^{(p)} \prod_{l=1}^{k-1} \frac{r_l^{(p)}}{r_l} = E_{p-1} \prod_{l=0}^{k-1} \frac{r_l^{(p)}}{r_l}. \quad (2.6)$$

Letting the polynomial R be given by

$$R = \prod_{l=0}^{k-1} r_l, \quad (2.7)$$

we see that (2.6) yields (2.2), so that P is a p -LLS factor. From the assumption (2.5) it follows that $R(1) = 1$.

We also have, by consecutively using (1.19b), (2.4a), (1.8b), (1.8a), (2.4b) and (2.7), that

$$\begin{aligned} E_{p^{k-1}} &= \prod_{l=0}^{k-1} E_{p-1}^{(p^l)} = \prod_{l=0}^{k-1} q_0^{(p^l)} r_0^{(p^l)} \\ &= r_0 q_0^{(p^{k-1})} \prod_{l=0}^{k-2} q_0^{(p^l)} r_0^{(p^{l+1})} \\ &= r_0 q_0^{(p^{k-1})} \prod_{l=0}^{k-2} (q_0 r_0^{(p)})^{(p^l)} \\ &= r_0 q_0^{(p^{k-1})} \prod_{l=0}^{k-2} (q_1 r_1)^{(p^l)} \\ &= r_0 q_0^{(p^{k-1})} r_1 q_1^{(p^{k-2})} \prod_{l=0}^{k-3} q_1^{(p^l)} r_1^{(p^{l+1})} \\ &= r_0 q_0^{(p^{k-1})} r_1 q_1^{(p^{k-2})} \prod_{l=0}^{k-3} (q_2 r_2)^{(p^l)} \\ &= \cdots \\ &= \prod_{l=0}^{k-1} r_l q_l^{(p^{k-1-l})} = R \prod_{l=0}^{k-1} q_l^{(p^{k-1-l})}, \end{aligned}$$

so that $W = E_{p^{k-1}}/R = \prod_{l=0}^{k-1} q_l^{(p^{k-1-l})}$ is a polynomial. To complete the proof of the lemma, we observe that $W(1) = E_{p^{k-1}}(1)/R(1) = 1$. \square

Remark 2.15. In Lemma 2.14, for the special case $P = E_{p-1}^{(p^k)}$ which was mentioned in Example 2.12, we have $q_l = E_0$ for $l = 0, \dots, k$ in (2.4a)–(2.4c), so that

$$r_l = E_{p-1}^{(p^l)}, \quad l = 0, \dots, k,$$

which by (2.7) and (1.19b) yields $R = E_{p^{k-1}}$, from which we obtain $W = E_{p^{k-1}}/R = E_0$.

Example 2.16. To illustrate Lemma 2.14 for non-trivial GBP factors, let $P_{\mathbb{C}}$ and $P_{\mathbb{R}}$ denote the polynomials given by

$$P_{\mathbb{C}}(z) = \frac{1}{2} (1 - iz^2) (1 + iz) = \frac{1}{2} (1 + iz - iz^2 + z^3), \quad z \in \mathbb{C},$$

and

$$\begin{aligned} P_{\mathbb{R}}(z) &= \frac{1}{2} (1 + \sqrt{2}z^2 + z^4) (1 - \sqrt{2}z + z^2) \\ &= \frac{1}{2} (1 - \sqrt{2}z + (1 + \sqrt{2})z^2 - 2z^3 + (1 + \sqrt{2})z^4 - \sqrt{2}z^5 + z^6), \quad z \in \mathbb{C}, \end{aligned}$$

respectively. As shown in Figure 2.1, $P_{\mathbb{C}}$ is a non-trivial 2-GBP factor of lowest level (namely level 2), while $P_{\mathbb{R}}$ is a non-trivial 2-GBP factor with real coefficients of lowest level (namely level 3). To calculate the respective corresponding 2-refinable step functions $\sigma_{\mathbb{C}}$ and $\sigma_{\mathbb{R}}$, the existence of which are guaranteed by Proposition 2.6 and Lemma 2.14, we use (2.7), (2.5) and (2.1). For $P_{\mathbb{C}}$, we have $r_0 = E_1$ and $r_1(z) = \frac{1+i}{2} (1 - iz)$, $z \in \mathbb{C}$, so that

$$R(z) = \frac{1+i}{4} (1 + (1-i)z - iz^2), \quad z \in \mathbb{C}.$$

Thus we obtain

$$\sigma_{\mathbb{C}}(x) = \begin{cases} \frac{1+i}{4} & \text{if } x \in [0, 1), \\ \frac{1}{2} & \text{if } x \in [1, 2), \\ \frac{1-i}{4} & \text{if } x \in [2, 3), \\ 0 & \text{otherwise.} \end{cases}$$

For $P_{\mathbb{R}}$, we have, for $z \in \mathbb{C}$,

$$r_0(z) = \frac{1}{2} (1 + z), \quad r_1(z) = \frac{1}{2} (1 + z^2) \quad \text{and} \quad r_2(z) = \frac{1}{2 + \sqrt{2}} (1 + \sqrt{2}z + z^2),$$

so that we obtain, for $z \in \mathbb{C}$,

$$R(z) = \frac{1}{8 + 4\sqrt{2}} (1 + (1 + \sqrt{2})z + (2 + \sqrt{2})z^2 + (2 + \sqrt{2})z^3 + (1 + \sqrt{2})z^4 + z^5),$$

which yields

$$\sigma_{\mathbb{R}}(x) = \begin{cases} \frac{2-\sqrt{2}}{8} & \text{if } x \in [0, 1) \cup [5, 6), \\ \frac{\sqrt{2}}{8} & \text{if } x \in [1, 2) \cup [4, 5), \\ \frac{1}{4} & \text{if } x \in [2, 4), \\ 0 & \text{otherwise.} \end{cases}$$

These two functions are shown in Figure 2.2. In Figure 2.2(a) we also plot the functions

$2[P_{\mathbb{C}}]_j \sigma_{\mathbb{C}}(2 \cdot -j)$, $j = 0, \dots, 3$, with the real and imaginary parts shown separately, to illustrate that the refinement equation is indeed satisfied. In Figure 2.2(b) only the function $\sigma_{\mathbb{R}}$ is plotted due to the length of the mask. Note in particular that $\sigma_{\mathbb{R}}$ is non-negative, i.e. $\sigma_{\mathbb{R}}(x) \geq 0$ for $x \in \mathbb{R}$, although the mask symbol $P_{\mathbb{R}}$ contains negative coefficients.

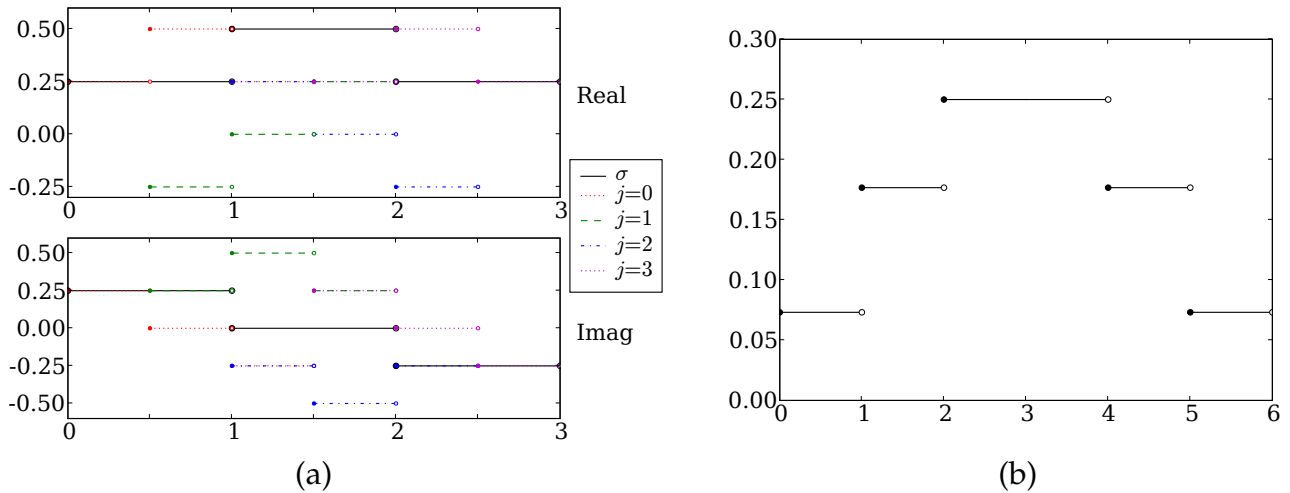


Figure 2.2: Plots of (a) $\sigma_{\mathbb{C}}$, as well as $2[P_{\mathbb{C}}]_j \sigma_{\mathbb{C}}(2 \cdot -j)$, $j = 0, \dots, 3$; and (b) $\sigma_{\mathbb{R}}$ of Example 2.16.

2.3 Equivalency of reduced masks

For a given refinement pair (A, ϕ) , the question sometimes arises whether ϕ can be expressed as a linear combination of shifts of a refinable function $\tilde{\phi}$ with smaller support. One reason why this could be useful, is for the purpose of graphing the function ϕ , since it may be that subdivision for the mask a does not converge or converges very slowly, but subdivision for the reduced mask \tilde{a} does converge or converges faster (see e.g. Neamtu [37]). Another reason is the issue of stability: if the function ϕ has linearly dependent integer shifts, for some applications one is interested in expressing it in terms of a function $\tilde{\phi}$ which has linearly independent integer shifts.

In this section we present some results on the existence of refinable functions for reduced masks for a general integer dilation factor and show how they relate to some of the results given in [37]. We start with the following result, as given in [3: Theorem 3.5].

Proposition 2.17. *Suppose A and \tilde{A} are polynomials of the form*

$$A = QR^{(2)} \quad \text{and} \quad \tilde{A} = QR$$

for some polynomials Q and R . Then (A, ϕ) is a 2-refinement pair if and only if $(\tilde{A}, \tilde{\phi})$ is a 2-

refinement pair, where the functions ϕ and $\tilde{\phi}$ are related by

$$\phi = \sum_j [R]_j \tilde{\phi}(\cdot - j). \quad (2.8)$$

The proof given in [3] uses Fourier transform methods. We next show how to generalise this result to a general integer dilation factor. Our proof really brings the usefulness of our polynomial power representation to the fore, allowing us to completely avoid the use of the Fourier transform.

Theorem 2.18. *Let $p \in \mathbb{Z}$, $p \geq 2$ and suppose that A and \tilde{A} are polynomials of the form*

$$A = QR^{(p)} \quad \text{and} \quad \tilde{A} = QR \quad (2.9)$$

for some rational function Q and polynomial R . Then (A, ϕ) is a p -refinement pair if and only if $(\tilde{A}, \tilde{\phi})$ is a p -refinement pair, where the functions ϕ and $\tilde{\phi}$ are related by (2.8).

Proof. By (1.8c) and (2.9) we have, for $k \in \mathbb{Z}$,

$$\sum_j [R]_j [\tilde{A}]_{k-pj} = [\tilde{A}R^{(p)}]_k = [AR]_k = \sum_j [R]_j [A]_{k-j}. \quad (2.10)$$

First suppose that $(\tilde{A}, \tilde{\phi})$ is a p -refinement pair and define ϕ by (2.8). We then obtain

$$\begin{aligned} \phi &= \sum_j [R]_j \tilde{\phi}(\cdot - j) \\ &= \sum_j [R]_j \sum_k p [\tilde{A}]_j \tilde{\phi}(p \cdot -pj - k) \\ &= p \sum_j [R]_j \sum_k [\tilde{A}]_{k-pj} \tilde{\phi}(p \cdot -k) \\ &= p \sum_k \left(\sum_j [R]_j [\tilde{A}]_{k-pj} \right) \tilde{\phi}(p \cdot -k) \\ &= p \sum_k \left(\sum_j [R]_j [A]_{k-j} \right) \tilde{\phi}(p \cdot -k) \\ &= p \sum_j [R]_j \sum_k [A]_{k-j} \tilde{\phi}(p \cdot -k) \\ &= p \sum_j [R]_j \sum_k [A]_k \tilde{\phi}(p \cdot -k - j) \\ &= \sum_k p [A]_k \sum_j [R]_j \tilde{\phi}(p \cdot -k - j) \\ &= \sum_k p [A]_k \phi(p \cdot -k), \end{aligned}$$

which shows that (A, ϕ) is a p -refinement pair.

Conversely, suppose that (A, ϕ) is a p -refinement pair. Let $M = \deg(\tilde{A})$ and define the function $\tilde{\phi}$ by

$$\tilde{\phi}(x) = \begin{cases} \frac{1}{[R]_0} \phi(x), & x < 1, \\ \frac{1}{a_0} \left(\tilde{\phi}\left(\frac{x}{p}\right) - \sum_{j=1}^M \tilde{a}_j \tilde{\phi}(x-j) \right), & x \geq 1. \end{cases} \quad (2.11)$$

Note that $\tilde{\phi}$ is indeed well-defined for all $x \in \mathbb{R}$, since the top line of (2.11) defines it for $x < 1$, while for a fixed $x \geq 1$, the inequalities $\frac{x}{p} \leq x - \frac{1}{p}$ and $x - j \leq x - 1$, $j \in \{1, \dots, M\}$ implies that the bottom line of (2.11) can be expanded recursively a finite number of times until all values of the arguments of $\tilde{\phi}$ in the right hand side are < 1 .

By noting that $a_0 = \tilde{a}_0$, after using the top line of (2.11), the refinability of ϕ and the fact that $\phi(x) = 0 = \tilde{\phi}(x)$, $x < 0$, we find for $x < \frac{1}{p}$ that

$$\sum_j \tilde{a}_j \tilde{\phi}(px - j) = \tilde{a}_0 \tilde{\phi}(px) = \frac{a_0}{[R]_0} \phi(px) = \frac{1}{[R]_0} \phi(x) = \tilde{\phi}(x),$$

while for $x \geq \frac{1}{p}$, the bottom line of (2.11) yields

$$\tilde{a}_0 \tilde{\phi}(px) = \tilde{\phi}(x) - \sum_{j \in \mathbb{Z} \setminus \{0\}} \tilde{a}_j \tilde{\phi}(px - j),$$

so that

$$\tilde{\phi} = \sum_j \tilde{a}_j \tilde{\phi}(p \cdot -j). \quad (2.12)$$

We next show that the equality

$$\phi(x) = \sum_j [R]_j \tilde{\phi}(x - j), \quad x < n, \quad (2.13)$$

holds for all $n \in \mathbb{N}$. If $n = 1$, the top line of (2.11) yields

$$\sum_j [R]_j \tilde{\phi}(x - j) = [R]_0 \tilde{\phi}(x) = \phi(x).$$

Now suppose that (2.13) holds for some $n \in \mathbb{N}$ and let $x < n + 1$. Using, amongst others, the refinability of ϕ , the inductive hypothesis, (2.12) and (2.10), we obtain

$$\begin{aligned} \phi(x) &= \frac{1}{a_0} \left(\phi\left(\frac{x}{p}\right) - \sum_{k=1}^{\uparrow a \uparrow} a_k \phi(x - k) \right) \\ &= \frac{1}{a_0} \left(\sum_j [R]_j \tilde{\phi}\left(\frac{x}{p} - j\right) - \sum_{k=1}^{\uparrow a \uparrow} a_k \sum_j [R]_j \tilde{\phi}(x - k - j) \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{p}{a_0} \sum_j [R]_j \left(\sum_k [\tilde{A}]_k \tilde{\phi}(x - pj - k) - \sum_k [A]_k \tilde{\phi}(x - k - j) \right) + \sum_j [R]_j \tilde{\phi}(x - j) \\
&= \frac{p}{a_0} \sum_j [R]_j \left(\sum_k [\tilde{A}]_{k-pj} \tilde{\phi}(x - k) - \sum_k [A]_{k-j} \tilde{\phi}(x - k) \right) + \sum_j [R]_j \tilde{\phi}(x - j) \\
&= \frac{p}{a_0} \sum_k \left(\sum_j [R]_j [\tilde{A}]_{k-pj} - \sum_j [R]_j [A]_{k-j} \right) \tilde{\phi}(x - k) + \sum_j [R]_j \tilde{\phi}(x - j) \\
&= \sum_j [R]_j \tilde{\phi}(x - j),
\end{aligned}$$

completing the inductive step, so that by induction we conclude that (2.8) holds.

Since $\phi \in M_0(\mathbb{R}) \setminus \{0\}$, we conclude from (2.8) that $\tilde{\phi} \in M_0(\mathbb{R}) \setminus \{0\}$, which together with (2.12) shows that $(\tilde{A}, \tilde{\phi})$ is a p -refinement pair. \square

Using Theorem 2.18 we can derive the following result, which generalises [3: Proposition 3.2] to the general integer dilation case. The proof is a straightforward adaptation of the one given in [3].

Theorem 2.19. *For $p \in \mathbb{Z}$, $p \geq 2$, suppose that the polynomial A satisfies $A = PB$, where B is a polynomial and P is a p -LLS factor having the characterisation (2.2) and define $\tilde{A} = E_{p-1}B$. Then (A, ϕ) is a p -refinement pair if and only if $(\tilde{A}, \tilde{\phi})$ is a p -refinement pair, where the functions ϕ and $\tilde{\phi}$ are related by (2.8).*

Proof. Set $Q = E_{p-1}B/R$. Then $\tilde{A} = QR$ and, by (2.2), we have $A = QR^{(p)}$. The result is now immediate from Theorem 2.18. \square

Remark 2.20. The result of Theorem 2.19 is consistent with our step function approach, as can be seen from the following argument. If we assume that $A(1) = 1$, then $B(1) = 1$, which by results in [18: Section 2] yields the existence of a distribution $\psi \in L^\infty(\mathbb{R})$ which is p -refinable with mask symbol B . Since the proof of Theorem 2.1 extends directly to such refinable distributions, we can apply Corollary 2.9 for both the mask symbols A and \tilde{A} . Doing so for \tilde{A} , we obtain

$$\tilde{\phi}(x) = \int_{x-1}^x \psi(s) ds, \quad x \in \mathbb{R}.$$

Combining this with the result (2.3) obtained for A , one obtains

$$\phi(x) = \sum_j [R]_j \int_{x-j-1}^{x-j} \psi(s) ds = \sum_j [R]_j \tilde{\phi}(x - j), \quad x \in \mathbb{R},$$

showing that (2.8) holds for the distributions ϕ and $\tilde{\phi}$.

Another special case of Theorem 2.18 appears in [37: Proposition 8.2], which states the following.

Proposition 2.21. *Suppose that R is a polynomial satisfying $R(-z)R(z) = R(z^2)$, $z \in \mathbb{C}$, as well as $R(1) \neq 0$, and that the polynomials A and \tilde{A} are related by $A(z) = R(-z)\tilde{A}(z)$, $z \in \mathbb{C}$. Then (A, ϕ) is a 2-refinement pair with $\phi \in C(\mathbb{R})$ if and only if $(\tilde{A}, \tilde{\phi})$ is a 2-refinement pair with $\tilde{\phi} \in C(\mathbb{R})$.*

By setting $Q = \tilde{A}/R$, we obtain that $A = QR^{(2)}$, showing that this proposition is indeed a special case of Theorem 2.18, except that we have not yet dealt with the continuity of the functions ϕ and $\tilde{\phi}$. This is what we proceed to do next.

Clearly it is not true in general that the linear combination of discontinuous functions is necessarily discontinuous: for instance $f + (-1)f = 0$ is continuous for any function f . However, our next result shows that in a special case, the linear combinations of shifts of a function are continuous if and only if the function itself is continuous in a sense made precise below.

Lemma 2.22. *Suppose $f, g \in M(\mathbb{R})$, with g continuous left of the origin, and that*

$$f = \sum_j r_j g(\cdot - j),$$

for some sequence $r \in M_0^+(\mathbb{Z})$. Then $f \in C(\mathbb{R})$ if and only if $g \in C(\mathbb{R})$.

Proof. Clearly, if $g \in C(\mathbb{R})$, then $g(\cdot - j) \in C(\mathbb{R})$, $j \in \mathbb{Z}$, so that $f \in C(\mathbb{R})$.

Conversely, suppose that $g \notin C(\mathbb{R})$, so that it has at least one point of discontinuity. Since also g is continuous left of the origin, the number

$$x_0 = \inf \{x \in \mathbb{R} : g \text{ is not continuous at } x\}$$

exists. Furthermore, from the definition of continuity, there is an $\varepsilon > 0$ such that for all $t > 0$ there exists a number $y_t \in (x_0 - t, x_0 + t)$ such that $|g(y_t) - g(x_0)| \geq \varepsilon$. Since x_0 is the infimum of the points of discontinuity of g , g is continuous at $x_0 - j$ for all $j \in \mathbb{N}$, so that the sum $\sum_{j=1}^{\infty} r_j g(\cdot - j)$ is continuous at x_0 . Thus there exists a $\tau_0 > 0$ such that

$$\left| \sum_{j=1}^{\infty} r_j (g(x - j) - g(x_0 - j)) \right| < \frac{|r_0|}{2} \varepsilon, \quad x \in (x_0 - \tau_0, x_0 + \tau_0).$$

Now let $\tau > 0$ be given. Set $t = \min\{\tau, \tau_0\}$. Then $t > 0$ so that there exists a number $y_t \in (x_0 - t, x_0 + t)$ such that $|g(y_t) - g(x_0)| \geq \varepsilon$. Suppose now that $g(y_t) - g(x_0) \geq \varepsilon$. If $r_0 > 0$, then since $y_t \in (x_0 - \tau, x_0 + \tau) \cap (x_0 - \tau_0, x_0 + \tau_0)$, we obtain

$$\begin{aligned} f(y_t) - f(x_0) &= r_0 (g(y_t) - g(x_0)) + \sum_{j=1}^{\infty} r_j (g(y_t - j) - g(x_0 - j)) \\ &\geq r_0 \varepsilon - \left| \sum_{j=1}^{\infty} r_j (g(x - j) - g(x_0 - j)) \right| > \frac{|r_0|}{2} \varepsilon. \end{aligned}$$

Similarly, if $r_0 < 0$, we obtain

$$\begin{aligned} f(y_t) - f(x_0) &= -r_0(g(x_0) - g(y_t)) + \sum_{j=1}^{\infty} r_j(g(y_t - j) - g(x_0 - j)) \\ &\leq -r_0(-\varepsilon) + \left| \sum_{j=1}^{\infty} r_j(g(x - j) - g(x_0 - j)) \right| < \frac{-|r_0|}{2}\varepsilon, \end{aligned}$$

so that we obtain the inequality $|f(y_t) - f(x_0)| > \frac{|r_0|}{2}\varepsilon$, after noting also that $r_0 \neq 0$ from the definition of $M_0^+(\mathbb{Z})$. The proof for the case $g(y_t) - g(x_0) \leq -\varepsilon$ is similar.

We conclude that, for any given $\tau > 0$, there is a number $y_t \in (x_0 - t, x_0 + t)$ such that $|f(y_t) - f(x_0)| > \frac{|r_0|}{2}\varepsilon$, which, in view of $r_0 \neq 0$, means that f is not continuous at x_0 , i.e. $f \notin C(\mathbb{R})$. \square

We immediately have the following result by virtue of our assumption that refinable functions vanish left of the origin.

Corollary 2.23. *In Theorem 2.18 or Theorem 2.19, $\phi \in C(\mathbb{R})$ if and only if $\tilde{\phi} \in C(\mathbb{R})$.*

2.4 Existence and subdivision convergence for positive masks

We next turn our attention to the question of refinable function existence and subdivision convergence in the case of positive masks. We start with a related result for non-negative masks, which is given by Goodman & Sun [27: Theorem 3.1]:

Theorem 2.24. *For $p \in \mathbb{Z}$, $p \geq 2$, suppose the mask symbol A is a polynomial of degree N having the form $A = E_{p-1}B$ with $B(1) = 1$, where $[B]_j \geq 0$ for $j \in \mathbb{Z}$ and $\sum_j b_{pj} < 1$. Then there exists a nonnegative function $\phi \in C(\mathbb{R})$ with the properties*

$$\phi(x) = 0, \quad x \notin \left(0, \frac{N}{p-1}\right), \quad (2.14)$$

$$\sum_j \phi(x - j) = 1, \quad x \in \mathbb{R}, \quad (2.15)$$

such that (A, ϕ) is a p -refinement pair. Moreover, if the mask a is positive, then we have

$$\phi(x) > 0, \quad x \in \left(0, \frac{N}{p-1}\right). \quad (2.16)$$

Remarks.

- (a) From Theorem 1.15 we see that the sum rules (1.11) hold for A .

(b) The proof of the theorem given in [27] shows that the cascade algorithm $(T_{a,p}, N_2)$ converges uniformly to ϕ . It can also be deduced from the proof that the subdivision scheme $(S_{a,p}, c)$ is convergent for $c \in \Delta^\infty(\mathbb{Z})$, although the issue of subdivision convergence was not explicitly considered in [27].

Although the above result holds for a class of non-negative masks which are not all positive masks, there are however positive masks for which the requirement that B be non-negative is not met (see Example 2.28). In order to deal with these cases, we present the following results, which generalise the results by De Villiers [20: Sections 2 & 3]. The proofs are similar to those used in [20].

First we extend the semi-norm κ defined in [20; 34] as follows. Define, for $N, p \in \mathbb{N}$ with $2 \leq p \leq N$, the semi-norm $\kappa_{N,p}$ by

$$\kappa_{N,p}(c) = \sup \left\{ |c_i - c_j| : i, j \in \mathbb{Z}, \quad |i - j| < \frac{N}{p-1} \right\}, \quad c \in \Delta^\infty(\mathbb{Z}). \quad (2.17)$$

Then we have

$$\|\Delta c\|_\infty \leq \kappa_{N,p}(c) \leq \frac{N}{p-1} \|\Delta c\|_\infty, \quad c \in \Delta^\infty(\mathbb{Z}). \quad (2.18)$$

Note. For notational convenience, we will henceforth write S_a for $S_{a,p}$ and T_a for $T_{a,p}$ in the proofs of theorems.

Lemma 2.25. *Suppose, for dilation factor $p \in \mathbb{Z}$, $p \geq 2$, that the mask a is positive, the sum rules (1.11) hold for a and $N = \deg(A) \geq p$. Then the subdivision operator $S_{a,p}$ maps $\Delta^\infty(\mathbb{Z})$ into itself and $\kappa = \kappa_{N,p}$ satisfies*

$$\kappa(S_{a,p}c) \leq \rho \kappa(c), \quad c \in \Delta^\infty(\mathbb{Z}), \quad (2.19)$$

where $\rho = \rho_{a,p,N}$ is defined by

$$\rho = \frac{1}{2} \sup \left\{ \sum_l |a_{i-pl} - a_{j-pl}| : i, j \in \mathbb{Z}, \quad |i - j| < \frac{N}{p-1} \right\} \quad (2.20)$$

and satisfies the inequalities

$$0 < \rho \leq 1 - \min \{a_0, a_1, \dots, a_N\} < 1. \quad (2.21)$$

Proof. Suppose $c \in \Delta^\infty(\mathbb{Z})$ and let $i, j \in \mathbb{Z}$ be such that $0 < j - i < \frac{N}{p-1}$. Then, because $a_j = 0$, $j \notin \{0, 1, \dots, N\}$, we have

$$(S_a c)_j - (S_a c)_i = \sum_{l=\mu}^{\nu} (a_{j-pl} - a_{i-pl}) c_l, \quad (2.22)$$

where $\mu = \left\lceil \frac{i-N}{p} \right\rceil$ and $\nu = \left\lfloor \frac{j}{p} \right\rfloor$. Define $\alpha = \min \{c_\mu, \dots, c_\nu\}$, $\beta = \max \{c_\mu, \dots, c_\nu\}$ and

$\gamma = \frac{\alpha+\beta}{2}$. Then, since

$$\nu - \mu \leq \frac{j}{p} - \frac{i-N}{p} = \frac{j-i}{p} + \frac{N}{p} < \frac{N}{p(p-1)} + \frac{N}{p} = \frac{N}{p-1},$$

it follows from (2.17) that

$$|c_l - \gamma| \leq \frac{1}{2}(\beta - \alpha) \leq \frac{1}{2}\kappa(c), \quad l \in \{\mu, \dots, \nu\}.$$

By using (2.22) and the sum rules (1.11), we then have

$$\left| (S_a c)_j - (S_a c)_i \right| = \left| \sum_{l=\mu}^{\nu} (a_{j-pl} - a_{i-pl}) (c_l - \gamma) \right| \leq \frac{1}{2}\kappa(c) \sum_l |a_{j-pl} - a_{i-pl}|.$$

A similar argument in the case $0 < i - j < \frac{N}{p-1}$ shows that

$$\left| (S_a c)_j - (S_a c)_i \right| \leq \frac{1}{2}\kappa(c) \sum_l |a_{j-pl} - a_{i-pl}| \quad (2.23)$$

for all $i, j \in \mathbb{Z}$ such that $|j - i| < \frac{N}{p-1}$.

Define $\varepsilon \in (0, 1)$ by $\varepsilon = \min \{a_0, a_1, \dots, a_N\}$. We proceed to prove that

$$\sum_l |a_{j-pl} - a_{i-pl}| \leq 2(1 - \varepsilon), \quad i, j \in \mathbb{Z}, \quad |j - i| < \frac{N}{p-1}. \quad (2.24)$$

To this end, first suppose $0 < j - i < \frac{N}{p-1}$ and choose $l_0 = \left\lfloor \frac{j-i}{p} \right\rfloor$. This yields

$$i - (p-1) \leq pl_0 \leq i, \quad (2.25)$$

so that we obtain, after recalling also $N \geq p \geq 2$,

$$\begin{aligned} 0 < j - i &\leq j - pl_0 \leq j - i + p - 1 < \frac{N}{p-1} + p - 1 \\ &= N - \frac{p-2}{p-1}(N-p) + \frac{1}{p-1} \\ &\leq N + \frac{1}{p-1} \leq N + 1. \end{aligned}$$

Since $j - pl_0 \in \mathbb{Z}$, we conclude that $1 \leq j - pl_0 \leq N$. From (2.25) we also obtain

$$0 \leq i - pl_0 \leq p - 1 \leq N - 1.$$

The positivity (1.9) of the mask a now yields $a_{j-pl_0} > 0$ and $a_{i-pl_0} > 0$. If $a_{j-pl_0} \geq a_{i-pl_0}$, it

follows from (1.9) and the sum rules (1.11) that

$$\begin{aligned} \sum_l |a_{j-pl} - a_{i-pl}| &\leq \sum_{l \neq l_0} (|a_{j-pl}| + |a_{i-pl}|) + (a_{j-pl_0} - a_{i-pl_0}) \\ &= \sum_l (a_{j-pl} + a_{i-pl}) - 2a_{i-pl_0} \leq 2 - 2\varepsilon. \end{aligned}$$

The proof for the case $a_{j-pl_0} < a_{i-pl_0}$ is similar. Thus (2.24) holds if $0 < j - i < \frac{N}{p-1}$. A similar proof shows the validity of (2.24) when $0 < i - j < \frac{N}{p-1}$.

From (2.23) and (2.20) we deduce that

$$\left| (S_a c)_j - (S_a c)_i \right| \leq \rho \kappa(c), \quad i, j \in \mathbb{Z}, \quad |j - i| < \frac{N}{p-1},$$

which, together with (2.20) and (2.24), implies

$$\sup \left\{ \left| (S_a c)_j - (S_a c)_i \right| : i, j \in \mathbb{Z}, \quad |j - i| < \frac{N}{p-1} \right\} \leq \rho \kappa(c) \leq (1 - \varepsilon) \kappa(c), \quad c \in \Delta^\infty(\mathbb{Z}),$$

so that by (2.18), we have $S_a c \in \Delta^\infty(\mathbb{Z})$. Also, by (2.17) and the definition of ε we see that the result (2.19) holds, where ρ satisfies the bounds (2.21). \square

We are now in a position to prove the main result of this section.

Theorem 2.26. *Under the conditions of Lemma 2.25, there exists a function $\phi \in C(\mathbb{R})$ such that (A, ϕ) is a p -refinement pair. The function ϕ has the properties (2.14), (2.15) and (2.16).*

Furthermore, for any $l \in \{2, 3, \dots, N\}$ the cascade algorithm $(T_{a,p}, N_l)$ converges uniformly to ϕ at a geometric rate, in the sense that

$$\left\| \phi - T_{a,p}^r N_l \right\|_\infty \leq (p-1) \frac{\rho^r}{1-\rho} \rightarrow 0, \quad r \rightarrow \infty, \quad (2.26)$$

with ρ defined by (2.20) and satisfying the bounds (2.21).

Also, for any $c \in \Delta^\infty(\mathbb{Z})$ the subdivision scheme $(S_{a,p}, c)$ converges at a geometric rate to the function $\Phi \in C(\mathbb{R})$ defined by

$$\Phi = \sum_j c_j \phi(\cdot - j), \quad (2.27)$$

in the sense that

$$\left\| \Delta S_{a,p}^r c \right\|_\infty \leq \frac{N}{p-1} \|\Delta c\|_\infty \rho^r \rightarrow 0, \quad r \rightarrow \infty, \quad (2.28)$$

and

$$\left\| \Phi \left(\frac{\cdot}{p^r} \right) - S_{a,p}^r c \right\|_\infty \leq \frac{N}{p-1} \|\Delta c\|_\infty \rho^r \rightarrow 0, \quad r \rightarrow \infty. \quad (2.29)$$

Proof. Let $l \in \{2, 3, \dots, N\}$. Then, for a fixed $r \in \mathbb{N}$, we have from (1.33) and (1.21) that

$$\begin{aligned} T_a^{r+1} N_l - T_a^r N_l &= \sum_j (S_a(S_a^r \delta))_j N_l(p^{r+1} \cdot -j) - \sum_i (S_a^r \delta)_i \sum_j a_j^{l,p} N_l(p^{r+1} \cdot -pi - j) \\ &= \sum_j \sum_{i=\mu_j}^{\nu_j} \left(a_{j-pi} - a_{j-pi}^{l,p} \right) (S_a^r \delta)_i N_l(p^{r+1} \cdot -j), \end{aligned} \quad (2.30)$$

where $\mu_j = \left\lceil \frac{j-N(p-1)}{p} \right\rceil$ and $\nu_j = \left\lfloor \frac{j}{p} \right\rfloor$ for $j \in \mathbb{Z}$. Set $\alpha_j = \min_{\mu_j \leq i \leq \nu_j} (S_a^r \delta)_i$, $\beta_j = \max_{\mu_j \leq i \leq \nu_j} (S_a^r \delta)_i$ and $\gamma_j = \frac{\alpha_j + \beta_j}{2}$ for $j \in \mathbb{Z}$. Since $\nu_j - \mu_j < (p-1) \frac{N}{p-1}$, we obtain

$$|(S_a^r \delta)_i - \gamma_j| \leq \frac{1}{2} (\beta_j - \alpha_j) \leq \frac{1}{2} (p-1) \kappa(S_a^r \delta), \quad i \in \{\mu_j, \dots, \nu_j\}, \quad j \in \mathbb{Z}.$$

By using (2.30), the sum rules (1.11) for a and $a^{l,p}$, (1.22b), the positivity of the masks a and $a^{l,p}$, (1.22c) and lastly Lemma 2.25, we now obtain

$$\begin{aligned} |(T_a^{r+1} N_l)(x) - (T_a^r N_l)(x)| &\leq \sum_j \left| \sum_{i=\mu_j}^{\nu_j} \left(a_{j-pi} - a_{j-pi}^{l,p} \right) ((S_a^r \delta)_i - \gamma_j) \right| N_l(p^{r+1}x - j) \\ &\leq \frac{1}{2} (p-1) \kappa(S_a^r \delta) \sum_j N_l(p^{r+1}x - j) \sum_i |a_{j-pi} - a_{j-pi}^{l,p}| \\ &\leq \frac{1}{2} (p-1) \kappa(S_a^r \delta) \sum_j N_l(p^{r+1}x - j) \sum_i \left(a_{j-pi} + a_{j-pi}^{l,p} \right) \\ &= (p-1) \kappa(S_a^r \delta) \sum_j N_l(p^{r+1}x - j) = (p-1) \kappa(S_a^r \delta) \\ &\leq (p-1) \rho \kappa(S_a^{r-1} \delta) \leq \dots \leq (p-1) \rho^r \kappa(\delta) = (p-1) \rho^r \end{aligned}$$

for all $x \in \mathbb{R}$. Thus

$$|T_a^{r+q} N_l(x) - T_a^r N_l(x)| \leq (p-1) \frac{\rho^r}{1-\rho}, \quad r \in \mathbb{N}, \quad q \in \mathbb{N}, \quad x \in \mathbb{R}.$$

We deduce that $\{T_a^r N_l : r \in \mathbb{Z}_+\}$ is a Cauchy sequence in the Banach space $C_u(\mathbb{R})$. It follows that a function $\phi \in C_u(\mathbb{R})$ exists such that (2.26) holds. From the continuity of T_a on $C_u(\mathbb{R})$ we have

$$\phi = \lim_{r \rightarrow \infty} T_a^{r+1} N_l = \lim_{r \rightarrow \infty} T_a(T_a^r N_l) = T_a \left(\lim_{r \rightarrow \infty} T_a^r N_l \right) = T_a \phi,$$

so that ϕ does indeed satisfy the refinement equation (1.1).

A proof that ϕ has the finite support property (2.14) appears in the proof of [27: Theorem 3.1]. Thus $\phi \in L^1(\mathbb{R})$. For a proof that ϕ satisfies the positivity condition (2.16), we again refer to the proof of [27: Theorem 3.1]. Hence $\phi \neq 0$ and it follows that (A, ϕ) is indeed a p -refinement pair.

If a function f has compact support and satisfies $\sum_j f(x-j) = 1$, $x \in \mathbb{R}$, then we have

from the sum rules (1.11) that, for $x \in \mathbb{R}$,

$$\begin{aligned} \sum_j (T_a f)(x - j) &= \sum_j \sum_l a_l f(px - pj - l) \\ &= \sum_j \sum_l a_{l-pj} f(px - l) = \sum_l \left[\sum_j a_{l-pj} \right] f(px - l) = 1. \end{aligned}$$

Then from (1.22c) it follows inductively that $\sum_j (T_a^r N_l)(x - j) = 1$, $x \in \mathbb{R}$, $r \in \mathbb{Z}_+$, so that by a limit argument based also on (2.26), it follows that (2.15) holds.

Furthermore, (2.28) follows from (2.18) and Lemma 2.25. From (2.27) and the refinability of ϕ we have, for $j \in \mathbb{Z}$ and $r \in \mathbb{Z}_+$, that

$$\begin{aligned} \Phi\left(\frac{j}{p^r}\right) &= \sum_i c_i \sum_n a_n \phi\left(\frac{j}{p^{r-1}} - pi - n\right) \\ &= \sum_n \sum_i a_{n-pi} c_i \phi\left(\frac{j}{p^{r-1}} - n\right) \\ &= \sum_n (S_a c)_n \phi\left(\frac{j}{p^{r-1}} - n\right) \\ &= \cdots = \sum_i (S_a^r c)_i \phi(j - i) = \sum_i \phi(i) (S_a^r c)_{j-i}. \end{aligned}$$

Then from (2.14), (2.15), (2.16) and the definition (2.17) of κ , we have, for $j \in \mathbb{Z}$ and $r \in \mathbb{Z}_+$,

$$\begin{aligned} \left| \Phi\left(\frac{j}{p^r}\right) - (S_a c)_j \right| &= \left| \sum_i \phi(i) \left[(S_a^r c)_{j-i} - (S_a^r c)_j \right] \right| \\ &\leq \sum_{i=1}^{\lceil \frac{N}{p-1} \rceil - 1} \phi(i) \left| (S_a^r c)_{j-i} - (S_a^r c)_j \right| \leq \kappa (S_a^r c), \end{aligned}$$

after noting also that $\lceil \frac{N}{p-1} \rceil - 1 < \frac{N}{p-1}$. Then (2.29) follows from (2.18) and Lemma 2.25. \square

Remark 2.27. By Theorem 1.15 the demands that $N \geq p$ and that the sum rules hold are equivalent to requiring that A has the form $A = E_{p-1}B$ with $\deg(B) \geq 1$ and $B(1) = 1$.

Example 2.28. Let $p = 3$ and consider the mask symbol

$$A(z) = \frac{1}{9} (2 + z + 3z^2 + z^3 + 2z^4) = \frac{1}{9} (1 + z + z^2) (2 - z + 2z^2), \quad z \in \mathbb{C}.$$

Observe in particular that A does not satisfy the conditions of Theorem 2.24, since the second factor has a negative coefficient. However, Theorem 2.26 guarantees the existence of a continuous function ϕ^{DR} (so named because it is a shifted version of the ‘‘De Rham function’’ considered in [18: Section 6.1], from where this example was obtained) such that (A, ϕ^{DR}) is a 3-refinement pair and that the subdivision algorithm $(S_{a,3}, \delta)$ will converge to ϕ^{DR} . In

[18] it is shown that the function ϕ^{DR} is continuous, but nowhere differentiable in its support interval $[0, 2]$, being the limit of a certain fractal process. The graph of ϕ^{DR} is shown in Figure 2.3.

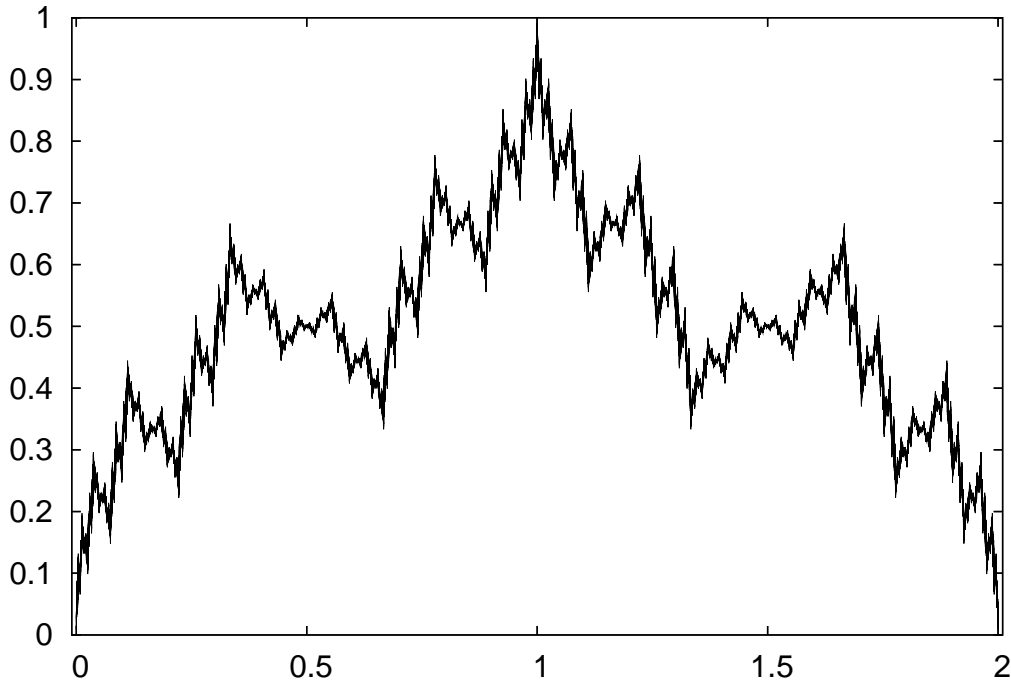


Figure 2.3: The shifted De Rham function ϕ^{DR} of Example 2.28.

Example 2.29. Another example (which is non-symmetric) for Theorem 2.26, also with dilation factor 3 and not covered by Theorem 2.24, is provided by the mask B , given by

$$B(z) = \frac{1}{12} (2 + z + 3z^2 + 2z^3 + 3z^4 + z^5) = \frac{1}{12} (1 + z + z^2) (2 - z + 2z^2 + z^3), \quad z \in \mathbb{C}.$$

The corresponding refinable function ψ has a most interesting shape, as shown in Figure 2.4.

2.5 Preservation of subdivision convergence

In this section we consider the effect with respect to subdivision convergence of adding step function mask symbols to a given mask symbol. The next theorem shows that adding generalised Berg-Plonka factors to the mask symbol preserves subdivision convergence. Our result provides a substantial generalisation of [34: Proposition 2.4], where the result was proved only for $p = 2$ and the factor $\frac{1}{2}(1 + z)$.

Theorem 2.30. For a dilation factor $p \in \mathbb{Z}$, $p \geq 2$, suppose that the mask symbol A has the form $A = PB$, where P is a p -GBP factor, and that the subdivision scheme $(S_{b,p}, \delta)$ converges to the function ψ . Then the subdivision scheme $(S_{a,p}, \delta)$ converges to the function ϕ given by (2.3).

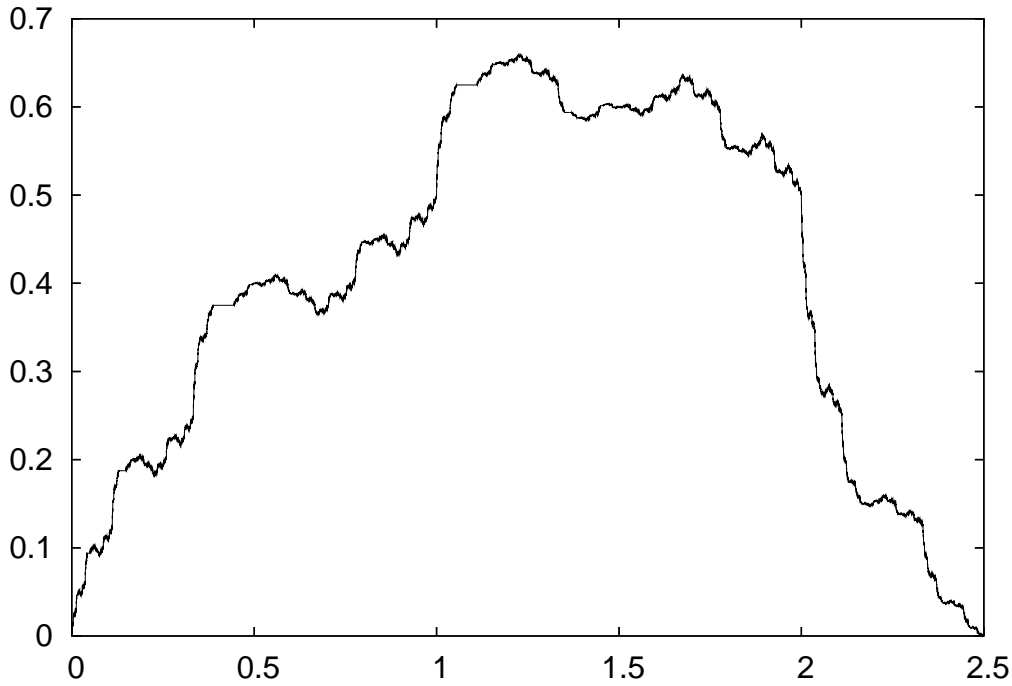


Figure 2.4: The interesting refinable function ψ of Example 2.29.

Proof. By Theorem 1.16, (B, ψ) is a p -refinement pair. It follows from Corollary 2.9 that (A, ϕ) , where ϕ is given by (2.3), is a p -refinement pair. Note that ϕ is uniformly continuous, being continuous and compactly supported.

Set $c^{(r)} = S_{a,p}^r \delta$ and $d^{(r)} = S_{b,p}^r \delta$ for $r \in \mathbb{Z}_+$ and suppose P is a p -GBP factor of level $k \in \mathbb{Z}_+$. The central idea in our proof is to use the convergence of a certain Riemann sum to the corresponding integral. To this end, we shall use the polynomials U_r , given by

$$U_r = E_{p^{r-k-1}}^{\langle p^k \rangle} \text{Lpol}(d^{(r)}), \quad r \in \{k, k+1, \dots\}.$$

From (1.26), (1.8b), (2.2), (1.19b), (1.19c) and Lemma 2.14 we obtain, for $r \in \{k, k+1, \dots\}$,

$$\begin{aligned} \text{Lpol}(c^{(r)}) &= p^r \prod_{j=0}^{r-1} A^{\langle p^j \rangle} = p^r \prod_{j=0}^{r-1} (P^{\langle p^j \rangle} B^{\langle p^j \rangle}) \\ &= p^r \prod_{j=0}^{r-1} \left(E_{p^{-1}}^{\langle p^j \rangle} \frac{R^{\langle p^{j+1} \rangle}}{R^{\langle p^j \rangle}} B^{\langle p^j \rangle} \right) \\ &= E_{p^{r-1}} \frac{R^{\langle p^r \rangle}}{R} p^r \prod_{j=0}^{r-1} B^{\langle p^j \rangle} \\ &= E_{p^{r-k-1}}^{\langle p^k \rangle} \frac{E_{p^{k-1}} R^{\langle p^r \rangle}}{R} \text{Lpol}(d^{(r)}) \\ &= E_{p^{r-k-1}}^{\langle p^k \rangle} W \text{Lpol}(d^{(r)}) R^{\langle p^r \rangle} = W U_r R^{\langle p^r \rangle}, \end{aligned} \tag{2.31}$$

with the polynomial W as in Lemma 2.14. Let $L = \deg(W)$. For $r \in \{k, k+1, \dots\}$ and $j \in \mathbb{Z}$, by (2.31) and the fact that $W(1) = 1$, we have

$$\begin{aligned} c_j^{(r)} - \phi\left(\frac{j}{p^r}\right) &= \sum_{l=0}^L [W]_l \left([U_r R^{(p^r)}]_{j-l} - \phi\left(\frac{j}{p^r}\right) \right) \\ &= \sum_{l=0}^L [W]_l \left([U_r R^{(p^r)}]_{j-l} - \phi\left(\frac{j-l}{p^r}\right) + \phi\left(\frac{j-l}{p^r}\right) - \phi\left(\frac{j}{p^r}\right) \right), \end{aligned} \quad (2.32)$$

so that, for $j \in \mathbb{Z}$ and $r \in \{k, k+1, \dots\}$,

$$\left| c_j^{(r)} - \phi\left(\frac{j}{p^r}\right) \right| \leq \sum_{l=0}^L |[W]_l| \left(\left| [U_r R^{(p^r)}]_{j-l} - \phi\left(\frac{j-l}{p^r}\right) \right| + \left| \phi\left(\frac{j-l}{p^r}\right) - \phi\left(\frac{j}{p^r}\right) \right| \right). \quad (2.33)$$

We have by (1.8c) and (1.17) that, for $j \in \mathbb{Z}$ and $r \in \{k, k+1, \dots\}$,

$$\begin{aligned} [U_r]_j &= \left[\text{Lpol}(d^{(r)}) E_{p^{r-k-1}}^{(p^k)} \right]_j = \sum_l d_{j-p^k l}^{(r)} [E_{p^{r-k-1}}]_l \\ &= \frac{1}{p^{r-k}} \sum_{l=0}^{p^{r-k-1}} d_{j-p^k l}^{(r)}. \end{aligned}$$

We thus obtain, for $r \in \{k, k+1, \dots\}$ and $j \in \mathbb{Z}$,

$$\begin{aligned} &\left| [U_r]_j - \int_{\frac{j}{p^r}-1}^{\frac{j}{p^r}} \psi(s) ds \right| \\ &= \left| \frac{1}{p^{r-k}} \sum_{l=0}^{p^{r-k-1}} \left(d_{j-p^k l}^{(r)} - \psi\left(\frac{j-p^k l}{p^r}\right) + \psi\left(\frac{j-p^k l}{p^r}\right) \right) - \int_{\frac{j}{p^r}-1}^{\frac{j}{p^r}} \psi(s) ds \right| \\ &\leq \left(\frac{1}{p^{r-k}} \sum_{l=0}^{p^{r-k-1}} \left| \psi\left(\frac{j-p^k l}{p^r}\right) - d_{j-p^k l}^{(r)} \right| \right) + \left| \frac{1}{p^{r-k}} \sum_{l=0}^{p^{r-k-1}} \psi\left(\frac{j}{p^r} - \frac{l}{p^{r-k}}\right) - \int_{\frac{j}{p^r}-1}^{\frac{j}{p^r}} \psi(s) ds \right| \\ &\leq \left(\frac{1}{p^{r-k}} \sum_{l=0}^{p^{r-k-1}} \left| \psi\left(\frac{j-p^k l}{p^r}\right) - d_{j-p^k l}^{(r)} \right| \right) + \sum_{l=0}^{p^{r-k-1}} \int_{\frac{j}{p^r} - \frac{l+1}{p^{r-k}}}^{\frac{j}{p^r} - \frac{l}{p^{r-k}}} \left| \psi\left(\frac{j}{p^r} - \frac{l}{p^{r-k}}\right) - \psi(s) \right| ds. \end{aligned} \quad (2.34)$$

Let $\varepsilon > 0$ be given. By the assumed subdivision convergence for the mask b , there exists a non-negative integer S_1 such that

$$\left| \psi\left(\frac{j-p^k l}{p^r}\right) - d_{j-p^k l}^{(r)} \right| < \varepsilon, \quad l \in \mathbb{Z}_{p^{r-k}}, \quad j \in \mathbb{Z}, \quad r > S_1. \quad (2.35)$$

Since ψ is uniformly continuous, being continuous and of compact support, there exists a

$\tau > 0$ such that

$$|\psi(x) - \psi(y)| < \varepsilon, \quad |x - y| < \tau, \quad x, y \in \mathbb{R}. \quad (2.36)$$

Now set

$$S_2 = \max \{k, \lceil -\log_p \tau \rceil + k\}.$$

Then for $r > S_2$ we obtain

$$\left| \left(\frac{j}{p^r} - \frac{l}{p^{r-k}} \right) - s \right| \leq \frac{1}{p^{r-k}} < \tau, \quad s \in \left[\frac{j}{p^r} - \frac{l+1}{p^{r-k}}, \frac{j}{p^r} - \frac{l}{p^{r-k}} \right], \quad l \in \mathbb{Z}_{p^{r-k}}, \quad j \in \mathbb{Z}.$$

Thus we conclude from (2.36) that

$$\sum_{l=0}^{p^{r-k}-1} \int_{\frac{j}{p^r} - \frac{l+1}{p^{r-k}}}^{\frac{j}{p^r} - \frac{l}{p^{r-k}}} \left| \psi \left(\frac{j}{p^r} - \frac{l}{p^{r-k}} \right) - \psi(s) \right| ds \leq \sum_{l=0}^{p^{r-k}-1} \int_{\frac{j}{p^r} - \frac{l+1}{p^{r-k}}}^{\frac{j}{p^r} - \frac{l}{p^{r-k}}} \varepsilon ds = \varepsilon, \quad j \in \mathbb{Z}, \quad r > S_2. \quad (2.37)$$

By substituting (2.35) and (2.37) into (2.34) and letting $S_3 = \max \{S_1, S_2\}$, we obtain

$$\left| [U_r]_j - \int_{\frac{j}{p^r}-1}^{\frac{j}{p^r}} \psi(s) ds \right| < \left(\frac{1}{p^{r-k}} \sum_{l=0}^{p^{r-k}-1} \varepsilon \right) + \varepsilon = 2\varepsilon, \quad j \in \mathbb{Z}, \quad r \geq S_3. \quad (2.38)$$

Next we obtain, by use of (1.8c) and (2.3), for $r \in \{k, k+1, \dots\}$ and $j \in \mathbb{Z}$,

$$[U_r R^{(p^r)}]_j - \phi \left(\frac{j}{p^r} \right) = \sum_k [R]_k \left([U_r]_{j-p^r k} - \int_{\frac{j}{p^r}-k-1}^{\frac{j}{p^r}-k} \psi(s) ds \right). \quad (2.39)$$

After setting $C_1 = \sum_k |[R]_k| < \infty$, we have by (2.38) and (2.39) that $r \geq S_3$ implies

$$\left| [U_r R^{(p^r)}]_j - \phi \left(\frac{j}{p^r} \right) \right| \leq \sum_k |[R]_k| \left| [U_r]_{j-p^r k} - \int_{\frac{j}{p^r}-k-1}^{\frac{j}{p^r}-k} \psi(s) ds \right| < 2C_1 \varepsilon, \quad j \in \mathbb{Z}. \quad (2.40)$$

We have, for $j \in \mathbb{Z}$ and $l \in \{0, \dots, L\}$, that $\left| \left(\frac{j-l}{p^r} \right) - \frac{j}{p^r} \right| \leq \frac{L}{p^r} \rightarrow 0$ independently of j and l as $r \rightarrow \infty$. By the uniform continuity of ϕ , there thus exists an integer $S_4 \in \mathbb{Z}_+$ such that

$$\left| \phi \left(\frac{j-l}{p^r} \right) - \phi \left(\frac{j}{p^r} \right) \right| < \varepsilon, \quad j \in \mathbb{Z}, \quad l \in \{0, \dots, L\}, \quad r \geq S_4. \quad (2.41)$$

Letting $C_2 = \sum_{l=0}^L |[W]_l| < \infty$, by substituting (2.40) and (2.41) into (2.33), we obtain

$$\left| c_j^{(r)} - \phi \left(\frac{j}{p^r} \right) \right| < \sum_{l=0}^L |[W]_l| (2C_1 \varepsilon + \varepsilon) = C_2 (2C_1 + 1) \varepsilon, \quad j \in \mathbb{Z}, \quad r \geq \max \{S_3, S_4\},$$

which yields the desired result. \square

An interesting question that presents itself at this stage, is whether, if subdivision for a specified mask has a geometric convergence rate, one will still obtain a geometric convergence rate after adding a GBP factor. To obtain some answers regarding this question, we will use the following result, which is given by [44: Theorem 1]. As shown there, the proof is immediate by using the mean value theorem for integrals on each of the intervals $\left[\frac{k-1}{n}, \frac{k}{n}\right]$, $k \in \{1, \dots, n\}$.

Lemma 2.31. *Let f be continuous in the interval $[0, 1]$ and suppose the function $\omega : [0, 1] \rightarrow \mathbb{R}$ is such that*

$$|f(x) - f(y)| \leq \omega(\tau), \quad |x - y| \leq \tau, \quad x, y \in [0, 1].$$

Then we have

$$\left| \int_0^1 f(x) dx - \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) \right| \leq \omega\left(\frac{1}{n}\right). \quad (2.42)$$

We can now prove the following extension to Theorem 2.30.

Theorem 2.32. *In Theorem 2.30, with $c^{(r)}$, $d^{(r)}$, k , C_1 , C_2 and L given as in the proof of that result, if we additionally have that $(S_{b,p}, \delta)$ converges to ψ at a geometric rate, in the sense that*

$$\sup_j \left| \psi\left(\frac{j}{p^r}\right) - d_j^{(r)} \right| \leq K_1 \rho^r, \quad r \in \mathbb{Z}_+, \quad (2.43)$$

for some constants $\rho \in (0, 1)$ and $K_1 \in \mathbb{R}$, as well as that ψ is α -Hölder continuous, i.e. there are constants $\alpha \in (0, 1]$ and $K_2 \in \mathbb{R}$ such that

$$|\psi(x) - \psi(y)| \leq K_2 |x - y|^\alpha, \quad x, y \in \mathbb{R}, \quad (2.44)$$

then $(S_{a,p}, \delta)$ will converge to ϕ at a geometric rate, in the sense that

$$\sup_j \left| \phi\left(\frac{j}{p^r}\right) - c_j^{(r)} \right| \leq K \tau^r, \quad r \geq k, \quad (2.45)$$

with the constants $K \in \mathbb{R}$ and $\tau \in [p^{-1}, 1)$ given by

$$K = C_2 C_1 (K_1 + K_2 (p^{\alpha k} + L)) \quad (2.46)$$

and

$$\tau = \max \{ \rho, p^{-\alpha} \}. \quad (2.47)$$

Proof. The proof follows the same line of argument as in Theorem 2.30, with the following changes. We use (2.44) and Lemma 2.31 to replace the estimate (2.37) by

$$\left| \frac{1}{p^{r-k}} \sum_{l=0}^{p^{r-k}-1} \psi\left(\frac{j}{p^r} - \frac{l}{p^{r-k}}\right) - \int_{\frac{j}{p^r}-1}^{\frac{j}{p^r}} \psi(s) ds \right| \leq K_2 \left(\frac{1}{p^{r-k}}\right)^\alpha, \quad j \in \mathbb{Z}, \quad r \geq k. \quad (2.48)$$

We use (2.34), (2.43), (2.48) and (2.47) to replace the estimate (2.38) by

$$\begin{aligned} \left| [U_r]_j - \int_{\frac{j}{p^r-1}}^{\frac{j}{p^r}} \psi(s) ds \right| &< \left(\frac{1}{p^{r-k}} \sum_{l=0}^{p^{r-k}-1} K_1 \rho^r \right) + K_2 \left(\frac{1}{p^{r-k}} \right)^\alpha \\ &= K_1 \rho^r + p^{\alpha k} K_2 (p^{-\alpha})^r \leq (K_1 + p^{\alpha k} K_2) \tau^r \end{aligned} \quad (2.49)$$

for $j \in \mathbb{Z}$ and $r \geq k$. This in turn yields that (2.40) becomes

$$\left| [U_r R^{(p^r)}]_j - \phi\left(\frac{j}{p^r}\right) \right| \leq C_1 (K_1 + p^{\alpha k} K_2) \tau^r, \quad j \in \mathbb{Z}. \quad (2.50)$$

From (2.3), (2.44) and the definition of C_1 we obtain, for $x \in \mathbb{R}$,

$$\begin{aligned} |\phi'(x)| &= \left| \sum_j [R]_j (\psi(x-j) - \psi(x-j-1)) \right| \\ &\leq \left| \sum_j [R]_j \right| K_2 = C_1 K_2, \end{aligned}$$

so that, since $p^{-1} \leq p^{-\alpha} \leq \tau$, (2.41) can here be replaced by

$$\left| \phi\left(\frac{j-l}{p^r}\right) - \phi\left(\frac{j}{p^r}\right) \right| \leq \frac{L}{p^r} \|\phi'\|_\infty \leq LC_1 K_2 (p^{-1})^r \leq LC_1 K_2 \tau^r \quad (2.51)$$

for $j \in \mathbb{Z}$, $l \in \{0, \dots, L\}$ and $r \in \mathbb{Z}_+$. Substituting (2.50) and (2.51) into (2.33) and noting the definition of C_2 yields

$$\left| c_j^{(r)} - \phi\left(\frac{j}{p^r}\right) \right| < C_2 (C_1 (K_1 + p^{\alpha k} K_2) \tau^r + LC_1 K_2 \tau^r)$$

for $j \in \mathbb{Z}$ and $r \geq k$, which by (2.46) yields the desired result (2.45). \square

2.6 Polynomial sections in refinable functions

In this section, we consider refinable functions which correspond to polynomials on certain intervals within their support, while not necessarily having polynomial behaviour in other intervals.

To obtain results in this direction, we consider a special class of step functions, which are merely stretched and appropriately scaled versions of the characteristic function. Given any $p \in \mathbb{N}$, $p \geq 2$, by Theorem 1.10 and the fact that (E_{p-1}, N_1) is a p -refinement pair, we know that $(E_{p-1}^{(m)}, \frac{1}{m} N_1(\frac{\cdot}{m}))$ is a p -refinement pair for $m \in \mathbb{N}$. This is consistent with

Proposition 2.6, since applying (1.19a) twice, once with m and p interchanged, yields

$$E_{p-1}^{(m)} = \frac{E_{pm-1}}{E_{m-1}} = E_{p-1} \frac{E_{m-1}^{(p)}}{E_{m-1}},$$

while it is also true that $\sum_j [E_{m-1}]_j \chi(\cdot - j) = \frac{1}{m} N_1\left(\frac{\cdot}{m}\right)$.

2.6.1 The operator L_m

To express convolution by stretched characteristic functions, we introduce, for $m \in \mathbb{N}$, the operator $L_m : L^1(\mathbb{R}) \rightarrow L^1(\mathbb{R})$ defined by

$$L_m f = \frac{1}{m} \int_0^m f(\cdot - t) dt, \quad f \in L^1(\mathbb{R}). \quad (2.52)$$

Now Theorem 2.1, along with the fact that $\left(E_{p-1}^{(m)}, \frac{1}{m} N_1\left(\frac{\cdot}{m}\right)\right)$ is a p -refinement pair for $m \in \mathbb{N}$, immediately yield the following.

Corollary 2.33. *For $p \in \mathbb{Z}$, $p \geq 2$, if (B, ψ) is a p -refinement pair and the polynomial A satisfies $A = E_{p-1}^{(m)} B$, then $(A, L_m \psi)$ is a p -refinement pair.*

It is easy to verify that, for $m \in \mathbb{N}$, the linear operator L_m is bounded on $C_u(\mathbb{R})$, with $\|L_m\|_\infty = 1$. For $m \in \mathbb{N}$, one also has $\|L_m f\|_1 = \|f\|_1$, $f \in L^1(\mathbb{R})$, so that L_m is also bounded on $L^1(\mathbb{R})$, with $\|L_m\|_1 = 1$. Moreover we have, for all $p, q, r \in \mathbb{N}$, that $L_p L_q = L_q L_p$ and $(L_p L_q) L_r = L_p (L_q L_r)$. Note also that, for $m \in \mathbb{N}$ and $l \in \mathbb{Z}_+$, we have

$$L_m f \in C^{l+1}(\mathbb{R}), \quad f \in C^l(\mathbb{R}). \quad (2.53)$$

The next lemma brings to light some useful relationships involving the operator family $\{L_m : m \in \mathbb{N}\}$, on which we shall later rely. Henceforth, for an integer $l \in \mathbb{N}$ and vectors $\mathbf{x} = (x_1, \dots, x_l)$, $\mathbf{y} = (y_1, \dots, y_l) \in \mathbb{Z}^l$, we shall use the definition $\sum \mathbf{x} = \sum_{i=1}^l x_i$ and shall write $\mathbf{x} < \mathbf{y}$ ($\mathbf{x} \leq \mathbf{y}$) if and only if $x_i < y_i$ ($x_i \leq y_i$), $i \in \{1, \dots, l\}$.

Lemma 2.34. *Suppose $f \in L^1$. Then the following holds:*

(a) *For $m \in \mathbb{N}$ we have*

$$m(L_m f) = \sum_{j=0}^{m-1} (L_1 f)(\cdot - j). \quad (2.54)$$

(b) *If, for $l \in \mathbb{N}$, we have $\{m_i : i \in \{1, \dots, l\}\} \subset \mathbb{N}$, then*

$$\left(\prod_{i=1}^l m_i L_{m_i} \right) f = \sum_{\mathbf{0} \leq \mathbf{j} < (m_1, \dots, m_l)} (L_1^l f)(\cdot - \sum \mathbf{j}). \quad (2.55)$$

Proof. (a) Let $f \in L^1(\mathbb{R})$. Then (2.52) implies that

$$m(L_m f)(x) = \int_{x-m}^x f(t) dt = \sum_{j=0}^{m-1} \int_{x-j-1}^{x-j} f(t) dt = \sum_{j=0}^{m-1} (L_1 f)(x-j), \quad x \in \mathbb{R},$$

thereby establishing (2.54).

(b) The above also establishes (2.55) for the case $l = 1$. Assuming now that (2.55) holds for a fixed $l \in \mathbb{N}$, we have by use of (2.54) that

$$\begin{aligned} \left(\prod_{i=1}^{l+1} m_i L_{m_i} \right) f &= m_{l+1} L_{m_{l+1}} \left(\sum_{\mathbf{0} \leq \mathbf{j} < (m_1, \dots, m_l)} (L_1^l f) (\cdot - \sum \mathbf{j}) \right) \\ &= \sum_{k=0}^{m_{l+1}-1} \left(L_1 \left(\sum_{\mathbf{0} \leq \mathbf{j} < (m_1, \dots, m_l)} (L_1^l f) \right) \right) (\cdot - \sum \mathbf{j} - k) \\ &= \sum_{k=0}^{m_{l+1}-1} \sum_{\mathbf{0} \leq \mathbf{j} < (m_1, \dots, m_l)} (L_1^{l+1} f) (\cdot - \sum \mathbf{j} - k) \\ &= \sum_{\mathbf{0} \leq \mathbf{j} < (m_1, \dots, m_l, m_{l+1})} (L_1^{l+1} f) (\cdot - \sum \mathbf{j}), \end{aligned}$$

which completes the inductive step, so that (2.55) holds for all $l \in \mathbb{N}$. \square

2.6.2 Constant sections : “Table Mountlets”

We now investigate the behaviour of the refinable function $L_m \psi$ of Corollary 2.33 as a function of the integer parameter m . To this end, we construct a family of refinable functions as follows. Suppose, for $p \in \mathbb{Z}$, $p \geq 2$, that (B, ϕ_B) is a p -refinement pair with $\deg(B) = M$. For $m \in \mathbb{N}$, let the mask A_m be given by

$$A_m = E_{p-1}^{(m)} B, \quad (2.56)$$

and let $\phi_m = L_m \psi$ be the p -refinable function associated to the mask A_m , as in Corollary 2.33.

Then $\deg(A_m) = M + m(p-1)$, so that Theorem 1.8(d) yields

$$\phi_m(x) = 0, \quad x \notin \left(0, \frac{M}{p-1} + m \right). \quad (2.57)$$

Some interesting relationships exist between the different members of the family of functions $\{\phi_m : m \in \mathbb{N}\}$. The next lemma will help us to uncover some of them.

Lemma 2.35. For all $m \in \mathbb{N}$, the function ϕ_m , as defined above, satisfies

$$\phi_m = \frac{1}{m} \sum_{j=0}^{m-1} \phi_1(\cdot - j). \quad (2.58)$$

Proof. It follows immediately from Corollary 2.33 and (2.54) that

$$m\phi_m = mL_m\phi_B = \sum_{j=0}^{m-1} (L_1\phi_B)(\cdot - j) = \sum_{j=0}^{m-1} \phi_1(\cdot - j),$$

thereby establishing (2.58). \square

Remark. The result (2.58) was already noted in [3: Theorem 3.6] for the case $p = 2$.

Lemma 2.35 allows us to establish the following relationship, the meaning of which should become clearer from the subsequent examples.

Theorem 2.36. Suppose, for $p \in \mathbb{Z}$, $p \geq 2$, that (B, ϕ_B) and (A_m, ϕ_m) are normalised p -refinement pairs with A_m defined by (2.56) for $m \in \mathbb{N}$ and $\deg(B) = M$. Then the family of refinable functions $\{\phi_m : m \in \mathbb{N}\}$ satisfies, for $m \in \mathbb{N}$,

$$m\phi_m(x) = \begin{cases} n\phi_n(x) & \text{if } x \leq n, \quad n \in \{1, \dots, m\} \\ 1 & \text{if } \frac{M}{p-1} \leq x \leq m, \text{ provided that } m \geq \frac{M}{p-1} \\ n\phi_n(x + n - m) & \text{if } x \geq \frac{M}{p-1} + m - n, \quad n \in \{1, \dots, m\}. \end{cases} \quad (2.59)$$

Proof. Let $m \in \mathbb{N}$ be fixed.

If $x \leq n$, with $n \in \{1, \dots, m\}$, we have from (2.58) that $m\phi_m(x) = \sum_{j=0}^{n-1} \phi_1(x - j)$, since $n - 1 \leq m - 1$ and if $j \geq n$, then $x - j \leq 0$, so that $\phi_1(x - j) = 0$ from (2.57). However, by (2.58), $\sum_{j=0}^{n-1} \phi_1(x - j) = n\phi_n(x)$. This proves the top line of (2.59).

If $m \geq \frac{M}{p-1}$ and $\frac{M}{p-1} \leq x \leq m$, then the inequalities $x - m \leq 0$ and $\frac{M}{p-1} \leq x$ imply that $\text{supp}(\phi_B) \subset \left[0, \frac{M}{p-1}\right] \subset [x - m, x]$, so that

$$m\phi_m(x) = m(L_m\phi_B)(x) = \int_{x-m}^x \phi_B(t)dt = \int_{-\infty}^{\infty} \phi_B(t)dt = 1,$$

since (B, ϕ_B) is a normalised p -refinement pair.

To prove the bottom line of (2.59), observe that, for $n \in \{1, \dots, m\}$, if $x \geq \frac{M}{p-1} + m - n$, then since $x - j \geq \frac{M}{p-1} + 1$ for $j \leq m - n - 1$, we have from (2.57) that $\phi_1(x - j) = 0$ for $j \leq m - n - 1$. It now follows from (2.58) that

$$m\phi_m(x) = \sum_{j=m-n}^{m-1} \phi_1(x - j).$$

By again using (2.58), we obtain

$$n\phi_n(x - m + n) = \sum_{j=0}^{n-1} \phi_1(x - m + n - j) = \sum_{j=m-n}^{m-1} \phi_1(x - j) = m\phi_m(x).$$

□

Remarks.

- (a) An important immediate implication of (2.59) is that all the members of the family $\{\phi_m : m > \frac{M}{p-1}\}$ are equally smooth, i.e. members of the same regularity class, since with increasing m there is only an increasingly long constant section inserted between scalar multiples of the same end sections of the function (see the examples below). The issue of regularity will be further considered in Chapter 3.
- (b) The interval of constancy appearing in the middle line of (2.59) was already noted in [4: Section 4] for the case $p = 2$.

Example 2.37. As an illustration of Theorem 2.36 for $p = 2$, consider the following example, derived from the cardinal B -splines. Let $B = (E_1)^2$, with corresponding 2-refinable function $\phi_B = N_2$. We have $\phi_B \in C(\mathbb{R}) \setminus C^1(\mathbb{R})$ and $\phi_B(x) = 0$, $x \notin (0, 2)$. We now form our family of 2-refinement pairs (A_m, ϕ_m) according to (2.56). In particular we get $A_1 = (E_1)^3$ and $\phi_1 = N_3$. Figure 2.5 shows the first 5 refinable functions of the obtained family.

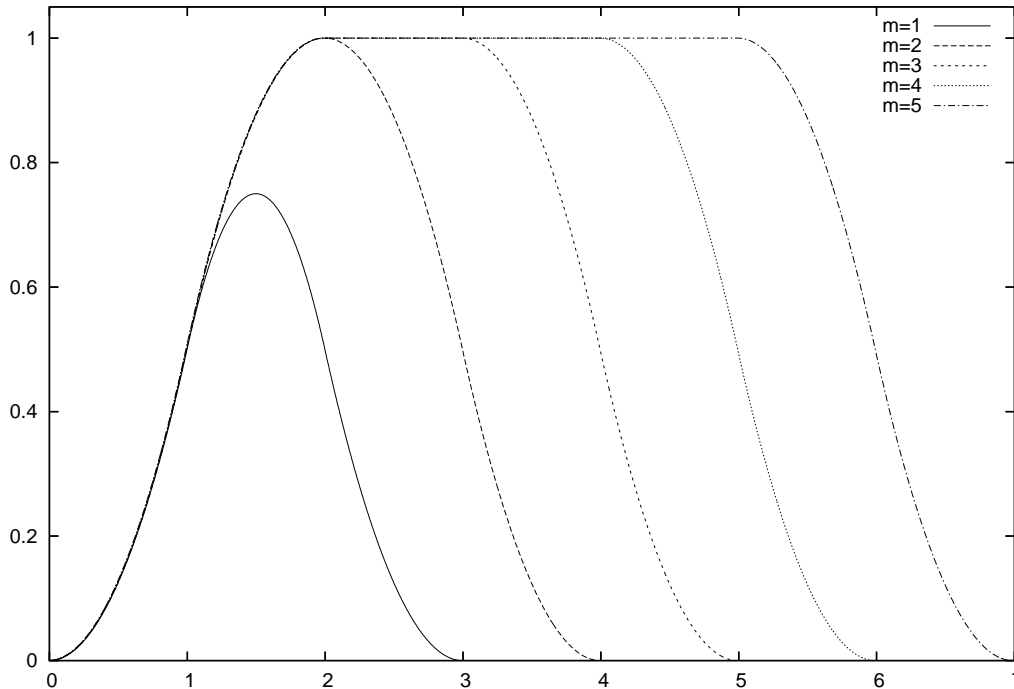


Figure 2.5: The plots of $m\phi_m$, $m = 1, 2, 3, 4, 5$ for Example 2.37.

From this figure, the meaning of (2.59) becomes more clear. All the (scaled) functions have the same values over the first and last parts of their respective support intervals. The function ϕ_1 agrees with the other functions over the first and last unit intervals of its support, that is

$$m\phi_m(x) = \begin{cases} \phi_1(x), & x \leq 1, \\ \phi_1(x - m + 1), & x \geq M + m - 1 = m + 1. \end{cases}$$

In general, for $n \in \mathbb{N}$, $n\phi_n$ agrees with all the functions $\{m\phi_m : m \geq n\}$ over the intervals of length n at the start and end of their respective supports (which are $[0, 2 + n]$ and $[0, 2 + m]$ for ϕ_n and ϕ_m respectively).

As mentioned before, we see that all the functions $\{m\phi_m : m \geq 2\}$ are in fact the same function, except that each has a horizontal section of length $m - 2$ inserted between its support's starting and ending sections (which both are of length 2).

Example 2.38. Our next example, again with $p = 2$, is derived from Example 8.6 of [37]. In this case, if we set $B(z) = \frac{1}{4}(1 + z)(3 - z)$, then the mask studied in [37] is A_3 in our notation (2.62). Note that B is not a nonnegative mask. If we vary m from 1 to 5, we see a similar situation to the previous example. This is illustrated in Figure 2.6. The graph vividly illustrates our choice of the term “Table Mountlets”, inspired by Cape Town’s famous Table Mountain.

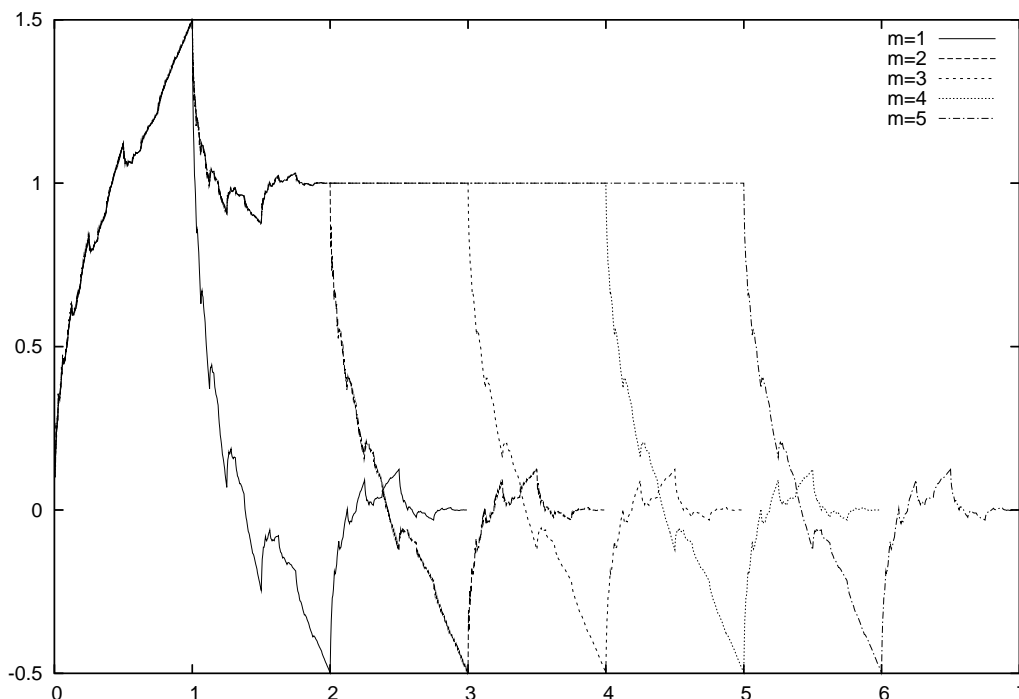


Figure 2.6: The plots of $m\phi_m$, $m = 1, 2, 3, 4, 5$ for Example 2.38.

It is not clear whether a function ϕ_B exists such that (B, ϕ_B) is a 2-refinement pair. If it does, numerical experiments suggest that ϕ_B is not continuous. Thus it is not surprising that

$\phi_m \notin C^1(\mathbb{R})$, as the plot suggests. Although $(S_{b,2}, \delta)$ is divergent (e.g. $(S_{b,2}^r \delta) \rightarrow \infty$ as $r \rightarrow \infty$, but $(S_{b,2}^r \delta)_{-1} = 0$, $r \in \mathbb{Z}_+$), if we take a finite number of subdivision steps for B (supposedly approximating the distribution ϕ_B in some sense) and then numerically approximate the integral $L_m \phi_B$ using the data points so generated, we get the same functions as those generated by (convergent) subdivision for the A_m masks. Also note that the subdivision convergence rate for $m = 3, 5$ is very slow (to the extent that drawing the limit curve by use of subdivision is not feasible), so the method described above or the use of (2.58) provides an alternative graphing technique which works better in this case.

2.6.3 Polynomial sections of any degree

After noting the constant sections mentioned in the previous section, a natural question that arises is whether one can get polynomial sections of any degree in a refinable function. We present here one set of sufficient conditions for this to happen.

We will find the following notation useful. Define, for any $I \subset \mathbb{R}$, $n \in \mathbb{N}$,

$$\Pi_n^I = \{f \in M(\mathbb{R}) : f(x) = p(x), x \in I, \text{ where } p \text{ is a polynomial of degree } n\}. \quad (2.60)$$

We have the following:

Theorem 2.39. For $p \in \mathbb{Z}$, $p \geq 2$, suppose that (B, ψ) is a normalised p -refinement pair with $\deg(B) = M$. For $l \in \mathbb{N}$ and integers $\{m_i : i = 0, 1, \dots, l\} \subset \mathbb{N}$ such that

$$\frac{M}{p-1} < m_0 \leq m_1 \leq \dots \leq m_l,$$

let the mask symbol A be defined by

$$A = \left(\prod_{i=0}^l E_{p-1}^{(m_i)} \right) B. \quad (2.61)$$

Then the refinable function ϕ in the p -refinement pair (A, ϕ) satisfies

$$\phi \in \Pi_l^{\left[\frac{M}{p-1}, m_0\right]} \cap \Pi_l^{\left[\mu + \frac{M}{p-1} - m_0, \mu\right]}$$

where $\mu = \sum_{i=0}^l m_i$. Furthermore, the leading coefficients of the two polynomials in the two intervals $\left[\frac{M}{p-1}, m_0\right]$ and $\left[\mu + \frac{M}{p-1} - m_0, \mu\right]$ are $\left[l! \prod_{i=0}^l m_i\right]^{-1}$ and $\left[(-1)^l l! \prod_{i=0}^l m_i\right]^{-1}$ respectively.

Proof. Define $\mu_j = \sum_{i=0}^j m_i$ for $j \in \{0, 1, \dots, l\}$, so that $\mu_l = \mu$ and let

$$A_j = \prod_{i=0}^j E_{p-1}^{(m_i)} B, \quad j = 0, 1, \dots, l, \quad (2.62)$$

so that $A_l = A$. Let $\phi_j = \left(\prod_{i=0}^j L_{m_i}\right) \psi$ be the refinable function associated with the mask A_j as guaranteed by repeated application of Corollary 2.33. By induction on j , we shall show, for all $j \in \{0, \dots, l\}$, that $\phi_j \in \Pi_j^{\left[\frac{M}{p-1}, m_0\right]} \cap \Pi_j^{\left[\mu_j + \frac{M}{p-1} - m_0, \mu_j\right]}$ and that the polynomials for the first and second intervals have the leading coefficients $\left(j! \prod_{i=0}^j m_i\right)^{-1}$ and $\left((-1)^j j! \prod_{i=0}^j m_i\right)^{-1}$ respectively, which then yields the desired result.

The case $j = 0$ has been proved already in Theorem 2.36, since $\mu_0 = m_0$ implies that $\left[\mu_0 + \frac{M}{p-1} - m_0, m_0\right] = \left[\frac{M}{p-1}, m_0\right]$ and by (2.59), $\phi_0(x) = \frac{1}{m_0}$, $x \in \left[\frac{M}{p-1}, m_0\right]$.

Assume now that for some fixed $j \in \{1, \dots, l\}$, the inductive hypothesis holds for $j - 1$. Thus $\phi_{j-1} \in \Pi_{j-1}^{\left[\frac{M}{p-1}, m_0\right]} \cap \Pi_{j-1}^{\left[\mu_{j-1} + \frac{M}{p-1} - m_0, \mu_{j-1}\right]}$, which implies the existence of polynomials $P, Q \in \Pi_{j-1}$ such that

$$\phi_{j-1}(x) = P(x), \quad x \in \left[\frac{M}{p-1}, m_0\right] \quad (2.63)$$

and

$$\phi_{j-1}(x) = Q(x), \quad x \in \left[\mu_{j-1} + \frac{M}{p-1} - m_0, \mu_{j-1}\right]. \quad (2.64)$$

We also have

$$\phi_{j-1}(x) = 0, \quad x \notin \left[0, \frac{M}{p-1} + \mu_{j-1}\right], \quad (2.65)$$

as well as

$$[P]_{j-1} = \left((j-1)! \prod_{i=0}^{j-1} m_i\right)^{-1} \quad \text{and} \quad [Q]_{j-1} = \left((-1)^{j-1} (j-1)! \prod_{i=0}^{j-1} m_i\right)^{-1}. \quad (2.66)$$

If now $x \in \left[\frac{M}{p-1}, m_0\right]$, then since $x - m_j \leq 0$, we have from Corollary 2.33, (2.63) and (2.65) that

$$\phi_j(x) = \frac{1}{m_j} \int_{x-m_j}^x \phi_{j-1}(t) dt = \frac{1}{m_j} \int_0^x \phi_{j-1}(t) dt = \frac{1}{m_j} \int_0^{m_0} \phi_{j-1}(t) dt + \frac{1}{m_j} \int_{m_0}^x P(t) dt. \quad (2.67)$$

But

$$\int_{m_0}^x P(t) dt = \int_{m_0}^x \sum_{k=0}^{j-1} [P]_k t^k dt = \sum_{k=0}^{j-1} \frac{[P]_k}{k+1} (x^{k+1} - m_0^{k+1}). \quad (2.68)$$

It follows from (2.66), (2.67) and (2.68) that on the interval $\left[\frac{M}{p-1}, m_0\right]$, the function ϕ_j equals a polynomial of degree j with leading coefficient $\left(j! \prod_{i=0}^j m_i\right)^{-1}$.

Also, if $x \in \left[\mu_j + \frac{M}{p-1} - m_0, \mu_j\right]$, it follows by using $\mu_j = \mu_{j-1} + m_j$ that

$$\mu_{j-1} + \frac{M}{p-1} - m_0 \leq x - m_j \leq \mu_{j-1}.$$

Since also $x \geq \mu_{j-1} + \frac{M}{p-1}$, we have by Corollary 2.33, (2.65) and (2.64) that

$$\begin{aligned}\phi_j(x) &= \frac{1}{m_j} \int_{x-m_j}^x \phi_{j-1}(t) dt = \frac{1}{m_j} \int_{x-m_j}^{\mu_{j-1} + \frac{M}{p-1}} \phi_{j-1}(t) dt \\ &= \frac{1}{m_j} \int_{x-m_j}^{\mu_{j-1}} Q(t) dt + \frac{1}{m_j} \int_{\mu_{j-1}}^{\mu_{j-1} + \frac{M}{p-1}} \phi_{j-1}(t) dt.\end{aligned}\quad (2.69)$$

But

$$\int_{x-m_j}^{\mu_{j-1}} Q(t) dt = \int_{x-m_j}^{\mu_{j-1}} \sum_{k=0}^{j-1} [Q]_k t^k dt = \sum_{k=0}^{j-1} \frac{[Q]_k}{k+1} \left[\mu_{j-1}^{k+1} - (x-m_j)^{k+1} \right]. \quad (2.70)$$

It follows from (2.66), (2.69) and (2.70) that on the interval $\left[\mu_j + \frac{M}{p-1} - m_0, \mu_j \right]$, the function ϕ_j equals a polynomial of degree j with leading coefficient $\left((-1)^j j! \prod_{i=0}^j m_i \right)^{-1}$. This completes the inductive step and the result follows. \square

Remarks.

- (a) We conjecture that a stronger result than the above theorem actually holds, namely that it is not necessary that $\psi \in L^1(\mathbb{R})$. Numerical experiments indicate that it might be sufficient that ψ is a distribution such that $\left(\prod_{i=0}^l L_{m_i} \right) \psi \in C(\mathbb{R})$.
- (b) Note that $\left[\frac{M}{p-1}, m_0 \right] = \left[\mu + \frac{M}{p-1} - m_0, \mu \right]$ if $l = 0$, as was mentioned in the proof, whereas if $l \geq 1$, then $\left[\frac{M}{p-1}, m_0 \right] \cap \left[\mu + \frac{M}{p-1} - m_0, \mu \right] = \emptyset$.

Example 2.40. To illustrate Theorem 2.39 for $p = 3$, we recall the mask symbol B of Example 2.29 and define the mask symbol A by

$$A(z) = \frac{1}{9} (1 + z^3 + z^6) (1 + z^4 + z^8) B(z), \quad z \in \mathbb{C},$$

so that in the notation of Theorem 2.39, we have here $l = 1$, $M = 5$, $m_0 = 3$, $m_1 = 4$ and thus $\mu = 7$. The corresponding refinable function ϕ is shown in Figure 2.7(a). According to Theorem 2.39, the refinable function ϕ has linear pieces in the intervals $\left[\frac{5}{2}, 3 \right]$ and $\left[\frac{13}{2}, 7 \right]$ with gradients $\pm \frac{1}{12}$ in the respective intervals. This is confirmed by Figure 2.7(b), where we show the graph of $12\phi'$. This plot also shows that there are no other linear pieces in the interval $\left[0, \frac{19}{2} \right]$.

By use of (2.55), we obtain the following generalisation of Lemma 2.35.

Lemma 2.41. For $p \in \mathbb{Z}$, $p \geq 2$, suppose (B, ψ) is a p -refinement pair and let, for $l \in \mathbb{N}$ and $\mathbf{m} = (m_1, \dots, m_l) \in \mathbb{N}^l$, the mask symbol $A_{\mathbf{m}}$ be defined by

$$A_{\mathbf{m}} = \left(\prod_{i=1}^l E_{p-1}^{(m_i)} \right) B,$$

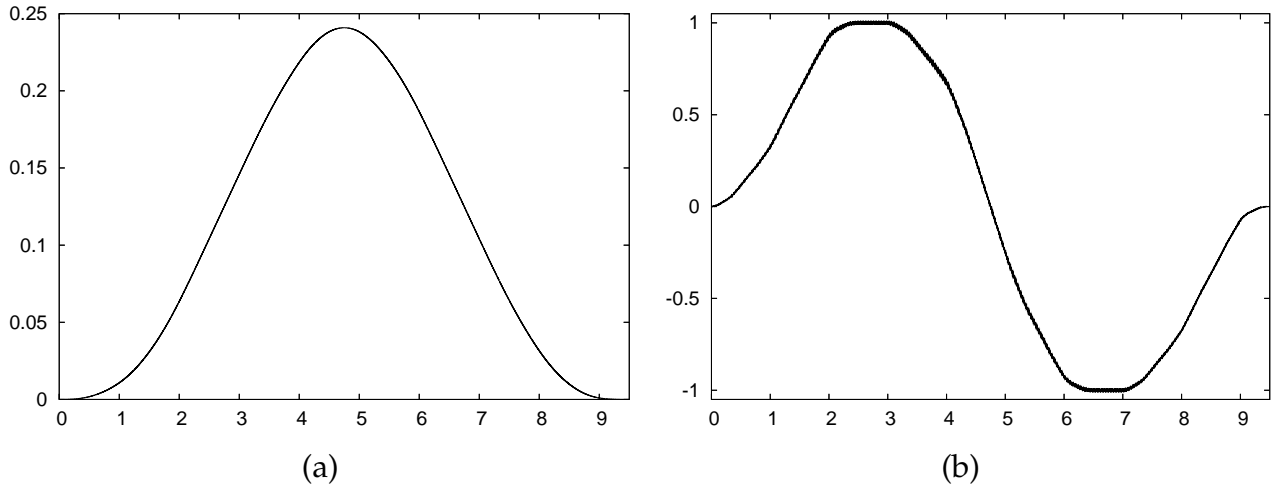


Figure 2.7: The graphs of (a) ϕ ; and (b) $12\phi'$ in Example 2.40.

and let $\phi_{\mathbf{m}}$ denote the p -refinable function corresponding to $A_{\mathbf{m}}$. The family of functions so formed satisfy the identity

$$\left(\prod_{i=1}^l m_i \right) \phi_{\mathbf{m}} = \sum_{\mathbf{0} \leq \mathbf{j} < \mathbf{m}} \phi_{\mathbf{1}_l} \left(\cdot - \sum \mathbf{j} \right), \quad \mathbf{m} \in \mathbb{N}^l, \quad l \in \mathbb{N}, \quad (2.71)$$

where $\mathbf{1}_l$ denotes the vector $(1, \dots, 1) \in \mathbb{N}^l$ for $l \in \mathbb{N}$.

Remark 2.42. In Lemma 2.41, the requirement $\psi \in L^1(\mathbb{R})$ is once again not essential. It can be verified that the result (2.71) will hold for a particular $l \in \mathbb{N}$ if ψ is a p -refinable distribution such that $L_1^l \psi \in M_0(\mathbb{R})$.

Lemma 2.41 can be used to prove relationships between the different members of the family $\{\phi_{\mathbf{m}} : \mathbf{m} \in \mathbb{N}^l\}$ for a given $l \in \mathbb{N}$ in a similar manner to what was done in Theorem 2.36. Of course the relationships will become more involved. Rather than giving all the tedious detail here, we illustrate some of these relationships in the following example.

Example 2.43. For another example in the context of Theorem 2.39, this time with $p = 2$, we continue with the mask from Example 2.38, but now we set $B(z) = \frac{1}{2}(3 - z)$, $z \in \mathbb{C}$, and form the mask symbols

$$A_m(z) = \frac{1}{8}(1 + z^m)^2(3 - z), \quad z \in \mathbb{C}, \quad m \in \mathbb{N},$$

so that, in the notation of Theorem 2.39, we have $M = 1$, $l = 2$, $m_0 = m_1 = m$ and $\mu = 2m$. By Theorem 2.11, there is no nonzero 2-refinable L^1 -function corresponding to B . However, since subdivision converges for A_1 (see e.g. [37: Example 8.5]), we have $\phi_1 \in C(\mathbb{R})$ and it follows from Remark 2.42 and (2.71) that, for each A_m , there is a 2-refinable function $\phi_m \in C(\mathbb{R})$, which is given by $\phi_m = \frac{1}{m^2} \sum_{j_1=0}^{m-1} \sum_{j_2=0}^{m-1} \phi_1(\cdot - j_1 - j_2)$.

We find that the result of Theorem 2.39, namely that $\phi_m \in \Pi_1^{[1,m]} \cap \Pi_1^{[m+1,2m]}$ with the

polynomials in the two intervals having the leading coefficients $\pm m^{-2}$, still holds. This is confirmed in Figure 2.8, in which we show two different plots of the family $\{m^2\phi_m : m \in \mathbb{N}\}$ to illustrate the relationship between the different members of the family. Note that, except for the case $m = 1$, subdivision does not converge. The other graphs were drawn using the algorithm discussed in [18: Theorem 4.1] after solving the eigenvalue problem described in Section 1.4.1 to obtain the function values at the integers.

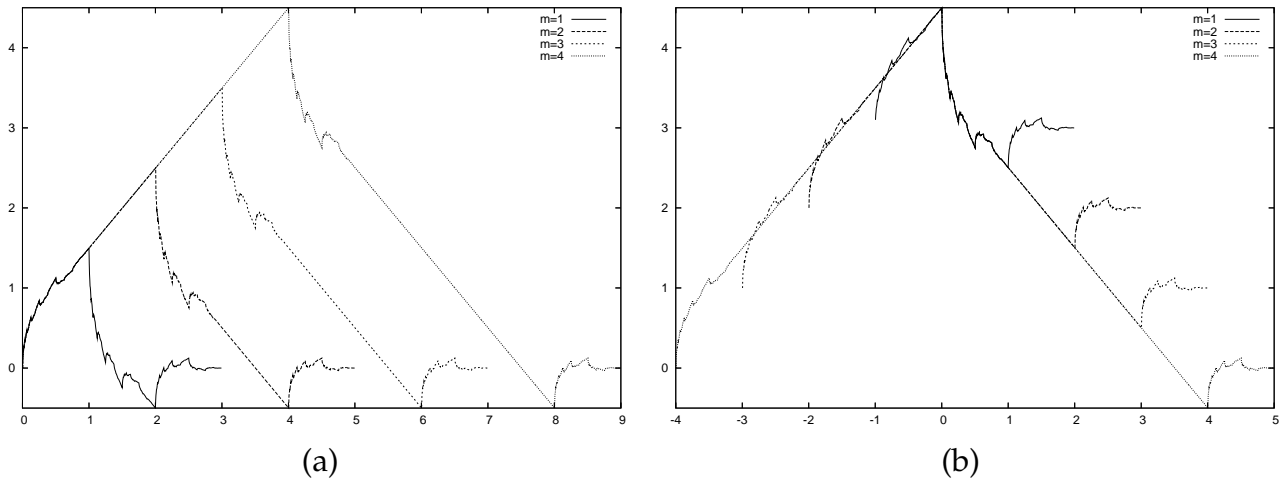


Figure 2.8: Plots of (a) $m^2\phi_m$ and (b) $m^2\phi_m(\cdot + m) + 4 - m$ for $m = 1, 2, 3, 4$ in Example 2.43.

We see a similar situation to what occurred in the “Table Mountlet” case, namely that certain sections of the scaled functions are in fact the same. All the functions consist of three similar pieces separated by two linear sections of length $m - 1$. From Figure 2.8 we derive, for $m \in \mathbb{N}$ and $n \in \{1, \dots, m\}$, the formal relationships

$$m^2\phi_m(x) = \begin{cases} n^2\phi_n(x), & x \leq n, \\ n^2\phi_n(x - m + n) + n - m, & x \in [m - n + 1, m + n], \\ n^2\phi_n(x - 2(m - n)), & x \geq 2m + 1 - n. \end{cases}$$

We can also give an exact expression for the linear sections of the functions which occur when $m \geq 2$, namely

$$m^2\phi_m(x) = \begin{cases} x + \frac{1}{2}, & x \in [1, m], \\ 2m - x - \frac{1}{2}, & x \in [m + 1, 2m]. \end{cases}$$

Chapter 3

Regularity analysis in terms of the mask symbol factorisation

The following result is well-known (see [40: Theorem 1] and the references listed there).

Proposition 3.1. *If (A, ϕ) is a 2-refinement pair, and ϕ has stable integer shifts in the sense of (1.16), then $\phi \in C^k(\mathbb{R})$ if and only if there exists a sequence $b \in M_0(\mathbb{Z})$ such that $A = \frac{1}{2} (E_1)^k \text{Lpol}(b)$ and the subdivision scheme for the mask b converges.*

If we do not assume stability for ϕ , more general conditions for its regularity in terms of the mask symbol factorisation can be obtained by employing some of the results of the previous chapter. In Section 3.1 we present sufficient conditions for a general integer dilation factor, while in Section 3.2, we present necessary conditions for dilation factor 2.

3.1 Sufficient conditions for regularity

We start with some known results for the case $p = 2$.

The next result was a long standing conjecture, being considered consecutively by Gonsor [26], Melkman [33] and Wang [45], who established it for a large class of special cases. It was finally proved by Zhou [47].

Proposition 3.2. *Suppose that the mask a has all the following properties:*

- (a) *the sum rules hold for a , i.e. $\sum_j a_{2j} = \sum_j a_{2j+1} = 1$,*
- (b) *a is a non-negative mask, i.e. $a_j \geq 0$, $j \in \mathbb{Z}$,*
- (c) *$0 < a_0 < 1$ and $0 < a_N < 1$,*
- (d) *the greatest common divisor of $\{j : a_j > 0\}$ is 1.*

Then there exists a function $\phi \in C(\mathbb{R})$ such that (A, ϕ) is a 2-refinement pair. Moreover, the subdivision scheme $(S_{a,2}, c)$ converges for $c \in l^\infty(\mathbb{Z})$.

Definition 3.3. We shall call a mask satisfying the requirements (a) to (d) of Proposition 3.2 a *Zhou mask*.

Remark 3.4. The conditions imposed on B in Theorem 2.24 imply that $M := \deg(B) \geq 1$ and that there exists an integer $n \in \mathbb{Z}_M$ such that $\{n, n+1, \dots, n+p-1\} \subset \text{supp}(a)$. Since the

mask a is also nonnegative with $\{M, M + 1, \dots, M + p - 1\} \subset \text{supp}(a)$, it follows that A is a Zhou mask in the case $p = 2$. In fact, the conditions on A in Theorem 2.24 are much stronger than for a Zhou mask.

In the case of positive masks, some existing results on a lower bound for the regularity of a 2-refinable function ϕ in terms of the corresponding mask symbol factorisation are as follows.

A Hurwitz polynomial is one for which all its zeros in \mathbb{C} have negative real parts and for which the coefficients are therefore all of the same sign. The next result appears in [34: pp.93-95]:

Theorem 3.5. *Suppose that the mask symbol A is given by*

$$A(z) = \frac{1}{2^{m+1}}(1+z)^{m+1}C(z), \quad z \in \mathbb{C}, \quad (3.1)$$

where $m \in \mathbb{N}$ and C is a Hurwitz polynomial with $C(1) = 1$ and $\deg(C) \geq 1$. Then the associated 2-refinable function ϕ satisfies $\phi \in C^m(\mathbb{R})$ and the subdivision scheme $(S_{a,2}, \delta)$ converges to ϕ .

Remark 3.6. In [34], the requirement $\deg(C) \geq 1$ is erroneously omitted, in which case the cardinal B -splines provide an immediate counterexample. Also, [34] states the result in an “if and only if” form, where A is assumed to be Hurwitz, but the proof given there that $\phi \in C^m(\mathbb{R})$ implies the form (3.1) for A seems erroneous. We shall present a short alternative proof in Remark 3.20.

The following generalisation of Theorem 3.5 is given in [20: Theorem 4.2]:

Theorem 3.7. *Suppose that C is a polynomial with positive coefficients, with $\deg(C) = d \geq 1$ and $C(1) = 1$. Let $B(z) = \frac{1}{2}(1+z)C(z)$, $z \in \mathbb{C}$. Suppose further that there exist, for $m \in \mathbb{N}$, integers q_1, q_2, \dots, q_m with $0 \leq q_1 \leq q_2 \leq \dots \leq q_m$ and*

$$q_1 \leq \log_2(d+1), \quad q_{r+1} \leq \log_2 \left(d+1 + \sum_{j=1}^r 2^{q_j} \right), \quad r = 1, 2, \dots, m-1,$$

such that the mask symbol A is given by

$$A(z) = \frac{1}{2^m} \prod_{r=1}^m (1+z^{2^{q_r}}) B(z), \quad z \in \mathbb{C}.$$

Then the associated 2-refinable function ϕ satisfies $\phi \in C^m(\mathbb{R})$ and the subdivision algorithm $(S_{a,2}, \delta)$ converges to ϕ .

The first paragraph of Example 2.12, together with the second paragraph in the introduction of Section 2.6, show that all factors of the form $(1+z^{2^q})$, $q \in \mathbb{Z}_+$, are 2-GBP factors corresponding to the refinable function $N_1(\frac{\cdot}{2^q})$. Thus the following result, combined with the subsequent Theorem 3.9, provides a substantial generalisation of Theorem 3.7.

Theorem 3.8. *Let $p \in \mathbb{Z}$ with $p \geq 2$. If, with $m \in \mathbb{N}$, the polynomial A can be factorised as*

$$A = \left(\prod_{i=1}^m P_i \right) B,$$

where (B, ψ) is a p -refinement pair with $\psi \in C(\mathbb{R})$ and P_i is a p -LLS factor for $i \in \{1, \dots, m\}$, then the function ϕ defined by

$$\phi = \psi * s_1 * \dots * s_m,$$

where each s_i is the p -refinable step function corresponding to P_i , as given by Proposition 2.6, is p -refinable with mask symbol A and satisfies $\phi \in C^m(\mathbb{R})$.

Proof. Define $\phi_0 = \psi$ and $\phi_i = \phi_{i-1} * s_i$ for $i \in \{1, \dots, m\}$. For $i \in \{1, \dots, m\}$ let the sequence $\{\sigma_{i,j} : j \in \mathbb{Z}\} \in M_0^+(\mathbb{Z})$ be defined by $s_i = \sum_j \sigma_{i,j} \chi(\cdot - j)$. Also define $A_0 = B$ and $A_i = P_i A_{i-1}$ for $i \in \{1, \dots, m\}$. We shall show inductively that (A_i, ϕ_i) is a p -refinement pair with $\phi_i \in C^i(\mathbb{R})$ for $i \in \{0, \dots, m\}$. By assumption this is true for $i = 0$. Now suppose that (A_i, ϕ_i) is a p -refinement pair with $\phi_i \in C^i(\mathbb{R})$ for some $i \in \{0, \dots, m-1\}$. Then, since (P_{i+1}, s_{i+1}) is a p -refinement pair, we have by Theorem 2.1 that (A_{i+1}, ϕ_{i+1}) is a p -refinement pair. Furthermore, we have that

$$\begin{aligned} \phi_{i+1} &= \phi_i * s_{i+1} = \phi_i * \left(\sum_j \sigma_{i+1,j} \chi(\cdot - j) \right) \\ &= \sum_j \sigma_{i+1,j} \int_{\cdot-j-1}^{\cdot-j} \phi_i(s) ds \\ &= \sum_j \sigma_{i+1,j} \left[\int_{-\infty}^{\cdot-j} \phi_i(s) ds - \int_{-\infty}^{\cdot-j-1} \phi_i(s) ds \right]. \end{aligned}$$

Since $\phi_i \in C^i(\mathbb{R})$, it follows that $\phi_{i+1} \in C^{i+1}(\mathbb{R})$. This completes the inductive step and the result follows by virtue of $A = A_m$ and $\phi = \phi_m$. \square

By repeated use of Theorem 2.30 we immediately obtain the following result.

Theorem 3.9. *In Theorem 3.8, if each P_i is a p -GBP factor and $(S_{b,p}, \delta)$ converges to ψ , then $(S_{a,p}, \delta)$ converges to ϕ .*

In the case $p = 2$, we can use Proposition 3.2 to obtain the following corollary.

Corollary 3.10. *In Theorem 3.8, if $p = 2$, each P_i is a 2-GBP factor and B is a Zhou mask symbol, then $(S_{a,p}, \delta)$ converges to ϕ .*

Remark. Since GBP factors can have negative coefficients, the mask a in Corollary 3.10 need not be non-negative, so that it is not necessarily a Zhou mask.

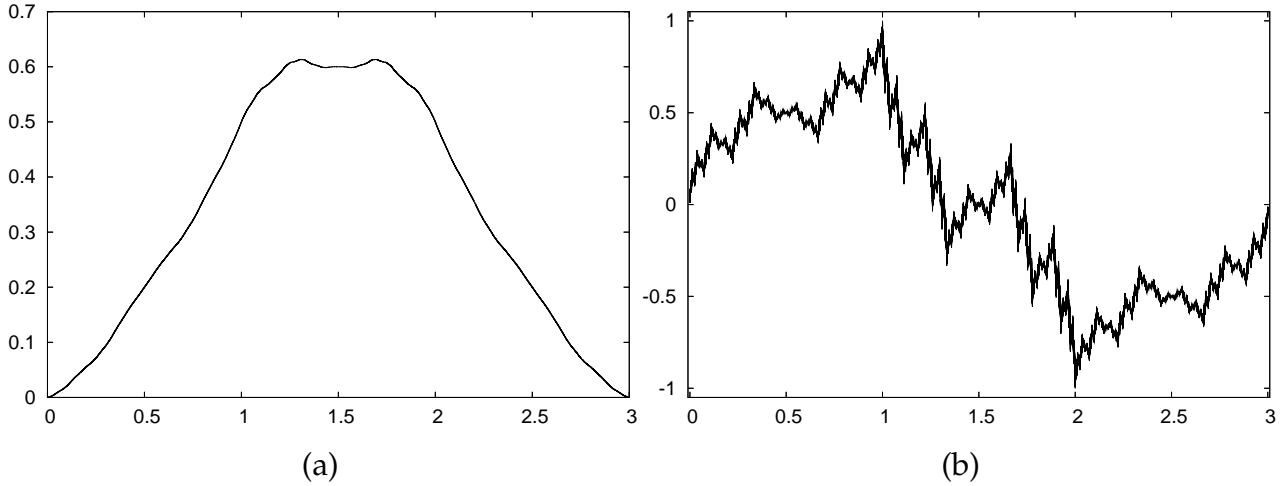


Figure 3.1: Plots of (a) $\tilde{\phi}$; and (b) $\tilde{\phi}'$ in Example 3.11.

Example 3.11. For an illustration of Theorems 3.8 and 3.9, we let A be the mask symbol corresponding to the function ϕ^{DR} as in Example 2.28 and define the mask \tilde{A} by

$$\tilde{A}(z) = (E_2A)(z) = \frac{1}{27} (2 + 3z + 6z^2 + 5z^3 + 6z^4 + 3z^5 + 2z^6), \quad z \in \mathbb{C}.$$

According to Theorem 3.8, the function $\tilde{\phi} = L_1\phi^{DR} \in C^1(\mathbb{R})$ is such that $(\tilde{A}, \tilde{\phi})$ is a 3-refinement pair. Also, by Theorem 3.9, the corresponding subdivision scheme $(S_{\tilde{a},3}, \delta)$ converges to $\tilde{\phi}$. The graphs of $\tilde{\phi}$ and $\tilde{\phi}'$ are shown in Figure 2.3 and illustrate the facts that $\tilde{\phi} \in C^1(\mathbb{R})$, whereas $\tilde{\phi}' = \phi^{DR} - \phi^{DR}(\cdot - 1)$ is nowhere differentiable in its support interval. Also note that $\tilde{\phi}$ provides a clear example of the fact that a continuously differentiable function need not appear smooth to the human eye.

3.2 Necessary conditions for regularity: dilation factor 2

By application of Theorem 2.4 combined with some of the results given in [3], we can derive necessary conditions for regularity in terms of the factorisation of the mask symbol. To do so, we shall first derive some auxiliary results regarding the inverse operation of convolution by a step function.

Definition 3.12. Given a sequence $h \in M_0^+(\mathbb{Z})$, define the sequence $g \in M(\mathbb{Z})$ recursively by

$$g_j = \begin{cases} 0 & \text{if } -j \in \mathbb{N}; \\ \frac{1}{h_0} & \text{if } j = 0; \\ \frac{1}{h_0} \sum_{k=1}^j g_{j-k} (h_{k-1} - h_k) & \text{if } j \in \mathbb{N}. \end{cases} \quad (3.2)$$

(Recall that the definition of $M_0^+(\mathbb{Z})$ implies $h_0 \neq 0$.) We call g the *inverse convoluant* of h and write $g = IC(h)$. The motivation for this name is made evident by the next lemma.

Lemma 3.13. *If $h \in M_0^+(\mathbb{Z})$ and $g = IC(h)$, then the following identity holds:*

$$\sum_k g_k h_{j-k} = \begin{cases} 1 & \text{if } j \in \mathbb{Z}_+; \\ 0 & \text{if } -j \in \mathbb{N}. \end{cases} \quad (3.3)$$

Proof. The bottom line of (3.3) is immediately clear from the top line of (3.2), coupled with the fact that $h_{-k} = 0$ for $k \in \mathbb{N}$.

For $j \geq 0$, note first that $\sum_k g_k h_{j-k} = \sum_{k=0}^j g_k h_{j-k}$, since $g, h \in M_+(\mathbb{Z})$. For $j = 0$, $\sum_k g_k h_{j-k} = g_0 h_0 = 1$ from the middle line of (3.2).

Proceeding inductively, let $j \in \mathbb{N}$ and suppose that $\sum_{k=0}^{j-1} g_k h_{j-1-k} = 1$. Then from (3.2) we have

$$\begin{aligned} g_j h_0 &= \sum_{k=1}^j g_{j-k} (h_{k-1} - h_k) \\ &= \sum_{k=0}^{j-1} g_k (h_{j-k-1} - h_{j-k}) \\ &= \sum_{k=0}^{j-1} g_k h_{j-1-k} - \sum_{k=0}^{j-1} g_k h_{j-k} = 1 - \sum_{k=0}^{j-1} g_k h_{j-k}. \end{aligned}$$

Thus $\sum_{k=0}^j g_k h_{j-k} = 1$ and the result follows for all $j > 0$ by induction. \square

Example 3.14. Consider, for $m \in \mathbb{N}$, the sequence $h = m[E_{m-1}]$, that is

$$h_j = \begin{cases} 1 & \text{if } j \in \mathbb{Z}_m; \\ 0 & \text{if } j \notin \mathbb{Z}_m, \end{cases}$$

so that $h_{k-1} - h_k = \delta_{k-m}$ for $k \in \{1, \dots, j\}$, $j \in \mathbb{N}$. Then from (3.2) we have $g_0 = 1$, $g_j = 0$ for $j \in \{1, \dots, m-1\}$ and $g_j = g_{j-m}$ for $j \geq m$, which yields $g_{jm+i} = \delta_i$, $j \in \mathbb{Z}_+$, $i \in \mathbb{Z}_m$. It can now easily be verified that (3.3) does indeed hold.

The following theorem shows that we can always find the inverse convolution of a step function if the result of the convolution is continuously differentiable.

Theorem 3.15. *For $m \in \mathbb{Z}_+$, if $F \in C_+^{m+1}(\mathbb{R})$ and $H = \sum_j h_j \chi(\cdot - j)$ for some $h \in M_0^+(\mathbb{Z})$, let the function G be defined by*

$$G = \sum_j g_j F'(\cdot - j) \quad (3.4)$$

where $g = IC(h)$. Then $G \in C_+^m(\mathbb{R})$ and $F = G * H$. Furthermore, G is the unique locally Lebesgue integrable function such that $F = G * H$.

Remark. Note that, since $g_j = 0$ if $-j \in \mathbb{N}$, while $F(x) = 0$ if $x \leq 0$, the sum in the right hand side of (3.4) is in fact a finite sum for any given value of the argument.

Proof. To see that indeed $G \in C_+^m(\mathbb{R})$, we differentiate (3.4) m times to obtain

$$G^{(m)} = \sum_j g_j F^{(m+1)}(\cdot - j),$$

which, since $F^{(m+1)} \in C_+(\mathbb{R})$, shows that $G^{(m)} \in C_+(\mathbb{R})$. This implies that G is locally Lebesgue integrable.

For $x \leq 0$, it is clear that $(G * H)(x) = \int_0^\infty G(s) H(x-s) ds = 0 = F(x)$. For $x > 0$, we obtain, by using amongst others (3.4), (3.3) and the fact that $F(x) = 0$ for $x < 0$, that

$$\begin{aligned} (G * H)(x) &= \int_{-\infty}^{\infty} \sum_{j=0}^{\uparrow h \uparrow} h_j \chi(s-j) G(x-s) ds \\ &= \sum_{j=0}^{\uparrow h \uparrow} h_j \int_j^{j+1} G(x-s) ds \\ &= \sum_{j=0}^{\uparrow h \uparrow} h_j \int_{x-1}^x G(s-j) ds \\ &= \sum_{j=0}^{\uparrow h \uparrow} h_j \int_{x-1}^x \sum_k g_k F'(s-j-k) ds \\ &= \int_{x-1}^x \sum_j h_j \sum_k g_k F'(s-j-k) ds \\ &= \int_{x-1}^x \sum_k g_k \sum_j h_{j-k} F'(s-j) ds \\ &= \int_{x-1}^x \sum_j \left[\sum_k h_{j-k} g_k \right] F'(s-j) ds \\ &= \int_{x-1}^x \sum_{j=0}^{\infty} F'(s-j) ds \\ &= \sum_{j=0}^{\infty} [F(x-j) - F(x-j-1)] \\ &= \sum_{j=0}^{\lfloor x \rfloor} [F(x-j) - F(x-j-1)] \\ &= F(x) - F(x - \lfloor x \rfloor - 1) = F(x), \end{aligned}$$

thereby establishing that $F = G * H$.

To prove the uniqueness part of the theorem, suppose that \tilde{G} is a locally Lebesgue integrable function such that $F = \tilde{G} * H$. This implies that $(G - \tilde{G}) * H = 0$, while $h \in M_0^+(\mathbb{Z})$ implies $H \neq 0$, so that it follows from Theorem 2.3 that $G - \tilde{G} = 0$. Thus $G = \tilde{G}$, yielding the desired result. \square

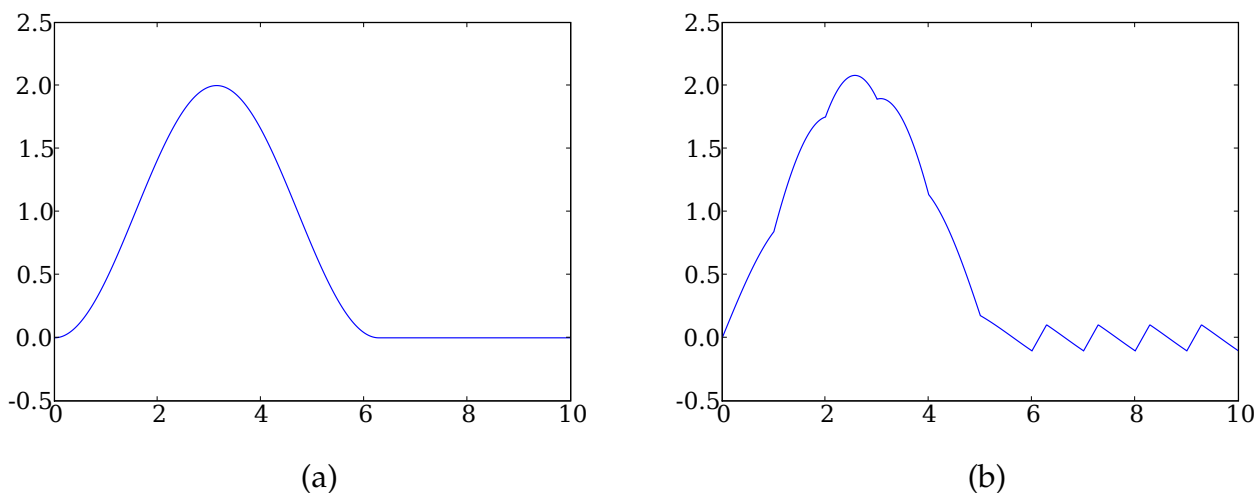


Figure 3.2: Plots of (a) F ; and (b) G in Example 3.16. The function G is such that $G * \chi = F$.

Example 3.16. To illustrate Theorem 3.15, consider the function F given by

$$F(x) = \begin{cases} 1 - \cos(x), & x \in [0, 2\pi], \\ 0, & x \notin [0, 2\pi], \end{cases}$$

and let $H = \chi$, that is $h_j = \delta_j$, $j \in \mathbb{Z}$. Example 3.14 with $m = 1$ yields $g_j = 1$, $j \in \mathbb{Z}_+$, so that, with $r(x) = \max\{0, \lceil x - 2\pi \rceil\}$, $x \in \mathbb{R}$, we obtain

$$G(x) = \sum_{j=0}^{\infty} \sin(x-j) \chi_{[0, 2\pi]} = \sum_{j=r(x)}^{\lfloor x \rfloor} \sin(x-j), \quad x \geq 0.$$

The plots of F and G are shown in Figure 3.2. This example illustrates an important point, namely that the function G need not be compactly supported, even though both F and H are compactly supported.

In order to show that this phenomenon of non-compact support for the constructed function G in Theorem 3.15 will not occur if we start with a 2-refinable function F , we shall employ Theorem 2.11, which allows us to show that all 2-refinable functions satisfy a certain modified partition of unity property, as stated in the following result.

Lemma 3.17. *Suppose (A, ϕ) is a 2-refinement pair and let P denote the 2-GBP factor implied by Theorem 2.11. Then ϕ satisfies the identity*

$$\sum_j g_j \phi(x-j) = \int_{-\infty}^{\infty} \phi(s) ds, \quad x > N - M - 1, \quad (3.5)$$

where $N = \deg(A)$, $M = \deg(R)$ and $g = IC([R])$, with the polynomial R given by (2.2).

Proof. Let $(\tilde{A}, \tilde{\phi})$ be the 2-refinement pair given by Theorem 2.19, so that

$$\phi = \sum_{j=0}^M [R]_j \tilde{\phi}(\cdot - j). \quad (3.6)$$

Since $\deg(\tilde{A}) = N - M$, we conclude from part (d) of Theorem 1.8 that

$$\tilde{\phi}(x) = 0, \quad x \notin [0, N - M]. \quad (3.7)$$

By Lemma 2.14, $R(1) = 1$, from which we conclude that

$$\begin{aligned} \int_{-\infty}^{\infty} \tilde{\phi}(s) ds &= \sum_j [R]_j \int_{-\infty}^{\infty} \tilde{\phi}(s) ds \\ &= \sum_j [R]_j \int_{-\infty}^{\infty} \tilde{\phi}(s - j) ds \\ &= \int_{-\infty}^{\infty} \sum_j [R]_j \tilde{\phi}(s - j) ds = \int_{-\infty}^{\infty} \phi(s) ds, \end{aligned}$$

so that, since the sum rules (1.11) hold for \tilde{A} , Lemma 1.12 yields

$$\sum_j \tilde{\phi}(x - j) = \int_{-\infty}^{\infty} \phi(s) ds, \quad x \in \mathbb{R}. \quad (3.8)$$

By use of (3.6), (3.3), (3.7) and (3.8), we obtain, for $x > N - M - 1$,

$$\begin{aligned} \sum_j g_j \phi(x - j) &= \sum_j g_j \sum_k [R]_k \tilde{\phi}(x - j - k) \\ &= \sum_j g_j \sum_k [R]_{k-j} \tilde{\phi}(x - k) \\ &= \sum_k \left(\sum_j g_j [R]_{k-j} \right) \tilde{\phi}(x - k) \\ &= \sum_{k=0}^{\infty} \tilde{\phi}(x - k) \\ &= \sum_k \tilde{\phi}(x - k) = \int_{-\infty}^{\infty} \phi(s) ds, \end{aligned}$$

which completes the proof. \square

Using this result, we can show that if we start with a 2-refinable function ϕ in Theorem 3.15, the constructed function will be compactly supported.

Corollary 3.18. *If, under the conditions and in the notation of Theorem 3.15 and Lemma 3.17, we have that $F = \phi$ and that (P, H) is a 2-refinement pair, then G is a compactly supported function with*

$$G(x) = 0, \quad x \notin [0, N - M - 1]. \quad (3.9)$$

Proof. Let R , N , M and g be as in Lemma 3.17. By noting that the sequence h in Theorem 3.15 is exactly $[R]$, we find, by differentiating (3.5), that

$$\sum_j g_j \phi'(x - j) = 0, \quad x > N - M - 1,$$

since $\int_{-\infty}^{\infty} \phi(s) ds$ is a constant. The desired result (3.9) now follows from (3.4), together with the fact that $G(x) = 0$ if $x < 0$. \square

We can now derive necessary conditions for a 2-refinable function to have m continuous derivatives in terms of the mask symbol factorization:

Theorem 3.19. *If (A, ϕ) is a 2-refinement pair and $\phi \in C^m(\mathbb{R})$ for some $m \in \mathbb{Z}_+$, then A has a factorization of the form*

$$A = \left(\prod_{i=0}^m P_i \right) B,$$

where each P_i is a 2-GBP factor and B is a polynomial with $B(1) = 1$ and $\deg(B) \geq 1$.

Proof. We prove by induction that, for $l \in \{0, \dots, m\}$, there exist 2-GBP factors P_i , for $i = 0, \dots, l$, as well as a 2-refinement pair (B_l, ϕ_l) such that

$$A = \left(\prod_{i=0}^{l-1} P_i \right) B_l \quad \text{and} \quad \phi_l \in C^{m-l}(\mathbb{R}),$$

where we define $\prod_{i=0}^{-1} P_i = 1$. Setting $B_0 = A$ and $\phi_0 = \phi$, we see that this does indeed hold for $l = 0$.

If $m \geq 1$, suppose now that the result holds for an arbitrary $l \in \mathbb{Z}_m$. By use of Theorem 2.11 we find that $B_l = P_l B_{l+1}$, where P_l is a 2-GBP factor and B_{l+1} is some polynomial. By Proposition 2.6 and Lemma 2.14, there is a step function σ_l such that (P_l, σ_l) is a 2-refinement pair. According to Corollary 3.18, there exists a compactly supported function $\phi_{l+1} \in C_+^{m-l-1}(\mathbb{R})$ such that $\phi_l = \phi_{l+1} * \sigma_l$. By Theorem 2.4, (B_{l+1}, ϕ_{l+1}) is a 2-refinement pair. This completes the inductive step.

We thus obtain inductively that $A = \left(\prod_{i=0}^{m-1} P_i \right) B_m$, where B_m forms a 2-refinement pair with a function $\phi_m \in C(\mathbb{R})$. By Theorem 2.11, B_m has the form $B_m = P_m B$ for some 2-GBP factor P_m and polynomial B . Then $A = \left(\prod_{i=0}^m P_i \right) B$. Also $B(1) = B_m(1) / P_m(1) = 1$. If B is the constant polynomial 1, we have $B_m = P_m$, which by Proposition 2.6 and Lemma 2.14 yields that ϕ_m is a step function, contradicting $\phi_m \in C(\mathbb{R})$. Thus $\deg(B) \geq 1$. \square

Remark 3.20. We now give an alternative proof for the converse direction of Theorem 3.5, as mentioned in Remark 3.6. From the construction algorithm of GBP factors, it can be verified that E_1 is the only 2-GBP factor which is a Hurwitz polynomial. Therefore, if we assume that A is a Hurwitz polynomial and that $\phi \in C^m(\mathbb{R})$, it follows immediately from Theorem 3.19 that A has a factorisation of the form (3.1).

As a corollary of Theorem 3.19, we obtain the following factorization property for continuously differentiable 2-refinable functions.

Corollary 3.21. *If (A, ϕ) is a 2-refinement pair and $\phi \in C^m(\mathbb{R})$ with $m \in \mathbb{N}$, then for every $l \in \{0, \dots, m-1\}$ there exist 2-refinable functions $\psi_l \in C^{l-1}(\mathbb{R})$ and $\tilde{\psi}_l \in C^{m-l-1}(\mathbb{R})$ such that $\phi = \psi_l * \tilde{\psi}_l$.*

Proof. Given $l \in \mathbb{Z}_m$, from the proof of Theorem 3.19 we see that

$$\phi = \phi_{l+1} * \sigma_0 * \dots * \sigma_l,$$

where (P_i, σ_i) is a 2-refinement pair with σ_i a step function for each $i \in \{0, \dots, l\}$, while (B_{l+1}, ϕ_{l+1}) is a 2-refinement pair with $\phi_{l+1} \in C^{m-l-1}(\mathbb{R})$. Setting $\psi_l = \sigma_0 * \dots * \sigma_l$, we have from Theorem 2.1 that $(\prod_{i=0}^l P_i, \psi_l)$ is a 2-refinement pair. Since all the σ_i are step functions, it follows that $\psi_l \in C^{l-1}(\mathbb{R})$. By setting $\tilde{\psi}_l = \phi_{l+1}$, we then obtain the desired result. \square

From Theorem 3.8 and the proof of Theorem 3.19, we derive the following full characterisation of m times continuously differentiable 2-refinable functions.

Corollary 3.22. *Suppose (A, ϕ) is a 2-refinement pair. Then, for $m \in \mathbb{N}$, we have $\phi \in C^m(\mathbb{R})$ if and only if the mask symbol A has a factorisation of the form*

$$A = \left(\prod_{i=1}^m P_i \right) B,$$

where (B, ψ) is a 2-refinement pair with $\psi \in C(\mathbb{R})$ and P_i is a 2-GBP factor for $i = 1, \dots, m$.

Remark 3.23. If Theorem 2.11 holds for dilation factor $p \in \mathbb{Z}$, $p \geq 2$, which appears to be a reasonable conjecture, it follows that all the results for refinable functions in this section hold with every occurrence of the dilation factor 2 replaced by p and with $N - M$ replaced by $\frac{N-M}{p-1}$ in (3.5).

Chapter 4

Subsequence convergence in subdivision

Theorem 1.14 states that if p -subdivision is convergent, the associated mask symbol must contain a factor E_{p-1} . During numerical experiments with masks not satisfying this requirement, it was detected that sometimes the subdivision algorithm “converges” to two or more limits. To make this clear, consider the following example, which is a modification of [37: Example 2.2].

Example 4.1. Let $p = 2$ and let the mask symbol A be given by

$$A(z) = \frac{1}{8} (3 - z + 6z^2 - 2z^3 + 3z^4 - z^5) = \frac{1}{8} (1 + z^2)^2 (3 - z), \quad z \in \mathbb{C}.$$

Observe that $A(1) = 1$ and $A(-1) \neq 0$. As can be seen from Figure 4.1(a), which shows the fourth iteration of subdivision, the subdivision algorithm is divergent and appears to oscillate between two functions on consecutive control points. In Figure 4.1(b), we show separate plots connecting the odd-indexed points and even-indexed points respectively of the eighth iteration of subdivision. We see that the two functions thus obtained seem to be scalar multiples of one another, with the one built from the even indices being minus three times the other one.

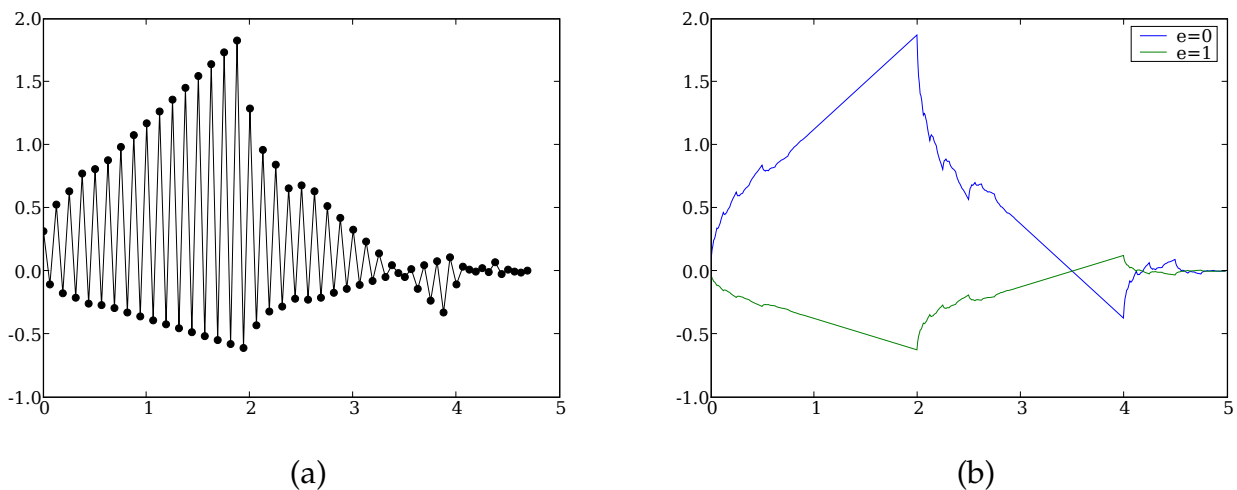


Figure 4.1: Plots of (a) $S_a^4 \delta$; and (b) $(S_a^8 \delta)_{2+e}$, $e = 0, 1$, in Example 4.1.

To make this notion formal, we have the following definition.

Definition 4.2. For $m \in \mathbb{N}$, we say that the subdivision scheme $(S_{a,p}, c)$ has m -subsequence convergence to Φ if there exists a function $\Phi \in C(\mathbb{R}) \setminus \{0\}$ and constants K_i , $i \in \mathbb{Z}_m$, satisfying $\sum_{i=0}^{m-1} K_i = 1$, such that

$$\max_{i \in \mathbb{Z}_m} \sup_j \left| K_i \Phi \left(\frac{mj+i}{p^r} \right) - c_{mj+i}^{(r)} \right| \rightarrow 0, \quad r \rightarrow \infty. \quad (4.1)$$

We call the K_i 's the subsequence convergence constants (SCC).

Note that the definition (4.1) has the equivalent formulation

$$\sup_j \left| K_{j \bmod m} \Phi \left(\frac{j}{p^r} \right) - c_j^{(r)} \right| \rightarrow 0, \quad r \rightarrow \infty, \quad (4.2)$$

which perhaps expresses the notion more clearly.

Remark 4.3. Subdivision convergence is equivalent to 1-subsequence convergence, since for $m = 1$, one obtains $K_{j \bmod m} = K_0 = 1$, $j \in \mathbb{Z}$, in which case (4.2) becomes (1.27). This shows that m -subsequence convergence is a generalisation of the concept of subdivision convergence.

Remark 4.4. Our concept of subsequence convergence must not be confused with the concept called "subconvergence" which is considered in [40: Section 3]. There the convergence of only a subsequence of the *iterations* of the subdivision scheme are considered, whereas we consider all iterations, but take a subsequence of the *entries* of every iteration. Symbolically, "subconvergence" considers the convergence of $c^{(r_k)}$, where $\{r_k : k \in \mathbb{Z}_+\} \subset \mathbb{Z}_+$ is a strictly increasing sequence, while our concept of "subsequence convergence" considers the convergence of $c_{m \cdot i}^{(r)}$ for appropriate integers m and i .

Returning to Example 4.1, we see that it appears that 2-subsequence convergence occurs with $K_0 = \frac{3}{2}$ and $K_1 = -\frac{1}{2}$, since $K_0 = 3K_1$ and $K_0 + K_1$ must equal 1. This will be shown to indeed be the case by Theorem 4.10.

We proceed to obtain some basic properties related to subsequence convergence.

Lemma 4.5. *Suppose, for $c \in M_0(\mathbb{Z})$, the subdivision scheme $(S_{a,p}, c)$ has m -subsequence convergence to the function Φ with SCC K_i , $i \in \mathbb{Z}_m$. Then the following statements are true:*

(a) Φ is compactly supported, with

$$\Phi(x) = 0, \quad x \notin \left(\downarrow c \downarrow, \frac{\uparrow a \uparrow}{p-1} + \uparrow c \uparrow \right). \quad (4.3)$$

(b) If $(S_{a,p}, c)$ has m -subsequence convergence to the function $\tilde{\Phi}$ with the corresponding SCC given by \tilde{K}_i , $i \in \mathbb{Z}_m$, then $\Phi = \tilde{\Phi}$ and $K_i = \tilde{K}_i$, $i \in \mathbb{Z}_m$.

(c) For any $n \in \mathbb{N}$, $(S_{a,p}, c)$ has mn -subsequence convergence to $n\Phi$ with the SCC given by $\frac{1}{n}K_{i \bmod m}$, $i \in \mathbb{Z}_{mn}$.

(d) If $m = nl$, with $l, n \in \mathbb{N}$, $n \geq 2$, then $(S_{a,p}, c)$ has l -subsequence convergence to $\frac{1}{n}\Phi$ if and only if $K_i = K_{i \bmod l}$, $i \in \mathbb{Z}_m$. In this case the SCC for the l -subsequence convergence are given by nK_i , $i \in \mathbb{Z}_l$.

Proof.

(a) From (1.25), (1.6) and (1.7), after recalling also $\downarrow a \downarrow = 0$, we obtain, for $r \in \mathbb{N}$,

$$\uparrow c^{(r)} \uparrow = \sum_{j=0}^{r-1} p^j \uparrow a \uparrow + p^r \uparrow c \uparrow = \frac{p^r - 1}{p - 1} \uparrow a \uparrow + p^r \uparrow c \uparrow \quad (4.4a)$$

and

$$\downarrow c^{(r)} \downarrow = \sum_{j=0}^{r-1} p^j \downarrow a \downarrow + p^r \downarrow c \downarrow = p^r \downarrow c \downarrow. \quad (4.4b)$$

Suppose now that $x \notin \left[\downarrow c \downarrow, \frac{\uparrow a \uparrow}{p-1} + \uparrow c \uparrow \right]$ and let the sequence $\{j_r : r \in \mathbb{Z}_+\} \subset \mathbb{R}$ be such that $\frac{j_r}{p^r} \rightarrow x$ as $r \rightarrow \infty$. Then there is an integer R such that $\frac{j_r}{p^r} \notin \left[\downarrow c \downarrow, \frac{\uparrow a \uparrow}{p-1} + \uparrow c \uparrow \right]$ if $r \geq R$, so that also $j_r \notin \left[p^r \downarrow c \downarrow, \frac{p^r - 1}{p-1} \uparrow a \uparrow + p^r \uparrow c \uparrow \right]$ whenever $r \geq R$, which, by (4.4a) and (4.4b), yields $c_{j_r}^{(r)} = 0$, $r \geq R$. But then (4.1), together with the fact that $\Phi \in C(\mathbb{R})$, yields the desired compact support property (4.3) of Φ .

(b) Suppose that $(S_{a,p}, c)$ has m -subsequence convergence to the function $\tilde{\Phi}$ with the corresponding SCC given by \tilde{K}_i , $i \in \mathbb{Z}_m$. It follows from (4.2) by the triangle inequality that

$$\sup_j \left| K_{j \bmod m} \Phi \left(\frac{j}{p^r} \right) - \tilde{K}_{j \bmod m} \tilde{\Phi} \left(\frac{j}{p^r} \right) \right| \rightarrow 0, \quad r \rightarrow \infty,$$

from which we conclude by the continuity of Φ and $\tilde{\Phi}$, together with the fact that the set $\left\{ \frac{j}{p^r} : j \in \mathbb{Z}, r \in \mathbb{Z}_+ \right\}$ is dense in \mathbb{R} , that $K_i \Phi = \tilde{K}_i \tilde{\Phi}$, $i \in \mathbb{Z}_m$. Together with the identities $\sum_{i=0}^{m-1} K_i = 1 = \sum_{i=0}^{m-1} \tilde{K}_i$, this yields

$$\Phi = \sum_{i \in \mathbb{Z}_m} K_i \Phi = \sum_{i \in \mathbb{Z}_m} \tilde{K}_i \tilde{\Phi} = \tilde{\Phi}.$$

Since $\Phi \neq 0$ by definition, there exists a $x_0 \in \mathbb{R}$ such that $\Phi(x_0) \neq 0$. Furthermore, since

$$K_i \Phi(x_0) = \tilde{K}_i \tilde{\Phi}(x_0) = \tilde{K}_i \Phi(x_0), \quad i \in \mathbb{Z}_m,$$

we obtain $K_i = \tilde{K}_i$, $i \in \mathbb{Z}_m$.

(c) Suppose $n \in \mathbb{N}$. Then we obtain, for $r \in \mathbb{Z}_+$,

$$\begin{aligned} & \max_{i \in \mathbb{Z}_{mn}} \sup_j \left| \frac{1}{n} K_{i \bmod m} n \Phi \left(\frac{mnj + i}{p^r} \right) - c_{mnj+i}^{(r)} \right| \\ &= \max_{i \in \mathbb{Z}_m} \max_{k \in \mathbb{Z}_n} \sup_j \left| K_{(km+i) \bmod m} \Phi \left(\frac{m(nj+k) + i}{p^r} \right) - c_{m(nj+k)+i}^{(r)} \right| \end{aligned}$$

$$= \max_{i \in \mathbb{Z}_m} \sup_j \left| K_i \Phi \left(\frac{mj+i}{p^r} \right) - c_{mj+i}^{(r)} \right|,$$

which yields the desired result by (4.1) after noting also that

$$\sum_{i=0}^{m-1} \frac{1}{n} K_{i \bmod m} = \sum_{i=0}^{m-1} \sum_{k=0}^{n-1} \frac{1}{n} K_{(i+mk) \bmod m} = \sum_{i=0}^{m-1} K_i = 1.$$

- (d) Assume $m = nl$, with $l, n \in \mathbb{N}$, $n \geq 2$. Now first suppose that $(S_{a,p}, c)$ has l -subsequence convergence to $\frac{1}{n}\Phi$ with the corresponding SCC given by \tilde{K}_i , $i \in \mathbb{Z}_l$. Then by the result (c), $(S_{a,p}, c)$ has m -subsequence convergence to Φ with the SCC given by $\frac{1}{n}\tilde{K}_{i \bmod l}$, $i \in \mathbb{Z}_m$. Now from (b) we have $K_i = \frac{1}{n}\tilde{K}_{i \bmod l}$, $i \in \mathbb{Z}_m$, from which we obtain

$$K_{i \bmod l} = \frac{1}{n}\tilde{K}_{(i \bmod l) \bmod l} = \frac{1}{n}\tilde{K}_{i \bmod l} = K_i, \quad i \in \mathbb{Z}_m.$$

Conversely, suppose that $K_i = K_{i \bmod l}$, $i \in \mathbb{Z}_m$, which has the equivalent formulation $K_i = K_{i+lk}$, $i \in \mathbb{Z}_l$, $k \in \mathbb{Z}_n$. From this identity, together with $m = ln$ and the assumed m -subsequence convergence of $(S_{a,p}, c)$ to Φ , we obtain

$$\begin{aligned} \max_{i \in \mathbb{Z}_l} \sup_j \left| nK_i \frac{1}{n} \Phi \left(\frac{l(j+i)}{p^r} \right) - c_{j^{l+i}}^{(r)} \right| &= \max_{i \in \mathbb{Z}_l} \sup_j \max_{k \in \mathbb{Z}_n} \left| K_i \Phi \left(\frac{l(nj+k)+i}{p^r} \right) - c_{l(nj+k)+i}^{(r)} \right| \\ &= \max_{i \in \mathbb{Z}_l} \max_{k \in \mathbb{Z}_n} \sup_j \left| K_{i+lk} \Phi \left(\frac{mj+lk+i}{p^r} \right) - c_{mj+lk+i}^{(r)} \right| \\ &= \max_{i \in \mathbb{Z}_m} \sup_j \left| K_i \Phi \left(\frac{mj+i}{p^r} \right) - c_{mj+i}^{(r)} \right| \rightarrow 0, \quad r \rightarrow \infty. \end{aligned}$$

This, together with the identity

$$\sum_{i=0}^{l-1} nK_i = \sum_{i=0}^{l-1} \sum_{k=0}^{n-1} K_i = \sum_{i=0}^{l-1} \sum_{k=0}^{n-1} K_{i+lk} = \sum_{i=0}^{m-1} K_i = 1,$$

yields that $(S_{a,p}, c)$ has l -subsequence convergence to $\frac{1}{n}\Phi$ with the SCC given by nK_i for $i \in \mathbb{Z}_l$, thereby completing the proof of the lemma. \square

The next lemma shows that, like in the case of normal subdivision convergence, it is sufficient to consider the the initial sequence δ .

Lemma 4.6. *The subdivision scheme $(S_{a,p}, c)$ has m -subsequence convergence for any initial sequence $c \in l^\infty(\mathbb{Z}) \setminus \{0\}$ with limit Φ if and only if $(S_{a,p}, \delta)$ has m -subsequence convergence with limit ϕ , where Φ and ϕ are related by (1.31).*

Proof. As before, let $N = \deg(A)$. Assume that $(S_{a,p}, \delta)$ has m -subsequence convergence to

ϕ with the SCC given by K_i , $i = 0, \dots, m-1$. We next prove inductively that

$$(S_a^r c)_j = \sum_l (S_a^r \delta)_{j-p^r l} c_l, \quad j \in \mathbb{Z}, \quad r \in \mathbb{Z}_+, \quad c \in M(\mathbb{Z}). \quad (4.5)$$

To do so, let $c \in M(\mathbb{Z})$ be given. Since $c_j = \sum_l \delta_{j-l} c_l$, $j \in \mathbb{Z}$, (4.5) holds for $r = 0$. If we suppose that (4.5) holds for some $r \in \mathbb{Z}_+$, we obtain, by use of the definition (1.24), that

$$\begin{aligned} (S_a^{r+1} c)_j &= \sum_k a_{j-pk} (S_a^r c)_k \\ &= \sum_k a_{j-pk} \sum_l (S_a^r \delta)_{k-p^r l} c_l \\ &= \sum_l \sum_k a_{j-p^{r+1}l-pk} (S_a^r \delta)_k c_l \\ &= \sum_l (S_a (S_a^r \delta))_{j-p^{r+1}l} c_l, \quad j \in \mathbb{Z}, \end{aligned}$$

thereby completing the inductive step.

Given any initial sequence $c \in l^\infty(\mathbb{Z})$ in the subdivision algorithm, we obtain by use of (1.31) and (4.5) that the identity

$$K_i \Phi \left(\frac{mj+i}{p^r} \right) - (S_a^r c)_{mj+i} = \sum_l c_l \left(K_i \phi \left(\frac{mj+i}{p^r} - l \right) - (S_a^r \delta)_{mj+i-p^r l} \right) \quad (4.6)$$

holds for $j \in \mathbb{Z}$, $r \in \mathbb{Z}_+$ and $i \in \{0, \dots, m-1\}$.

From (1.26), (1.6) and (1.7) we obtain, for $r \in \mathbb{N}$,

$$\uparrow S_a^r \delta \uparrow = \frac{p^r - 1}{p - 1} N \quad \text{and} \quad \downarrow S_a^r \delta \downarrow = 0. \quad (4.7)$$

If we define, for $r \in \mathbb{Z}_+$, $j \in \mathbb{Z}$, the constants $\mu_{r,j}, \nu_{r,j} \in \mathbb{Z}$ by

$$\mu_{r,j} = \left\lceil \frac{mj}{p^r} \right\rceil - N \quad \text{and} \quad \nu_{r,j} = \left\lfloor \frac{mj+m-1}{p^r} \right\rfloor, \quad (4.8)$$

it follows from (4.7) and Lemma 4.5(a) applied to the initial sequence δ , after recalling also $\uparrow \delta \uparrow = 0 = \downarrow \delta \downarrow$ and $\uparrow a \uparrow = N$, that the equalities

$$\phi \left(\frac{mj+i}{p^r} - l \right) = 0 = (S_a^r \delta)_{mj+i-p^r l}, \quad l \in \mathbb{Z} \setminus \{\mu_{r,j}, \dots, \nu_{r,j}\} \quad (4.9)$$

hold for $i \in \{0, \dots, m-1\}$. Note from (4.8) that we have $0 \leq \nu_{r,j} - \mu_{r,j} \leq N$ for $j \in \mathbb{Z}$ if $r > \log_p(m-1)$, from which, together with (4.6), (4.9) and (4.1) applied to the initial

sequence δ , we find that, for any $\varepsilon > 0$, the inequalities

$$\begin{aligned} \left| K_i \Phi \left(\frac{mj+i}{p^r} \right) - (S_a^r c)_{mj+i} \right| &\leq \sum_{l=\mu_{r,j}}^{\nu_{r,j}} |c_l| \left| K_i \phi \left(\frac{mj+i}{p^r} - l \right) - (S_a^r \delta)_{mj+i-p^r l} \right| \\ &\leq \sum_{l=\mu_{r,j}}^{\nu_{r,j}} |c_l| \varepsilon \leq (N+1) \|c\|_\infty \varepsilon \end{aligned}$$

hold for $j \in \mathbb{Z}$ and $i \in \{0, \dots, m-1\}$ by taking r large enough. This shows that m -subsequence convergence occurs with limit Φ .

The converse direction of the proof is trivial, since $\delta \in l^\infty(\mathbb{Z})$ and in this case (1.31) reduces to $\Phi = \phi$. \square

The next result extends another well-known property of standard subdivision convergence to the case of subsequence convergence, namely that the limit function must be refinable if one starts with the delta sequence.

Theorem 4.7. *If there is a $k \in \mathbb{Z}_+$ such that the subdivision scheme $(S_{a,p}, \delta)$ has p^k -subsequence convergence with limit function ϕ , then (A, ϕ) is a p -refinement pair.*

Proof. By application of (1.26), we obtain

$$\text{Lpol}(S_a^{r+1} \delta) = p^{r+1} \prod_{j=0}^r A^{\langle p^j \rangle} = p A^{\langle p^r \rangle} \text{Lpol}(S_a^r \delta), \quad r \in \mathbb{Z}_+,$$

which by (1.8c) is equivalent to

$$(S_a^{r+1} \delta)_j = \sum_l a_l (S_a^r \delta)_{j-p^r l}, \quad j \in \mathbb{Z}, \quad r \in \mathbb{Z}_+. \quad (4.10)$$

Since $\sum_{i=0}^{p^k-1} K_i = 1$, there is an index $i \in \{0, \dots, p^k-1\}$ such that $K_i \neq 0$. Let now $x \in \mathbb{R}$ be fixed and choose a sequence $\{j_r : r \in \mathbb{Z}_+\}$ such that $\frac{j_r}{p^r} \rightarrow p^{1-k}x$ as $r \rightarrow \infty$ and let $\varepsilon > 0$ be given.

We have, by use of (4.10), that, for $r \in \mathbb{Z}_+$, $r \geq k$,

$$\begin{aligned} &K_i \left(\phi \left(\frac{p^k j_r + i}{p^{r+1}} \right) - \sum_l a_l \phi \left(\frac{p^k j_r + i}{p^r} - l \right) \right) = \\ &\left(K_i \phi \left(\frac{p^k j_r + i}{p^{r+1}} \right) - (S_a^{r+1} \delta)_{p^k j_r + i} \right) + \sum_l a_l \left((S_a^r \delta)_{p^k(j_r - p^{r-k}l) + i} - K_i \phi \left(\frac{p^k(j_r - p^{r-k}l) + i}{p^r} \right) \right). \end{aligned}$$

Together with the definition (4.1) of subsequence convergence, this yields the existence of

an integer $R \geq k$ such that

$$|K_i| \left| \phi \left(\frac{p^k j_r + i}{p^{r+1}} \right) - \sum_l a_l \phi \left(\frac{p^k j_r + i}{p^r} - l \right) \right| < \varepsilon + \sum_l |a_l| \varepsilon = C\varepsilon, \quad r \geq R, \quad (4.11)$$

where

$$C = 1 + \sum_l |a_l| < \infty. \quad (4.12)$$

According to Lemma 4.5(a), ϕ is compactly supported, so that $\phi \in C(\mathbb{R})$ implies that ϕ is uniformly continuous on \mathbb{R} . Hence there is a $\tau > 0$ such that

$$|\phi(x) - \phi(y)| < \varepsilon, \quad x, y \in \mathbb{R}, \quad |x - y| < \tau. \quad (4.13)$$

Since $\lim_{r \rightarrow \infty} \frac{p^k j_r + i}{p^{r+1}} = \lim_{r \rightarrow \infty} \left(p^{k-1} \frac{j_r}{p^r} + \frac{i}{p^{r+1}} \right) = x$, there is an integer R' such that $r \geq R'$ implies $\left| \frac{p^k j_r + i}{p^{r+1}} - x \right| < \frac{\tau}{p}$. Thus

$$\left| \left(\frac{p^k j_r + i}{p^r} - l \right) - (px - l) \right| = p \left| \frac{p^k j_r + i}{p^{r+1}} - x \right| < \tau, \quad l \in \mathbb{Z}, \quad r \geq R'.$$

Since also $\frac{\tau}{p} < \tau$, we now have, by (4.13), for $r \geq R'$,

$$\left| \phi \left(\frac{p^k j_r + i}{p^{r+1}} \right) - \phi(x) \right| < \varepsilon \quad (4.14)$$

and

$$\left| \phi \left(\frac{p^k j_r + i}{p^r} - l \right) - \phi(px - l) \right| < \varepsilon, \quad l \in \mathbb{Z}. \quad (4.15)$$

If we take $r \geq \max\{R, R'\}$, we have from (4.11), (4.12), (4.14) and (4.15) that

$$\begin{aligned} & \left| \phi(x) - \sum_l a_l \phi(px - l) \right| \\ &= \left| \phi(x) - \phi \left(\frac{p^k j_r + i}{p^{r+1}} \right) + \phi \left(\frac{p^k j_r + i}{p^{r+1}} \right) - \sum_l a_l \phi \left(\frac{p^k j_r + i}{p^r} - l \right) \right. \\ & \quad \left. + \sum_l a_l \left(\phi \left(\frac{p^k j_r + i}{p^r} - l \right) - \phi(px - l) \right) \right| \\ &\leq \left| \phi(x) - \phi \left(\frac{p^k j_r + i}{p^{r+1}} \right) \right| + \left| \phi \left(\frac{p^k j_r + i}{p^{r+1}} \right) - \sum_l a_l \phi \left(\frac{p^k j_r + i}{p^r} - l \right) \right| \\ & \quad + \sum_l |a_l| \left| \phi \left(\frac{p^k j_r + i}{p^r} - l \right) - \phi(px - l) \right| \\ &< \varepsilon + \frac{C\varepsilon}{|K_i|} + \sum_l |a_l| \varepsilon = C\varepsilon \left(1 + \frac{1}{|K_i|} \right). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we deduce that $\phi(x) = \sum_l a_l \phi(px - l)$ for any given $x \in \mathbb{R}$, which, together with the compact support of ϕ and the fact that ϕ is not identically zero, yields the desired result. \square

One trivial set of conditions which yields subsequence convergence occurs in the case of “stretched” masks as in Theorem 1.10. This is the subject of the next theorem.

Theorem 4.8. *Let $p \in \mathbb{N}$, $p \geq 2$. Suppose, for $m \in \mathbb{N}$, that the masks a and \tilde{a} are related by*

$$a_{jm+l} = \delta_l \tilde{a}_j, \quad j \in \mathbb{Z}, \quad l \in \mathbb{Z}_m$$

and that the subdivision algorithm $(S_{\tilde{a},p}, \delta)$ converges to the function ϕ . Then $(S_{a,p}, \delta)$ has m -subsequence convergence to the function $\phi\left(\frac{\cdot}{m}\right)$ with SCC given by $K_i = \delta_i$, $i \in \mathbb{Z}_m$.

Proof. Set, as usual, $c^{(r)} = S_{a,p}^r \delta$ and $\tilde{c}^{(r)} = S_{\tilde{a},p}^r \delta$. We show by induction that

$$c_{jm+l}^{(r)} = \delta_l \tilde{c}_j^{(r)}, \quad j \in \mathbb{Z}, \quad l \in \mathbb{Z}_m, \quad (4.16)$$

for $r \in \mathbb{Z}_+$. For $r = 0$ we obtain, for $l \in \mathbb{Z}_m$, $c_{jm+l}^{(0)} = \delta_{jm+l} = \delta_l \delta_j = \delta_l \tilde{c}_j^{(0)}$. Supposing now that (4.16) holds for a given $r \in \mathbb{Z}_+$, we obtain, for $j \in \mathbb{Z}$ and $l \in \mathbb{Z}_m$,

$$\begin{aligned} c_{jm+l}^{(r+1)} &= \sum_k a_{jm+l-pk} c_k^{(r)} = \sum_k \sum_{n=0}^{m-1} a_{jm+l-p(km+n)} c_{km+n}^{(r)} \\ &= \sum_k \sum_{n=0}^{m-1} a_{(j-pk)m+l-pn} \delta_n \tilde{c}_k^{(r)} = \sum_k a_{(j-pk)m+l} \tilde{c}_k^{(r)} \\ &= \sum_k \delta_l \tilde{a}_{j-pk} \tilde{c}_k^{(r)} = \delta_l \tilde{c}_j^{(r+1)}, \end{aligned}$$

which completes the inductive step.

Denoting $\phi\left(\frac{\cdot}{m}\right)$ by f , we obtain, from (4.16) and the convergence of $(S_{\tilde{a},p}, \delta)$ to ϕ , that

$$\begin{aligned} \max_{i \in \mathbb{Z}_m} \sup_j \left| \delta_i f\left(\frac{mj+i}{p^r}\right) - c_{mj+i}^{(r)} \right| &= \max_{i \in \mathbb{Z}_m} \sup_j \left| \delta_i \left(f\left(\frac{mj+i}{p^r}\right) - \tilde{c}_j^{(r)} \right) \right| \\ &= \sup_j \left| f\left(\frac{mj}{p^r}\right) - \tilde{c}_j^{(r)} \right| \\ &= \sup_j \left| \phi\left(\frac{j}{p^r}\right) - \tilde{c}_j^{(r)} \right| \rightarrow 0, \quad r \rightarrow \infty, \end{aligned}$$

which completes the proof of the theorem. \square

Example 4.9. In order to illustrate Theorem 4.8 for $p = 2$, consider the mask with symbol

$$A(z) = \frac{1}{4} (1 + z^3)^2, \quad z \in \mathbb{C},$$

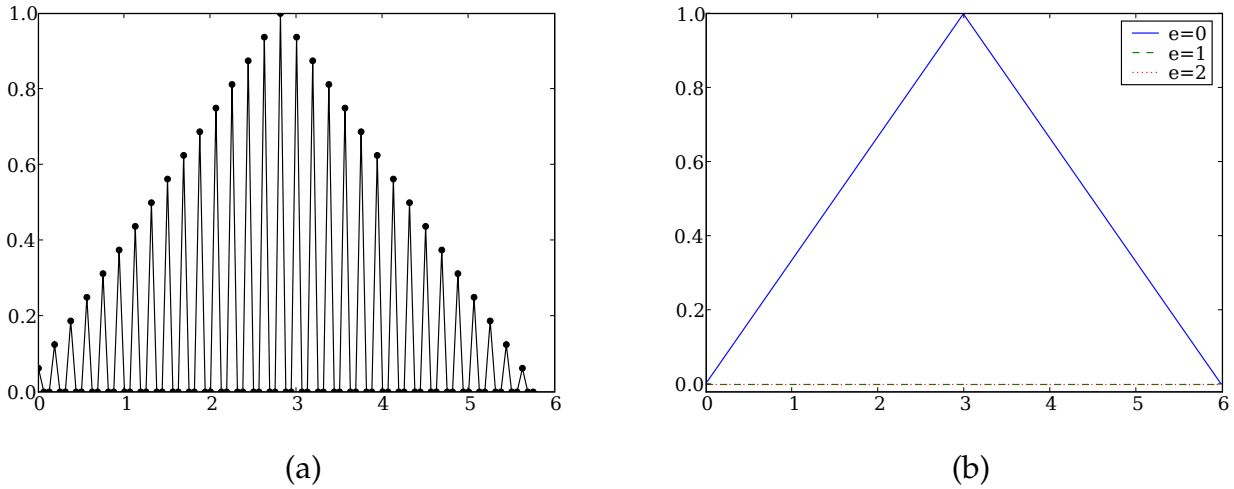


Figure 4.2: Plots of (a) $S_a^4 \delta$; and (b) $(S_a^8 \delta)_{3+e}$, $e = 0, 1, 2$ in Example 4.9.

which was also considered in [37: Example 2.1]. In the notation of Theorem 4.8, we have here $m = 3$ and $\tilde{A} = (E_1)^2$. Since $(S_{\tilde{a},2}, \delta)$ converges to N_2 , Theorem 4.8 tells us that $(S_{a,2}, \delta)$ has 3-subsequence convergence to $N_2(\frac{\cdot}{3})$, i.e. the linear B -spline with knots $\{0, 3, 6\}$, with subsequence convergence constants $K_0 = 1$, $K_1 = K_2 = 0$. This is illustrated in Figure 4.2.

This elementary case is well-known (see e.g. [37]), although the notion of subsequence convergence has not, to our knowledge, been formally defined before. We are of course interested in less elementary cases, as for instance in Example 4.1, which is not covered by Theorem 4.8. In order to derive such less trivial sufficient conditions for subsequence convergence to occur, we shall again make use of the GBP-factors of Section 2.2. This leads us to the following result.

Theorem 4.10. For dilation factor $p \in \mathbb{Z}$, $p \geq 2$, suppose the polynomial A satisfies $A = PB$, where P is a p -GBP factor of level k . Let the polynomials R, W be as in Lemma 2.14 and let the polynomial \tilde{A} be given by

$$\tilde{A} = E_{p-1}B. \tag{4.17}$$

Then, if the subdivision algorithm $(S_{\tilde{a},p}, c)$ converges to $\tilde{\Phi}$, the subdivision algorithm $(S_{a,p}, c)$ has p^k -subsequence convergence to the function Φ defined by

$$\Phi = p^k \sum_j [R]_j \tilde{\Phi}(\cdot - j), \tag{4.18}$$

with the SCC given by

$$K_i = \sum_l \left[W \prod_{j=0}^{k-1} B^{(p^j)} \right]_{i-p^k l}, \quad i = 0, 1, \dots, p^k - 1. \tag{4.19}$$

Proof. By Lemma 4.6, it is sufficient to consider the initial sequence $c = \delta$; hence, letting $\tilde{\phi}$ denote the limit of the subdivision scheme $(S_{\tilde{a},p}, \delta)$, we have to show that $(S_{a,p}, \delta)$ has p^k -subsequence convergence to the function ϕ , defined, in accordance with (4.18), by

$$\phi = p^k \sum_j [R]_j \tilde{\phi}(\cdot - j), \quad (4.20)$$

with the SCC given by (4.19).

From (1.26) we obtain

$$\text{Lpol}(S_a^r \delta) = p^r \prod_{j=0}^{r-1} \tilde{A}^{\langle p^j \rangle}, \quad r \in \mathbb{N}. \quad (4.21)$$

By using (2.2), (1.26), (1.8b), (1.8a) and (1.19b), we now obtain, for $r \in \mathbb{N}$, $r > k$,

$$\begin{aligned} \text{Lpol}(S_a^r \delta) &= p^r \prod_{j=0}^{r-1} E_{p-1}^{\langle p^j \rangle} \frac{R^{\langle p^{j+1} \rangle}}{R^{\langle p^j \rangle}} B^{\langle p^j \rangle} \\ &= p^r \frac{R^{\langle p^r \rangle}}{R} \left(\prod_{j=0}^{k-1} E_{p-1}^{\langle p^j \rangle} B^{\langle p^j \rangle} \right) \left(\prod_{j=k}^{r-1} E_{p-1}^{\langle p^j \rangle} B^{\langle p^j \rangle} \right) \\ &= p^r \frac{R^{\langle p^r \rangle}}{R} \left(\prod_{j=0}^{k-1} E_{p-1}^{\langle p^j \rangle} B^{\langle p^j \rangle} \right) \left(\prod_{j=0}^{r-k-1} E_{p-1}^{\langle p^{j+k} \rangle} B^{\langle p^{j+k} \rangle} \right) \\ &= p^r \frac{R^{\langle p^r \rangle}}{R} \left(\prod_{j=0}^{k-1} E_{p-1}^{\langle p^j \rangle} B^{\langle p^j \rangle} \right) \left(\prod_{j=0}^{r-k-1} E_{p-1}^{\langle p^j \rangle} B^{\langle p^j \rangle} \right)^{\langle p^k \rangle} \\ &= p^r \frac{E_{p^{k-1}}}{R} \left(\prod_{j=0}^{k-1} B^{\langle p^j \rangle} \right) \left(R^{\langle p^{r-k} \rangle} \prod_{j=0}^{r-k-1} \tilde{A}^{\langle p^j \rangle} \right)^{\langle p^k \rangle} \\ &= UV_r^{\langle p^k \rangle}, \end{aligned} \quad (4.22)$$

where, by noting (4.21) and the definition of W , the polynomials U and V are defined by

$$U = p^k W \prod_{j=0}^{k-1} B^{\langle p^j \rangle} \quad (4.23)$$

and

$$V_r = R^{\langle p^{r-k} \rangle} \text{Lpol}(S_a^{r-k} \delta).$$

We thus find, by also using (1.8c), that, for $j \in \mathbb{Z}$ and $i \in \{0, 1, \dots, p^k - 1\}$,

$$\begin{aligned} (S_a^r \delta)_{p^k j + i} &= \sum_l [U]_{p^k(j-l)+i} [V_r]_l \\ &= \sum_l [U]_{i-p^k l} [V_r]_{j+l} \end{aligned}$$

$$\begin{aligned}
&= \sum_l [U]_{i-p^k l} \left[\text{Lpol} (S_a^{r-k} \delta) R^{\langle p^{r-k} \rangle} \right]_{j+l} \\
&= \sum_l [U]_{i-p^k l} \sum_n (S_a^{r-k} \delta)_{j+l-p^{r-k} n} [R]_n.
\end{aligned} \tag{4.24}$$

Let $\lambda = -\lfloor p^{-k} \deg(U) \rfloor$. Using (4.19), (4.23), (4.20) and (4.24) we obtain

$$\begin{aligned}
&\left| K_i \phi \left(\frac{p^k j + i}{p^r} \right) - (S_a^r \delta)_{p^k j + i} \right| \\
&= \left| \sum_l [U]_{i-p^k l} \sum_n [R]_n \left(\tilde{\phi} \left(\frac{p^k j + i}{p^r} - n \right) - (S_a^{r-k} \delta)_{j+l-p^{r-k} n} \right) \right| \\
&\leq \sum_n |[R]_n| \sum_{l=\lambda}^0 |[U]_{i-p^k l}| \left(\left| \tilde{\phi} \left(\frac{p^k j + i}{p^r} - n \right) - \tilde{\phi} \left(\frac{j+l}{p^{r-k}} - n \right) \right| \right. \\
&\quad \left. + \left| \tilde{\phi} \left(\frac{j+l}{p^{r-k}} - n \right) - (S_a^{r-k} \delta)_{j+l-p^{r-k} n} \right| \right).
\end{aligned}$$

We have, for $l \in \{\lambda, \dots, 0\}$ and $i \in \{0, 1, \dots, p^k - 1\}$, that

$$\left| \frac{p^k j + i}{p^r} - n - \left(\frac{j+l}{p^{r-k}} - n \right) \right| = \frac{i - p^k l}{p^r} < \frac{p^k + p^k \lambda}{p^r} = \frac{\lambda + 1}{p^{r-k}} \rightarrow 0$$

independently of i, j, l and n as $r \rightarrow \infty$ and thus, by the uniform continuity of $\tilde{\phi}$, we find that $\left| \tilde{\phi} \left(\frac{p^k j + i}{p^r} - n \right) - \tilde{\phi} \left(\frac{j+l}{p^{r-k}} - n \right) \right| \rightarrow 0$ independently of i, j, l and n as $r \rightarrow \infty$. By the convergence of subdivision for \tilde{A} , we have $\left| \tilde{\phi} \left(\frac{j+l-p^{r-k} n}{p^{r-k}} \right) - (S_a^{r-k} \delta)_{j+l-p^{r-k} n} \right| \rightarrow 0$ independently of j, l and n as $r \rightarrow \infty$. Since both $\sum_n |[R]_n|$ and $\sum_{l=\lambda}^0 |[U]_{i-p^k l}|$ are finite constants and not dependent on r , we conclude that indeed (4.1) holds with $m = p^k$.

We also have, by use of (4.19), Lemma 2.14, as well as the fact that $B(1) = 1$, that the SCC satisfy

$$\sum_{i=0}^{p^k-1} K_i = \sum_{i=0}^{p^k-1} \sum_l \left[W \prod_{j=0}^{k-1} B^{\langle p^j \rangle} \right]_{i-p^k l} = \sum_j \left[W \prod_{j=0}^{k-1} B^{\langle p^j \rangle} \right]_j = W(1) \prod_{j=0}^{k-1} B^{\langle p^j \rangle}(1) = 1,$$

which completes the proof of the theorem. \square

We can now verify the results conjectured in Example 4.1. We have, in the notation of Theorem 4.10, $p = 2$ and $A(z) = \frac{1}{8}(1+z^2)^2(3-z)$, $z \in \mathbb{C}$, so that $P(z) = \frac{1}{2}(1+z^2)$, which implies $k = 1$, while

$$B(z) = \frac{1}{4}(1+z^2)(3-z) = \frac{1}{4}(3-z+3z^2-z^3), \quad z \in \mathbb{C},$$

and
$$\tilde{A}(z) = \frac{1}{8}(1+z)(1+z^2)(3-z) = \frac{1}{8}(3+5z+z^2-z^3), \quad z \in \mathbb{C}.$$

To show subdivision convergence for \tilde{A} , we shall again apply Theorem 4.10 and also employ Lemma 4.5. Consider the mask symbol

$$\tilde{A}(z) = \frac{1}{8}(1+z)^2(3-z), \quad z \in \mathbb{C}.$$

It is known (see e.g. [37: Example 8.5]) that subdivision for the mask symbol \tilde{A} converges. Hence by Theorem 4.10 subdivision for the mask symbol \tilde{A} has 2-subsequence convergence. We have $\tilde{B}(z) = \frac{1}{4}(1+z)(3-z) = \frac{1}{4}(3+2z-z^2)$, $z \in \mathbb{C}$, while Remark 2.15 yields that $\tilde{W} = E_0$. Thus

$$\tilde{W}(z)\tilde{B}(z) = \frac{1}{4}(3+2z-z^2), \quad z \in \mathbb{C}.$$

Consequently, the SCC are given, according to (4.19), by

$$\tilde{K}_0 = \sum_l [\tilde{W}\tilde{B}]_{-2l} = \frac{1}{4}(3-1) = \frac{1}{2}$$

and

$$\tilde{K}_1 = \sum_l [\tilde{W}\tilde{B}]_{1-2l} = \frac{1}{4}(2) = \frac{1}{2}.$$

Thus $\tilde{K}_0 = \tilde{K}_1$, which by Lemma 4.5(d) with $m = n = 2$ and $l = 1$, implies that subdivision for the mask symbol \tilde{A} has 1-subsequence convergence, i.e. subdivision for \tilde{A} converges.

Returning to the mask symbol A , we now have from Theorem 4.10 that indeed 2-subsequence convergence occurs. To calculate the SCC, we again have from Remark 2.15 that $W = E_0$, hence

$$W(z)B(z) = B(z) = \frac{1}{4}(3-z+3z^2-z^3), \quad z \in \mathbb{C}.$$

Then from (4.19) we obtain

$$K_0 = \sum_l [WB]_{-2l} = \frac{1}{4}(3+3) = \frac{3}{2}$$

and

$$K_1 = \sum_l [WB]_{1-2l} = \frac{1}{4}(-1-1) = -\frac{1}{2},$$

as conjectured previously.

Example 4.11. Our next examples illustrates Theorem 4.10 for the case of complex-valued masks. Let $p = 2$ and set

$$A_{\mathbb{C}}(z) = \frac{1}{6}(2 + (1 - 2i)z + iz^2 + (2 + i)z^3 + z^4) = \frac{1}{3}P(z)(2+z), \quad z \in \mathbb{C},$$

where $P(z) = \frac{1}{2}(1 - iz)(1 + iz^2)$, $z \in \mathbb{C}$, so that, according to Figure 2.1, P is a 2-GBP factor

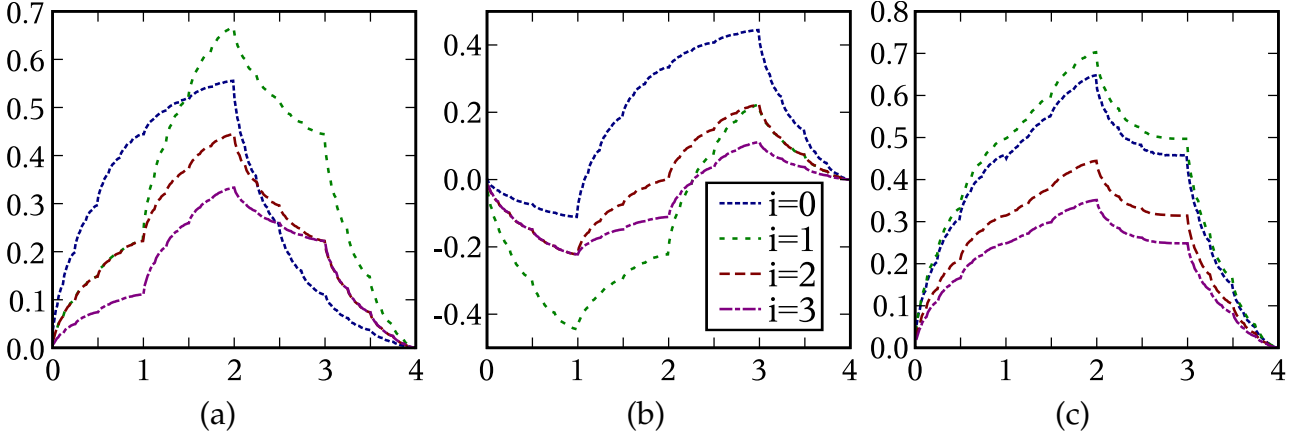


Figure 4.3: Plots of (a) the real parts; (b) imaginary parts; and (c) magnitudes of $\left(c_{\mathbb{C}}^{(8)}\right)_{4+i}$, $i \in \mathbb{Z}_4$, in Example 4.11.

of level $k = 2$. The corresponding reduced mask \tilde{A} is given by

$$\tilde{A}(z) = \frac{1}{6}(1+z)(2+z) = \frac{1}{6}(2+3z+z^2), \quad z \in \mathbb{C}.$$

By Theorem 2.26, the subdivision algorithm $(S_{\tilde{a}, p}, \delta)$ converges to the corresponding refinable function $\tilde{\phi}$. (This function is well known, being considered in e.g. [34: Chapter 2].) Since P is a 2-GBP factor of level 2, it follows from Theorem 4.10 that subdivision with the mask $a_{\mathbb{C}}$ has 4-subsequence convergence to the corresponding 2-refinable function $\phi_{\mathbb{C}}$. In the notation of Remark 2.13, we have here $r_0 = E_1$ and $r_1(z) = \frac{1-i}{2}(1+iz)$, $z \in \mathbb{C}$, so that

$$R(z) = \frac{1-i}{4}(1+(1+i)z+iz^2), \quad z \in \mathbb{C}.$$

Thus we find that $\phi_{\mathbb{C}}$ is given by

$$\phi_{\mathbb{C}} = (1-i)\tilde{\phi} + 2\tilde{\phi}(\cdot-1) + (1+i)\tilde{\phi}(\cdot-2).$$

To calculate the SCC, observe that

$$B(z) = \frac{1}{3}(2+z) \quad \text{and} \quad W(z) = \frac{E_3(z)}{R(z)} = \frac{1+i}{2}(1-iz)$$

for $z \in \mathbb{C}$. Thus we obtain

$$(WBB^{(2)})(z) = \frac{1}{18}(4+4i+(6-2i)z+4z^2+(3-i)z^3+(1-i)z^4), \quad z \in \mathbb{C},$$

which yields, together with the formula $K_i = \sum_l [WBB^{(2)}]_{i-4l}$, $i \in \mathbb{Z}_4$, obtained from (4.19), the SCC values

$$K_0 = \frac{5+3i}{18}, \quad K_1 = \frac{3-i}{9}, \quad K_2 = \frac{2}{9} \quad \text{and} \quad K_3 = \frac{3-i}{18}.$$

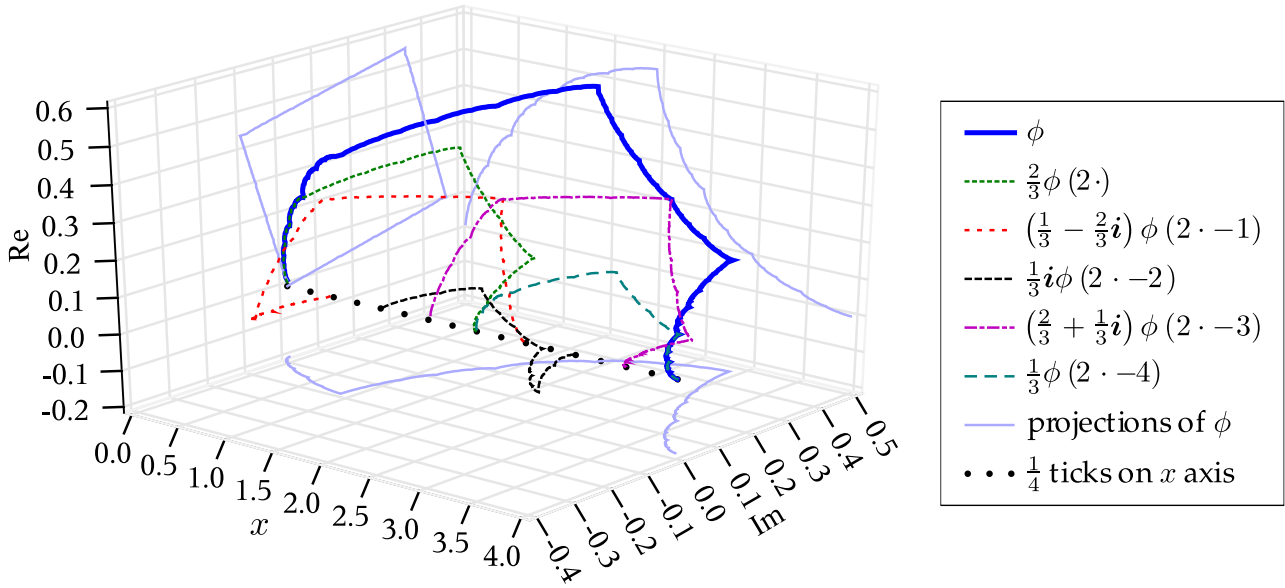


Figure 4.4: A further plot to illustrate the structure of the complex-valued refinable function $\phi_{\mathbb{C}}$ in Example 4.11.

Note in particular that $K_1 = 2K_3$. This result is graphically illustrated in Figure 4.3, where we adopt the notation $c_{\mathbb{C}}^{(r)} = (S_{a_{\mathbb{C}}, 2}^r \delta)$, $r \in \mathbb{Z}_+$, while Figure 4.4 illustrates the structure and refinability of the limit function $\phi_{\mathbb{C}}$.

4.1 Applications

It is important to realise that normal subdivision algorithms can easily be adapted to make use of subsequence convergence, as one simply uses normal subdivision and then subsamples the resulting control points to obtain the desired subdivided curve. It must be noted that in practice one must take extra care at the endpoints to account for subsequence convergence.

Example 4.12. We now show another novel (albeit less serious) application of subdivision convergence, namely how it can be used for decorative effects. Fix the dilation factor at 2 and let, for $\beta \in (0, 1)$, the mask symbol A_{β} be defined by

$$A_{\beta}(z) = \frac{1}{4} (1 + z^2)^2 (\beta + (1 - \beta)z) = \frac{\beta}{4} (1 + 2z^2 + z^4) + \frac{1 - \beta}{4} (z + 2z^3 + z^5), \quad z \in \mathbb{C}, \tag{4.25}$$

and take $P = E_1^{(2)}$. The corresponding reduced mask symbol \tilde{A}_{β} of Theorem 4.10 is given by

$$\tilde{A}_{\beta}(z) = \frac{1}{4} (\beta + z + z^2 + z^3 + (1 - \beta)z^4), \quad z \in \mathbb{C},$$

so that \tilde{a}_{β} is a positive mask. By Theorem 2.26, subdivision with the mask \tilde{a}_{β} is convergent. Since P is a 2-GBP factor of level 1, by Theorem 4.10 it follows that 2-subsequence

convergence will occur for the mask a_β . Since

$$B(z) = \frac{1}{2} (\beta + (1 - \beta)z + \beta z^2 + (1 - \beta)z^3), \quad z \in \mathbb{C},$$

and, according to Remark 2.15, $W = E_0$, we obtain that

$$(WB)(z) = \frac{1}{2} (\beta + (1 - \beta)z + \beta z^2 + (1 - \beta)z^3), \quad z \in \mathbb{C}.$$

Thus, from (4.19) with $k = 1$, the SCC are given by

$$K_0 = \frac{1}{2} (\beta + \beta) = \beta$$

and

$$K_1 = \frac{1}{2} (1 - \beta + 1 - \beta) = 1 - \beta.$$

In this example, we actually have

$$c_{2^r}^{(r)} = \frac{1 - \beta}{\beta} c_{2^{r+1}}^{(r)} = \frac{K_1}{K_0} c_{2^{r+1}}^{(r)}, \quad r \in \mathbb{N}, \quad (4.26)$$

as we will show by induction. From (4.25), we see that

$$(a_\beta)_{2j} = \frac{1 - \beta}{\beta} (a_\beta)_{2j+1}, \quad j \in \mathbb{Z}. \quad (4.27)$$

Since $c^{(1)} = a_\beta$, we conclude that (4.26) holds for $r = 1$. Assuming now that (4.26) holds for an arbitrary $r \in \mathbb{N}$, we obtain, by use of (4.27), for $j \in \mathbb{Z}$,

$$c_{2j}^{(r+1)} = \sum_j (a_\beta)_{2j-2k} c_k^{(r)} = \sum_j \frac{1 - \beta}{\beta} (a_\beta)_{2j-2k+1} c_k^{(r)} = \frac{1 - \beta}{\beta} c_{2j+1}^{(r+1)},$$

completing the inductive step.

(4.26) means that if we take a small number of iterations, say 5 or 6, we obtain a curve which jumps back and forth between the two curves formed by the even and odd indexed entries of the curve. This can give rise to interesting patterns. We illustrate this in Figure 4.5, where we subdivide a simple “diamond” shape six times using the mask obtained for $\beta = \frac{3}{4}$.

Note that the appearance of the resulting shape is not translation independent, as can be seen from Figure 4.6, where we again use $\beta = \frac{3}{4}$ and subdivide 6 times as in Figure 4.5, but shifted the initial curve a $\frac{1}{4}$ unit right and an $\frac{1}{8}$ unit up. This provides us with some parameters we can use to change the appearance of our decoration.

By varying the value of β , we can also control the ratio between the “even” and “odd” curves according to (4.26). This is illustrated in Figure 4.7, where we used the same initial curve and plot the curves of the fifth iteration of subdivision for $\beta = \frac{8}{9}, \frac{2}{3}, \frac{4}{9}$.

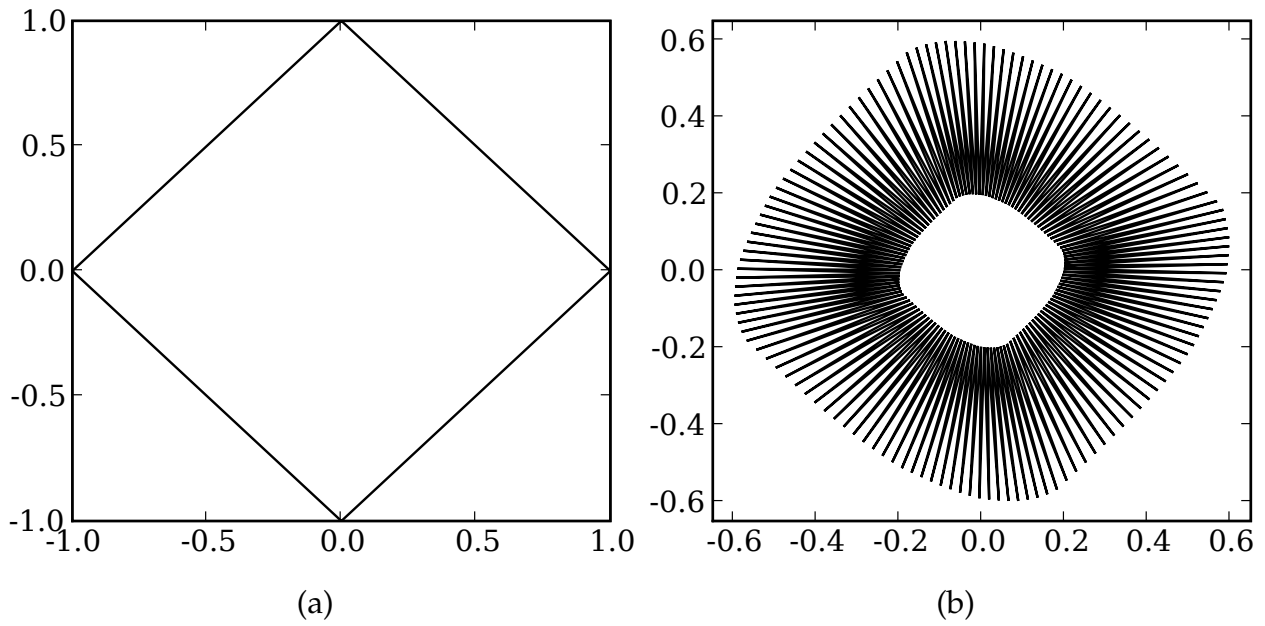


Figure 4.5: Plots of (a) the initial curve; and (b) the sixth iteration of subdivision with $\beta = \frac{3}{4}$ in Example 4.12.

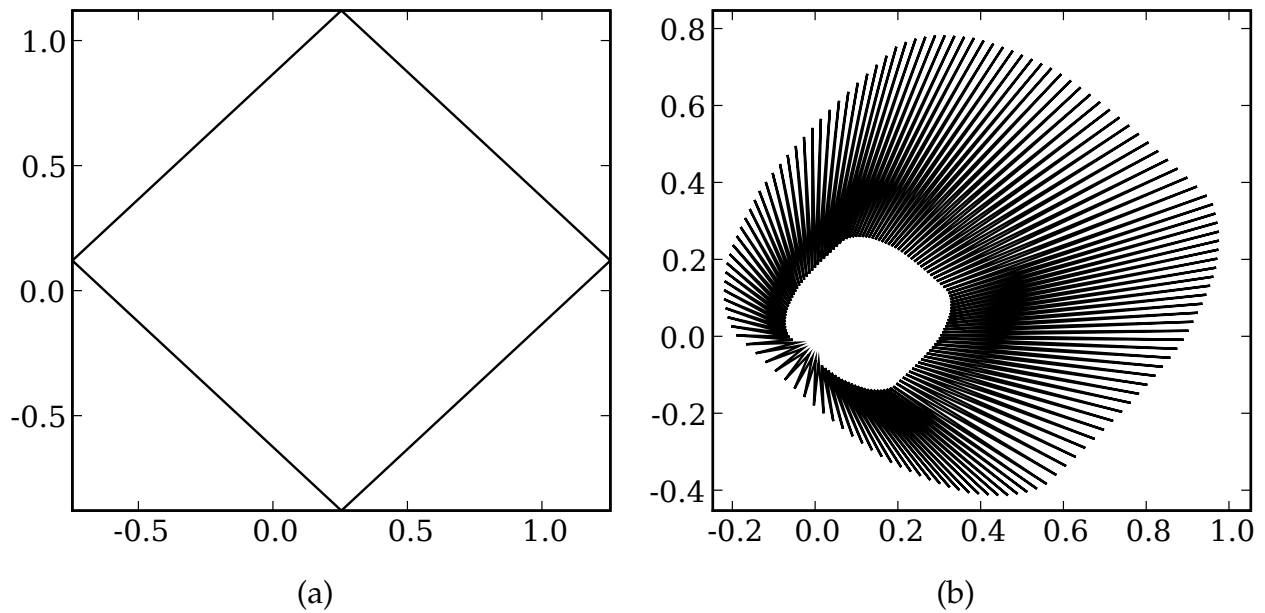


Figure 4.6: Plots of (a) the shifted initial curve; and (b) the sixth iteration of the resulting subdivision with $\beta = \frac{3}{4}$ in Example 4.12.

Of course one can use the same techniques to construct more complex decorations based on m -subsequence convergence where $m > 2$.

An issue that might be of interest for future research is the consideration of subsequence convergence in the matrix subdivision case as considered in [31] and the other references listed in the introduction.

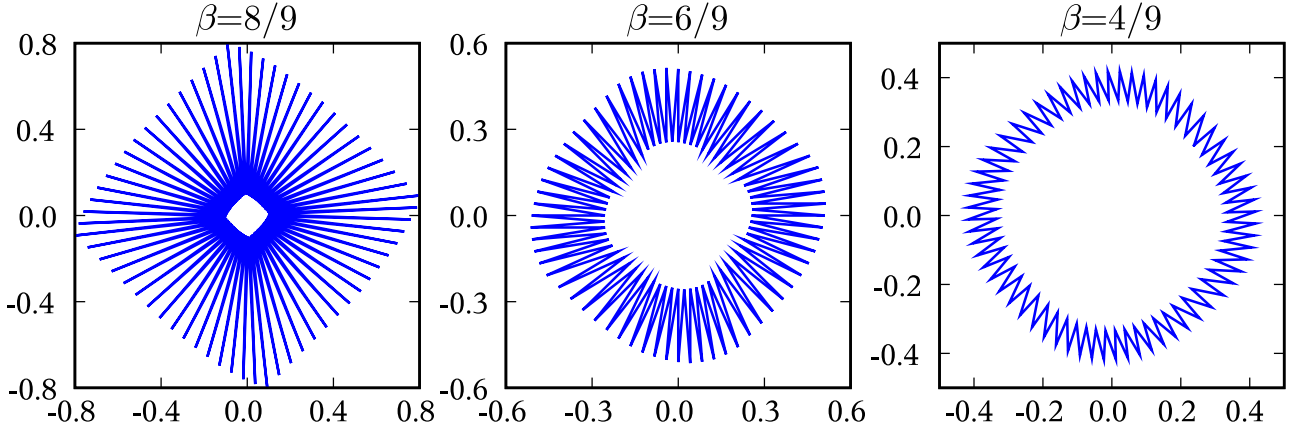


Figure 4.7: Plots of the fifth iteration of subdivision for 3 different values of β in Example 4.12.

4.2 Nested sets of refinement masks

The results of this chapter allow us to build a nested sequence of sets of refinement masks. Let $RM(p)$ denote the set of all p -refinement mask symbols corresponding to continuous p -refinable functions. Let $RMS(p, k)$ denote the set of mask symbols for which the corresponding p -subdivision algorithm has p^k -subsequence convergence. Specifically, $RMS(p, 0)$ denotes the set of mask symbols for which p -subdivision converges.

It is well known (see e.g. [37]) that $RMS(2, 0) \subsetneq RM(2)$. The following results give further insight into the nature of the set $RM(p) \setminus RMS(p, 0)$.

In view of Theorem 4.7, we know that $RMS(p, k) \subset RM(p)$ for any $k \in \mathbb{Z}_+$. Furthermore, from Lemma 4.5(c) it follows that $RMS(p, k-1) \subset RMS(p, k)$ for $k \in \mathbb{N}$. Since $(E_{p-1})^m \in RMS(p, 0)$ for $p \in \mathbb{N}$, $p \geq 2$ and $m \in \mathbb{N}$, we thus obtain, for $p \in \mathbb{N}$, $p \geq 2$,

$$\emptyset \neq RMS(p, 0) \subset RMS(p, 1) \subset \cdots \subset RMS(p, k) \subset \cdots \subset RM(p). \quad (4.28)$$

The next result shows that $\{RMS(p, k) : k \in \mathbb{Z}_+\}$ is a properly nested sequence.

Theorem 4.13. *For $k \in \mathbb{N}$, the relation $RMS(p, k-1) \subsetneq RMS(p, k)$ holds.*

Proof. Let the dilation factor $p \in \mathbb{Z}$, $p \geq 2$ be given and let $B(z) = \frac{2}{3} + \frac{1}{3}z$, $z \in \mathbb{C}$. Define, for $k \in \mathbb{Z}_+$, $A_k = E_{p-1}^{\langle p^k \rangle} B$. Then A_0 is the symbol of a positive mask for which the sum rules hold, so that Theorem 2.26 guarantees the convergence of the corresponding subdivision algorithm. Then for any $k \in \mathbb{Z}_+$, since $E_{p-1}^{\langle p^k \rangle}$ is a p -GBP factor of level k and the reduced mask \tilde{A}_k is exactly A_0 , we know from Theorem 4.10 that the subdivision algorithm for A_k has p^k -subsequence convergence to the associated refinable function ϕ_k , so that $A_k \in RMS(p, k)$. To show that p^{k-1} -subsequence does not occur for A_k when $k \geq 1$, we first show that the SCC satisfy $K_0 \neq K_{p^{k-1}}$. To do so, define

$$M_k = \prod_{j=0}^{k-1} B^{\langle p^j \rangle}, \quad k \in \mathbb{N}.$$

Then for $k \in \mathbb{Z}$, $k \geq 2$ it follows that $M_k = BM_{k-1}^{(p)}$, from which we deduce by (1.8c) and the definition of B that

$$[M_k]_{pj} = \sum_l [M_{k-1}]_l [B]_{pj-pl} = \frac{2}{3} [M_{k-1}]_j, \quad k \in \mathbb{Z}, k \geq 2. \quad (4.29)$$

Since $[B]_{pj} = 2[B]_{pj+1}$ and $B = M_1$, we obtain, by repeated application of (4.29), for $k \in \mathbb{Z}$, $k \geq 2$,

$$[M_k]_{p^k j} = \left(\frac{2}{3}\right)^{k-1} [M_1]_{pj} = 2 \left(\frac{2}{3}\right)^{k-1} [M_1]_{pj+1} = 2 [M_k]_{p^k j + p^{k-1}}, \quad j \in \mathbb{Z},$$

allowing us, in view also of Remark 2.15, to deduce that

$$K_0 = \sum_j [M_k]_{-p^k j} = 2 \sum_j [M_k]_{p^{k-1} - p^k j} = 2K_{p^{k-1}}$$

for any given $k \in \mathbb{N}$. Since it can also be verified that $K_0 \neq 0$, it follows that $K_0 \neq K_{p^{k-1}}$, so that by part 4 of Lemma 4.5 we conclude that the subdivision algorithm for A_k does not have p^{k-1} -subsequence convergence to $p^{-1}\phi_k$. Now suppose that $A_k \in RMS(p, k-1)$. Then there is a p -refinable function ψ_k such that the subdivision algorithm for A_k has p^{k-1} -subsequence convergence to ψ_k . Then by part 3 of Lemma 4.5, subdivision for A_k has p^k -subsequence convergence to $p\psi_k$, which by part 2 of Lemma 4.5 means that $\psi_k = p^{-1}\phi_k$, which contradicts the fact that subdivision for A_k does not have p^{k-1} -subsequence convergence to $p^{-1}\phi_k$. Thus $A_k \notin RMS(p, k-1)$, showing that $A_k \in RMS(p, k) \setminus RMS(p, k-1)$. With (4.28), this gives the desired result. \square

Using subsequence convergence in subdivision allows us to plot all the p -refinable functions corresponding to masks in $RMS(p, \infty) := \bigcup_{k \in \mathbb{Z}_+} RMS(p, k)$, which, by Theorem 4.13 and the uniqueness results Theorem 1.8(a) and Lemma 1.9, is a proper superset of the p -refinable functions that can be plotted by normal subdivision. An interesting question that remains open, is what the nature of the set $RM(p) \setminus RMS(p, \infty)$ is.

Chapter 5

A note on explicit formulas for masks of length 3

The issue of explicit formulas to calculate the values of refinable functions is addressed by Daubechies & Lagarias [19], who present the formulas in terms of infinite products of matrices. It is shown in [19] how these formulas can be used to calculate the (local) Hölder continuity and fractal dimension of a given refinable function.

In this chapter, we consider a very special case, where the dilation factor is fixed at 2 and the mask is of length 3 and present our own formula for this case. What we point out specifically, is how this formula can be used to calculate the exact value of the refinable function at any rational point.

Given $x \in [0, 1]$, let $d_j(x)$ denote the j 'th digit in a binary expansion of x , i.e.

$$x = \sum_{j=1}^{\infty} d_j(x) 2^{-j}, \quad x \in [0, 1],$$

with $d_j(x) \in \{0, 1\}$, $j \in \mathbb{N}$. Subsequently define $D_j(x)$ to be the number of 1's to the left of the j 'th digit in the chosen binary expansion above, that is

$$D_1(x) = 0, \quad D_j(x) = \sum_{i=1}^{j-1} d_i(x), \quad j \geq 2.$$

Consider, for $a_0 \in (0, 1)$, the refinement equation

$$\phi = a_0\phi(2\cdot) + \phi(2\cdot - 1) + (1 - a_0)\phi(2\cdot - 2). \quad (5.1)$$

According to Theorem 1.8(a) and (d), Lemma 1.12, Remark 1.13 and Theorem 2.26, we know that there exists a unique 2-refinable function $\phi \in C(\mathbb{R}) \cap M_0(\mathbb{R})$ satisfying the normalising condition (1.15), with ϕ also having the property

$$\phi(x) = 0, \quad x \notin (0, 2). \quad (5.2)$$

Independently of [19], and by intensive analysis of the graph of ϕ for the special case

$a_0 = \frac{2}{3}$, we derived the explicit formula

$$\phi(x) = \sum_{j=1}^{\infty} d_j(x) a_0^{j-D_j(x)} (1-a_0)^{D_j(x)}, \quad x \in (0, 1). \quad (5.3)$$

Since (1.15) and (5.2) give

$$\phi(x) = 1 - \phi(x-1), \quad x \in [1, 2), \quad (5.4)$$

we see that the formula (5.3) can in fact also be used to explicitly compute ϕ on the interval $[1, 2)$. Observe in particular from (5.2) and (5.4) that $\phi(1) = 1$.

We proceed to verify our formula (5.3) by showing that, on $(0, 1)$, if ϕ is given by (5.3), then the refinement equation (5.1) is satisfied.

Let $x \in (0, 1)$ and suppose $d_1(x) = 0$. This means that $0 < x \leq \frac{1}{2}$, so that $0 < 2x \leq 1$, and we find

$$d_j(2x) = d_{j+1}(x), \quad j \in \mathbb{N}, \quad (5.5)$$

yielding

$$D_j(2x) = D_{j+1}(x), \quad j \in \mathbb{N}. \quad (5.6)$$

From (5.3), (5.5) and (5.6) we obtain

$$\begin{aligned} a_0\phi(2x) + \phi(2x-1) + (1-a_0)\phi(2x-2) &= a_0 \sum_{j=1}^{\infty} d_j(2x) a_0^{j-D_j(2x)} (1-a_0)^{D_j(2x)} + 0 + 0 \\ &= \sum_{j=1}^{\infty} d_{j+1}(x) a_0^{j+1-D_{j+1}(x)} (1-a_0)^{D_{j+1}(x)} \\ &= \sum_{j=2}^{\infty} d_j(x) a_0^{j-D_j(x)} (1-a_0)^{D_j(x)} \\ &= \sum_{j=1}^{\infty} d_j(x) a_0^{j-D_j(x)} (1-a_0)^{D_j(x)} = \phi(x), \end{aligned}$$

where we used $d_1(x) = 0$ in the last line.

Now suppose $d_1(x) = 1$, which means that $\frac{1}{2} \leq x < 1$. Then $0 \leq 2x-1 < 1$, and we obtain

$$d_j(2x-1) = d_{j+1}(x), \quad j \in \mathbb{N}, \quad (5.7)$$

which gives

$$D_j(2x-1) = D_{j+1}(x) - 1, \quad j \in \mathbb{N}. \quad (5.8)$$

Since $1 \leq 2x < 2$, (5.4), (5.3), (5.7) and (5.8), together with the facts that $d_1(x) = 1$ and

$D_0(x) = 0$, now yield

$$\begin{aligned}
a_0\phi(2x) + \phi(2x-1) + (1-a_0)\phi(2x-2) &= a_0(1-\phi(2x-1)) + \phi(2x-1) + 0 \\
&= a_0 + (1-a_0)\phi(2x-1) \\
&= a_0 + (1-a_0) \sum_{j=1}^{\infty} d_j(2x-1) a_0^{j-D_j(2x-1)} (1-a_0)^{D_j(2x-1)} \\
&= a_0 + \sum_{j=1}^{\infty} d_{j+1}(x) a_0^{j-D_{j+1}(x)+1} (1-a_0)^{1+D_{j+1}(x)-1} \\
&= a_0 + \sum_{j=2}^{\infty} d_j(x) a_0^{j-D_j(x)} (1-a_0)^{D_j(x)} \\
&= \sum_{j=1}^{\infty} d_j(x) a_0^{j-D_j(x)} (1-a_0)^{D_j(x)} = \phi(x),
\end{aligned}$$

thereby concluding the verification of the formula (5.3) for ϕ .

The formula (5.3) can be efficiently computed using the following very simple algorithm, which is especially suited to the binary floating point representation common in computers today. In this algorithm, J is some pre-chosen integer depending on the desired accuracy of the computed value or the precision with which x is stored in the computer.

Algorithm 5.1.

```

y ← a0
f ← 0
for j = 1 to J do
  if dj(x) = 0
    y ← ya0
  else
    f ← f + y
    y ← y(1 - a0)
return f

```

It is clear that Algorithm 5.1 computes $\sum_{j=1}^J d_j(x) y_j(x)$, where $y_j(x)$ represents the value of y in the algorithm at the beginning of the j 'th iteration. To prove the correctness of the algorithm, it is thus sufficient to prove that the equality

$$y_j(x) = a_0^{j-D_j(x)} (1-a_0)^{D_j(x)} \quad (5.9)$$

holds for $j \in \{1, \dots, J\}$. Since $D_1 = 0$ and $y_1 = a_0$, it is clear that (5.9) holds for $j = 1$. Now assume that (5.9) holds for some $j \in \{1, \dots, J-1\}$. If $d_j(x) = 0$, we have $D_{j+1}(x) = D_j(x)$,

so that

$$y_{j+1}(x) = a_0 y_j(x) = a_0 a_0^{j-D_j(x)} (1-a_0)^{D_j(x)} = a_0^{j+1-D_{j+1}(x)} (1-a_0)^{D_{j+1}(x)},$$

while $d_{j+1}(x) = 1$ yields $D_{j+1}(x) = D_j(x) + 1$, so that

$$y_{j+1}(x) = (1-a_0) y_j(x) = (1-a_0) a_0^{j-D_j(x)} (1-a_0)^{D_j(x)} = a_0^{j+1-D_{j+1}(x)} (1-a_0)^{D_{j+1}(x)},$$

showing that (5.9) holds with j replaced by $j+1$. By induction, this yields the desired result.

Remark 5.2. In practice, Algorithm 5.1 might have to be adapted slightly to avoid the “vanishing tail” problem in the summation: since y can get very small relative to f for large values of j , a floating point addition of f and y might cause a large number of the significant digits of y to be lost. This problem can be overcome by standard numerical techniques, like appropriate grouping of the iterations.

Note that for the special case $a_0 = \frac{1}{2}$, which means that $\phi = N_2$, we obtain from (5.3), for $x \in (0, 1)$,

$$\phi(x) = \sum_{j=1}^{\infty} d_j(x) 2^{D_j(x)-j} 2^{-D_j(x)} = \sum_{j=1}^{\infty} d_j(x) 2^{-j} = x,$$

as expected.

The formula (5.3) is particularly useful to calculate exact values of the refinable function for non-dyadic rational fractions. In this case we exploit the fact that the binary expansion of a non-dyadic rational has a non-ending repetition of a fixed sequence of digits as its tail. This allows us to partition the infinite sum in (5.3) to calculate the value exactly. As an example, we compute $\phi(\frac{1}{3})$. In this case we have

$$x = \frac{1}{3} = .0\dot{1}_2 = \sum_{k=1}^{\infty} 2^{-2k},$$

so that $d_{2k-l}(\frac{1}{3}) = 1-l$, $k \in \mathbb{N}$, $l = 0, 1$. Hence $D_{2k-1}(\frac{1}{3}) = D_{2k}(\frac{1}{3}) = k-1$, $k \in \mathbb{N}$, so that, in view also of $0 < a_0(1-a_0) < \frac{1}{4}$, (5.3) yields

$$\phi\left(\frac{1}{3}\right) = \sum_{k=1}^{\infty} a_0^{2k-(k-1)} (1-a_0)^{k-1} = a_0^2 \sum_{k=0}^{\infty} (a_0(1-a_0))^k = \frac{a_0^2}{a_0^2 - a_0 + 1}.$$

For the choice $a_0 = \frac{2}{3}$, we get $\phi(\frac{1}{3}) = \frac{4}{7}$, as illustrated in Figure 5.1, where we show the function ϕ and a blow-up of the same plot around the point $(\frac{1}{3}, \frac{4}{7})$.

As was already done in [19] in a more general setting, we can also use the formula (5.3) to calculate the Hölder continuity of ϕ and to show that ϕ is not differentiable at every dyadic point within its support by making use of the binary expansions of $x \pm \frac{1}{2^r}$ for x a dyadic rational and r a large enough integer. We do not pursue this topic further here.

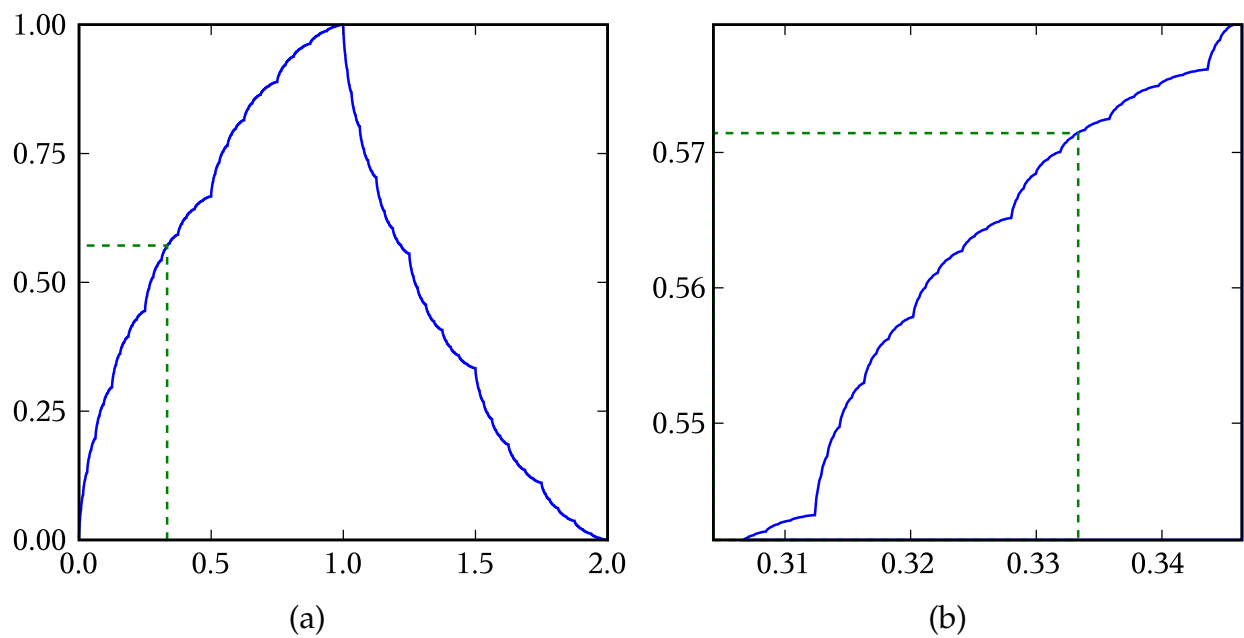


Figure 5.1: (a) Plot of the refinable function ϕ corresponding to the choice $a_0 = \frac{2}{3}$. (b) The same plot zoomed in the area of the point $(\frac{1}{3}, \frac{4}{7})$. In both plots, the dashed lines are drawn on $x = \frac{1}{3}$ and $y = \frac{4}{7}$ respectively.

Bibliography

- [1] Baron, K. & Jarczyk, W. 2001. Recent results on functional equations in a single variable, perspectives and open problems. *Aequationes Math.*, 61:1–48. 1, 2
- [2] Belogay, E. & Wang, Y. 1999. Compactly supported orthogonal symmetric scaling functions. *Appl. Comput. Harmon. Anal.*, 7(2):137–150. 2
- [3] Berg, L. & Plonka, G. 1998. Compactly supported solutions of two-scale difference equations. *Lin. Alg. Appl.*, 275:49–75. 14, 20, 24, 25, 27, 43, 54
- [4] Berg, L. & Plonka, G. 2000. Some notes on two-scale difference equations. In T. Rassias (ed.), *Functional Equations & Inequalities*. Kluwer Academic Publishers. 20, 44
- [5] Bi, N., Dai, X. & Sun, Q. 1999. Construction of compactly supported m -band wavelets. *Appl. Comput. Harmon. Anal.*, 6:113–131. 2
- [6] Bi, N., Debnath, L. & Sun, Q. 1998. Asymptotic behaviour of m -band scaling functions of daubechies type. *Z. Anal. Anwendungen*, 17:813–830. 2
- [7] Blu, T. 1998. A new design algorithm for two-band orthonormal rational filter banks and orthonormal rational wavelets. *IEEE Trans. Signal Proc.*, 46:1494–1504. 15
- [8] Borwein, J.M. & Girgensohn, R. 1994. Functional equations and distribution functions. *Results Math.*, 26:229–237. 1, 2
- [9] Bownik, M. 2003. On a problem of daubechies. *Constr. Approx.*, 19:179–190. 15
- [10] Cavaretta, A.S., Dahmen, W. & Micchelli, C.A. 1991. Stationary subdivision. *Mem. Amer. Math. Soc.*, 93(453). 1, 10, 11, 12, 13, 16
- [11] Chui, C.K. 1992. *Introduction to Wavelet Analysis*. Academic Press. 1, 10
- [12] Cohen, A. & Daubechies, I. 1993. Orthonormal bases of compactly supported wavelets III. better frequency resolution. *SIAM J. Math. Anal.*, 24(2):520–527. 2, 15
- [13] Dahlke, S., Dahmen, W. & Latour, V. 1995. Smooth refinable functions and wavelets obtained by convolution products. *Appl. Comput. Harmon. Anal.*, 2:68–84. 2
- [14] Dahmen, W. & Micchelli, C.A. 1997. Biorthogonal wavelet expansions. *Constr. Approx.*, 13:293–328. 2, 10, 12
- [15] Dai, X.-R., Feng, D.-J. & Wang, Y. 2007. Structure of refinable splines. *Appl. Comput. Harmon. Anal.*, 22:374–381. 2, 16

- [16] Dalhke, S., Latour, V. & Neeb, M. 1997. Generalized cardinal B-splines: Stability, linear independence, and appropriate scaling matrices. *Constr. Approx.*, 13:29–56. 2
- [17] Daubechies, I. 1992. *Ten Lectures on Wavelets*. SIAM, Philadelphia. 1, 2
- [18] Daubechies, I. & Lagarias, J.C. 1991. Two-scale difference equations I. Existence and global regularity of solutions. *SIAM J. Math. Anal.*, 22(5):1388–1410. 1, 2, 5, 15, 27, 34, 35, 50
- [19] Daubechies, I. & Lagarias, J.C. 1992. Two-scale difference equations II. Local regularity, infinite products of matrices and fractals. *SIAM J. Math. Anal.*, 23(4):1031–1079. 2, 79, 82
- [20] De Villiers, J. 2006. On refinable functions and subdivision with positive masks. *Adv. Comput. Math.* 14, 30, 52
- [21] De Villiers, J. 2006. *Subdivision, Wavelets and Splines*. Department of Mathematical Sciences, University of Stellenbosch. Lecture notes for Honours course. 7
- [22] Deliu, A. & Spruill, M.C. 2000. Existence results for refinement equations. *Aequationes Math.*, 59:20–37. 2, 11
- [23] Derfel, G. & Schilling, R. 1996. Spatially chaotic configurations and functional equations with rescaling. *J. Phys. A: Math. Gen.*, 29:4537–4547. 1, 2
- [24] Dyn, N., Gregory, J.A. & Levin, D. 1991. Analysis of uniform binary subdivision schemes for curve design. *Constr. Approx.* 1
- [25] Girgensohn, R. 2002. A survey of results and open problems on the Schilling equation. In Z. Daróczy & Z. Páles (eds.), *Functional Equations – Results and Advances*, pages 159–174. Kluwer. 1
- [26] Gonsor, D.E. 1993. Subdivision algorithms with nonnegative masks generally converge. *Adv. Comput. Math.*, 1:215–221. 51
- [27] Goodman, T.N.T. & Sun, Q. 2004. Total positivity and refinable functions with general dilation. *Appl. Comput. Harmon. Anal.*, 16:69–89. 2, 13, 29, 30, 33
- [28] Han, B. 2002. Symmetry property and construction of wavelets with a general dilation matrix. *Lin. Alg. Appl.*, 353(1). 2
- [29] Huang, D., Sun, Q. & Zhang, Z. 1997. Integral representation of M -band filters of Daubechies type. *Chinese Sci. Bull.*, 42:803–807. 2
- [30] Jia, R.-Q. & Zhou, D.-X. 1999. Convergence of subdivision schemes associated with nonnegative masks. *SIAM J. Matrix Anal. Appl.*, 21(2):418–430. 10

- [31] Latour, V., Müller, J. & Nickel, W. 1998. Stationary subdivision for general scaling matrices. *Math. Z.*, 227:645–661. 2, 10, 11, 12, 76
- [32] Lawton, W., Lee, S.L. & Shen, Z. 1995. Characterization of compactly supported refinable splines. *Adv. Comp. Math.*, 3:137–145. 2, 9, 18, 19
- [33] Melkman, A.A. 1997. Subdivision schemes with non-negative masks always converge—unless they obviously cannot? *Ann. Numer. Math.*, 4:451–460. 12, 51
- [34] Micchelli, C.A. 1995. *Mathematical Aspects of Geometric Modeling*. SIAM. 1, 30, 35, 52, 73
- [35] Mikusiński, J. 1959. *Operational Calculus*. Pergamon Press. 17
- [36] Morawiec, J. 2006. Remarks on L^1 -solutions of dilation equations. *Aequationes Math.*, 71:19–28. 2, 5, 16
- [37] Neamtu, M. 1999. Convergence of subdivision versus solvability of refinement equations. *East J. Approx.*, 5(2):183–210. 24, 27, 45, 49, 61, 69, 72, 77
- [38] Peres, Y., Schlag, W. & Solomyak, B. 2000. Sixty years of Bernoulli convolutions. In C. Bandt, S. Graf & M. Zähle (eds.), *Fractal Geometry and Stochastics II*, pages 39–65. Birkhäuser. 1
- [39] Rioul, O. & Blu, T. Simple regularity criteria for subdivision schemes. II. The rational case. Obtained from <http://citeseer.ist.psu.edu/rioul97simple.html>. 15
- [40] Sauer, T. 2006. Differentiability of multivariate refinable functions and factorization. *Adv. Comput. Math.*, 25:211–235. 10, 51, 62
- [41] Sun, Q. 1996. Refinable functions with compact support. *J. Approx. Th.*, 86:240–252. 2
- [42] Sun, Q. & Zhang, Z. 1998. M -band scaling function with filter having vanishing moments two and minimal length. *J. Math. Anal. Appl.*, 222(1):225–243. 2
- [43] Titchmarsh, E.C. 1926. The zeros of certain integral functions. *Proc. London Math. Soc.*, 25:283–302. 17
- [44] Wals, J.L. & Sewell, W.E. 1937. Note on degree of approximation to an integral by Riemann sums. *Amer. Math. Monthly*, 44(3):155–160. 39
- [45] Wang, Y. 2001. Subdivision schemes and refinement equations with nonnegative masks. *J. Approx. Theory*, 113:207–220. 51
- [46] Wang, Y.-P. & Lee, S.L. 1998. Scale-space derived from B -splines. *IEEE Trans. Pattern Anal. Mach. Intel.*, 20(10):1040–1055. 1
- [47] Zhou, X. 2005. Subdivision schemes with nonnegative masks. *Math. Comput.*, 74(250):819–839. 51