# ARITHMETICAL RANK OF THE CYCLIC AND BICYCLIC GRAPHS

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ABSTRACT. We show that for the edge ideals of the graphs consisting of one cycle or two cycles of any length connected through a vertex or a path, the arithmetical rank equals the projective dimension.

#### 1. INTRODUCTION

For any homogeneous ideal I of a polynomial ring  $R = K[x_1, \ldots, x_n]$  there exists a graded minimal finite free resolution

$$0 \to \bigoplus_{j} R(-j)^{\beta_{pj}} \to \ldots \to \bigoplus_{j} R(-j)^{\beta_{1j}} \to R \to R/I \to 0$$

of R/I, in which R(-j) denotes the graded free module obtained by shifting the degrees of elements in R by j. The numbers  $\beta_{ij}$ , which we shall refer to as the *i*th Betti numbers of degree j of R/I, are independent of the choice of the graded minimal finite free resolution. We also define the *i*th Betti number of I as  $\beta_i := \sum \beta_{ij}$ .

Given a polynomial ring R over a field, and a graph G having the set of indeterminates as its vertex set V(G) and the set of edges E(G), one can associate with G a monomial ideal of R: this ideal is generated by the products of the vertices of each edge in E(G), and is hence generated by squarefree quadratic monomials. It is called the *edge ideal* I(G) of G, and has been intensively studied by Simis, Vasconcelos and Villarreal in [17]. The arithmetical rank (ara), i.e., the least number of elements of R which generate a given monomial ideal up to radical, is in general bounded below by its projective dimension (pd), i.e., by the length of every minimal free resolution of the quotient of R with respect to the ideal. The simplicial complex  $\Delta_G$  of a graph G is defined by

 $\Delta_G = \{ A \subseteq V(G) | A \text{ is an independent set in } G \},\$ 

where A is an independent set in G if none of its elements are adjacent. Note that  $\Delta_G$  is precisely the Stanley-Reisner simplicial complex of I(G).

For any simplicial complex  $\Delta$  on the vertex set  $V(\Delta)$ , the Alexander dual of  $\Delta$  is the simplicial complex defined by

$$\Delta^* := \{ F \subseteq V(\Delta) \mid V(\Delta) \setminus F \notin \Delta \}.$$

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The link of a face  $F \in \Delta$  is defined as the simplicial complex

$$Link_{\Delta}F := \{ G \in \Delta | G \cup F \in \Delta \text{ and } G \cap F = \emptyset \}.$$

In recent times, the projective dimension has been determined for large classes of edge ideals, where it is independent of the ground field: in Jacques' thesis it was computed for acyclic graphs (see also [12]), but also for the graphs  $C_n$ , consisting of one cycle of length n. Jacques, in [11, Theorem 6.1.8], using Hochster's formula [9], showed that for a graph G, the Betti numbers are

(\*) 
$$\beta_{i,d}(G) = \sum_{H \subset G, |V(H)| = d} \dim_k \tilde{H}_{i-2}(\varepsilon(H); K).$$

Then he used (\*) for providing formulas for the graded Betti numbers of special classes of graphs including lines, cycles and complete graphs. He proved the following theorems.

**Theorem A** [11, Lemma 8.2.7] The reduced homology of the disjoint union of the cyclic graph  $C_n$  and any non empty graph G may be expressed as follows:

$$\tilde{H}_i(\varepsilon(C_n \cup G); K) = \begin{cases} \tilde{H}_{i-\frac{2n+1}{3}}(\varepsilon(G); K), & \text{if } n \equiv 1 \mod 3\\ \tilde{H}_{i-\frac{2n-1}{3}}(\varepsilon(G); K), & \text{if } n \equiv 2 \mod 3 \end{cases}$$

**Theorem B** [11, Corollary 7.6.30] The non zero Betti numbers in degree n and the projective dimension of  $C_n$  in degree n are the following:

$$\beta_{\frac{2n}{3},n} = 2, \text{ and } pd \ I(C_n) = \frac{2n}{3}, \text{ if } n \equiv 0 \mod 3, \\ \beta_{\frac{2n+1}{3},n} = 1, \text{ and } pd \ I(C_n) = \frac{2n+1}{3}, \text{ if } n \equiv 1 \mod 3, \\ \beta_{\frac{2n-1}{3},n} = 1, \text{ and } pd \ I(C_n) = \frac{2n-1}{3}, \text{ if } n \equiv 2 \mod 3.$$

**Theorem C** [11, Lemma 8.1.3] The reduced homology of the disjoint union of the line graph  $L_n$  and any non empty graph G may be expressed as follows:

$$\tilde{H}_i(\varepsilon(L_n \cup G); K) = \begin{cases} \tilde{H}_{i-\frac{2n}{3}}(\varepsilon(G); K), & \text{if } n \equiv 0 \mod 3\\ 0, & \text{if } n \equiv 1 \mod 3\\ \tilde{H}_{i-\frac{2n-1}{3}}(\varepsilon(G); K), & \text{if } n \equiv 2 \mod 3. \end{cases}$$

From the proof of [11, Corollary 7.7.35] one can derive the following result.

**Theorem D** [11, Corollary 7.7.35] The projective dimension of the line graph is independent of the characteristic of the chosen field and is

$$pd \ I(L_n) = \begin{cases} \frac{2n}{3} & \text{if } n \equiv 0 \mod 3\\ \frac{2n-2}{3} & \text{if } n \equiv 1 \mod 3\\ \frac{2n-1}{3} & \text{if } n \equiv 2 \mod 3 \end{cases}$$

All Betti numbers of  $L_n$  in degree n are zero if  $n \equiv 1 \mod 3$ . Otherwise the non zero Betti numbers of degree n of  $L_n$  are

$$\beta_{\frac{2n}{3},n} I(L_n) = 1, \text{if} \quad n \equiv 0 \mod 3,$$
  
$$\beta_{\frac{2n-1}{3},n} I(L_n) = 1, \text{if} \quad n \equiv 2 \mod 3.$$

In [6] an explicit formula is given for the Betti numbers of a special kind of bipartite graphs, the so-called *Ferrers graphs*. In [2] it is shown that the arithmetical rank equals the projective dimension for a special class of acyclic graphs, in [3] that

this is also true for all Ferrers graphs. In the present paper we prove that the same equality holds for all cyclic and bicyclic graphs. By *bicyclic graph* we mean a graph which consists of two cycles that have exactly one vertex in common or are connected by a path. In particular, we will see that the projective dimension of the edge ideals of these graphs does not depend on the characteristic of the ground field.

### 2. The arithmetical rank of cyclic graphs

Let K be a field, and consider the polynomial ring  $R = K[x_1, \ldots, x_n]$ , where  $n \geq 3$ . Let  $C_n$  be the graph on the vertex set  $\{x_1, \ldots, x_n\}$  whose set of edges is  $\{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}, \{x_1, x_n\}\}$ . Then its edge ideal is the following ideal of R:

$$I(C_n) = (x_1 x_2, x_2 x_3, \dots, x_{n-1} x_n, x_1 x_n).$$

We will show that for all n,  $pd I(C_n) = ara I(C_n)$ . In general, for any ideal I of R we have that  $cd I \leq ara I$ , where cd denotes the local cohomological dimension (see [8], Example 2, p. 414) and, whenever I is a monomial ideal, pd I = cd I (see [14], Theorem 1). Hence it will suffice to show that, for all n,  $ara I(C_n) \leq pd I(C_n)$ , i.e., to produce  $pd I(C_n)$  elements of R generating  $I(C_n)$ , up to radical. Among the available tools, we have, on the one hand, Jacques' result providing explicit formulas for the projective dimension of  $I(C_n)$ .

On the other hand, we know that a finite set of elements of R which generate a given ideal up to radical can be constructed according to the following criterion, which is due to Schmitt and Vogel.

**Lemma 2.1.** ([16], p. 249) Let P be a finite subset of elements of R. Let  $P_0, \ldots, P_r$  be subsets of P such that

(i)  $\bigcup_{i=0}^{r} P_i = P$ ; (ii)  $P_0$  has exactly one element; (iii) if p and p' are different elements of  $P_i$  ( $0 < i \le r$ ) there is an integer i' with  $0 \le i' < i$  and an element in  $P_{i'}$  which divides pp'.

We set  $q_i = \sum_{p \in P_i} p^{e(p)}$ , where  $e(p) \ge 1$  are arbitrary integers. We will write (P) for the ideal of R generated by the elements of P. Then we get

$$\sqrt{(P)} = \sqrt{(q_0, \dots, q_r)}.$$

We have to distinguish between three cases, depending on the residue of n modulo 3. The cases  $n \equiv 0, 1 \mod 3$  can be settled by a direct application of Lemma 2.1. The case  $n \equiv 2 \mod 3$  is more interesting, since it needs some additional non trivial computations on the generators.

**Proposition 2.2.** Suppose that n = 3m, for some integer m. Set  $q_0 = x_1x_2$ ,  $q_1 = x_1x_{3m} + x_2x_3$ , and, for  $1 \le i \le m - 1$ , set

$$q_{2i} = x_{3i+1}x_{3i+2}$$
$$q_{2i+1} = x_{3i}x_{3i+1} + x_{3i+2}x_{3i+3}$$

Then

$$I(C_n) = \sqrt{(q_0, \dots, q_{2m-1})}.$$

In particular, ara  $I(C_n) \leq 2m$ .

*Proof.* For all i = 0, ..., m - 1, the monomial  $q_{2i}$  divides the product of the two summands of  $q_{2i+1}$ . By Lemma 2.1 it follows that

$$(x_{3i}x_{3i+1}, x_{3i+1}x_{3i+2}, x_{3i+2}x_{3i+3}) = \sqrt{(q_{2i}, q_{2i+1})}.$$

This implies the claim.

Using the same arguments as in the proof of Proposition 2.2, from Lemma 2.1 we can deduce the next result.

**Proposition 2.3.** Suppose that n = 3m + 1, for some integer m. Set  $q_0 = x_1x_2$ ,  $q_1 = x_1x_{3m+1} + x_2x_3$ , and, for  $1 \le i \le m - 1$ , set

$$q_{2i} = x_{3i+1}x_{3i+2}$$
  

$$q_{2i+1} = x_{3i}x_{3i+1} + x_{3i+2}x_{3i+3},$$

and, finally,  $q_{2m} = x_{3m}x_{3m+1}$ . Then

$$I(C_n) = \sqrt{(q_0, \ldots, q_{2m})}.$$

In particular, ara  $I(C_n) \leq 2m + 1$ .

**Proposition 2.4.** Suppose that n = 3m + 2, for some integer m. Set  $q_0 = x_1x_2$ ,  $q_1 = x_2x_3 + x_4x_5$ , and, for  $1 \le i \le m - 1$ , set

$$q_{2i} = x_{3i}x_{3i+1} + x_{3i+2}x_{3i+3}$$
$$q_{2i+1} = x_{3i+2}x_{3i+3} + x_{3i+4}x_{3i+5},$$

and, finally,  $q_{2m} = x_1 x_{3m+2} + x_{3m} x_{3m+1}$ . Then

$$I(C_n) = \sqrt{(q_0, \dots, q_{2m})}$$

In particular, ara  $I(C_n) \leq 2m + 1$ .

*Proof.* The claim for m = 1 was proven in [2], Example 1. So let  $m \ge 2$ . Set  $J_m = (q_0, \ldots, q_{2m})$ . It suffices to show that  $I(C_n) \subset \sqrt{J_m}$ . In this proof, for all  $f, g \in R$ , by abuse of notation we will write  $f \equiv g$  whenever f - g or f + g belongs to  $J_m$ , and,  $f \equiv_{q_i} g$  whenever f - g or f + g is divisible by  $q_i$ . In this way,  $f \equiv g$  or  $f \equiv_{q_i} g$  assures that  $f \in J_m$  occurs if and only if  $g \in J_m$ . We first show that

(2.1) 
$$x_1^{2^m} x_{3m+2}^{2^{m+1}} \in J_m$$

Set

$$u_m = x_1^{2^{m-1}} x_{3m+2}^{2^m},$$
  

$$v_m = x_3 x_4 x_5 \prod_{i=2}^m x_{3i}^{3 \cdot 2^{i-2}},$$
  

$$w_m = (x_{3m} x_{3m+1} x_{3m+2})^{2^{m-1}}.$$

We prove that

 $(2.2) u_m \equiv v_m \equiv w_m.$ 

Note that  $x_1x_{3m+2}v_m$  is a multiple of  $x_1x_{3m+2}x_4x_5$ , and

$$x_1 x_{3m+2} x_4 x_5 \equiv_{q_0} x_1 x_{3m+2} (x_2 x_3 + x_4 x_5) \in J_m$$

whence we deduce that  $x_1x_{3m+2}v_m \in J_m$ . Thus (2.2) will imply that

 $x_1^{2^m} x_{3m+2}^{2^{m+1}} = x_1 x_{3m+2} u_m \in J_m,$ 

as claimed in (2.1). We prove (2.2) by induction on  $m \ge 2$ . First take m = 2. We have  $q_2 = x_3x_4 + x_5x_6$ ,  $q_3 = x_5x_6 + x_7x_8$  and  $q_4 = x_1x_8 + x_6x_7$ , so that

$$v_2 = x_3 x_4 x_5 x_6^3 \equiv_{q_2} x_5^2 x_6^2 x_6^2 \equiv_{q_3} x_6^2 x_7^2 x_8^2 = w_2 \equiv_{q_4} x_1^2 x_8^4 = u_2$$

which shows (2.2) for m = 2. Now suppose that m > 2 and that the claim is true for m - 1. We have:

 $2 n^{m-2}$ 

$$v_{m} = v_{m-1}x_{3m}^{5/2} \equiv w_{m-1}x_{3m}^{5/2}$$

$$= (x_{3m-3}x_{3m-2}x_{3m-1})^{2^{m-2}}x_{3m}^{3\cdot 2^{m-2}} = (x_{3m-3}x_{3m-2})^{2^{m-2}}x_{3m-1}^{2^{m-2}}x_{3m}^{3\cdot 2^{m-2}}$$

$$\equiv_{q_{2m-2}} (x_{3m-1}x_{3m})^{2^{m-2}}x_{3m-1}^{2^{m-2}}x_{3m}^{3\cdot 2^{m-2}} = (x_{3m-1}x_{3m})^{2^{m-1}}x_{3m}^{2\cdot 2^{m-2}}$$

$$\equiv_{q_{2m-1}} (x_{3m+1}x_{3m+2})^{2^{m-1}}x_{3m}^{2^{m-1}} = (x_{3m}x_{3m+1})^{2^{m-1}}x_{3m+2}^{2^{m-1}} = w_{m}$$

$$\equiv_{q_{2m}} (x_{1}x_{3m+2})^{2^{m-1}}x_{3m+2}^{2^{m-1}} = x_{1}^{2^{m-1}}x_{3m+2}^{2^{m}} = u_{m}.$$

This completes the proof of (2.2) and of (2.1). We have thus shown that

$$(2.3) x_1 x_{3m+2} \in \sqrt{J_m}.$$

 $2 n^{m-2}$ 

But then

(2.4) 
$$x_{3m}x_{3m+1} = q_{2m} - x_1x_{3m+2} \in \sqrt{J_m}.$$

In general, whenever, for some  $i \in \{2, \ldots, m\}$ ,

$$(2.5) x_{3i}x_{3i+1} \in \sqrt{J_m},$$

from the fact that  $x_{3i}x_{3i+1}$  divides  $x_{3i-1}x_{3i} \cdot x_{3i+1}x_{3i+2}$ , i.e., the product of the summands of  $q_{2i-1}$ , by Lemma 2.1 one deduces that

$$(2.6) x_{3i-1}x_{3i} \in \sqrt{J_m}$$

Since  $x_{3i-3}x_{3i-2} = q_{2i-2} - x_{3i-1}x_{3i}$ , this in turn implies that

(2.7) 
$$x_{3i-3}x_{3i-2} \in \sqrt{J_m}$$

Finally, since  $x_{3i-3}x_{3i-2}$  divides  $x_{3i-4}x_{3i-3} \cdot x_{3i-2}x_{3i-1}$ , i.e., the product of the summands of  $q_{2i-3}$ , by Lemma 2.1 we again conclude that

$$(2.8) x_{3i-2}x_{3i-1} \in \sqrt{J_m}$$

Therefore, since (2.5) implies (2.6), (2.7) and (2.8), for all  $i = 2, \ldots, m$ , from (2.4) one can derive by descending induction on h, that  $x_h x_{h+1} \in \sqrt{J_m}$  for all  $h = 3, \ldots, 3m + 1$ . In particular we have that  $x_3 x_4 \in \sqrt{J_m}$ , which, together with  $q_1 \in J_m$ , yields  $x_2 x_3 \in \sqrt{J_m}$  by Lemma 2.1. This, together with (2.4) and  $q_0 \in J_m$ , shows that  $I(C_n) \subset \sqrt{J_m}$ , as claimed.

Theorem B and Propositions 2.2, 2.3, 2.4 imply our main result.

**Theorem 2.5.** Let  $n \geq 3$  be an integer. Then

ara 
$$I(C_n) = pd \ I(C_n) = \begin{cases} \frac{2n}{3} & \text{if} \quad n \equiv 0 \mod 3\\ \frac{2n+1}{3} & \text{if} \quad n \equiv 1 \mod 3\\ \frac{2n-1}{3} & \text{if} \quad n \equiv 2 \mod 3. \end{cases}$$

Every ideal  $I(C_n)$  is of pure height  $\lceil \frac{n}{2} \rceil$ , where  $\lceil a \rceil$  denotes the least integer not less than a. Recall that an ideal is called a *set-theoretic complete intersection* if its arithmetical rank equals its height. In view of Theorem 2.5 we thus have the following.

**Corollary 2.6.**  $I(C_n)$  is a set-theoretic complete intersection only for n = 3 and n = 5.

#### 3. The arithmetical rank of bicyclic graphs

In this section by  $\equiv$ , we mean  $\equiv \pmod{3}$  and all equivalence relations will be considered modulo 3. Let  $a_1, \ldots, a_s$  be subsets of the finite set V. Define  $\varepsilon(a_1,\ldots,a_s;V)$  to be the simplicial complex which has vertex set  $\bigcup_{i=1}^s (V \setminus a_i)$  and maximal faces  $V \setminus a_1, \ldots, V \setminus a_s$ . Let  $\Delta = \Delta_G$ , and let  $F \in \Delta^*$  and  $e_1, \ldots, e_r$  be all the edges of G which are disjoint from F. Then  $Link_{\Delta^*}F = \varepsilon(e_1, \ldots, e_r; V(G) \setminus F)$ by [12, Proposition 3.3]. According to [11, Proposition 6.1.6], associating F with the induced subgraph H of G on the vertex set  $V(G) \setminus F$  defines a bijection between the faces of  $\Delta^*$  and the set of induced subgraphs of G which have at least one edge. Let H be an induced subgraph of the graph G. If H is associated with the face F of  $\Delta^*$  as described above, we write  $\varepsilon(H)$  for  $\varepsilon(e_1,\ldots,e_s;V)$ , where  $e_1,\ldots,e_s$ are the edges of H and V is the vertex set  $V(G) \setminus F$  (or equivalently the vertex set of H). In this section, using (\*), we find explicit descriptions of the projective dimension of all bicyclic graphs. For every vertex u of a graph G we denote by  $N_G(u)$  the set of vertices adjacent to u. In the proof of our main results we will use the Mayer-Vietoris sequence for the reduced homology of simplicial complexes, which, for any pair  $\Delta_1$ ,  $\Delta_2$  of simplicial complexes, has the following form (see [11, Remark 6.2.13]):

$$\dots \to \tilde{H}_i(\Delta_1 \cap \Delta_2; K) \to \tilde{H}_i(\Delta_1; K) \oplus \tilde{H}_i(\Delta_2; K) \to \tilde{H}_i(\Delta_1 \cup \Delta_2; K) \to \\ \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2; K) \to \dots$$

**Lemma 3.1.** For a graph G with an edge  $\{u, v\}$  such that deg(v) = 1, we have  $\tilde{H}_i(\varepsilon(G); K) = 0$ , if some vertex in  $N_G(u)$  has an adjacent vertex of degree one in G. Otherwise,  $\tilde{H}_i(\varepsilon(G); K) = \tilde{H}_{i-t}(\varepsilon(H); K)$ , where  $t = |N_G(u)|$  and H is the induced subgraph on  $V(G) \setminus (\{u\} \cup N_G(u))$ , provided H is non empty.

*Proof.* In this and in the following proofs we will omit the coefficient field in the homology groups. We set V = V(G). Let  $N_G(u) = \{v, u_1, \ldots, u_{t-1}\}$  and  $\{u, v\}, \{u, u_1\}, \ldots, \{u, u_{t-1}\}, e_1, \ldots, e_r$  be the edges of G. We can write  $\varepsilon(G) = E_1 \cup E_2$ , where  $E_1 = \varepsilon(\{u, u_1\}, \ldots, \{u, u_{t-1}\}, e_1, \ldots, e_r; V)$  and  $E_2 = \varepsilon(\{u, v\}; V)$ . The intersection of these simplicial complexes is:  $E_1 \cap E_2 = \varepsilon(\{u, v, u_1\}, \ldots, \{u, v, u_{t-1}\}, \{u, v\} \cup e_1, \ldots, \{u, v\} \cup e_r; V)$  $= \varepsilon(\{u_1\}, \ldots, \{u_{t-1}\}, e_1, \ldots, e_r; V \setminus (\{u, v\}))$  (see [12, Lemma 3.4]).

If there exists a vertex  $v_i$  of degree one such that  $\{u_i, v_i\} \in E(G)$ , then without loss

of generality we can assume that  $e_1 = \{u_i, v_i\}$ . Then  $E_1 \cap E_2 = \varepsilon(\{u_1\}, \ldots, \{u_{t-1}\}, \{u_i, v_i\}, e_2, \ldots, e_r; V \setminus (\{u, v\})) = \varepsilon(\{u_1\}, \ldots, \{u_{t-1}\}, e_2, \ldots, e_r; V \setminus (\{u, v\}))$ , whose reduced homology is identically zero, since  $v_i \in V \setminus (\{u, v\})$  and  $v_i$  belongs to all faces of  $E_1 \cap E_2$ . Otherwise, by [12, Lemma 3.5] we have  $\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-t+1}(\varepsilon(H))$ , for all i, where H is the induced subgraph on  $V \setminus (\{u\} \cup N_G(u))$ . Since  $E_2$  is a simplex,  $\tilde{H}_i(E_2) = 0$  for all i. Also,  $\tilde{H}_i(E_1) = 0$  for all i, since v belongs to all faces of  $E_1$ . Using the Mayer-Vietoris sequence (for  $\Delta_i = E_i$ ) we deduce that  $\tilde{H}_i(\varepsilon(G)) = \tilde{H}_{i-1}(E_1 \cap E_2)$ , which completes the proof.

The next result can be deduced from Lemma 3.1 by a trivial inductive argument.

**Corollary 3.2.** Let  $n \equiv 0$ . Suppose that  $L_n$  intersects graph G only at one of its endpoints. Then, for all i, we have  $\tilde{H}_i(\varepsilon(G \cup L_n)) = \tilde{H}_{i-\frac{2n}{2}}(\varepsilon(G \setminus L_n))$ .

**Theorem 3.3.** Let G be the graph which is a joint of two cycles  $C_n$  and  $C_m$  in a common vertex. Then the following hold:

(a) If  $|V(G)| \equiv 1$ , then  $pd \ I(G) = ara \ I(G) = \frac{2|V(G)|+1}{3}$ . (b) If  $|V(G)| \equiv 0$ , then  $pd \ I(G) = ara \ I(G) = \frac{2|V(G)|}{3}$ . (c) If  $|V(G)| \equiv 2$ , then  $pd \ I(G) = ara \ I(G) = \frac{2|V(G)|+2}{3}$ , for  $m \equiv 0$ , whereas  $pd \ I(G) = ara \ I(G) = \frac{2|V(G)|-1}{3}$  otherwise.

*Proof.* We will prove the claim by showing that the desired number is, on the one hand, a lower bound for pd I(G), on the other hand, an upper bound for ara I(G).

Let V = V(G). Consider the labeling for V such that  $V(C_n) = \{y_1, y_2, \ldots, y_n\}$ , and  $V(C_m) = \{x_1, x_2, \ldots, x_m\}$ , where  $x_1 = y_1$ . Up to exchanging m and n we have the following cases.

Case 1.  $|V| \equiv 0$  or 1.

First let n = 3. Then  $m \equiv 1$  or  $m \equiv 2$ . In view of (\*) the *i*th Betti number of degree |V| is  $\beta_{i,|V|}(G) = dim_k \tilde{H}_{i-2}(\varepsilon(G); K)$ . So we compute the reduced homology of G of degree |V|. We can write  $\varepsilon(G) = \varepsilon(\{x_1, x_2\}, \ldots, \{x_m, x_1\}, \{x_1, y_2\}, \{y_2, y_3\}, \{y_3, x_1\}; V) = E_1 \cup E_2$ , where  $E_1 = \varepsilon(\{x_1, x_2\}, \ldots, \{x_m, x_1\}, \{x_1, y_2\}, \{y_3, x_1\}; V)$  and  $E_2 = \varepsilon(\{y_2, y_3\}; V)$ .

By [12, Lemma 3.4], the intersection of these simplicial complexes is:

$$E_1 \cap E_2 = \varepsilon(\{x_1, x_2, y_2, y_3\}, \dots, \{x_1, x_m, y_2, y_3\}, \{x_1, y_2, y_3\}; V)$$
  
=  $\varepsilon(\{x_1\}, \{x_2, x_3\}, \dots, \{x_{m-1}, x_m\}; V \setminus \{y_2, y_3\}).$ 

By [12, Lemma 3.5],  $\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-1}(\varepsilon(L_{m-1}))$ , for all *i*. Since  $E_2$  is a simplex,  $\tilde{H}_i(E_2) = 0$  for all *i*. Applying Lemma 3.1 to  $E_1$  for  $v = y_2$  (and  $u = x_1$ , so that  $N(u) = \{x_2, x_m, y_2, y_3\}$ ), we obtain  $\tilde{H}_i(E_1) = \tilde{H}_{i-4}(\varepsilon(L_{m-3}))$ , for all *i*.

If  $m \equiv 1$ , then  $\tilde{H}_i(E_1) = 0$  for all *i* by Theorem D and (\*) (since  $m - 3 \equiv 1$ ). By the Mayer-Vietoris sequence we deduce that, for all *i*,  $\tilde{H}_i(\varepsilon(G)) = \tilde{H}_{i-1}(E_1 \cap E_2)$ . Theorem D and (\*) then show that  $\tilde{H}_{i-2}(\varepsilon(L_{m-1})) \neq 0$  (i.e.,  $\tilde{H}_i(\varepsilon(G)) \neq 0$ ) if and only if  $i - 2 + 2 = \frac{2(m-1)}{3}$ , if and only if  $i = \frac{2|V|}{3} - 2$ . In view of (\*) we deduce that  $pd I(G) \geq \frac{2|V|}{3}$ .

If  $m \equiv 2$ , then  $\hat{H}_i(E_1 \cap E_2) = 0$  for all *i* by Theorem D and (\*) (since  $m-1 \equiv 1$ ). By the Mayer-Vietoris sequence we deduce that  $\tilde{H}_i(\varepsilon(G)) = \tilde{H}_i(E_1)$  for all *i*. Theorem D and (\*) show that  $\tilde{H}_i(\varepsilon(G)) \neq 0$  if and only if  $i = \frac{2|V|+1}{3} - 2$ . In view of (\*) we deduce that  $pd I(G) \ge \frac{2|V|+1}{3}$ .

So we can assume that  $n \ge 4$ . Moreover, since n and m cannot be both divisible by 3, we may assume that  $m \equiv 1$  or  $m \equiv 2$ . In view of (\*) we compute the reduced homology of G of degree |V|. We can write

$$\begin{split} \varepsilon(G) &= \varepsilon(\{x_1, x_2\}, \dots, \{x_m, x_1\}, \{x_1, y_2\}, \{y_2, y_3\}, \dots, \{y_{n-1}, y_n\}, \{y_n, x_1\}; V) = \\ E_1 \cup E_2, \text{ where } E_1 &= \varepsilon(\{x_2, x_3\}, \dots, \{x_m, x_1\}, \{x_1, y_2\}, \dots, \{y_{n-1}, y_n\}, \{y_n, x_1\}; V) \\ \text{and } E_2 &= \varepsilon(\{x_1, x_2\}; V). \text{ We have that } E_1 &= \varepsilon(L_m \cup C_n), \text{ where } L_m : x_2 x_3 \dots x_m x_1. \\ \text{The intersection of these simplicial complexes is } E_1 \cap E_2 &= \varepsilon(\{x_1, x_2, x_3\}, \{x_1, x_2, x_3, x_4\}, \dots, \{x_1, x_2, x_{m-1}, x_m\}, \{x_m, x_1, x_2\}, \{x_1, y_2, x_2\}, \dots, \{y_{n-1}, y_n, x_1, x_2\}, \{y_n, x_1, x_2\}; V) &= \varepsilon(\{x_3\}, \{x_m\}, \{y_2\}, \{y_n\}, \{x_4, x_5\}, \dots, \{x_{m-2}, x_{m-1}\}, \{y_3, y_4\}, \dots, \{y_{n-2}, y_{n-1}\}; V \setminus \{x_1, x_2\}) \text{ (see [12, Lemma 3.4]).} \end{split}$$

By [12, Lemma 3.5],  $H_i(E_1 \cap E_2) = H_{i-4}(\varepsilon(H))$  for all *i*, where *H* is the induced subgraph on  $V \setminus \{x_1, x_2, x_3, x_m, y_2, y_n\}$ . We have  $\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-4}(\varepsilon(L_{m-4} \cup L_{n-3}))$  for all *i*. Since  $E_2$  is a simplex,  $\tilde{H}_i(E_2) = 0$  for all *i*.

## Case 1.1. Let $m \equiv 2$ .

By Theorem C,  $\tilde{H}_i(E_1 \cap E_2) = 0$  for any i, since  $m - 4 \equiv 1$ . Applying Corollary 3.2 to the path  $L_{m-2}: x_2x_3 \dots x_{m-1}$ , we get that, for all i,  $\tilde{H}_i(E_1) = \tilde{H}_{i-\frac{2(m-2)}{3}}(\varepsilon(L_2 \cup C_n))$ , where  $L_2: x_1x_m$ . If we apply Lemma 3.1 once again for  $v = x_m$  (and  $u = x_1$ , so that  $N(u) = \{x_m, y_2, y_n\}$ ), we then obtain  $\tilde{H}_i(E_1) = \tilde{H}_{i-\frac{2(m-2)}{3}-3}(\varepsilon(L_{n-3}))$ , for all i, where  $L_{n-3}: y_3 \dots y_{n-1}$ . In part (a), we have  $n \equiv 0$ . Theorem D and (\*) show that  $\tilde{H}_i(E_1) \neq 0$  if and only if  $i = \frac{2|V|+1}{3} - 2$ . The Mayer-Vietoris sequence implies that  $\tilde{H}_i(\varepsilon(G)) \neq 0$  if and only if  $i = \frac{2|V|+1}{3} - 2$ . By (\*) it follows that  $pd I(G) \geq \frac{2|V|+1}{3}$ , as claimed. In part (b), we have  $n \equiv 2$ . Theorem D and (\*) show that  $\tilde{H}_i(E_1) \neq 0$  if and only if  $i = \frac{2|V|}{3} - 2$ . As above, it follows that  $\tilde{H}_i(\varepsilon(G)) \neq 0$  if and only if  $i = \frac{2|V|}{3} - 2$ . As above, it follows that  $\tilde{H}_i(\varepsilon(G)) \neq 0$  if and only if  $i = \frac{2|V|}{3} - 2$ . In view of (\*) we deduce that  $pd I(G) \geq \frac{2|V|}{3}$ , as claimed.

**Case** 1.2. Let  $m \equiv 1$ . By Theorem C,  $\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-4-\frac{2(m-4)}{3}}(\varepsilon(L_{n-3}))$  for any *i*. Moreover, applying Corollary 3.2 to the path  $L_{m-1}: x_2x_3 \dots x_m$ , we get that, for all  $i, \tilde{H}_i(E_1) = \tilde{H}_{i-\frac{2(m-1)}{3}}(\varepsilon(C_n))$ . In part (*a*), we have  $n \equiv 1$ . Hence, by Theorem C,  $\tilde{H}_i(E_1 \cap E_2) = 0$  for any *i*, since  $n-3 \equiv 1$ . On the other hand, Theorem B and (\*) show that  $\tilde{H}_i(E_1) \neq 0$  if and only if  $i = \frac{2|V|+1}{3} - 2$ . We deduce that  $\tilde{H}_i(\varepsilon(G)) \neq 0$  if and only if  $i = \frac{2|V|+1}{3} - 2$ . By (\*) it follows that  $pd I(G) \geq \frac{2|V|+1}{3}$ , as claimed. In part (*b*), we have  $n \equiv 0$ . By Theorem D and (\*) it follows that  $\tilde{H}_i(E_1 \cap E_2) \neq 0$  if and only if  $i = \frac{2|V|}{3} - 2$ , in which case the *i*th homology module is equal to *K*. Theorem B and (\*) show that  $\tilde{H}_i(E_1) \neq 0$  if and only if  $i = \frac{2|V|}{3} - 2$  in which case it is equal to  $K^2$ . Thus the Mayer-Vietoris sequence implies that  $\tilde{H}_i(\varepsilon(G)) \neq 0$  for  $i = \frac{2|V|}{3} - 2$ . In view of (\*) we deduce that  $pd I(G) \geq \frac{2|V|}{3}$ , as claimed.

Case 2. Let  $|V| \equiv 2$ .

Case 2.1.  $m \equiv 0$ .

We have  $n \equiv 0$ . First assume that n = m = 3. We first show that in this case  $pd I(G) \geq 4$ . We use the fact that pd I(G) = cd I(G), (see [15, Theorem 1]), where cd denotes the local cohomological dimension, i.e., for any ideal I of

R, cd I is the maximum index *i* for which the local cohomology module  $H_I^i(R)$ (of *R* with respect to *I*) does not vanish. We have that  $I(G) = I \cap J$ , where  $I = (x_1, x_2x_3, y_2y_3)$  and  $J = (x_2, x_3, y_2, y_3)$ . It is well-known that, whenever an ideal is a complete intersection, its height is the only index for which the cohomology module of *R* with respect to this ideal does not vanish (see [10, Proposition 3.8] together with [10, Theorem 4.4], or together with [8, Example 2, p. 414]). Since  $I + J = (x_1, x_2, x_3, y_2, y_3)$  we thus have that  $H_{I+J}^i(R) \neq 0$  if and only if i = 5. We also have that cd J = 4. In the Mayer-Vietoris sequence for local cohomology (see [10], Section 3)

$$\dots \to H^4_{I+J}(R) \to H^4_I(R) \oplus H^4_J(R) \to H^4_{I\cap J}(R) \to \dots,$$

the left term is zero, whereas the middle term is not. It follows that the right term is non zero, too. This implies that  $pd I(G) = cd I(G) \ge 4$ . On the other hand, by Lemma 2.1, the elements  $x_1x_2, x_2x_3 + x_1x_3, x_1y_2, x_1y_3 + y_2y_3$  generate I(G), up to radical, which shows that ara  $I(G) \le 4$ . It follows that pd I(G) = ara I(G) = 4, which proves the claim in this case.

Without loss of generality we can thus assume that  $m \ge 6$ . We can write  $\varepsilon(G) = E_1 \cup E_2$ , where  $E_1 = \varepsilon(\{x_1, x_2\}, \{x_3, x_4\}, \dots, \{x_m, x_1\}, \{x_1, y_2\}, \dots, \{y_n, x_1\}; V)$ and  $E_2 = \varepsilon(\{x_2, x_3\}; V)$ . We have that  $E_1 \cap E_2 = \varepsilon(\{x_1\}, \{x_4\}, \{x_5, x_6\}, \dots, \{x_{m-1}, x_m\}, \{y_2, y_3\}, \dots, \{y_{n-1}, y_n\})$ . By [12, Lemma 3.5]) it follows that  $\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-2}(\varepsilon(L_{m-4} \cup L_{n-1}))$ , for all *i*. We also have that  $E_1 = \varepsilon(H_1)$ , where  $H_1$  is the union of  $C_n$  and the paths  $x_3 \dots x_m x_1$  and  $x_1 x_2$ . Applying Lemma 3.1 for  $v = x_2$  (and  $u = x_1$ , so that  $N(u) = \{x_2, x_m, y_2, y_n\}$ , we obtain that, for all *i*,  $\tilde{H}_i(E_1) = \tilde{H}_{i-4}(\varepsilon(L_{m-3} \cup L_{n-3}))$ . Thus, by Theorem C, we deduce that, for all *i*,  $\tilde{H}_i(E_1) = \tilde{H}_{i-4-2} \frac{2(n-3)}{3}(\varepsilon(L_{m-3}))$ . According to Theorem D and (\*) it is non zero if and only if  $i = \frac{2|V|+2}{3} - 2$ . Applying Theorem C, Theorem D and (\*) we also get that  $\tilde{H}_i(E_1 \cap E_2) \neq 0$  if and only if  $i = \frac{2|V|+2}{3} - 4$ . By the Mayer-Vietoris sequence we conclude that  $\tilde{H}_i(\varepsilon(G)) \neq 0$  for  $i = \frac{2|V|+2}{3} - 2$ . In view of (\*) we deduce that  $pd I(G) \geq \frac{2|V|+2}{3}$ , as claimed.

Case 2.2. Let  $m \equiv 1$ .

We have  $n \equiv 2$ . Consider the induced subgraph H' on  $V \setminus \{y_2\}$ . Then H' is the union of  $C_m$  and the path  $L_{n-1} : y_3 \ldots y_n y_1$ . Applying Corollary 3.2 to the path  $L_{n-2} : y_3 \ldots y_n$ , we obtain  $\tilde{H}_i(\varepsilon(H')) = \tilde{H}_{i-\frac{2(n-2)}{3}}(\varepsilon(C_m))$ , for all *i*. By Theorem B and (\*) it is non zero if and only if  $i = \frac{2|V|-1}{3} - 2$ . In view of (\*) we deduce that  $pd I(G) \geq \frac{2|V|-1}{3}$ , as claimed.

Now we find an upper bound for the arithmetical rank in each case. In the rest of the proof, we will refer to the polynomials  $q_i$  introduced in Propositions 2.2, 2.3 and 2.4; in each case, the polynomial  $q'_i$  will be the one obtained from  $q_i$  by replacing each variable  $x_j$  with  $y_j$ .

In part (a), for  $m \equiv 2$ , by Proposition 2.4 the sequence  $A_m : q_0, \ldots, q_{\frac{2(m-2)}{3}}$ , generates  $I(C_m)$ , up to radical and by Proposition 2.2 the sequence  $A_n : q'_0, \ldots, q'_{\frac{2n}{3}-1}$ , generates  $I(C_n)$ , up to radical. Since  $I(G) = I(C_m) + I(C_n)$ , the following sequence generates I(G), up to radical:  $B : q_0, \ldots, q_{\frac{2(m-2)}{3}}, q'_0, \ldots, q'_{\frac{2n}{3}-1}$ . This implies that ara  $I(G) \leq \frac{2|V|+1}{3}$ .

If  $m \equiv 1$ , then, by Proposition 2.3 the sequence  $A_m : q_0, \ldots, q_{\frac{2(m-1)}{3}}$  generates  $I(C_m)$ , up to radical and the sequence  $A_n : q'_0, \ldots, q'_{\frac{2(n-1)}{3}}$  generates  $I(C_n)$ , up to radical. The summand  $x_1x_m$  of  $q_1$  divides the product of the monomials  $q_{\frac{2(m-1)}{3}} = x_{m-1}x_m$  and  $q'_0 = y_1y_2 = x_1y_2$ . Thus by Lemma 2.1 the sequence  $B: q_0, q_1, q_{\frac{2(m-1)}{3}} + q'_0, q_2, \ldots, q_{\frac{2(m-1)}{3}-1}, q'_1, \ldots, q'_{\frac{2(n-1)}{3}}$  of length  $\frac{2|V|+1}{3}$  generates I(G) up to radical. This implies that  $ara \ I(G) \leq \frac{2|V|+1}{3}$ .

In part (b), for  $m \equiv 2$ , according to Proposition 2.4, the sequence  $A_m : q_0, \ldots, q_{\frac{2(m-2)}{3}}$ , generates  $I(C_m)$ , up to radical and the sequence  $A_n : q'_0, \ldots, q'_{\frac{2(n-2)}{3}}$ , generates  $I(C_n)$ , up to radical. The sequence B formed by the union of these two sequences generates I(G), up to radical. This implies that ara  $I(G) \leq \frac{2|V|}{3}$ .

If  $m \equiv 1$ , then, by Proposition 2.3 the sequence  $A_m : q_0, \ldots, q_{\frac{2(m-1)}{3}}$  generates  $I(C_m)$ , up to radical and by Proposition 2.2 the sequence  $A_n : q'_0, \ldots, q'_{\frac{2n}{3}-1}$  generates  $I(C_n)$ , up to radical. Thus by Lemma 2.1 the sequence  $B : q_0, q_1, q_{\frac{2(m-1)}{3}} + q'_0, q_2, \ldots, q_{\frac{2(m-1)}{3}-1}, q'_1, \ldots, q'_{\frac{2n}{3}-1}$  of length  $\frac{2|V|}{3}$ , generates I(G), up to radical. So ara  $I(G) \leq \frac{2|V|}{3}$ .

In part (c), if  $m \equiv 0$ , then consider the sequence  $B: q_0, \ldots, q_{\frac{2m}{3}-1}, q'_0, \ldots, q'_{\frac{2n}{3}-1}$ , where  $A_m: q_0, \ldots, q_{\frac{2m}{3}-1}$  generates  $I(C_m)$  and  $A_n: q'_0, \ldots, q'_{\frac{2n}{3}-1}$  generates  $I(C_n)$ , up to radical, by Proposition 2.2. This implies that ara  $I(G) \leq \frac{2|V|+2}{3}$ . If  $m \equiv 1$ , then, by Proposition 2.3, the sequence  $A_m: q_0, \ldots, q_{\frac{2(m-1)}{3}}$ , generates  $I(C_m)$ , up to radical. By Proposition 2.4, the sequence  $A_n: q'_0, \ldots, q'_{\frac{2(n-2)}{3}}$ , generates  $I(C_n)$ , up to radical. Thus by Lemma 2.1 the sequence  $B: q_0, q_1, q_{\frac{2(m-1)}{3}} + q'_0, q_2, \ldots, q_{\frac{2(m-1)}{3}-1}, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}$ , generates I(G), up to radical. This implies that ara  $I(G) \leq \frac{2|V|-1}{3}$ .

**Theorem 3.4.** Let G be the graph formed by two cycles  $C_n$  and  $C_m$  with a path joining a vertex of  $C_n$  to a vertex of  $C_m$ . Then the following hold:

(a) If  $|V(G)| \equiv 1$ , then  $pd \ I(G) = ara \ I(G) = \frac{2|V(G)|-2}{3}$ , whenever  $m \equiv 2$ ,  $n \equiv 2$ . Otherwise,  $pd \ I(G) = ara \ I(G) = \frac{2|V(G)|+1}{3}$ .

(b) If  $|V(G)| \equiv 0$ , then  $pd I(G) = ara I(G) = \frac{2|V(G)|}{3}$ .

(c) If  $|V(G)| \equiv 2$  and  $m, n \equiv 0$  or 1, then  $pd I(G) = ara I(G) = \frac{2|V(G)|+2}{3}$ . Otherwise,  $pd I(G) = ara I(G) = \frac{2|V(G)|-1}{3}$ .

*Proof.* Let V = V(G). Consider the labeling for V such that  $V(C_n) = \{y_1, y_2, \ldots, y_n\}$ ,  $V(C_m) = \{x_1, x_2, \ldots, x_m\}$  and let  $P : z_0 z_1 \ldots z_k z_{k+1}$  be the path in G, where  $z_0 = x_1$  and  $z_{k+1} = y_1$ . We compute the reduced homology of G of degree |V|. Up to exchanging m and n, we have the following cases.

Case 1. Let  $k \equiv 2$ . We can write  $\varepsilon(G) = \varepsilon(\{x_1, x_2\}, \dots, \{x_m, x_1\}, \{y_1, y_2\}, \dots, \{y_n, y_1\}, \{x_1, z_1\}, \{z_1, z_2\}, \dots, \{z_{k-1}, z_k\}, \{z_k, y_1\}; V) = E_1 \cup E_2$ , where  $E_1 = \varepsilon(\{x_1, x_2\}, \dots, \{x_m, x_1\}, \{y_1, y_2\}, \dots, \{y_n, y_1\}, \{x_1, z_1\}, \{z_1, z_2\}, \dots, \{z_{k-1}, z_k\}; V)$  and  $E_2 = \varepsilon(\{z_k, y_1\}; V)$ . The intersection of these simplicial complexes is  $E_1 \cap E_2 = \varepsilon(\{y_2\}, \{y_n\}, \{z_{k-1}\}, \{x_1, x_2\}, \dots, \{x_m, x_1\}, \{y_3, y_4\}, \dots, \{y_{n-2}, y_{n-1}\}, \{x_1, z_1\}, \{z_1, z_2\}, \dots, \{z_{k-3}, z_{k-2}\}; V)$  (see [12, Lemma 3.4]). By [12, Lemma 3.5] it follows that

$$H_i(E_1 \cap E_2) = H_{i-3}(\varepsilon(H_1 \cup L_{n-3})),$$

for all *i*, where  $H_1$  is the induced subgraph on  $V \setminus (V(C_n) \cup \{z_{k-1}, z_k\})$ . Applying Corollary 3.2 to the path  $L_{k-2}$ :  $z_1 \dots z_{k-2}$ , we have

$$\ddot{H}_i(E_1 \cap E_2) = \dot{H}_{i-\frac{2(k-2)}{3}-3}(\varepsilon(C_m \cup L_{n-3})),$$

for all *i*. Since  $E_2$  is a simplex,  $\tilde{H}_i(E_2) = 0$  for all *i*. Applying Corollary 3.2 to the path  $L_{k+1} : z_0 \dots z_k$ , we have that, for all *i*,

$$\tilde{H}_i(E_1) = \tilde{H}_{i-\frac{2(k+1)}{2}}(\varepsilon(L_{m-1} \cup C_n)).$$

Case 1.1 Let  $n \equiv 1$ .

By Theorem C, since  $n-3 \equiv 1$ , we have that

 $\tilde{H}_i(E_1 \cap E_2) = 0$ 

for all *i*. The Mayer-Vietoris sequence then implies that  $\tilde{H}_i(\varepsilon(G)) = \tilde{H}_i(E_1)$  for all *i*. Moreover, in view of Proposition 2.3,  $I(C_n)$  is generated, up to radical, by the sequence  $A_n : q'_0, \ldots, q'_{\frac{2(n-1)}{\alpha}}$ .

Case 1.1.1 Let  $m \equiv 1$  or  $m \equiv 0$ .

First suppose that  $m \equiv 1$ . Then  $|V| \equiv 1$ . From Theorem C we have that, for all i,  $\tilde{H}_i(E_1) = \tilde{H}_{i-\frac{2(k+1)}{3}-\frac{2(m-1)}{3}}(\varepsilon(C_n))$ . In view of Theorem B and (\*) it follows that  $\tilde{H}_i(E_1) \neq 0$  if and only if  $i = \frac{2|V|+1}{3} - 2$ . Thus  $\tilde{H}_i(\varepsilon(G)) \neq 0$  for  $i = \frac{2|V|+1}{3} - 2$ , which, by (\*), implies that  $pd I(G) \geq \frac{2|V|+1}{3}$ .

Which, by ( ), implies that  $P^{-1}(z) = -3$ By Lemma 2.1, the sequence  $B: q_0, q_1, q_{\frac{2(m-1)}{3}} + x_1 z_1, q_2, \dots, q_{\frac{2(m-1)}{3}-1}, z_k y_1, z_{k-1} z_k + q'_0, q'_1, \dots, q'_{\frac{2(n-1)}{3}}, z_2 z_3, z_1 z_2 + z_3 z_4, \dots, z_{k-3} z_{k-2}, z_{k-4} z_{k-3} + z_{k-2} z_{k-1}$  of length  $\frac{2|V|+1}{3}$ , generates I(G), up to radical, where  $A_m: q_0, \dots, q_{\frac{2(m-1)}{3}}$  generates  $I(C_m)$ , up to radical by Proposition 2.3. Thus ara  $I(G) \leq \frac{2|V|+1}{3}$ .

Now suppose that  $m \equiv 0$ . Then  $|V| \equiv 0$ . From Theorem C, Theorem B and (\*) we deduce that  $\tilde{H}_i(E_1) \neq 0$  if and only if  $i = \frac{2|V|}{3} - 2$ . Thus  $\tilde{H}_i(\varepsilon(G)) \neq 0$  for  $i = \frac{2|V|}{3} - 2$ , which, by (\*), implies that  $pd I(G) \geq \frac{2|V|}{3}$ . By Lemma 2.1, the sequence  $B: q'_0, q'_1, q'_{2(n-1)} + y_1 z_k, q'_2, \dots, q'_{2(n-1)} - 1, z_1 x_1, z_1 z_2 + y_1 z_k, q'_2, \dots, q'_{2(n-1)} - 1, z_1 x_1, z_1 z_2 + y_1 z_k, q'_2, \dots, q'_{2(n-1)} - 1, z_1 x_1, z_1 z_2 + y_1 z_k, q'_2, \dots, q'_{2(n-1)} - 1, z_1 z_1, z_1 z_1 z_1 + y_1 z_k, q'_2, \dots, q'_{2(n-1)} - 1, z_1 z_1, z_1 z_1 + y_1 z_k, q'_2, \dots, q'_{2(n-1)} - 1, z_1 z_1 + y_1 z_k, q'_2, \dots, q'_{2(n-1)} - 1, z_1 z_1 + y_1 z_k, q'_2, \dots, q'_{2(n-1)} - y_1 z_1, z_1 z_1 + y_1 z_k, q'_2, \dots, q'_{2(n-1)} - y_1 z_1, z_1 + y_1 z_k, q'_2, \dots, q'_{2(n-1)} - y_1 z_1, z_1 + y_1 z_k, q'_2, \dots, q'_{2(n-1)} - y_1 z_1, \dots, q'_{2(n-1)} - y_1 z_1, \dots, q'_{2(n-1)} - y_1 z_1, \dots, q$ 

By Lemma 2.1, the sequence  $B: q'_0, q'_1, q'_{\frac{2(n-1)}{3}} + y_1 z_k, q'_2, \dots, q'_{\frac{2(n-1)}{3}-1}, z_1 x_1, z_1 z_2 + q_0, q_1, \dots, q_{\frac{2m}{3}-1}, z_3 z_4, z_2 z_3 + z_4 z_5, \dots, z_{k-2} z_{k-1}, z_{k-3} z_{k-2} + z_{k-1} z_k$  of length  $\frac{2|V|}{3}$ , generates I(G), up to radical, where  $A_m: q_0, \dots, q_{\frac{2m}{3}}$  generates  $I(C_m)$ , up to radical by Proposition 2.2. Therefore, we have  $pd I(G) = ara I(G) = \frac{2|V|}{3}$ .

#### Case 1.1.2 Let $m \equiv 2$ .

In this case  $|V| \equiv 2$ . Consider the induced subgraph  $H_2$  on  $V \setminus \{z_k\}$ . We have

1MARGHERITA BARILE, DARIUSH KIANI, FATEMEH MOHAMMADI, AND SIAMAK YASSEMI

 $\begin{aligned} H_2 &= H' \cup C_n, \text{ where } H' \text{ is the induced subgraph on } V \setminus (V(C_n) \cup \{z_k\}). \text{ By Theorem} \\ \text{A we have that, for all } i, \quad \tilde{H}_i(\varepsilon(H_2)) &= \tilde{H}_{i-\frac{2n+1}{3}}(\varepsilon(H')). \text{ If we apply Corollary 3.2} \\ \text{along the path } L_{k-2} : z_2 \dots z_{k-1}, \text{ and then Lemma 3.1 for } v = z_1, \text{ we further get,} \\ \text{for all } i, \quad \tilde{H}_i(\varepsilon(C_n \cup H')) &= \tilde{H}_{i-\frac{2n+1}{3}-\frac{2(k-2)}{3}-3}(\varepsilon(L_{m-3})), \text{ which, by Theorem D and} \\ (*), \text{ is non zero in } i &= \frac{2|V|-1}{3} - 2. \text{ By } (*) \text{ this implies } pd \ I(G) \geq \frac{2|V|-1}{3}. \\ \text{By Lemma 2.1, the sequence } B : z_1x_1, z_1z_2+q_0, q_1, \dots, q_{\frac{2(m-2)}{3}}, z_3z_4, z_2z_3+z_4z_5, \dots, z_{k-2}z_{k-1}, z_{k-3}z_{k-2}+z_{k-1}z_k, q'_0, q'_1, z_ky_1 + q'_{\frac{2(n-1)}{3}}, q'_2, \dots, q'_{\frac{2(n-1)}{3}-1} \text{ of length } \frac{2|V|-1}{3} \text{ generates } I(G), \text{ up to radical, where the sequence } A_n : q'_0, \dots, q'_{\frac{2(n-1)}{3}} \text{ generates } I(C_n), \text{ up to radical,} \\ \text{ by Proposition 2.3. This shows that } ara \ I(G) \leq \frac{2|V|-1}{3}. \end{aligned}$ 

Case 1.2 Let  $n \equiv 2$ .

By Theorem C, since  $n-3 \equiv 2$ , we have that

$$\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-\frac{2(k-2)}{3}-3-\frac{2(n-3)-1}{3}}(\varepsilon(C_m))$$

for all *i*. Moreover, by Theorem A,

$$\tilde{H}_i(E_1) = \tilde{H}_{i-\frac{2(k+1)}{2}-\frac{2n-1}{2}}(\varepsilon(L_{m-1}))$$

for all *i*. Moreover, in view of Proposition 2.4,  $I(C_n)$  is generated, up to radical, by the sequence  $A_n: q'_0, \ldots, q'_{2(n-2)}$ .

## Case 1.2.1 Let $m \equiv 0$ .

In this case  $|V| \equiv 1$ . In view of Theorem B and (\*) we have that  $\tilde{H}_i(E_1 \cap E_2) \neq 0$  if and only if  $i = \frac{2|V|+1}{3} - 3$ , in which case the homology group is  $K^2$ , and, according to Theorem D and (\*), we have that  $\tilde{H}_i(E_1) \neq 0$  if and only if  $i = \frac{2|V|+1}{3} - 3$ , in which case the homology group is K. From the Mayer-Vietoris sequence it then follows that  $\tilde{H}_i(\varepsilon(G)) \neq 0$  for  $i = \frac{2|V|+1}{3} - 2$ , which, by (\*), implies that  $pd I(G) \geq \frac{2|V|+1}{3}$ . The sequence  $B : q'_0, \ldots, q'_{\frac{2(n-2)}{3}}, q_0, \ldots, q_{\frac{2m}{3}-1}, z_1z_2, z_0z_1+z_2z_3, \ldots, z_{k-1}z_k, z_{k-2}z_{k-1}$  $+ z_k z_{k+1}$  of length  $\frac{2|V|+1}{3}$  generates I(G), up to radical, by Lemma 2.1, where the sequence  $A_m : q_0, \ldots, q_{\frac{2m}{3}-1}$  generates  $I(C_m)$ , up to radical, by Proposition 2.2. This implies that  $ara I(G) \leq \frac{2|V|+1}{3}$ .

## Case 1.2.2 Let $m \equiv 2$ .

In this case  $|V| \equiv 0$ . In view of Theorem B and (\*) we have that  $\hat{H}_i(E_1 \cap E_2) \neq 0$ if and only if  $i = \frac{2|V|}{3} - 3$ , and, according to Theorem C, since  $m - 1 \equiv 1$ , we have that  $\tilde{H}_i(E_1) = 0$  for all *i*. From the Mayer-Vietoris sequence it then follows that  $\tilde{H}_i(\varepsilon(G)) \neq 0$  for  $i = \frac{2|V|}{3} - 2$ , which, by (\*), implies that  $pd I(G) \geq \frac{2|V|}{3}$ . By Lemma 2.1, the sequence  $B: q_0, \ldots, q_{\frac{2(m-2)}{3}}, q'_0, \ldots, q'_{\frac{2(n-2)}{3}}, z_1z_2, z_0z_1+z_2z_3, \ldots$ ,

By Lemma 2.1, the sequence  $D: q_0, \ldots, q_{\frac{2(m-2)}{3}}, q_0, \ldots, q_{\frac{2(n-2)}{3}}, z_{1}z_2, z_0z_1+z_2z_3, \ldots, z_{k-1}z_k, z_{k-2}z_{k-1} + z_kz_{k+1}$  of length  $\frac{2|V|}{3}$  generates I(G), up to radical, where the sequence  $A_m: q_0, \ldots, q_{\frac{2(m-2)}{3}}$  generates  $I(C_m)$ , up to radical, by Proposition 2.4. This implies that ara  $I(G) \leq \frac{2|V|}{3}$ .

Case 1.3 Let  $n \equiv m \equiv 0$ .

In this case  $|V| \equiv 2$ . In view of Theorem C, since  $n-3 \equiv 0$ , we have that, for all i,

$$\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i - \frac{2(k-2)}{3} - 3 - \frac{2(n-3)}{3}}(\varepsilon(C_m)),$$

and by Theorem A, for all i,

$$\tilde{H}_i(E_1) = \tilde{H}_{i-\frac{2(k+1)}{2}-\frac{2n}{2}}(\varepsilon(L_{m-1})).$$

According to Theorem B and (\*) it follows that  $\tilde{H}_i(E_1 \cap E_2) \neq 0$  if and only if  $i = \frac{2|V|+2}{3} - 3$ , in which case it is equal to  $K^2$ , and, in view of Theorem D and (\*),  $\tilde{H}_i(E_1) \neq 0$  if and only if  $i = \frac{2|V|+2}{3} - 3$ , in which case it is equal to K. From the Mayer-Vietoris sequence it then follows that  $\tilde{H}_i(\varepsilon(G)) \neq 0$  for  $i = \frac{2|V|+2}{3} - 2$ , which, by (\*), implies that  $pd I(G) \ge \frac{2|V|+2}{3}$ . The sequence  $B: q'_0, \ldots, q'_{\frac{2n}{2}-1}, q_0, \ldots, q_{\frac{2m}{3}-1}, z_1 z_2, x_1 z_1 + z_2 z_3, \ldots, z_{k-1} z_k, z_{k-2} z_{k-1}$  $+ z_k y_1$ , generates I(G), up to radical, by Lemma 2.1, where  $A_m : q_0, \ldots, q_{\frac{2m}{3}-1}$ generates  $I(C_m)$ , up to radical, and  $A_n : q'_0, \ldots, q'_{\frac{2n}{2}-1}$  generates  $I(C_n)$ , up to

radical, by Proposition 2.2. Therefore, we have ara  $I(G) \leq \frac{2|V|+2}{3}$ .

### Case 2 Let $k \equiv 0$ .

As in Case 1, we can write  $\varepsilon(G) = \varepsilon(\{x_1, x_2\}, \dots, \{x_m, x_1\}, \{y_1, y_2\}, \dots, \{y_n, y_1\}, \{y_n, y_1\}, \dots, \{y_n, y_n\}, \dots, \{y_n$  $\{x_1, z_1\}, \{z_1, z_2\}, \dots, \{z_{k-1}, z_k\}, \{z_k, y_1\}; V\} = E_1 \cup E_2$ , where  $E_1 = \varepsilon(\{x_1, x_2\}, \dots, \{x_1, x_m\}, \{y_1, y_2\}, \dots, \{y_1, y_n\}, \{x_1, z_1\}, \dots, \{z_{k-1}, z_k\})$  and  $E_2 = \varepsilon(\{z_k, y_1\}).$  $\{y_3, y_4\}, \ldots, \{y_{n-2}, y_{n-1}\}; V \setminus \{x_1, y_1\})$ , so that, by [11, Lemma 3.5], Ĥ

$$H_i(E_1 \cap E_2) = H_{i-4}(\varepsilon(L_{m-3} \cup L_{n-3}))$$

for all *i*. If  $k \ge 3$ , then  $E_1 \cap E_2 = \varepsilon(\{z_{k-1}\}, \{y_2\}, \{y_n\}, \{x_1, x_2\}, \dots, \{x_m, x_1\}, \{y_3, y_4\}, \{y_4, y_4\}, \{y_4$  $\ldots, \{y_{n-2}, y_{n-1}\}, \{x_1, z_1\}, \{z_1, z_2\}, \ldots, \{z_{k-3}, z_{k-2}\}; V \setminus \{z_k, y_1\})$ , so that, by [11, Lemma 3.5],  $\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-3}(\varepsilon(H'' \cup L_{n-3}))$ , for all *i*, where H'' is the induced subgraph on  $V \setminus (V(C_n) \cup \{z_{k-1}, z_k\})$ , i.e., it is the union of  $C_m$  and the path  $L_{k-1}: x_1z_1\ldots z_{k-2}$ . If we apply Corollary 3.2 along the path  $L_{k-3}: z_2\ldots z_{k-2}$ and then Lemma 3.1 for  $v = z_1$ , we deduce that, for all i,

$$\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-6-\frac{2(k-3)}{2}}(\varepsilon(L_{m-3} \cup L_{n-3})),$$

which is evidently also true for k = 0. If k = 0, we have that  $E_1 = \varepsilon(C_m \cup C_n)$ , otherwise, if we apply Corollary 3.2 along the path  $L_k : z_1 \dots z_k$ , we obtain that, for all i,

$$\tilde{H}_i(E_1) = \tilde{H}_{i-\frac{2k}{n}}(C_m \cup C_n).$$

This equality is evidently also true for k = 0. Since  $E_2$  is a simplex, we also have that  $\tilde{H}_i(E_2) = 0$  for all *i*.

Case 2.1 Let  $n \equiv 1$ .

In view of Theorem D (for m = 3) and of Theorem C (for  $m \ge 4$ ), since  $n - 3 \equiv 1$ , we have that  $H_i(E_1 \cap E_2) = 0$  for all *i*, so that  $H_i(\varepsilon(G)) = H_i(E_1)$  for all *i*. Moreover, in view of Theorem A, for all  $i, \hat{H}_i(E_1) = \hat{H}_{i-\frac{2k}{2}-\frac{2n+1}{2}}(\varepsilon(C_m)).$ 

By Proposition 2.3, the sequence  $A_n : q'_0, q'_1, \ldots, q_{\frac{2(n-1)}{2}}$  generates  $I(C_n)$ , up to radical.

Let  $m \equiv 1$ . Then  $|V| \equiv 2$ . In view of Theorem B and (\*),  $\tilde{H}_i(E_1) \neq 0$  if and only if  $i = \frac{2|V|+2}{3} - 2$ . Hence  $\tilde{H}_i(\varepsilon(G)) \neq 0$  for  $i = \frac{2|V|+2}{3} - 2$ , which, by (\*), implies that  $pd I(G) \geq \frac{2|V|+2}{3}$ .

If k = 0, then the sequence  $q_0, q_1, q_{\frac{2(m-1)}{3}} + x_1y_1, q_2, \dots, q_{\frac{2(m-1)}{3}-1}, q'_0, \dots, q'_{\frac{2(n-1)}{3}}$ of length  $\frac{2|V|+2}{3}$ , generates I(G), up to radical, by Lemma 2.1, where the sequence  $A_m: q_0, \dots, q_{\frac{2(m-1)}{3}}$  generates  $I(C_m)$ , up to radical, by Proposition 2.3. If  $k \ge 3$ , then the sequence  $B: q_0, q_1, q_{\frac{2(m-1)}{3}} + x_1z_1, q_2, \dots, q_{\frac{2(m-1)}{3}-1}, z_2z_3, z_1z_2 +$ 

 $\begin{array}{l} 1 \ k \geq 0, \text{ then the sequence } D \ \cdot \ q_0, q_1, q_2(\underline{m-1}) \\ z_3 z_4, \dots, z_{k-1} z_k, z_{k-2} z_{k-1} + z_k y_1, q'_0, \dots, q'_{\frac{2(n-1)}{3}} \text{ of length } \frac{2|V|+2}{3}, \text{ generates } I(G), \\ \text{up to radical. Hence we have } ara \ I(G) \leq \frac{2|V|+2}{3}. \end{array}$ 

Let  $m \equiv 2$ . Then  $|V| \equiv 0$ . In view of Theorem B and (\*),  $\tilde{H}_i(E_1) \neq 0$  if and only if  $i = \frac{2|V|}{3} - 2$ . Hence  $\tilde{H}_i(\varepsilon(G)) \neq 0$  for  $i = \frac{2|V|}{3} - 2$ , which, by (\*), implies that  $pd I(G) \geq \frac{2|V|}{3}$ .

If k = 0, then by Lemma 2.1, the sequence  $B: q'_0, q'_1, q'_{\frac{2(n-1)}{3}} + x_1y_1, q'_2, \dots, q'_{\frac{2(n-1)}{3}-1}, q_0, \dots, q_{\frac{2(m-2)}{3}}, of length <math>\frac{2|V|}{3}$ , generates I(G) up to radical, where the sequence  $A_m: q_0, \dots, q_{\frac{2(m-2)}{3}}$  generates  $I(C_m)$ , up to radical, by Proposition 2.4.

If  $k \ge 3$ , then the sequence  $B: q'_0, q'_1, q'_{\frac{2(n-1)}{3}} + y_1 z_k, q'_2, \dots, q'_{\frac{2(n-1)}{3}-1}, z_1 z_2, x_1 z_1 + z_2 z_3, \dots, z_{k-2} z_{k-1}, z_{k-3} z_{k-2} + z_{k-1} z_k, q_0, \dots, q_{\frac{2(m-2)}{3}}$ , of length  $\frac{2|V|}{3}$ , generates I(G) up to radical. This implies that ara  $I(G) \le \frac{2|V|}{3}$ .

Let  $m \equiv 0$ . Then  $|V| \equiv 1$ . In view of Theorem B and (\*),  $\tilde{H}_i(E_1) \neq 0$  if and only if  $i = \frac{2|V|+1}{3} - 2$ . Hence  $\tilde{H}_i(\varepsilon(G)) \neq 0$  for  $i = \frac{2|V|+1}{3} - 2$ , which, by (\*), implies that  $pd I(G) \geq \frac{2|V|+1}{3}$ .

If k = 0, then the sequence  $B: q'_0, q'_1, q'_{\frac{2(n-1)}{3}} + x_1y_1, q'_2, \dots, q'_{\frac{2(n-1)}{3}-1}, q_0, \dots, q_{\frac{2m}{3}-1}$ of length  $\frac{2|V|+1}{3}$ , generates I(G), up to radical, by Lemma 2.1, where the sequence  $A_m: q_0, \dots, q_{\frac{2m}{3}-1}$  generates  $I(C_m)$ , up to radical, by Proposition 2.2.

If  $k \ge 3$ , then the sequence  $B: q'_0, q'_1, q'_{\frac{2(n-1)}{3}} + y_1 z_k, q'_2, \dots, q'_{\frac{2(n-1)}{3}-1}, z_1 z_2, x_1 z_1 + z_2 z_3, \dots, z_{k-2} z_{k-1}, z_{k-3} z_{k-2} + z_{k-1} z_k, q_0, \dots, q_{\frac{2m}{3}-1}$  of length  $\frac{2|V|+1}{3}$ , generates I(G), up to radical, by Lemma 2.1. This shows that ara  $I(G) \le \frac{2|V|+1}{3}$ .

## **Case 2.2** Let $n \equiv 2$ and $m \equiv 0$ .

In this case  $|V| \equiv 2$ . In view of Theorem C, Theorem D and (\*),  $\tilde{H}_i(E_1 \cap E_2) \neq 0$ if and only if  $i = \frac{2|V|-1}{3} - 2$ , in which case the homology group is K. Moreover, in view of Theorem A,  $\tilde{H}_i(E_1) = \tilde{H}_{i-\frac{2k}{3}-\frac{2m}{3}}(\varepsilon(C_n))$ , for all *i*. In view of Theorem B and (\*),  $\tilde{H}_i(E_1) \neq 0$  if and only if  $i = \frac{2|V|-1}{3} - 2$ , in which case the homology group is  $K^2$ . Hence  $\tilde{H}_i(\varepsilon(G)) \neq 0$  for  $i = \frac{2|V|-1}{3} - 2$ , which, by (\*), implies that  $pd I(G) \geq \frac{2|V|-1}{2}$ .

 $pd I(G) \geq \frac{2|V|-1}{3}.$ Let k = 0. The sequence  $B: x_1y_1, q_0 + q'_0, q_1, \dots, q_{\frac{2m}{3}-1}, q'_1, \dots, q'_{\frac{2(n-2)}{3}}$  of length  $\frac{2|V|-1}{3}$  generates I(G), up to radical, by Lemma 2.1, where the sequence  $A_m: q_0, q_1, \dots, q_{\frac{2m}{3}-1}$  generates  $I(C_m)$ , up to radical, by Proposition 2.2.

Let  $k \geq 3$ . The sequence  $B: x_1z_1, q_0 + z_1z_2, q_1, \dots, q_{\frac{2m}{3}-1}, z_3z_4, z_2z_3 + z_4z_5, \dots, z_{k-3}z_{k-2}, z_{k-4}z_{k-3} + z_{k-2}z_{k-1}, z_ky_1, z_{k-1}z_k + q'_0, q'_1, \dots, q'_{\frac{2(n-2)}{3}}$  of length  $\frac{2|V|-1}{3}$  generates I(G), up to radical. Hence, in view of (\*), we conclude that ara  $I(G) \leq \frac{2|V|-1}{3}$ .

#### Case 2.3 Let $n \equiv m \equiv 2$ .

In this case  $|V| \equiv 1$ . Consider the induced subgraph  $H_3$  on  $V \setminus \{x_2\}$ . Applying Corollary 3.2 to the path  $L_{m+k-2} : x_3x_4 \dots x_1z_1 \dots z_{k-1} \ (L_{m-2} : x_3x_4 \dots x_1)$ , if k = 0 and Lemma 3.1 for  $v = z_k \ (u = y_1)$ , we obtain that, for all  $i, \ \tilde{H}_i(\varepsilon(H_3)) = \tilde{H}_{i-\frac{2(m+k-2)}{3}-3}(\varepsilon(L_{n-3}))$ , which, by Theorem D and (\*), is non zero in  $i = \frac{2|V|-2}{3}-2$ . So  $\tilde{H}_{2|V|-2}_{-2}(\varepsilon(G)) \neq 0$  and by (\*) we have  $pd \ I(G) \geq \frac{2|V|-2}{3}$ .

If k = 0, then, by Lemma 2.1, the sequence  $B : x_1y_1, q_0 + q'_0, q_1, \ldots, q_{\frac{2(m-2)}{3}}, q'_1, \ldots, q'_{\frac{2(m-2)}{3}}$  generates I(G), up to radical, where the sequence  $A_m : q_0, \ldots, q_{\frac{2(m-2)}{3}}$  generates  $I(C_m)$ , up to radical, and the sequence  $A_n : q'_0, \ldots, q'_{\frac{2(n-2)}{3}}$  generates  $I(C_n)$ , up to radical, and the sequence  $A_n : q'_0, \ldots, q'_{\frac{2(n-2)}{3}}$  generates  $I(C_n)$ , up to radical, by Proposition 2.4.

If  $k \ge 3$ , then, by Lemma 2.1, the sequence  $B: z_k y_1, z_{k-1} z_k + q'_0, q'_1, \dots, q'_{\frac{2(n-2)}{3}}, z_0 z_1, z_1 z_2 + q_0, q_1, \dots, q_{\frac{2(m-2)}{3}}, z_3 z_4, z_2 z_3 + z_4 z_5, \dots, z_{k-3} z_{k-2}, z_{k-4} z_{k-3} + z_{k-2} z_{k-1}^3$  generates I(G), up to radical. Hence we have ara  $I(G) \le \frac{2|V|-2}{3}$ .

## Case 2.4 Let $n \equiv m \equiv 0$ .

In this case  $|V| \equiv 0$ . First assume that n = m = 3. We have that  $I(G) = I \cap J$ , where  $I = I(G) + (x_1y_1z_3z_6 \dots z_k)$  and  $J = (x_2, x_3, y_2, y_3, z_1, z_2, z_4, z_5, \dots, z_{k-5}, z_{k-4}, z_{k-2}, z_{k-1})$ . Since J is a complete intersection ideal, we have that  $cd \ J = 4 + \frac{2k}{3}$ . Moreover,  $I+J = (x_1y_1z_3z_6 \dots z_k, x_2, x_3, y_2, y_3, z_1, z_2, z_4, z_5, \dots, z_{k-5}, z_{k-4}, z_{k-2}, z_{k-1})$ . Since I + J has grade equal to  $5 + \frac{2k}{3}$ , by [5, Theorem 6.2.7] we have  $H_{I+J}^i(R) \neq 0$ in  $i = 5 + \frac{2k}{3}$  and  $H_{I+J}^i(R) = 0$  for any  $i < 5 + \frac{2k}{3}$ . In the Mayer-Vietoris sequence for local cohomology (see [10], Section 3)

$$\dots \to H_{I+J}^{4+\frac{2k}{3}}(R) \to H_{I}^{4+\frac{2k}{3}}(R) \oplus H_{J}^{4+\frac{2k}{3}}(R) \to H_{I\cap J}^{4+\frac{2k}{3}}(R) \to \dots,$$

the left term is zero, whereas the middle term is not. It follows that the right term is non zero, too. This implies that  $pd \ I(G) = cd \ I(G) \ge 4 + \frac{2k}{3} = \frac{2|V|}{3}$ . So without loss of generality we may assume that n > 3. Then from Theorem C, since  $m - 3 \equiv 0$ , we have that  $\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-6-\frac{2(k-3)}{3}-\frac{2(m-3)}{3}}(\varepsilon(L_{n-3}))$ , for all *i*. Hence, in view of Theorem D and (\*), we have that  $\tilde{H}_i(E_1 \cap E_2) \neq 0$  only if  $i = \frac{2|V|}{3} - 2$ , in which case this homology group is K. In view of Theorem A,

Theorem B and (\*) we also have that  $\tilde{H}_i(E_1) \neq 0$  only if  $i = \frac{2|V|}{3} - 2$ , in which case this homology group is  $K^2$ . The Mayer-Vietoris sequence shows that  $\tilde{H}_i(\varepsilon(G)) \neq 0$  in  $i = \frac{2|V|}{3} - 2$ . Thus in view of (\*) we deduce that  $pd I(G) \geq \frac{2|V|}{3}$ .

If k = 0, then, by Lemma 2.1, the sequence  $B : x_1y_1, q_0 + q'_0, q_1, \ldots, q_{\frac{2m}{3}-1}, q'_1, \ldots, q'_{\frac{2n}{3}-1}$  of length  $\frac{2|V|}{3}$  generates I(G), up to radical, where the sequence  $A_m : q_0, \ldots, q_{\frac{2m}{3}-1}$  generates  $I(C_m)$ , up to radical, and the sequence  $A_n : q'_0, \ldots, q'_{\frac{2n}{3}-1}$  generates  $I(C_n)$ , up to radical, by Proposition 2.2.

If  $k \ge 3$ , then, by Lemma 2.1 the sequence  $B: z_1x_1, z_1z_2 + q_0, q_1, \dots, q_{\frac{2m}{2}-1}, z_3z_4,$ 

 $z_{2}z_{3} + z_{4}z_{5}, \dots, z_{k-3}z_{k-2}, z_{k-4}z_{k-3} + z_{k-2}z_{k-1}, z_{k}y_{1}, z_{k-1}z_{k} + q'_{0}, q'_{1}, \dots, q'_{\frac{2n}{3}-1} \text{ of } length \ \frac{2|V|}{3} \text{ generates } I(G), \text{ up to radical. We thus have } ara \ I(G) \le \frac{2|V|}{3}.$ 

Case 3 Let  $k \equiv 1$ .

We can write  $\varepsilon(G) = \varepsilon(\{x_1, x_2\}, \dots, \{x_m, x_1\}, \{y_1, y_2\}, \dots, \{y_n, y_1\}, \{x_1, z_1\}, \{z_1, z_2\}, \dots, \{z_{k-1}, z_k\}, \{z_k, y_1\}; V) = E_1 \cup E_2$ , where

 $E_1 = \varepsilon(\{x_2, x_3\}, \dots, \{x_m, x_1\}, \{y_1, y_2\}, \dots, \{y_n, y_1\}, \{x_1, z_1\}, \{z_1, z_2\}, \dots, \{z_{k-1}, z_k\}, \{z_k, y_1\}; V) \text{ and } E_2 = \varepsilon(\{x_1, x_2\}; V). We have that <math>E_1 = \varepsilon(L_{m+k} \cup C_n)$ , where  $L_{m+k} : x_2 \dots x_m x_1 z_1 \dots z_k$ .

If m = 3, then, by [12, Lemma 3.4],  $E_1 \cap E_2 = \varepsilon(\{x_3\}, \{z_1\}, \{z_2, z_3\}, \dots, \{z_k, y_1\}, \{y_1, y_2\}, \dots, \{y_n, y_1\}; V)$ , so that, by [12, Lemma 3.5],

$$\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-2}(\varepsilon(H_4)),$$

for all *i*, where  $H_4$  is the induced subgraph on  $V \setminus (V(C_m) \cup \{z_1\})$ , i.e., the union of  $C_n$  and the path  $L_k : z_2 \ldots y_1$ . If  $m \ge 4$ , then  $E_1 \cap E_2 = \varepsilon(\{x_3\}, \{x_m\}, \{z_1\}, \{x_4, x_5\}, \ldots, \{x_{m-2}, x_{m-1}\}, \{z_2, z_3\}, \ldots, \{z_k, y_1\}, \{y_1, y_2\}, \ldots, \{y_n, y_1\}; V)$ , so that, by [12, Lemma 3.5],

$$H_i(E_1 \cap E_2) = H_{i-3}(\varepsilon(L_{m-4} \cup H_4)),$$

for all *i*. Since  $E_2$  is a simplex,  $H_i(E_2) = 0$  for all *i*.

Case 3.1 Let  $n \equiv 1$ .

The sequence  $A_n$ :  $q'_0, \ldots, q'_{\frac{2(n-1)}{3}}$  generates  $I(C_n)$ , up to radical, by Proposition 2.2.

Case 3.1.1 Let  $m \equiv 0$  or  $m \equiv 2$ .

First let  $m \equiv 0$ . Then  $|V| \equiv 2$ . If we apply Corollary 3.2 to the path  $L_{m+k-1}$ :  $x_2 \dots x_m x_1 \dots z_{k-1}$ , and then Lemma 3.1 for  $v = z_k$  we get, that, for all i,  $\tilde{H}_i(E_1) = \tilde{H}_{i-\frac{2(m+k-1)}{3}-3}(\varepsilon(L_{n-3}))$ , which is zero for all i by Theorem D and (\*). If  $m \ge 6$ , applying Theorem C  $(m - 4 \equiv 2)$  and Corollary 3.2 to  $E_1 \cap E_2$  along the path  $L_{k-1}: z_2 \dots z_k$ , we deduce that, for all i,

$$H_i(E_1 \cap E_2) = H_{i - \frac{2(m-4)-1}{3} - \frac{2(k-1)}{3} - 3}(\varepsilon(C_n)),$$

which is also true for m = 3.

By Theorem B and (\*),  $\tilde{H}_{\frac{2|V|+2}{3}-3}(E_1 \cap E_2) \neq 0$ . So by the Mayer-Vietoris sequence  $\tilde{H}_{\frac{2|V|+2}{3}-2}(\varepsilon(G)) \neq 0$  and in view of (\*) we conclude that  $pd \ I(G) \geq \frac{2|V|+2}{3}$ . By Lemma 2.1,  $B: \ q'_0, \ldots, q'_{\frac{2(n-1)}{3}}, x_1z_1, z_1z_2 + q_0, q_1, \ldots, q_{\frac{2m}{3}-1}, z_3z_4, z_2z_3 + z_4z_5, \ldots, z_{k-1}z_k, z_{k-2}z_{k-1} + z_ky_1$  of length  $\frac{2|V|+2}{3}$  generates I(G), up to radical, where sequence  $A_m: \ q_0, \ldots, q_{\frac{2m}{3}-1}$  generates  $I(C_m)$ , up to radical, by Proposition 2.2. Therefore, we have  $ara \ I(G) \leq \frac{2|V|+2}{3}$ . Now let  $m \equiv 2$ . In this case  $|V| \equiv 1$ , and  $m + k \equiv 0$ . Moreover, by Theorem C, since  $m - 4 \equiv 1$ , we have that  $\tilde{H}_i(E_1 \cap E_2) = 0$  for all i. Hence  $\tilde{H}_i(\varepsilon(G)) = \tilde{H}_i(E_1)$ 

since  $m-4 \equiv 1$ , we have that  $H_i(E_1 + E_2) \equiv 0$  for all *i*. Hence  $H_i(\varepsilon(G)) \equiv H_i(E_1)$ for all *i*. Applying Corollary 3.2 to the path  $L_{m+k} : x_2 x_3 \dots x_1 z_1 \dots z_k$  we obtain that, for all *i*,  $\tilde{H}_i(E_1) = \tilde{H}_{i-\frac{2(m+k)}{3}}(\varepsilon(C_n))$ . By Theorem B and (\*),  $\tilde{H}_i(E_1) \neq 0$  in  $i = \frac{2|V|+1}{3} - 2$ . Thus by (\*) we have  $pd I(G) \geq \frac{2|V|+1}{3}$ .

By Lemma 2.1, the sequence  $B: x_1z_1, q_0+z_1z_2, q_1, \ldots, q_{\frac{2(m-2)}{2}}, z_3z_4, z_2z_3+z_4z_5, \ldots, z_{k-1}z_k, z_{k-2}z_{k-1}+z_{k-2}z_{k-2}+z_{$ 

 $z_k y_1, q'_0, \ldots, q'_{2(n-1)}$  of length  $\frac{2|V|+1}{3}$  generates I(G), up to radical, where the sequence  $A_m: q'_0, \ldots, q'_{\frac{2(m-2)}{2}}$  generates  $I(C_m)$ , up to radical, by Proposition 2.4. Thus ara  $I(G) \leq \frac{2|V|+1}{3}$ .

#### Case 3.1.2 Let $m \equiv 1$ .

In this case  $|V| \equiv 0$ . Consider the induced subgraph  $H_5$  on  $V \setminus \{z_k\}$ . We have, for all i,  $\tilde{H}_i(\varepsilon(H_5)) = \tilde{H}_i(H''' \cup C_n)$ , where H''' is the induced subgraph on  $V \setminus (V(C_n) \cup \{z_k\})$ , i.e., the union of  $C_m$  and the path  $L_k : x_1 z_1 \dots z_{k-1}$ . Applying Theorem A and then Corollary 3.2 to H''' along the path  $L_{k-1}: z_1 \dots z_{k-1}$ we have  $\tilde{H}_i(\varepsilon(H_5)) = \tilde{H}_{i-\frac{2n+1}{2}-\frac{2(k-1)}{2}}(C_m)$ , for all *i*, and this homology group, by Theorem B and (\*), is non zero in  $i = \frac{2|V|}{3} - 2$ . So  $\tilde{H}_{\frac{2|V|}{3}-2}(\varepsilon(H_5)) \neq 0$ . In view of (\*) we deduce that  $pd \ I(G) \ge \frac{2|V|}{3}$ . The sequence  $B: \ q_0, q_1, q_{\frac{2(m-1)}{3}} + z_1 x_1, \dots, q_{\frac{2(m-1)}{3}-1}, q'_0, q'_1, q'_{\frac{2(n-1)}{3}} + z_k y_1, \dots,$ 

 $q'_{\frac{2(n-1)}{2}-1}, z_2 z_3, z_1 z_2 + z_3 z_4, \dots, z_{k-2} z_{k-1}, z_{k-3} z_{k-2} + z_{k-1} z_k$  of length  $\frac{2|V|}{3}$ , generates I(G), up to radical, by Lemma 2.1, where the sequence  $A_m: q_0, \ldots, q_{\frac{2(m-1)}{2}}$ generates  $I(C_m)$ , up to radical, by Proposition 2.3. Therefore, we have that ara  $I(G) \leq \frac{2|V|}{3}$ .

Case 3.2 Let  $m \equiv 2$ , and  $n \equiv 0$  or 2.

In this case  $m+k \equiv 0$ . Applying Corollary 3.2 to the path  $L_{m+k}: x_2x_3 \dots x_1z_1 \dots z_k$ we obtain that, for all i,  $H_i(E_1) = H_{i-\frac{2(m+k)}{2}}(\varepsilon(C_n))$ .

Moreover, the sequence  $A_m$ :  $q_0, \ldots, q_{\frac{2(m-2)}{3}}$  generates  $I(C_m)$ , up to radical, by Proposition 2.4.

First let  $n \equiv 0$ . Then  $|V| \equiv 0$  and, by Theorem B and (\*),  $\tilde{H}_i(E_1) \neq 0$  in  $i = \frac{2|V|}{3} - 2$ . Thus by (\*) we have  $pd I(G) \ge \frac{2|V|}{3}$ 

By Lemma 2.1, the sequence  $B: x_1z_1, q_0+z_1z_2, q_1, \ldots, q_{2(m-2)}, z_3z_4, z_2z_3+z_4z_5, \ldots, z_{k-1}z_k, z_{k-2}z_{k-1}+z_{k-2}z_{k-2}+z_$  $z_k y_1, q'_0, q'_1, \ldots, q'_{\frac{2n}{2}-1}$  of length  $\frac{2|V|}{3}$  generates I(G), up to radical, where the sequence  $A_n: q'_0, \ldots, q'_{\frac{2n}{2}-1}$  generates  $I(C_n)$ , up to radical by Proposition 2.2. Thus ara  $I(G) \leq \frac{2|V|}{3}$ .

If  $n \equiv 2$ , then  $|V| \equiv 2$  and, by Theorem B and (\*),  $\tilde{H}_i(E_1) \neq 0$  in  $i = \frac{2|V|-1}{3} - 2$ . Thus by (\*) we have  $pd I(G) \ge \frac{2|V|-1}{3}$ . By Lemma 2.1, the sequence  $B: q_0, \ldots, q_{\frac{2(m-2)}{3}}, z_k y_1, z_{k-1} z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_{k-1} z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_{k-1} z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_{k-1} z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_{k-1} z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_k - z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_k - z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_k - z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_k - z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_k - z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_k - z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_k - z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_k - z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_k - z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_k - z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_k - z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_k - z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_k - z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_k - z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_k - z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_k - z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k y_1, z_k - z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k + z_k + q'_0, q'_1, \ldots, q'_{\frac{2(n-2)}{3}}, z_k + z_k +$  $z_1z_2, x_1z_1+z_2z_3, \ldots, z_{k-3}z_{k-2}, z_{k-4}z_{k-3}+z_{k-2}z_{k-1}$  of length  $\frac{2|V|-1}{3}$  generates I(G), up to radical, where the sequence  $A_n: q'_0, \ldots, q'_{\frac{2(n-2)}{3}}$  generates  $I(C_n)$ , up to radi-

cal by Proposition 2.4. Thus ara  $I(G) \leq \frac{2|V|-1}{3}$ .

## Case 3.3 Let $n \equiv m \equiv 0$ .

In this case  $|V| \equiv 1$ . As in Case 1, we can write  $\varepsilon(G) = E_1 \cup E_2$ , where  $E_1 =$  $\varepsilon(\{x_1, x_2\}, \dots, \{x_m, x_1\}, \{y_1, y_2\}, \dots, \{y_n, y_1\}, \{x_1, z_1\}, \{z_1, z_2\}, \dots, \{z_{k-1}, z_k\}; V)$  and  $E_2 = \varepsilon(\{z_k, y_1\}; V).$ 

Applying Corollary 3.2 to the path  $L_{k-1}: x_1z_1 \dots z_{k-2}$ , we have that, for all *i*,

#### 18MARGHERITA BARILE, DARIUSH KIANI, FATEMEH MOHAMMADI, AND SIAMAK YASSEMI

 $\tilde{H}_i(E_1 \cap E_2) = \tilde{H}_{i-\frac{2(k-1)}{3}-3}(\varepsilon(L_{m-1} \cup L_{n-3})).$  Theorem C  $(m-1 \equiv 2)$ , Theorem D and (\*) show that  $\tilde{H}_i(E_1 \cap E_2) \neq 0$  only if  $i = \frac{2|V|+1}{3} - 3$ . Since  $E_2$  is a simplex,  $\tilde{H}_i(E_2) = 0$  for all *i*. Applying Corollary 3.2 to  $E_1$  along the path  $L_{k-1} : z_2 \dots z_k$ , and once again Lemma 3.1 for  $v = z_1$ , we obtain that, for all i,  $\tilde{H}_i(E_1) = \tilde{H}_{i-\frac{2(k-1)}{3}-3}(\varepsilon(L_{m-3} \cup C_n))$ , which by Theorem C, Theorem B and (\*), is non zero only in  $i = \frac{2|V|+1}{3} - 2$ . The Mayer- Vietoris sequence shows that  $\tilde{H}_i(\varepsilon(G)) \neq 0$  in  $i = \frac{2|V|+1}{3} - 2$ . Thus, in view of (\*), we have that  $pd I(G) \geq \frac{2|V|+1}{3}$ . By Lemma 2.1, the sequence  $B : x_1z_1, z_1z_2+q_0, q_1, \dots, q_{\frac{2m}{3}-1}, z_3z_4, z_2z_3+z_4z_5, \dots, z_{k-1}z_k, z_{k-2}z_{k-1} + z_ky_1, q'_0, \dots, q'_{\frac{2m}{3}-1}$ , generates I(G), up to radical, where the sequence  $A_m : q_0, \dots, q_{\frac{2m}{3}-1}$  generates  $I(C_m)$ , up to radical, and the sequence  $A_n : q'_0, \dots, q'_{\frac{2m}{3}-1}$  generates  $I(C_n)$ , up to radical, by Proposition 2.2. This implies that  $pd I(G) = ara I(G) = \frac{2|V|+1}{3}$  in this case. This completes the proof.

From Theorem 3.3 and Theorem 3.4 we deduce the following corollary.

**Corollary 3.5.** Let G be a bicyclic graph, then ara  $I(C_n) = pd I(C_n)$ .

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