



RESEARCH PAPER

MULTIDIMENSIONAL VAN DER CORPUT-TYPE  
ESTIMATES INVOLVING  
MITTAG-LEFFLER FUNCTIONS

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Abstract

The paper is devoted to study multidimensional van der Corput-type estimates for the intergrals involving Mittag-Leffler functions. The generalisation is that we replace the exponential function with the Mittag-Leffler-type function, to study multidimensional oscillatory integrals appearing in the analysis of time-fractional evolution equations. More specifically, we study two types of integrals with functions  $E_{\alpha,\beta}(i\lambda\phi(x))$ ,  $x \in \mathbb{R}^N$  and  $E_{\alpha,\beta}(i^\alpha\lambda\phi(x))$ ,  $x \in \mathbb{R}^N$  for the various range of  $\alpha$  and  $\beta$ . Several generalisations of the van der Corput-type estimates are proved. As an application of the above results, the Cauchy problem for the multidimensional time-fractional Klein-Gordon and time-fractional Schrödinger equations are considered.

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*Key Words and Phrases:* van der Corput-type estimates; Mittag-Leffler function; time-fractional Schrödinger equation; time-fractional Klein-Gordon equation

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### 1. Introduction

One of the most important estimates in the theory of harmonic analysis is the van der Corput lemma, which is a decay estimate of the oscillatory integrals. This estimate was obtained by the Dutch mathematician Johannes Gautherus van der Corput [19] and named in his honour. Let us state the well-known classical van der Corput's lemma:

- **van der Corput lemma.** Let  $\phi$  be a real valued differentiable function such that the  $\phi'$  is monotonic and  $|\phi'(x)| \geq 1$ ,  $k \geq 1$ , for all  $x \in (a, b)$ , then

$$\left| \int_a^b e^{i\lambda\phi(x)} \psi(x) dx \right| \leq C\lambda^{-1}, \quad \lambda \rightarrow \infty, \quad (1.1)$$

where  $C$  does not depend on  $\lambda$ .

A multidimensional version of van der Corput's results would be of great value, but presents many difficulties. Let us consider an integral operator called an oscillatory integral defined by

$$I(\lambda) = \int_{\mathbb{R}^N} e^{i\lambda\phi(x)} \psi(x) dx, \quad (1.2)$$

where  $\phi(x)$  and  $\psi(x)$  are two functions that map  $\mathbb{R}^N$  to  $\mathbb{R}$  and are called the phase and the amplitude, respectively, and  $\lambda$  is a positive real number that can vary. It is well known that, if  $|\nabla\phi| \geq 1$  on the support of  $\psi$ , then the following estimate is true

$$|I(\lambda)| \leq C\lambda^{-1}. \quad (1.3)$$

The decay rate here is sharp, but the constant  $C$  may depend on phase and the van der Corput's estimate does not scale well. Again, such an estimate is closely related to the problem of multilinear sublevel set estimates [10], one of the fundamental problems of harmonic analysis. Certain parameter dependent sublevel set estimates were obtained and used by Kamotski and Ruzhansky [7] in the analysis of elliptic and hyperbolic systems with multiplicities, to yield Sobolev space estimates for relevant classes of oscillatory integrals and for the solutions of the hyperbolic systems of equations. There are various versions of the van der Corput estimate but most with one-dimensional rate of decay. Furthermore, Carbery, Christ and Wright [2] and Ruzhansky [12] proposed multidimensional versions of the van der Corput lemma, in formulations where also the constant in the estimate is independent of the phase function.

Recently, the attention of many mathematicians has been attracted by various generalizations of Van der Corput-type estimates for integrals involving special functions [3, 13, 14, 20, 21, 22].

The main goal of the present paper is to obtain multidimensional van der Corput-type estimates for the integrals defined by

$$I_{\alpha,\beta}^1(\lambda) = \int_{\mathbb{R}^N} E_{\alpha,\beta}(i\lambda\phi(x)) \psi(x) dx, \tag{1.4}$$

and

$$I_{\alpha,\beta}^2(\lambda) = \int_{\mathbb{R}^N} E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) \psi(x) dx, \tag{1.5}$$

where  $0 < \alpha, \beta \in \mathbb{R}$ ,  $\phi$  is a phase and  $\psi$  is an amplitude, and  $\lambda$  is a positive real number that can vary. Here  $E_{\alpha,\beta}(z)$  is the Mittag-Leffler function defined as (see e.g. [8, 5])

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \quad \beta \in \mathbb{R}, \tag{1.6}$$

for which we note that

$$E_{1,1}(z) = e^z. \tag{1.7}$$

This present paper is a continuation of [13, 14], where a variety of van der Corput type lemmas were obtained for the integrals (1.4) and (1.5) in the case  $N = 1$ .

Such integrals as in (1.4) and (1.5) arise in the study of decay estimates of solutions of the time-fractional Schrödinger and the time-fractional wave equations (for example see [4, 6, 9, 16, 17, 18]). In Section 4 we will give several immediate applications of the obtained estimates to time-fractional Schrödinger and time-fractional Klein-Gordon equations.

As the Mittag-Leffler functions have a very rich analytic structure, and, philosophically, give a generalisation of the usual exponent, the obtained estimate will also involve the dependence on the parameters  $\alpha$  and  $\beta$ .

**1.1. Preliminaries.** We will often make use of the following estimate.

**PROPOSITION 1.1** ([11]). *If  $0 < \alpha < 2$ ,  $\beta$  is an arbitrary real number,  $\mu$  is such that  $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$ , then there is  $C_1, C_2 > 0$ , such that we have*

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1 + |z|}, \quad z \in \mathbb{C}, \quad \mu \leq |\arg(z)| \leq \pi, \tag{1.8}$$

and

$$|E_{\alpha,\beta}(z)| \leq C_1(1 + |z|)^{(1-\beta)/\alpha} \exp(\operatorname{Re}(z^{1/\alpha})) + \frac{C_2}{1 + |z|}, \quad z \in \mathbb{C}, \quad |\arg(z)| \leq \mu. \tag{1.9}$$

PROPOSITION 1.2 ([1]). *The following optimal estimates are valid for the real-valued Mittag-Leffler function*

$$\frac{1}{1 + \Gamma(1 - \alpha)x} \leq E_{\alpha,1}(-x) \leq \frac{1}{1 + \frac{1}{\Gamma(1+\alpha)}x}, \quad x \geq 0, \quad 0 < \alpha < 1; \quad (1.10)$$

$$\begin{aligned} \frac{1}{\left(1 + \sqrt{\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}}x\right)^2} &\leq \Gamma(\alpha)E_{\alpha,\alpha}(-x) \\ &\leq \frac{1}{\left(1 + \sqrt{\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}}x\right)^2}, \quad x \geq 0, \quad 0 < \alpha < 1; \end{aligned} \quad (1.11)$$

$$\begin{aligned} \frac{1}{1 + \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta)}x} &\leq \Gamma(\beta)E_{\alpha,\beta}(-x) \\ &\leq \frac{1}{1 + \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}x}, \quad x \geq 0, \quad 0 < \alpha \leq 1, \quad \beta > \alpha. \end{aligned} \quad (1.12)$$

PROPOSITION 1.3. ([13]) *Let  $\alpha, \beta > 0$  and  $\phi : \mathbb{R}^N \rightarrow \mathbb{C}$ . Then for all  $\lambda \in \mathbb{C}$  we have*

$$E_{\alpha,\beta}(i\lambda\phi(x)) = E_{2\alpha,\beta}(-\lambda^2\phi^2(x)) + i\lambda\phi(x)E_{2\alpha,\beta+\alpha}(-\lambda^2\phi^2(x)). \quad (1.13)$$

**2. Van der Corput-type estimates for the integral  $I_{\alpha,\beta}^1(\lambda)$**

In this section we consider the integral operator defined by

$$I_{\alpha,\beta}^1(\lambda) = \int_{\mathbb{R}^N} E_{\alpha,\beta}(i\lambda\phi(x)) \psi(x) dx,$$

where  $0 < \alpha \leq 1, \beta > 0$ ,  $\phi$  is a phase and  $\psi$  is an amplitude, and  $\lambda$  is a positive real number that can vary. We are interested in particular in the behavior of  $I_{\alpha,\beta}^1(\lambda)$  when  $\lambda$  is large, as for small  $\lambda$  the integral is just bounded.

THEOREM 2.1. *Let  $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$  be a measurable function and let  $\psi \in L^1(\mathbb{R}^N)$ . If  $0 < \alpha < 1, \beta > 0$ , and  $m = \text{ess inf}_{x \in \mathbb{R}^N} |\phi(x)| > 0$ , then we have the estimate*

$$|I_{\alpha,\beta}^1(\lambda)| \leq \frac{M\|\psi\|_{L^1(\mathbb{R}^N)}}{1 + m\lambda}, \quad \lambda \geq 0, \quad (2.1)$$

where  $M$  does not depend on  $\phi, \psi$  and  $\lambda$ .

**P r o o f.** As for small  $\lambda$  the integral (1.4) is just bounded, we give the proof for  $\lambda \geq 1$ . Let  $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$  be a measurable function and  $\psi \in L^1(\mathbb{R}^N)$ . Then

$$|I_{\alpha,\beta}^1(\lambda)| = \left| \int_{\mathbb{R}^N} E_{\alpha,\beta}(i\lambda\phi(x)) \psi(x) dx \right| \leq \int_{\mathbb{R}^N} |E_{\alpha,\beta}(i\lambda\phi(x))| |\psi(x)| dx.$$

Using formula (1.13) and estimate (1.9) we have that

$$\begin{aligned} |E_{\alpha,\beta}(i\lambda\phi(x))| &\leq |E_{2\alpha,\beta}(-\lambda^2\phi^2(x))| + \lambda|\phi(x)| |E_{2\alpha,\alpha+\beta}(-\lambda^2\phi^2(x))| \\ &\leq \frac{C}{1 + \lambda^2\phi^2(x)} + \frac{C\lambda|\phi(x)|}{1 + \lambda^2\phi^2(x)} \leq C \frac{1 + \lambda|\phi(x)|}{1 + \lambda^2\phi^2(x)}. \end{aligned} \tag{2.2}$$

As  $\phi$  and  $\psi$  do not depend on  $\lambda$ , and  $m = \text{ess inf}_{x \in \mathbb{R}^N} |\phi(x)| > 0$ , then from (2.2) we have

$$\begin{aligned} |I_{\alpha,\beta}^1(\lambda)| &\leq \int_{\mathbb{R}^N} |E_{\alpha,\beta}(i\lambda\phi(x))| |\psi(x)| dx \leq C \int_{\mathbb{R}^N} \frac{1 + \lambda|\phi(x)|}{1 + \lambda^2\phi^2(x)} |\psi(x)| dx \\ &\leq 2C \int_{\mathbb{R}^N} \frac{1 + \lambda|\phi(x)|}{(1 + \lambda|\phi(x)|)^2} |\psi(x)| dx \leq C_1 \int_{\mathbb{R}^N} \frac{|\psi(x)|}{1 + \lambda|\phi(x)|} dx \leq \frac{M\|\psi\|_{L^1(\mathbb{R}^N)}}{1 + m\lambda}. \end{aligned}$$

The proof is complete. □

Now we find upper and lower bounds in estimates for

$$I_{\alpha,\beta}^{1,\Omega}(\lambda) = \int_{\Omega} E_{\alpha,\beta}(i\lambda\phi(x)) \psi(x) dx, \tag{2.3}$$

where  $\Omega \subset \mathbb{R}^N$  is a bounded domain,  $0 < \alpha \leq \frac{1}{2}$  and  $\beta \geq 2\alpha$ .

**THEOREM 2.2.** *Let  $0 < \alpha \leq 1/2$ ,  $\beta > 2\alpha$ . Let  $\phi \in L^\infty(\Omega)$  be a real-valued function and let  $\psi \in L^\infty(\Omega)$ .*

*Suppose that  $m_1 = \inf_{x \in \Omega} |\phi(x)| > 0$  and  $m_2 = \inf_{x \in \Omega} |\psi(x)| > 0$ . Then we have*

$$|I_{\alpha,\beta}^{1,\Omega}(\lambda)| \leq K_1 |\Omega| \|\psi\|_{L^\infty(\Omega)} \frac{1 + \lambda\|\phi\|_{L^\infty(\Omega)}}{1 + k_1\lambda^2 m_1^2}, \quad \lambda \geq 0, \tag{2.4}$$

where  $|\Omega|$  is a Lebesgue measure of  $\Omega$ ,

$$K_1 = \max \left\{ \frac{1}{\Gamma(\beta)}, \frac{1}{\Gamma(\alpha + \beta)} \right\}$$

and

$$k_1 = \min \left\{ \frac{\Gamma(\beta)}{\Gamma(2\alpha + \beta)}, \frac{\Gamma(\alpha + \beta)}{\Gamma(3\alpha + \beta)} \right\}.$$

We also have

$$|I_{\alpha,\beta}^{1,\Omega}(\lambda)| \geq \frac{m_2|\Omega|}{\Gamma(\alpha + \beta)} \frac{\lambda m_1}{1 + \frac{\Gamma(\beta-\alpha)}{\Gamma(\alpha+\beta)}\lambda^2\|\phi\|_{L^\infty(\Omega)}^2}, \quad \lambda \geq 0. \tag{2.5}$$

If  $m_1 = \inf_{x \in \Omega} |\phi(x)| = 0$  and  $m_2 = \inf_{x \in \Omega} |\psi(x)| > 0$ . Then we have

$$|I_{\alpha,\beta}^{1,\Omega}(\lambda)| \geq \frac{m_2|\Omega|}{\Gamma(\beta)} \frac{1}{1 + \frac{\Gamma(\beta-2\alpha)}{\Gamma(\beta)}\lambda^2\|\phi\|_{L^\infty(\Omega)}^2}, \quad \lambda \geq 0. \tag{2.6}$$

**P r o o f.** First, we prove estimate (2.4). Let  $\phi \in L^\infty(\Omega)$  and  $\psi \in L^\infty(\Omega)$ . Then

$$|I_{\alpha,\beta}^{1,\Omega}(\lambda)| \leq \int_{\Omega} |E_{\alpha,\beta}(i\lambda\phi(x))| |\psi(x)| dx.$$

Using formula (1.13) and estimate (1.9) we have that

$$|E_{\alpha,\beta}(i\lambda\phi(x))| \leq E_{2\alpha,\beta}(-\lambda^2\phi^2(x)) + \lambda|\phi(x)|E_{2\alpha,\alpha+\beta}(-\lambda^2\phi^2(x)). \tag{2.7}$$

The properties of functions  $\phi$  and  $\psi$ , and the use of estimate (1.12) lead to the result

$$\begin{aligned} |I_{\alpha,\beta}^{1,\Omega}(\lambda)| &\leq \int_{\Omega} |E_{\alpha,\beta}(i\lambda\phi(x))| |\psi(x)| dx \\ &\leq \int_{\Omega} (E_{2\alpha,\beta}(-\lambda^2\phi^2(x)) + \lambda|\phi(x)|E_{2\alpha,\alpha+\beta}(-\lambda^2\phi^2(x))) |\psi(x)| dx \\ &\leq \frac{\|\psi\|_{L^\infty(\Omega)}}{\Gamma(\beta)} \int_{\Omega} \frac{1}{1 + \frac{\Gamma(\beta)}{\Gamma(2\alpha+\beta)}\lambda^2\phi^2(x)} dx \\ &\quad + \frac{\|\psi\|_{L^\infty(\Omega)}}{\Gamma(\alpha + \beta)} \int_{\Omega} \frac{\lambda|\phi(x)|}{1 + \frac{\Gamma(\alpha+\beta)}{\Gamma(3\alpha+\beta)}\lambda^2\phi^2(x)} dx \\ &\leq \|\psi\|_{L^\infty} \max \left\{ \frac{1}{\Gamma(\beta)}, \frac{1}{\Gamma(\alpha + \beta)} \right\} \times \\ &\quad \times \left[ \int_{\Omega} \frac{1}{1 + \frac{\Gamma(\beta)}{\Gamma(2\alpha+\beta)}\lambda^2\phi^2(x)} dx + \int_{\Omega} \frac{\lambda|\phi(x)|}{1 + \frac{\Gamma(\alpha+\beta)}{\Gamma(3\alpha+\beta)}\lambda^2\phi^2(x)} dx \right] \end{aligned}$$

$$\begin{aligned} &\leq |\Omega| \|\psi\|_{L^\infty} \max \left\{ \frac{1}{\Gamma(\beta)}, \frac{1}{\Gamma(\alpha+\beta)} \right\} \left[ \frac{1}{1 + \frac{\Gamma(\beta)}{\Gamma(2\alpha+\beta)} \lambda^2 m_1^2} + \frac{\lambda \|\phi\|_{L^\infty}}{1 + \frac{\Gamma(\alpha+\beta)}{\Gamma(3\alpha+\beta)} \lambda^2 m_1^2} \right] \\ &\leq |\Omega| \|\psi\|_{L^\infty} \max \left\{ \frac{1}{\Gamma(\beta)}, \frac{1}{\Gamma(\alpha+\beta)} \right\} \frac{1 + \lambda \|\phi\|_{L^\infty}}{1 + \min \left\{ \frac{\Gamma(\beta)}{\Gamma(2\alpha+\beta)}, \frac{\Gamma(\alpha+\beta)}{\Gamma(3\alpha+\beta)} \right\} \lambda^2 m_1^2}. \end{aligned}$$

Now, we prove the lower bound estimate (2.6). As  $|x + iy| \geq |x|$ , we obtain

$$|E_{\alpha,\beta}(i\lambda\phi(x))| \geq E_{2\alpha,\beta}(-\lambda^2\phi^2(x)). \tag{2.8}$$

Hence, by estimate (1.12), it follows that

$$\begin{aligned} |I_{\alpha,\beta}^{1,\Omega}(\lambda)| &\geq m_2 \int_{\Omega} E_{2\alpha,\beta}(-\lambda^2\phi^2(x)) \, dx \\ &\geq \frac{m_2}{\Gamma(\beta)} \int_{\Omega} \frac{1}{1 + \frac{\Gamma(\beta-2\alpha)}{\Gamma(\beta)} \lambda^2 \phi^2(x)} \, dx \\ &\geq \frac{m_2 |\Omega|}{\Gamma(\beta)} \frac{1}{1 + \frac{\Gamma(\beta-2\alpha)}{\Gamma(\beta)} \lambda^2 \|\phi\|_{L^\infty(\Omega)}^2}. \end{aligned}$$

Now, we prove the lower bound estimate (2.5). As  $|x + iy| \geq |y|$ , we obtain

$$|E_{\alpha,\beta}(i\lambda\phi(x))| \geq \lambda |\phi(x)| E_{2\alpha,\alpha+\beta}(-\lambda^2\phi^2(x)). \tag{2.9}$$

Hence, by estimate (1.12), it follows

$$\begin{aligned} |I_{\alpha,\beta}^{1,\Omega}(\lambda)| &\geq m_2 \lambda \int_{\Omega} |\phi(x)| E_{2\alpha,\alpha+\beta}(-\lambda^2\phi^2(x)) \, dx \\ &\geq \frac{m_2 \lambda}{\Gamma(\alpha + \beta)} \int_{\Omega} \frac{|\phi(x)|}{1 + \frac{\Gamma(\beta-\alpha)}{\Gamma(\alpha+\beta)} \lambda^2 \phi^2(x)} \, dx \\ &\geq \frac{m_2 |\Omega|}{\Gamma(\alpha + \beta)} \frac{\lambda m_1}{1 + \frac{\Gamma(\beta-\alpha)}{\Gamma(\alpha+\beta)} \lambda^2 \|\phi\|_{L^\infty(\Omega)}^2}. \end{aligned}$$

The proof is complete. □

**THEOREM 2.3.** *Let  $0 < \alpha < 1/2$ ,  $\beta = 2\alpha$ . Let  $\phi \in L^\infty(\Omega)$  be a real-valued function and let  $\psi \in L^\infty(\Omega)$ .*

*Let  $m_1 = \inf_{x \in \Omega} |\phi(x)| > 0$  and  $m_2 = \inf_{x \in \Omega} |\psi(x)| > 0$ , then we have the following estimates*

$$|I_{\alpha,2\alpha}^{1,\Omega}(\lambda)| \leq K |\Omega| \|\psi\|_{L^\infty(\Omega)} \times \frac{1 + \lambda \|\phi\|_{L^\infty(\Omega)} \left(1 + \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}} \lambda^2 \|\phi\|_{L^\infty(\Omega)}^2\right)}{\left(1 + \min\left\{\frac{\Gamma(3\alpha)}{\Gamma(5\alpha)}, \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}}\right\} \lambda^2 m_1^2\right)^2}, \quad \lambda \geq 0, \quad (2.10)$$

and

$$|I_{\alpha,2\alpha}^{1,\Omega}(\lambda)| \geq \frac{m_2 |\Omega|}{\Gamma(3\alpha)} \frac{\lambda m_1}{1 + \frac{\Gamma(\alpha)}{\Gamma(3\alpha)} \lambda^2 \|\phi\|_{L^\infty(\Omega)}^2}, \quad \lambda \geq 0, \quad (2.11)$$

where  $K = \max\left\{\frac{1}{\Gamma(2\alpha)}, \frac{1}{\Gamma(3\alpha)}\right\}$ .

If  $m_1 = \inf_{x \in \Omega} |\phi(x)| = 0$  and  $m_2 = \inf_{x \in \Omega} |\psi(x)| > 0$ , then we have

$$|I_{\alpha,2\alpha}^{1,\Omega}(\lambda)| \geq \frac{m_2 |\Omega|}{\Gamma(2\alpha)} \frac{1}{\left(1 + \sqrt{\frac{\Gamma(1-2\alpha)}{\Gamma(1+2\alpha)}} \lambda^2 \|\phi\|_{L^\infty(\Omega)}^2\right)^2}, \quad \lambda \geq 0. \quad (2.12)$$

*P r o o f.* First, we prove the estimate (2.10). Let  $\phi \in L^\infty(\Omega)$  and  $\psi \in L^\infty(\Omega)$ . Then

$$|I_{\alpha,2\alpha}^{1,\Omega}(\lambda)| \leq \int_{\Omega} |E_{\alpha,2\alpha}(i\lambda\phi(x))| |\psi(x)| dx.$$

Using formula (1.13), we have that

$$|E_{\alpha,2\alpha}(i\lambda\phi(x))| \leq E_{2\alpha,2\alpha}(-\lambda^2\phi^2(x)) + \lambda|\phi(x)|E_{2\alpha,3\alpha}(-\lambda^2\phi^2(x)). \quad (2.13)$$

The properties of functions  $\phi$  and  $\psi$ , using estimates (1.11) and (1.12), imply

$$\begin{aligned} |I_{\alpha,2\alpha}^{1,\Omega}(\lambda)| &\leq \int_{\Omega} |E_{\alpha,2\alpha}(i\lambda\phi(x))| |\psi(x)| dx \\ &\leq \int_{\Omega} (E_{2\alpha,2\alpha}(-\lambda^2\phi^2(x)) + \lambda|\phi(x)|E_{2\alpha,3\alpha}(-\lambda^2\phi^2(x))) |\psi(x)| dx \\ &\leq \frac{\|\psi\|_{L^\infty(\Omega)}}{\Gamma(2\alpha)} \int_{\Omega} \frac{1}{\left(1 + \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}} \lambda^2 \phi^2(x)\right)^2} dx \\ &\quad + \frac{\|\psi\|_{L^\infty(\Omega)}}{\Gamma(3\alpha)} \int_{\Omega} \frac{\lambda|\phi(x)|}{1 + \frac{\Gamma(3\alpha)}{\Gamma(5\alpha)} \lambda^2 \phi^2(x)} dx \end{aligned}$$



$$\begin{aligned}
 &\leq \|\psi\|_{L^\infty(\Omega)} K \int_{\Omega} \frac{1}{\left(1 + \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}} \lambda^2 \phi^2(x)\right)^2} dx \\
 &+ \|\psi\|_{L^\infty(\Omega)} K \int_{\Omega} \frac{\lambda|\phi(x)|}{1 + \frac{\Gamma(3\alpha)}{\Gamma(5\alpha)} \lambda^2 \phi^2(x)} dx \\
 &\leq \frac{K|\Omega| \|\psi\|_{L^\infty(\Omega)}}{\left(1 + \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}} \lambda^2 m_1^2\right)^2} \\
 &+ K|\Omega| \|\psi\|_{L^\infty(\Omega)} \frac{\lambda \|\phi\|_{L^\infty(\Omega)} \left(1 + \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}} \lambda^2 \|\phi\|_{L^\infty(\Omega)}^2\right)}{\left(1 + \frac{\Gamma(3\alpha)}{\Gamma(5\alpha)} \lambda^2 m_1^2\right) \left(1 + \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}} \lambda^2 m_1^2\right)} \\
 &\leq K|\Omega| \|\psi\|_{L^\infty(\Omega)} \frac{1 + \lambda \|\phi\|_{L^\infty(\Omega)} \left(1 + \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}} \lambda^2 \|\phi\|_{L^\infty(\Omega)}^2\right)}{\left(1 + \min\left\{\frac{\Gamma(3\alpha)}{\Gamma(5\alpha)}, \sqrt{\frac{\Gamma(1+2\alpha)}{\Gamma(1+4\alpha)}}\right\} \lambda^2 m_1^2\right)^2},
 \end{aligned}$$

where  $K = \max\left\{\frac{1}{\Gamma(2\alpha)}, \frac{1}{\Gamma(3\alpha)}\right\}$ .

Now, we prove estimate (2.12). We have

$$|E_{\alpha,2\alpha}(i\lambda\phi(x))| \geq E_{2\alpha,2\alpha}(-\lambda^2\phi^2(x)). \tag{2.14}$$

Hence, by estimate (1.11), it follows that

$$\begin{aligned}
 |I_{\alpha,2\alpha}^{1,\Omega}(\lambda)| &\geq m_2 \int_{\Omega} E_{2\alpha,2\alpha}(-\lambda^2\phi^2(x)) dx \\
 &\geq \frac{m_2}{\Gamma(2\alpha)} \int_{\Omega} \frac{1}{\left(1 + \sqrt{\frac{\Gamma(1-2\alpha)}{\Gamma(1+2\alpha)}} \lambda^2 \phi^2(x)\right)^2} dx \\
 &\geq \frac{m_2|\Omega|}{\Gamma(2\alpha)} \frac{1}{\left(1 + \sqrt{\frac{\Gamma(1-2\alpha)}{\Gamma(1+2\alpha)}} \lambda^2 \|\phi\|_{L^\infty(\Omega)}^2\right)^2}.
 \end{aligned}$$

The estimate (2.11) can be proved similarly as estimate (2.5) by replacing  $\beta = 2\alpha$ . In fact,

$$\begin{aligned}
 |I_{\alpha,2\alpha}^{1,\Omega}(\lambda)| &\geq m_2 \lambda \int_{\Omega} |\phi(x)| E_{2\alpha,3\alpha}(-\lambda^2\phi^2(x)) dx \\
 &\geq \frac{m_2 \lambda}{\Gamma(3\alpha)} \int_{\Omega} \frac{|\phi(x)|}{1 + \frac{\Gamma(\alpha)}{\Gamma(3\alpha)} \lambda^2 \phi^2(x)} dx \\
 &\geq \frac{m_2|\Omega|}{\Gamma(3\alpha)} \frac{\lambda m_1}{1 + \frac{\Gamma(\alpha)}{\Gamma(3\alpha)} \lambda^2 \|\phi\|_{L^\infty(\Omega)}^2}.
 \end{aligned}$$

The proof is complete. □

**3. Van der Corput-type estimates for the integral  $I_{\alpha,\beta}^2(\lambda)$ .**

In this section we consider  $I_{\alpha,\beta}^2$  defined by (1.4), that is,

$$I_{\alpha,\beta}^2(\lambda) = \int_{\mathbb{R}^N} E_{\alpha,\beta}(i^\alpha \lambda \phi(x)) \psi(x) dx.$$

As for small  $\lambda$  the integral (1.4) is just bounded, we consider the case  $\lambda \geq 1$ .

**THEOREM 3.1.** *Let  $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$  be a measurable function and let  $\psi \in L^1(\mathbb{R}^N)$ . Suppose that  $0 < \alpha \leq 2$ ,  $\beta > 1$ , and  $m = \text{ess inf}_{x \in \mathbb{R}^N} |\phi(x)| > 0$ , then*

**(i):** for  $0 < \alpha < 2$  and  $\beta \geq \alpha + 1$  we have

$$|I_{\alpha,\beta}^2(\lambda)| \leq \frac{M_1}{1 + \lambda m} \|\psi\|_{L^1(\mathbb{R}^N)}, \quad \lambda \geq 1, \tag{3.1}$$

where  $M_1$  does not depend on  $\phi$ ,  $\psi$  and  $\lambda$ ;

**(ii):** for  $0 < \alpha < 2$  and  $1 < \beta < \alpha + 1$  we have

$$|I_{\alpha,\beta}^2(\lambda)| \leq \frac{M_2}{(1 + \lambda m)^{\frac{\beta-1}{\alpha}}} \|\psi\|_{L^1(\mathbb{R}^N)}, \quad \lambda \geq 1, \tag{3.2}$$

where  $M_2$  does not depend on  $\phi$ ,  $\psi$  and  $\lambda$ ;

**(iii):** for  $\alpha = 2$  and  $1 < \beta < 3$  we have

$$|I_{2,\beta}^2(\lambda)| \leq \frac{M_3}{(1 + \lambda m)^{\frac{\beta-1}{2}}} \|\psi\|_{L^1(\mathbb{R}^N)}, \quad \lambda \geq 1, \tag{3.3}$$

where  $M_3$  does not depend on  $\phi$ ,  $\psi$  and  $\lambda$ .

**P r o o f.** Let  $\phi : \mathbb{R}^N \rightarrow \mathbb{R}$  be a measurable function and  $\psi \in L^1(\mathbb{R}^N)$ . As

$$|\arg(i^\alpha \lambda \phi(x))| = \frac{\pi\alpha}{2},$$

and

$$\text{Re}(i\lambda^{1/\alpha}(\phi(x))^{1/\alpha}) = 0,$$

then using estimate (1.9) we have that

$$\begin{aligned} |I_{\alpha,\beta}^2(\lambda)| &\leq \int_{\mathbb{R}^N} |E_{\alpha,\beta}(i^\alpha \lambda \phi(x))| |\psi(x)| dx \\ &\leq C_1 \int_{\mathbb{R}^N} (1 + \lambda|\phi(x)|)^{(1-\beta)/\alpha} |\psi(x)| dx \\ &\quad + C_2 \int_{\mathbb{R}^N} \frac{|\psi(x)|}{1 + \lambda|\phi(x)|} dx. \end{aligned}$$

As  $\phi$  and  $\psi$  do not depend on  $\lambda$ , and  $m = \text{ess inf}_{x \in \mathbb{R}^N} |\phi(x)| > 0$ , then for  $\beta \geq \alpha + 1$  we have

$$\begin{aligned} |I_{\alpha,\beta}^2(\lambda)| &\leq \int_{\mathbb{R}^N} |E_{\alpha,\beta}(i^\alpha \lambda \phi(x))| |\psi(x)| dx \\ &\leq \max\{C_1, C_2\} \int_{\mathbb{R}^N} \frac{|\psi(x)|}{1 + \lambda|\phi(x)|} dx \\ &\leq \frac{M_1}{1 + \lambda m} \|\psi\|_{L^1(\mathbb{R}^N)}. \end{aligned}$$

In the case  $1 < \beta < \alpha + 1$  we have that

$$\begin{aligned} |I_{\alpha,\beta}^2(\lambda)| &\leq \int_{\mathbb{R}^N} |E_{\alpha,\beta}(i^\alpha \lambda \phi(x))| |\psi(x)| dx \\ &\leq \max\{C_1, C_2\} \int_{\mathbb{R}^N} \frac{|\psi(x)|}{(1 + \lambda|\phi(x)|)^{\frac{\beta-1}{\alpha}}} dx \\ &\leq \frac{M_2}{(1 + \lambda m)^{\frac{\beta-1}{\alpha}}} \|\psi\|_{L^1(\mathbb{R}^N)}. \end{aligned}$$

The cases (i) and (ii) are proved.

Now we will prove the case (iii). Applying the asymptotic estimate (see [8, page 43])

$$\begin{aligned} E_{2,\beta}(z) &= \frac{1}{2} z^{(1-\beta)/2} \left( e^{\sqrt{z}} + e^{-\sqrt{z} - \pi i(1-\beta)\text{sign}(\arg z)} \right) \\ &\quad - \sum_{k=1}^N \frac{z^{-k}}{\Gamma(\beta - 2k)} + O\left(\frac{1}{z^{N+1}}\right), \quad |z| \rightarrow \infty, \quad |\arg(z)| \leq \pi, \end{aligned}$$

we have

$$\begin{aligned} |I_{2,\beta}^2(\lambda)| &\leq \int_{\mathbb{R}^N} |E_{2,\beta}(-\lambda \phi(x))| |\psi(x)| dx \\ &\leq M_3 \lambda^{(1-\beta)/2} \int_{\mathbb{R}^N} |\phi(x)|^{(1-\beta)/2} |\psi(x)| dx \\ &\leq M_3 m^{(1-\beta)/2} \lambda^{(1-\beta)/2} \int_{\mathbb{R}^N} |\psi(x)| dx \\ &\leq \frac{M_3 \|\psi\|_{L^1(\mathbb{R}^N)}}{(1 + m\lambda)^{(\beta-1)/2}}. \end{aligned}$$

Here  $M_3$  is a constant that does not depend on  $\lambda$ . The proof is complete.  $\square$

### 4. Applications

In this section we give some applications of multidimensional van der Corput-type estimates involving Mittag-Leffler function.

#### 4.1. Decay estimates for the time-fractional Schrödinger equation.

Consider the time-fractional Schrödinger equation

$$i\mathcal{D}_{0+,t}^\alpha u(t, x) - \Delta_x u(t, x) + \mu u(t, x) = 0, \quad t > 0, \quad x \in \mathbb{R}^N, \quad (4.1)$$

with Cauchy data

$$u(0, x) = \psi(x), \quad x \in \mathbb{R}^N, \quad (4.2)$$

where  $\lambda, \mu > 0$  and

$$\mathcal{D}_{0+,t}^\alpha u(t, x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u_s(s, x) ds$$

is the Caputo fractional derivative of order  $0 < \alpha < 1$ .

By using the direct and inverse Fourier and Laplace transforms, we can obtain a solution to problem (4.1)-(4.2) in the form

$$u(t, x) = \int_{\mathbb{R}^N} e^{ix\xi} E_{\alpha,1}(i(|\xi|^2 + \mu)t^\alpha) \hat{\psi}(\xi) d\xi, \quad (4.3)$$

where  $\hat{\psi}(\xi) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-iy\xi} \psi(y) dy$ . Suppose that  $\psi \in L^1(\mathbb{R}^N)$  and  $\hat{\psi} \in L^1(\mathbb{R}^N)$ . As

$$\inf_{\xi \in \mathbb{R}^N} (|\xi|^2 + \mu) > 0,$$

then using Theorem 2.1 we obtain the dispersive estimate

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R}^N)} \leq C(1+t)^{-\alpha} \|\hat{\psi}\|_{L^1(\mathbb{R}^N)}, \quad t \geq 0.$$

#### 4.2. Decay estimates for the time-fractional Klein-Gordon equation.

Consider the time-fractional Klein-Gordon equation

$$D_{0+,t}^\alpha u(t, x) + i^\alpha \Delta_x u(t, x) - i^\alpha \mu u(t, x) = 0, \quad t > 0, \quad x \in \mathbb{R}^N, \quad (4.4)$$

with initial data

$$I_{0+,t}^{2-\alpha} u(0, x) = 0, \quad x \in \mathbb{R}^N, \quad (4.5)$$

$$\partial_t I_{0+,t}^{2-\alpha} u(0, x) = \psi(x), \quad x \in \mathbb{R}^N, \quad (4.6)$$

where  $\mu > 0$ , and

$$I_{0+,t}^\alpha u(t, x) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s, x) ds$$

and

$$D_{0+,t}^\alpha u(t, x) = \frac{1}{\Gamma(2 - \alpha)} \partial_t^2 \int_0^t (t - s)^{1-\alpha} u(s, x) ds$$

is the Riemann-Liouville fractional integral and derivative of order  $1 < \alpha \leq 2$ .

If  $\alpha = 2$ , then from (4.4) we obtain a classical Klein-Gordon equation.

Applying the Fourier transform  $\mathcal{F}$  to problem (4.4)-(4.6) with respect to space variable  $x$  yields

$$D_{0+,t}^\alpha \hat{u}(t, \xi) - i^\alpha (|\xi|^2 + \mu) \hat{u}(t, \xi) = 0, \quad t > 0, \quad \xi \in \mathbb{R}^N, \tag{4.7}$$

$$I_{0+,t}^{2-\alpha} \hat{u}(0, \xi) = 0, \quad \xi \in \mathbb{R}^N, \tag{4.8}$$

$$\partial_t I_{0+,t}^{2-\alpha} \hat{u}_t(0, \xi) = \hat{\psi}(\xi), \quad \xi \in \mathbb{R}^N, \tag{4.9}$$

due to  $\mathcal{F} \{ \Delta_x u(t, x) \} = -|\xi|^2 \hat{u}(t, \xi)$ . The general solution of equation (4.7) can be represented as

$$\hat{u}(t, \xi) = C_1(\xi) t^{\alpha-1} E_{\alpha,\alpha} (i^\alpha (|\xi|^2 + \mu) t^\alpha) + C_2(\xi) t^{\alpha-2} E_{\alpha,\alpha-1} (i^\alpha (|\xi|^2 + \mu) t^\alpha),$$

where  $C_1(\xi)$  and  $C_2(\xi)$  are unknown coefficients. Then by initial conditions (4.8)-(4.9) we have

$$\hat{u}(t, \xi) = \hat{\psi}(\xi) t^{\alpha-1} E_{\alpha,\alpha} (i^\alpha (|\xi|^2 + \mu) t^\alpha).$$

By applying the inverse Fourier transform  $\mathcal{F}^{-1}$  we have

$$u(t, x) = \int_{\mathbb{R}^N} e^{ix\xi} t^{\alpha-1} E_{\alpha,\alpha} (i^\alpha (|\xi|^2 + \mu) t^\alpha) \hat{\psi}(\xi) d\xi, \tag{4.10}$$

where  $\hat{\psi}(\xi) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-iy\xi} \psi(y) dy$ .

Suppose that  $\psi \in L^1(\mathbb{R}^N)$  and  $\hat{\psi} \in L^1(\mathbb{R}^N)$ . As  $\inf_{\xi \in \mathbb{R}^N} (|\xi|^2 + \mu) > 0$ , then using Theorem 3.1 (ii) we obtain the dispersive estimate

$$\begin{aligned} \|u(t, \cdot)\|_{L^\infty(\mathbb{R}^N)} &\leq C t^{\alpha-1} (1 + t^\alpha)^{\frac{1-\alpha}{\alpha}} \|\hat{\psi}\|_{L^1(\mathbb{R}^N)} \\ &\leq C t^{\alpha-1} (1 + t)^{1-\alpha} \|\hat{\psi}\|_{L^1(\mathbb{R}^N)}, \quad t > 0. \end{aligned}$$

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