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Abstract

We consider assignment problems where heterogeneous indivisible goods are to be assigned to individuals so that each individual receives at most one good. Individuals have single-peaked preferences over the goods. In this setting, first we show that there is no strategy-proof, non-bossy, Pareto efficient, and strongly pairwise reallocation-proof assignment rule on a minimally rich single-peaked domain when there are at least three individuals and at least three objects in the market. Next, we characterize all strategy-proof, Pareto efficient, top-envy-proof, non-bossy, and pairwise reallocation-proof assignment rules on a minimally rich single-peaked domain as hierarchical exchange rules. We additionally show that strategy-proofness and non-bossiness together are equivalent to group strategy-proofness on a minimally rich single-peaked domain, and every hierarchical exchange rule satisfies group-wise reallocation-proofness on a minimally rich single-peaked domain.

Keywords: Assignment problem; Single-peaked preferences; Strategy-proofness; Pareto efficiency; Non-bossiness; Top-envy-proofness; Strong reallocation-proofness; Pairwise/group-wise reallocation-proofness
JEL Classification: C78; D82

1 Introduction

We consider the well-known assignment problem where heterogeneous indivisible goods are to be assigned to individuals so that each individual receives at most one good. Such problems arise when, for instance, the Government wants to assign houses to the citizens, or hospitals to doctors, or a manager wants to allocate offices to employees, or tasks to workers, or a professor wants to assign projects to students. Individuals are asked to report their preferences over the goods and the designer decides the allocation based on these reports. We analyze the structure of such decision process satisfying some desirable properties such as (group) strategy-proofness, efficiency, non-bossiness, (top-)envy-proofness, and (pairwise/group-wise) reallocation-proofness.

(Group) strategy-proofness ensures that a (a group of) dishonest individual(s) cannot improve her (their) assignment(s) by misreporting her (their) preference(s).[1] Efficiency says that the assignments cannot be improved in the sense of Pareto (that is, everyone is weakly better off and someone is strictly better off). Non-bossiness says that a person cannot change the assignment of any other person without changing her own assignment. Envy-proofness says that if an individual is envious at another individual (that is, if she strictly prefers the assignment of the individual to her own assignment), then she cannot harm the individual by misreporting her preference. Top-envy-proofness, in a sense, can be viewed as envy-proofness with respect to the top-ranked object of the envious individual. Pairwise/group-wise reallocation-proofness rules out the possibility of an obvious case of manipulation where a pair/group of individuals misreport their preferences and become better off by redistributing the objects they obtain at the misreported profile.

Svensson (1999) shows that the set of strategy-proof, non-bossy, and neutral assignment rules on the unrestricted domain is the set of serial dictatorships, if every individual is assumed to be assigned an object.[2,3] Pápai (2000) characterizes strategy-proof, Pareto efficient, non-bossy, and reallocation-proof assignment rules on the unrestricted domain as *hierarchical exchange rules*. These rules can be regarded as generalizations of Gale's well-known top trading cycle (TTC) procedure.[4] Pycia and Ünver (2017) characterizes strategy-proof, Pareto efficient, and non-bossy assignment rules on the unrestricted domain as *trading cycles rules*.[5]

[1]A group of individuals improve their assignments if each member in it is weakly better-off and some member is strictly better-off.

[2]An assignment rule is neutral if its outcomes do not depend on the identities of the objects.

[3]Whenever it is clear from the context, we use the term "domain" to refer to a set of preferences or a set of preference profiles.

[4]Top trading cycle (TTC) is due to David Gale and discussed in Shapley and Scarf (1974).

[5]Ergin (2000) shows that an assignment rule satisfies Pareto efficiency, neutrality, and consistency if and only if it is a simple serial dictatorship rule (he uses somewhat weaker properties to show his result). Ehlers and Klaus (2006) characterize all Pareto efficient, strategy-proof, and reallocation-consistent assignment rules as *efficient priority rules*. Later, Ehlers and Klaus (2007) and Velez (2014) characterize a slightly larger class of assignment rules by weakening these characterizing properties. Karakaya et al. (2019) analyze TTC rules in the context of house allocation problem with existing tenants.

1.1 Our motivation and contribution

As we have mentioned, Svensson (1999), Pápai (2000), and Pycia and Ünver (2017) assume that the individuals can have arbitrary preferences over the goods. However, it is well-known that in many circumstances preferences of individuals are restricted in a particular way. *Single-peakedness* is known as one of the most common such restrictions. It arises when goods can be ordered based on certain criteria and individuals' preferences respect that ordering in the sense that as one moves away from her top-ranked (peak) good, her preference declines. For instance, in the problem of assigning hospitals (houses) to doctors (citizens), hospitals (houses) can be ordered based on their locations on a street and an individual may like to be assigned as close as possible to her favorite location, in the problem of assigning tasks to students, tasks can be ordered based on their technical difficulties and an individual may like to get a task that she is technically more comfortable with, etc. This motivates us to explore the structure of strategy-proof assignment rules when individuals have single-peaked preferences. Instead of focusing only on the maximal single-peaked domain, we do our analysis on a class of single-peaked domains that we call *minimally rich*. A single-peaked domain is minimally rich if it contains all left single-peaked and all right single-peaked preferences.[6]

There are two main results in this paper. The first one says that there is no strategy-proof, non-bossy, Pareto efficient, and strongly pairwise reallocation-proof assignment rule on a minimally rich single-peaked domain, when there are at least three individuals and three objects in the market (Theorem 5.1). The second result characterizes all strategy-proof, Pareto efficient, top-envy-proof, non-bossy, and pairwise reallocation-proof assignment rules on a minimally rich single-peaked domain as hierarchical exchange rules (Theorem 7.1). We additionally show that strategy-proofness and non-bossiness together are equivalent to group strategy-proofness on a minimally rich single-peaked domain (Proposition 4.1), and every hierarchical exchange rule satisfies group-wise reallocation-proofness on a minimally rich single-peaked domain (Proposition 7.1).[7]

Ours is not the first paper to deal with single-peaked domains, Damamme et al. (2015) and Bade (2019) consider single-peaked domains in the context of housing markets.[8] Damamme et al. (2015) provide an algorithm which is Pareto efficient on a single-peaked domain and Bade (2019) introduces the notion of the *crawler* algorithm and shows that it is Pareto efficient, strategy-proof, and individually rational on the maximal single-peaked domain.[9] To the best of our knowledge, the present paper is the first paper to analyze the structure of assignment rules on the single-peaked domains.

[6]A single-peaked preference is left (right) if every alternative on the left (right) of the peak is preferred to every alternative on the right (left) of the peak.

[7]This, in particular, implies that if we replace pairwise reallocation-proofness by its stronger version group-wise reallocation-proofness, the conclusion of Theorem 7.1 does not change.

[8]Shapley and Scarf (1974) introduce the housing market, a model (with equal number of individuals and objects) in which each individual owns a unique indivisible object (a house) initially.

[9]In fact, Bade (2019) shows that the crawler algorithm satisfies a stronger version of strategy-proofness called OSP-implementability.

1.2 Organization of the paper

The organization of this paper is as follows. In Section 2, we introduce basic notions and notations that we use throughout the paper. In Section 3, we define domains and discuss their properties. In Section 4, we define assignment rules and discuss their standard properties. We present an impossibility result (non-existence of strategy-proof, non-bossy, Pareto efficient, and strongly pairwise reallocation-proof assignment rules on a minimally rich single-peaked domain) in Section 5. Section 6 introduces the notion of hierarchical exchange rules. In Section 7, we present our main result: a characterization of all strategy-proof, Pareto efficient, top-envy-proof, non-bossy, and pairwise reallocation-proof assignment rules on a minimally rich single-peaked domain as hierarchical exchange rules, and in Section 8, we discuss the independence of these characterizing properties. All the proofs are collected in the Appendix.

2 Basic notions and notations

Let $N = \{1, \ldots, n\}$ be a (finite) set of individuals and A be a (non-empty and finite) set of objects. We denote the set of all strict linear orders over the elements of A by $\mathbb{L}(A)$.[10] An element P of $\mathbb{L}(A)$ is called a ***preference*** over A. For a preference $P \in \mathbb{L}(A)$, by R we denote the weak part of P, that is, for all $a, b \in A$, aRb if and only if $[aPb$ or $a = b]$. For $P \in \mathbb{L}(A)$ and non-empty $B \subseteq A$, we define $\tau(P, B) = a$ if and only if $a \in B$ and aPb for all $b \in B \setminus \{a\}$. For ease of presentation, we denote $\tau(P, A)$ by $\tau(P)$.

We introduce the notion of an ***allocation*** of a (non-empty) set of objects $B \subseteq A$ over a (non-empty) set of individuals $S \subseteq N$. If $|S| \leq |B|$, then an allocation assigns a unique object to each individual (some objects will be left unassigned if $|S| < |B|$). More formally, an allocation in this scenario is a one-to-one function $\mu : S \to B$. On the other hand, if $|B| < |S|$, then an allocation assigns each object to a unique individual (some individuals will not be assigned any object). More formally, an allocation in this scenario is an onto function $\mu : S \to B \cup \{\emptyset\}$ such that $\mu^{-1}(a)$ is singleton for all $a \in B$.

Here, $\mu(i) = a$ for some element a of A means individual i is assigned object a in allocation μ, and $\mu(i) = \emptyset$ means individual i is not assigned any object in μ. For $S \subseteq N$ and $B \subseteq A$ with $|S|, |B| \neq 0$, we denote by $\mathcal{M}(S, B)$ the set of all allocations of B over S. For ease of presentation, we denote $\mathcal{M}(N, A)$ by $\mathcal{M}$.

For ease of presentation we use the following convention throughout the paper: for a set $\{1, \ldots, g\}$ of integers, whenever we refer to the number $g + 1$, we mean 1. For instance, if we write $s_t \geq r_{t+1}$ for all $t = 1, \ldots, g$, we mean $s_1 \geq r_2, \ldots, s_{g-1} \geq r_g$, and $s_g \geq r_1$.

[10] A ***strict linear order*** is a semiconnex, asymmetric, and transitive binary relation.

3 Domains and their properties

Each $i \in N$ has a preference $P_i \in \mathbb{L}(A)$ over A. We denote by $\mathcal{P}_i \subseteq \mathbb{L}(A)$ the set of all admissible preferences of individual i, and by $P_N = (P_1, \ldots, P_n)$ a n-vector of all the individuals' preferences, which will be referred to as a ***preference profile***. By $\mathcal{P}_N = \prod_{i=1}^{n} \mathcal{P}_i$ we denote the set of all admissible preference profiles.

Given a preference profile P_N, we denote by (P_i', P_{-i}) the preference profile obtained from P_N by changing the preference of individual i from P_i to P_i' and keeping all other preferences unchanged.

Definition 3.1. A preference $P \in \mathbb{L}(A)$ is called ***single-peaked*** with respect to an ordering $\prec \in \mathbb{L}(A)$ if

(i) for all $a_j, a_k \in A$ with $a_j \prec a_k \prec \tau(P)$, we have $a_k P a_j$, and

(ii) for all $a_j, a_k \in A$ with $\tau(P) \prec a_j \prec a_k$, we have $a_j P a_k$.

A single-peaked preference (with respect to $\prec$) is called ***left (right) single-peaked*** if for all $a_j, a_k \in A$, $a_j \prec \tau(P) \prec a_k$ implies $a_j P a_k$ ($a_k P a_j$). A domain of preferences is called ***single-peaked*** (with respect to $\prec$) if each preference in it is single-peaked. A single-peaked domain of preferences is called ***minimally rich*** if it contains *all* left single-peaked and all right single-peaked preferences.

In the rest of the paper we assume that for all $i \in N$, $\mathcal{P}_i$ is a minimally rich single-peaked domain (with respect to some (fixed) ordering $\prec$).

4 Assignment rules and their properties

In this section, we introduce the notion of assignment rules and discuss a few properties of those.

Definition 4.1. A function $f : \mathcal{P}_N \to \mathcal{M}$ is called an ***assignment rule*** on $\mathcal{P}_N$.

For an assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ and a preference profile $P_N \in \mathcal{P}_N$, we denote by $f_i(P_N)$ the object that is assigned to individual i by the assignment rule f at P_N.

An allocation μ ***Pareto dominates*** another allocation ν at a preference profile P_N if $\mu(i) R_i \nu(i)$ for all $i \in N$ and $\mu(j) P_j \nu(j)$ for some $j \in N$.

Definition 4.2. An assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ is called *Pareto efficient at a preference profile* $P_N \in \mathcal{P}_N$ if there is no allocation that Pareto dominates $f(P_N)$ at P_N, and it is called ***Pareto efficient*** if it is Pareto efficient at every preference profile in $\mathcal{P}_N$.

Remark 4.1. If an assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ satisfies Pareto efficiency, then $\tau(P_j) \in \bigcup_{i \in N} \{f_i(P_N)\}$ for all $j \in N$. In other words, every object that is ranked at the top position by some individual must not be left unassigned. To see this, note that if $\tau(P_j) \notin \bigcup_{i \in N} \{f_i(P_N)\}$ for some $j \in N$, then the allocation μ defined by $\mu(j) = \tau(P_j)$ and $\mu(k) = f_k(P_N)$ for all $k \neq j$ Pareto dominates $f(P_N)$ at P_N.

Non-bossiness is a standard notion in matching theory which says that if an individual misreports her preference and her assignment does not change by the same, then the assignment of any other individual cannot change.[11]

Definition 4.3. An assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ is ***non-bossy*** if for all $P_N \in \mathcal{P}_N$, all $i \in N$, and all $\tilde{P}_i \in \mathcal{P}_i$, $f_i(P_N) = f_i(\tilde{P}_i, P_{-i})$ implies $f(P_N) = f(\tilde{P}_i, P_{-i})$.

Definition 4.4. An assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ is ***strategy-proof*** if for all $P_N \in \mathcal{P}_N$, all $i \in N$ and all $\tilde{P}_i \in \mathcal{P}_i$, we have $f_i(P_N) R_i f_i(\tilde{P}_i, P_{-i})$.

Note that if an assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ is not strategy-proof, then there exist $P_N \in \mathcal{P}_N$, $i \in N$ and $\tilde{P}_i \in \mathcal{P}_i$ such that $f_i(\tilde{P}_i, P_{-i}) P_i f_i(P_N)$. In such cases, we say that *the individual i manipulates f at P_N via $\tilde{P}_i$*.

Definition 4.5. An assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ is ***group strategy-proof*** if for all $P_N \in \mathcal{P}_N$, there do not exist a set of individuals $S \subseteq N$, and a preference profile $\tilde{P}_S$ of the individuals in S such that $f_i(\tilde{P}_S, P_{-S}) R_i f_i(P_N)$ for all $i \in S$ and $f_j(\tilde{P}_S, P_{-S}) P_j f_j(P_N)$ for some $j \in S$.

Proposition 4.1. *An assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ is group strategy-proof if and only if it is strategy-proof and non-bossy.*

The proof of this proposition is relegated to Appendix B.

5 An impossibility result

We introduce the notion of *strongly pairwise reallocation-proof* assignment rules. It says that no pair of individuals can misreport their preferences and be better off redistributing their assignments ex post.[12]

Definition 5.1. An assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ is *weakly manipulable through pairwise reallocation* if there exist $P_N \in \mathcal{P}_N$, distinct individuals $i, j \in N$, and $\tilde{P}_i \in \mathcal{P}_i$, $\tilde{P}_j \in \mathcal{P}_j$ such that

(i) $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) R_i f_i(P_N)$, and

(ii) $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j(P_N)$.

An assignment rule is ***strongly pairwise reallocation-proof*** if it is not weakly manipulable through pairwise reallocation.

Pápai (2000) mentions that there is no strategy-proof, non-bossy, Pareto efficient, and strongly pairwise reallocation-proof assignment rule on the unrestricted domain, where there are at least three individuals and three objects. Our next result says that the result holds if we restrict the domain to be minimally rich single-peaked.

[11] The concept of non-bossiness is due to Satterthwaite and Sonnenschein (1981).

[12] Here, we say a group of individuals is better-off if each member in it is weakly better-off and some member is strictly better-off.

Theorem 5.1. *Suppose $|N| \geq 3$ and $|A| \geq 3$. Then, there does not exist a strategy-proof, non-bossy, Pareto efficient, and strongly pairwise reallocation-proof assignment rule on $\mathcal{P}_N$.*

The proof of this theorem is relegated to Appendix C.

Since group strategy-proofness is equivalent to strategy-proofness and non-bossiness (see Proposition 4.1), we obtain the following corollary from Theorem 5.1.

Corollary 5.1. *Suppose $|N| \geq 3$ and $|A| \geq 3$. Then, there does not exist a group strategy-proof, Pareto efficient, and strongly pairwise reallocation-proof assignment rule on $\mathcal{P}_N$.*

6 Hierarchical exchange rules

We introduce the notion of *hierarchical exchange rules* in this section. These rules are introduced in Pápai (2000) and are well-known in the literature. We present a description of these rules for the sake of completeness. The description in Section 6 is taken from Mandal and Roy (2020).

We introduce some basic definitions from graph theory which we will use in defining hierarchical exchange rules. We denote a rooted (directed) tree by T. For a tree T, we denote its set of nodes by $V(T)$, set of all edges by $E(T)$, and root by $r(T)$. For a node $v \in V(T)$, we denote the set of all outgoing edges from v by $E^{out}(v)$. For an edge $e \in E(T)$, we denote its source node by $s(e)$. A path in a tree is a sequence of nodes such that every two consecutive nodes form an edge.

First we explain the notion of a *TTC procedure* with respect to a given endowments of the objects over the individuals. Suppose that each object is owned by exactly one individual. Note that an individual may own more than one objects. A directed graph is constructed in the following manner. The set of nodes is the same as the set of individuals. There is a directed edge from individual i to individual j if and only if individual j owns individual i's most preferred object. Note that such a graph will have exactly one outgoing edge from every node (though possibly many incoming edges to a node). Further, there may be an edge from a node to itself. It is clear that such a graph will always have a cycle. This cycle is called a *top trading cycle (TTC)*. After forming a TTC, the individuals in the TTC are assigned their most preferred objects.

6.1 Verbal description of hierarchical exchange rules

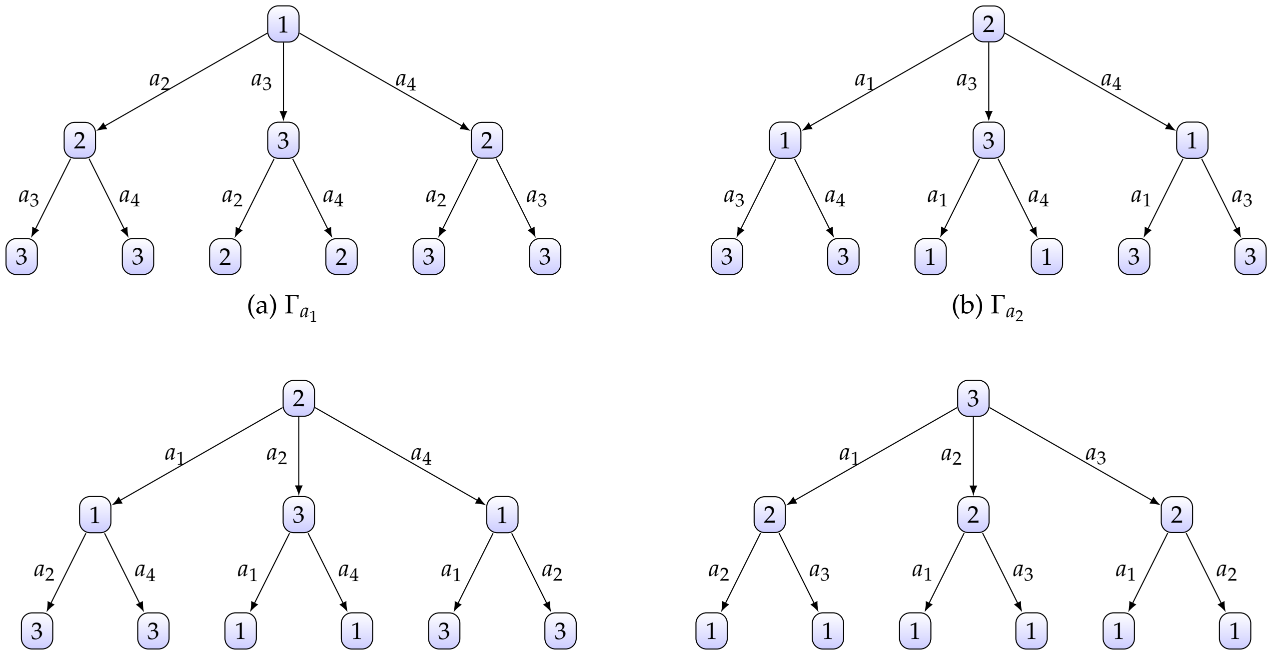
The following verbal description of hierarchical exchange rules is taken from Pápai (2000). The allocation obtained by a hierarchical exchange rule can be described by the following iterative procedure. Individuals have an initial individual "endowment" of objects such that each object is exactly one individual's endowment. It is important to note that some individuals may not be endowed with any objects. Now apply the TTC procedure to this market with individual endowments. Notice that individuals who don't have endowments cannot be part of a top trading cycle, since nobody points to them, and therefore they need not point. Given that multiple endowments are allowed, after the individuals in top trading cycles leave the market with their most preferred objects, unassigned objects in the initial endowment sets of individuals who received their assignment may be left behind. These objects are reassigned as endowments to individuals who are still in the market, that is, they are "inherited" by individuals who have not yet received their assignments. Furthermore, the objects in the initial endowment sets of individuals who are still in the market remain the individual endowments of these individuals. Thus, notice that each unassigned object is the endowment of exactly one individual who is still in the market. Now apply the TTC procedure to this reduced market with the new endowments.[13] Repeat this procedure until every individual has her assignment or all the objects are assigned. Since there exists at least one top trading cycle in every stage, this procedure leads to an allocation of the objects in a finite number of stages. In particular, there are at most as many stages as there are individuals or objects, whichever number is smaller, since in each stage at least one person receives her assignment. Furthermore, for any strict preferences of the individuals, the resulting allocation is unique.

A hierarchical exchange rule is determined by the initial endowments and the hierarchical endowment inheritance in later stages. While the initial endowment sets are given a priori, the hierarchical endowment inheritance may be endogenous. In particular, the inheritance of endowments may depend on the assignments made in earlier stages.

We explain how a hierarchical exchange rule works by means of the following example.

Example 6.1. Suppose $N = \{1, 2, 3\}$ and $A = \{a_1, a_2, a_3, a_4\}$ with a prior order $a_1 \prec a_2 \prec a_3 \prec a_4$. A hierarchical exchange rule is based on a collection of *inheritance trees*, one tree for each object. We will define this notion formally; for the time being we explain it through the current example. Figure 6.1 presents a collection of inheritance trees $\Gamma_{a_1}, \ldots, \Gamma_{a_4}$. To understand their structure, let us look at one of them, say Γ_{a_1}. Each maximal path of this tree has $\min\{|N|, |A|\} - 1 = 2$ edges. In any maximal path, each individual appears *at most* once at the nodes. For instance, individuals $1, 2$ and 3 appear at the nodes (in that order) in the left most path of Γ_{a_1}. Each object other than a_1 appears *exactly* once at the outgoing edges from the root (thus there are three edges from the root). For every subsequent node which is not the end node of a maximal path, each object other than a_1, that has *not* already appeared in the path from the root to that node, appears *exactly* once at the outgoing edges from that node. For instance, consider the node marked with 2 in the left most path of Γ_{a_1}. Since this node is not the end node of the left most maximal path and object a_2 has already appeared at the edge from the root to this node, objects a_3 and a_4 appear exactly once at the outgoing edges from this node. Thus, each object other than a_1 appears *at most* once at the edges in any maximal path of Γ_{a_1}. For instance, objects a_2 and a_3 appear at the edges (in that order) in the left most path of Γ_{a_1}. It can be verified that other inheritance trees have the same structure.

[13]In this TTC procedure, an individual i point to an individual j if j owns i's most preferred object among the remaining objects.



(a) Γ_{a_1} (b) Γ_{a_2} (c) Γ_{a_3} (d) Γ_{a_4}

Figure 6.1: Inheritance trees for Example 6.1

Consider the hierarchical exchange rule based on the collection of inheritance trees given in Figure 6.1. We explain how to compute the outcome of the rule at a given preference profile. Consider the preference profile P_N as given below:

P_1	P_2	P_3
a_2	a_1	a_1
a_1	a_2	a_2
a_3	a_3	a_3
a_4	a_4	a_4

Table 6.1: Preference profile for Example 6.1

The outcome is computed through a number of stages. In each stage, endowments of the individuals are determined by means of the inheritance trees and TTC procedure is performed with respect to the endowments.

Stage 1.

In Stage 1, the "owner" of an object a is the individual who is assigned to the root-node of the inheritance tree Γ_a. Thus, object a_1 is owned by individual 1, objects a_2 and a_3 are owned by individual 2, and object a_4 is owned by individual 3.

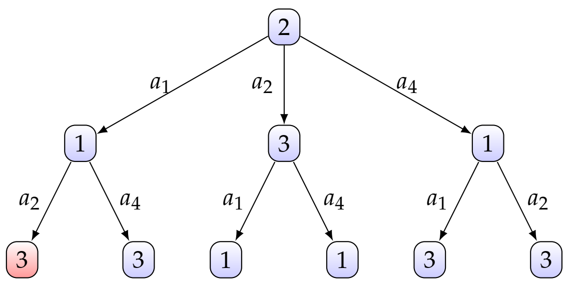
Once the endowments of the individuals are decided, TTC procedure is performed with respect to the endowments to decide the outcome of Stage 1. Individuals who are assigned some object in Stage 1 leave the market with the corresponding objects. It can be verified that for the preference profile P_N given in

Table 6.1, individual 1 gets object a_2 and individual 2 gets object a_1 at the outcome of TTC procedure in this stage. So, individuals 1 and 2 leave the market with objects a_2 and a_1, respectively.

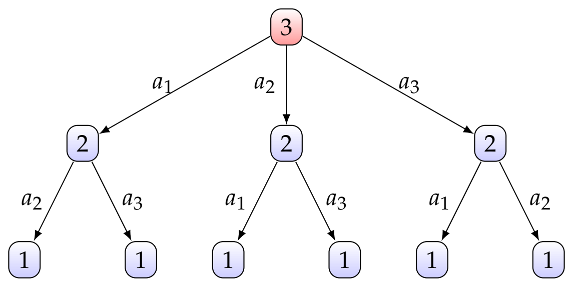
Stage 2.

As in Stage 1, the endowments of the individuals are decided first and then TTC procedure is performed with respect to the endowments. To decide the owner of a (remaining) object a, look at the root of the inheritance tree Γ_a. If the individual who appears there, say individual i, is remained in the market, then i becomes the owner of a. Otherwise, that is, if i is assigned an object in Stage 1, say b, then follow the edge from the root that is marked with b. If the individual appearing at the node following this edge, say j, is remained in the market, then j becomes the owner of a. Otherwise, that is, if j is assigned an object in Stage 1, say c, then follow the edge that is marked with c from the current node. As before, check whether the individual appearing at the end of this edge is remained in the market or not. Continue in this manner until an individual is found in the particular path who is not already assigned an object and decide that individual as the owner of a.

For the example at hand, the remaining market in Stage 2 consists of objects a_3 and a_4, and individual 3. Consider object a_3. Individual 2 appears at the root of Γ_{a_3}. Since individual 2 is assigned object a_1 in Stage 1, we follow the edge from the root that is marked with a_1 and come to individual 1. Since individual 1 is assigned object a_2, we follow the edge marked with a_2 from this node and come to individual 3. Since individual 3 is remained in the market, she becomes the owner of a_3. For object a_4, individual 3 appears at the root of Γ_{a_4} and she is remained in the market. So, individual 3 becomes the owner of a_4 in Stage 2. To emphasize the process of deciding the owner of an object, we have highlighted the node in red in the corresponding inheritance tree in Figure 6.2.



(a) Γ_{a_3}



(b) Γ_{a_4}

Figure 6.2: Stage 2

Once the endowments are decided for Stage 2, TTC procedure is performed with respect to the endowments to decide the outcome of this stage. As in Stage 1, individuals who are assigned some object in Stage 2 leave the market with the corresponding objects. It can be verified that for the current example, individual 3 gets object a_3 in this stage. So, individual 3 leave the market with objects a_3.

Stage 3 is followed on the remaining market in a similar way as Stage 2. For the current example, everybody is assigned some object by the end of Stage 2 and hence the algorithm stops in this stage. Thus, individuals 1, 2, and 3 get objects a_2, a_1, and a_3, respectively, at the outcome of the hierarchical exchange rule.

6.2 Formal definition of hierarchical exchange rules

In what follows, we present a formal description of hierarchical exchange rules.

6.2.1 Inheritance trees

For a rooted tree T, the *level* of a node $v \in V(T)$ is defined as the number of edges appearing in the (unique) path from $r(T)$ to v.

Definition 6.1. For an object $a \in A$, an ***inheritance tree for*** $a \in A$ is defined as a tuple $\Gamma_a = \langle T_a, \zeta_a^{NI}, \zeta_a^{EO} \rangle$, where

(i) T_a is a rooted tree with

  (a) $\max_{v \in V(T_a)} level(v) = \min\{|N|, |A|\} - 1$, and

  (b) $|E^{out}(v)| = |A| - level(v) - 1$ for all $v \in V(T_a)$ with $level(v) < \min\{|N|, |A|\} - 1$,

(ii) $\zeta_a^{NI} : V(T_a) \to N$ is a nodes-to-individuals function with $\zeta_a^{NI}(v) \neq \zeta_a^{NI}(\tilde{v})$ for all distinct $v, \tilde{v} \in V(T_a)$ that appear in same path, and

(iii) $\zeta_a^{EO} : E(T_a) \to A \setminus \{a\}$ is an edges-to-objects function with $\zeta_a^{EO}(e) \neq \zeta_a^{EO}(\tilde{e})$ for all distinct e, $\tilde{e} \in E(T_a)$ that appear in same path or have same source node (that is, $s(e) = s(\tilde{e})$).

In what follows, we provide two examples (for two different scenarios) of inheritance trees.

Example 6.2. Suppose $N = \{1, 2, 3\}$ and $A = \{a_1, a_2, a_3, a_4\}$ with a prior order $a_1 \prec a_2 \prec a_3 \prec a_4$.[14] Figure 6.3 presents an example of Γ_{a_1}.

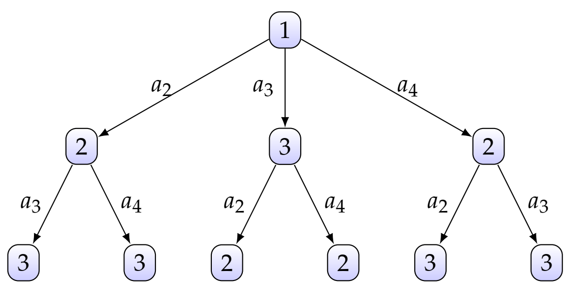


Figure 6.3: Example of Γ_{a_1}

Example 6.3. Suppose $N = \{1, 2, 3, 4\}$ and $A = \{a_1, a_2, a_3\}$ with a prior order $a_1 \prec a_2 \prec a_3$. Figure 6.4 presents another example of Γ_{a_1}.

[14]The ordering $\prec$ over A does not play any role in the definition of an inheritance tree.

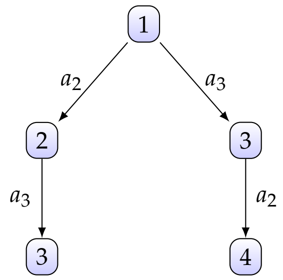


Figure 6.4: Example of Γ_{a_1}

6.2.2 Endowments

A hierarchical exchange rule works in several stages and in each stage, endowments of individuals are determined by using a (fixed) collection of inheritance trees.

Given a collection of inheritance trees $\Gamma = (\Gamma_a)_{a \in A}$, one for each object $a \in A$, we define a class of endowments $\mathcal{E}^\Gamma$ as follows:

(i) The ***initial endowment*** $\mathcal{E}_i^\Gamma(\emptyset)$ of individual i is given by

$$\mathcal{E}_i^\Gamma(\emptyset) = \{a \in A \mid \zeta_a^{NI}(r(T_a)) = i\}.$$

(ii) For all $S \subseteq N \setminus \{i\}$ and $B \subseteq A$ with $|S| = |B| \neq 0$, and all $\hat{\mu} \in \mathcal{M}(S, B)$, the ***endowment*** $\mathcal{E}_i^\Gamma(\hat{\mu})$ of individual i is given by

$$\begin{aligned}\mathcal{E}_i^\Gamma(\hat{\mu}) =& \{a \in A \setminus B \mid \zeta_a^{NI}(r(T_a)) = i, \text{ or} \\ & \text{there exists a path } (v_a^1, \ldots, v_a^{r_a}) \text{ from } r(T_a) \text{ to } v_a^{r_a} \text{ in } \Gamma_a \text{ such that } \zeta_a^{NI}(v_a^{r_a}) = i \\ & \text{and for all } s = 1, \ldots, r_a - 1, \text{ we have } \zeta_a^{NI}(v_a^s) \in S \text{ and } \hat{\mu}(\zeta_a^{NI}(v_a^s)) = \zeta_a^{EO}(v_a^s, v_a^{s+1})\}.\end{aligned}$$

6.2.3 Iterative procedure to compute the outcome of a hierarchical exchange rule

For a given collection of inheritance trees $\Gamma = (\Gamma_a)_{a \in A}$, the ***hierarchical exchange rule*** f^Γ ***associated with*** Γ is defined by an iterative procedure with at most $\min\{|N|, |A|\}$ number of stages. Consider a preference profile $P_N \in \mathcal{P}_N$.

Stage 1.

Hierarchical Endowments (Initial Endowments): For all $i \in N$, $E_1(i, P_N) = \mathcal{E}_i^\Gamma(\emptyset)$.

Top Choices: For all $i \in N$, $T_1(i, P_N) = \tau(P_i)$.

Trading Cycles: For all $i \in N$,

$$C_1(i, P_N) = \begin{cases} \{j_1, \ldots, j_g\} & \text{if there exist } j_1, \ldots, j_g \in N \text{ such that} \\ & \text{for all } s = 1, \ldots, g,\ T_1(j_s, P_N) \in E_1(j_{s+1}, P_N), \text{ and} \\ & \text{for some } \hat{s} = 1, \ldots, g,\ j_{\hat{s}} = i; \\ \emptyset & \text{otherwise.} \end{cases}$$

Since each individual can be in at most one trading cycle, $C_1(i, P_N)$ is well-defined for all $i \in N$. Furthermore, since both the number of individuals and the number of objects are finite, there is always at least one trading cycle. Note that $C_1(i, P_N) = \{i\}$ if $T_1(i, P_N) \in E_1(i, P_N)$.

Assigned Individuals: $W_1(P_N) = \{i \mid C_1(i, P_N) \neq \emptyset\}$.

Assignments: For all $i \in W_1(P_N)$, $f_i^\Gamma(P_N) = T_1(i, P_N)$.

Assigned Objects: $F_1(P_N) = \{T_1(i, P_N) \mid i \in W_1(P_N)\}$.

This procedure is repeated iteratively in the remaining reduced market. For each stage t, define $W^t(P_N) = \bigcup_{u=1}^{t} W_u(P_N)$ and $F^t(P_N) = \bigcup_{u=1}^{t} F_u(P_N)$. In what follows, we present Stage $t+1$ of f^Γ.

$\vdots$

Stage $t + 1$**.**

Hierarchical Endowments (Non-initial Endowments): Let $\mu^t \in \mathcal{M}(W^t(P_N), F^t(P_N))$ such that for all $i \in W^t(P_N)$,

$$\mu^t(i) = f_i^\Gamma(P_N).$$

For all $i \in N \setminus W^t(P_N)$, $E_{t+1}(i, P_N) = \mathcal{E}_i^\Gamma(\mu^t)$.

Top Choices: For all $i \in N \setminus W^t(P_N)$, $T_{t+1}(i, P_N) = \tau(P_i, A \setminus F^t(P_N))$.

Trading Cycles: For all $i \in N \setminus W^t(P_N)$,

$$C_{t+1}(i, P_N) = \begin{cases} \{j_1, \ldots, j_g\} & \text{if there exist } j_1, \ldots, j_g \in N \setminus W^t(P_N) \text{ such that} \\ & \text{for all } s = 1, \ldots, g,\ T_{t+1}(j_s, P_N) \in E_{t+1}(j_{s+1}, P_N), \text{ and} \\ & \text{for some } \hat{s} = 1, \ldots, g,\ j_{\hat{s}} = i; \\ \emptyset & \text{otherwise.} \end{cases}$$

Assigned Individuals: $W_{t+1}(P_N) = \{i \mid C_{t+1}(i, P_N) \neq \emptyset\}$.

Assignments: For all $i \in W_{t+1}(P_N)$, $f_i^\Gamma(P_N) = T_{t+1}(i, P_N)$.

Assigned Objects: $F_{t+1}(P_N) = \{T_{t+1}(i, P_N) \mid i \in W_{t+1}(P_N)\}$.

$$\vdots$$

This procedure is repeated iteratively until either all individuals are assigned or all objects are assigned. The hierarchical exchange rule f^Γ associated with Γ is defined as follows. For all $i \in N$,

$$f_i^\Gamma(P_N) = \begin{cases} T_t(i, P_N) & \text{if } i \in W_t(P_N) \text{ for some stage } t; \\ \emptyset & \text{otherwise.} \end{cases}$$

Since for every preference profile P_N and every individual i, there exists at most one stage t such that $i \in W_t(P_N)$, f^Γ is well-defined.

Remark 6.1. Note that a collection of inheritance trees do not uniquely identify a hierarchical exchange rule. More formally, two different collections of inheritance trees Γ and $\overline{\Gamma}$ may give rise to the same hierarchical exchange rule, that is, $f^\Gamma \equiv f^{\overline{\Gamma}}$.

7 A characterization of hierarchical exchange rules

We introduce the notion of *top-envy-proofness* for an assignment rule. It says that if an individual i is assigned the most preferred object of another individual j, then no matter how the individual j misreports her preference, individual i cannot be worse-off. Thus, if an individual (here, j) is envious at another individual (here, i) for getting her (here, j's) top-ranked object, then the former one can never harm the latter. As the name suggests, top-envy-proofness is weaker than *envy-proofness* (that is, envy-proofness implies top-envy-proofness).[15] Loosely speaking, top-envy-proofness can be viewed as envy-proofness with respect to the top-ranked object of the envious individual.

Definition 7.1. An assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ satisfies ***top-envy-proofness*** condition if for all $P_N \in \mathcal{P}_N$ and all distinct $i, j \in N$, $\tau(P_j) = f_i(P_N)$ implies $f_i(\tilde{P}_j, P_{-j}) R_i f_i(P_N)$ for all $\tilde{P}_j \in \mathcal{P}_j$.

Next, we introduce the notion of an assignment rule being *manipulable through pairwise reallocation*. It captures the idea of manipulation where two individuals simultaneously misreport their preferences and finally benefit (with respect to their original assignments) by reshuffling their assignments that they obtain at the misreported preference profile. It further says that if any one of the two individuals misreports her preference as "planned", then her assignment will not depend whether the other individual misreports her preference as planned or reports truthfully.

[15] An assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ satisfies ***envy-proofness*** condition if for all $P_N \in \mathcal{P}_N$ and all distinct $i, j \in N$, $f_i(P_N) P_j f_j(P_N)$ implies $f_i(\tilde{P}_j, P_{-j}) R_i f_i(P_N)$ for all $\tilde{P}_j \in \mathcal{P}_j$.

Definition 7.2. An assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ is *manipulable through pairwise reallocation* if there exist $P_N \in \mathcal{P}_N$, individuals $i, j \in N; i \neq j$, and $\tilde{P}_i \in \mathcal{P}_i$, $\tilde{P}_j \in \mathcal{P}_j$ such that

(i) $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) R_i f_i(P_N)$,

(ii) $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j(P_N)$, and

(iii) $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = f_i(\tilde{P}_i, P_j, P_{-i,j})$ and $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = f_j(P_i, \tilde{P}_j, P_{-i,j})$.

An assignment rule is ***pairwise reallocation-proof*** if it is not manipulable through pairwise reallocation.

Our next result provides a characterization of hierarchical exchange rules.

Theorem 7.1. *An assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ is strategy-proof, Pareto efficient, top-envy-proof, non-bossy, and pairwise reallocation-proof if and only if it is a hierarchical exchange rule.*

The proof of this theorem is relegated to Appendix D.

Since group strategy-proofness is equivalent to strategy-proofness and non-bossiness (see Proposition 4.1), we obtain the following corollary from Theorem 7.1.

Corollary 7.1. *An assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ is group strategy-proof, Pareto efficient, top-envy-proof, and pairwise reallocation-proof if and only if it is a hierarchical exchange rule.*

We now strengthen the notion of pairwise reallocation-proof by *group-wise reallocation-proof*. As the name suggests, instead of a pair of individuals, arbitrary groups of individuals are considered in group-wise reallocation-proof. Thus, group-wise reallocation-proof ensures that no group of individuals can be better off by misreporting their preferences and redistributing the objects they obtain at the misreported preference profile. Condition (iii) in Definition 7.2 is suitably modified for group of individuals.

To ease our presentation, for an assignment rule f, a preference profile P_N, and a set of individuals S, we denote by $f_S(P_N)$ the allocation over S according to $f(P_N)$. More formally, $f_S(P_N)$ is the allocation μ over S such that $\mu(i) = f_i(P_N)$ for all $i \in S$. With slight abuse of notation, by $\{f_S(P_N)\}$ we denote the set of objects which are assigned to the individuals in S at P_N, that is, $\{f_S(P_N)\} := \{a \in A \mid f_i(P_N) = a \text{ for some } i \in S\}$.

Definition 7.3. An assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ is *manipulable through group-wise reallocation* if there exist $P_N \in \mathcal{P}_N$, a set of individuals $S \subseteq N$, a preference profile $\tilde{P}_S$ of the individuals in S, and an allocation $\hat{\mu}$ of $\{f_S(\tilde{P}_S, P_{-S})\}$ over S where $\hat{\mu} \neq f_S(\tilde{P}_S, P_{-S})$ such that

(i) $\hat{\mu}(i) R_i f_i(P_N)$ for all $i \in S$,

(ii) $\hat{\mu}(j) P_j f_j(P_N)$ for some $j \in S$, and

(iii) $f_i(\tilde{P}_i, \tilde{P}_{S\setminus\{i\}}, P_{-S}) = f_i(\tilde{P}_i, P_{S\setminus\{i\}}, P_{-S})$ for all $i \in S$.

An assignment rule is ***group-wise reallocation-proof*** if it is not manipulable through group-wise reallocation.

Proposition 7.1. *Every hierarchical exchange rule satisfies group-wise reallocation-proofness.*

The proof of this proposition is relegated to Appendix E.

We obtain the following corollary from Theorem 7.1 and Proposition 7.1.

Corollary 7.2. *An assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ is strategy-proof, Pareto efficient, top-envy-proof, non-bossy, and group-wise reallocation-proof if and only if it is a hierarchical exchange rule.*

The next corollary is obtained by combining Corollary 7.1 and Proposition 7.1.

Corollary 7.3. *An assignment rule $f : \mathcal{P}_N \to \mathcal{M}$ is group strategy-proof, Pareto efficient, top-envy-proof, and group-wise reallocation-proof if and only if it is a hierarchical exchange rule.*

8 Independence of the conditions in Theorem 7.1

In this section, we show that strategy-proofness, Pareto efficiency, top-envy-proofness, non-bossiness and pairwise reallocation-proofness are all independent for a hierarchical exchange rule. In particular, we show that no four of those conditions imply the fifth one.

Example 8.1. In this example, we show that Pareto efficiency, top-envy-proofness, non-bossiness, and pairwise reallocation-proofness do *not* imply strategy-proofness. Consider an allocation problem with three individuals $N = \{1, 2, 3\}$ and three objects $A = \{a_1, a_2, a_3\}$ with a prior order $a_1 \prec a_2 \prec a_3$. Consider the assignment rule f such that

$$f = \begin{cases} \text{Serial dictatorship with priority } (1 \succ 3 \succ 2) & \text{if } \tau(P_1) = \tau(P_2) = a_1, \text{ and } \tau(P_3) = a_2; \\ \text{Serial dictatorship with priority } (1 \succ 2 \succ 3) & \text{otherwise.} \end{cases}$$

Consider the preference profiles $P_N = (a_1a_2a_3, a_1a_2a_3, a_2a_1a_3)$ and $\tilde{P}_N = (a_1a_2a_3, a_2a_1a_3, a_2a_1a_3)$.[16] Note that only individual 2 changes her preference from P_N to $\tilde{P}_N$. This, together with the facts $f_2(P_N) = a_3$, $f_2(\tilde{P}_N) = a_2$, and $a_2P_2a_3$, implies f is not strategy-proof. It can be easily verified that f is Pareto efficient, top-envy-proof, non-bossy, and pairwise reallocation-proof.

Example 8.2. In this example, we show that strategy-proofness, top-envy-proofness, non-bossiness, and pairwise reallocation-proofness do *not* imply Pareto efficiency. Define f such that $f_i(P_N) = \emptyset$ for all $i \in N$ and all P_N. It is easy to verify that f satisfies strategy-proofness, top-envy-proofness, non-bossiness, and pairwise reallocation-proofness. However, from Remark 4.1, it follows that f does *not* satisfy Pareto efficiency.

[16]Here, we denote by $(a_1a_2a_3, a_2a_3a_1, a_3a_2a_1)$ a preference profile where individuals 1, 2 and 3 have preferences $a_1a_2a_3$, $a_2a_3a_1$, and $a_3a_2a_1$, respectively.

Example 8.3. In this example, we show that strategy-proofness, Pareto efficiency, non-bossiness, and pairwise reallocation-proofness do *not* imply top-envy-proofness condition. Consider an allocation problem with three individuals $N = \{1, 2, 3\}$ and four objects $A = \{a_1, a_2, a_3, a_4\}$ with a prior order $a_1 \prec a_2 \prec a_3 \prec a_4$. Consider the assignment rule f such that

$$f = \begin{cases} \text{Serial dictatorship with priority } (2 \succ 1 \succ 3) & \text{if } \tau(P_1) = \tau(P_2) = a_1, \text{ and } \tau(P_3) = a_4; \\ \text{Serial dictatorship with priority } (1 \succ 2 \succ 3) & \text{otherwise.} \end{cases}$$

Consider the preference profiles $P_N = (a_1a_2a_3a_4, a_1a_2a_3a_4, a_1a_2a_3a_4)$ and $\tilde{P}_N = (a_1a_2a_3a_4, a_1a_2a_3a_4, a_4a_3a_2a_1)$. Note that only individual 3 changes her preference from P_N to $\tilde{P}_N$. This, together with the facts $f_1(P_N) = a_1$, $\tau(P_3) = a_1$, $f_1(\tilde{P}_N) = a_2$, and $a_1P_1a_2$, implies f is not top-envy-proof. It can be easily verified that f is strategy-proof, Pareto efficient, non-bossy, and pairwise reallocation-proof.

Example 8.4. In this example, we show that strategy-proofness, Pareto efficiency, top-envy-proofness, and pairwise reallocation-proofness do *not* imply non-bossiness. Consider an allocation problem with three individuals $N = \{1, 2, 3\}$ and three objects $A = \{a_1, a_2, a_3\}$ with a prior order $a_1 \prec a_2 \prec a_3$. Consider the assignment rule f such that

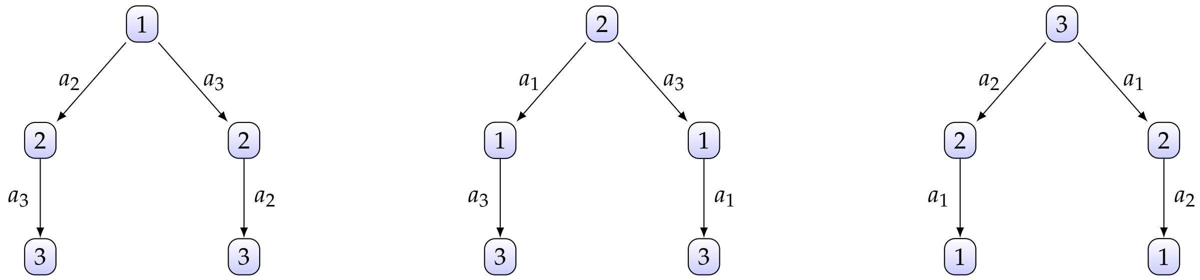
$$f = \begin{cases} \text{Serial dictatorship with priority } (1 \succ 2 \succ 3) & \text{if } a_1P_1a_3; \\ \text{Serial dictatorship with priority } (1 \succ 3 \succ 2) & \text{if } a_3P_1a_1. \end{cases}$$

Consider the preference profiles $P_N = (a_2a_1a_3, a_2a_1a_3, a_2a_1a_3)$ and $\tilde{P}_N = (a_2a_3a_1, a_2a_1a_3, a_2a_1a_3)$. Note that only individual 1 changes her preference from P_N to $\tilde{P}_N$. This, together with the facts $f(P_N) = [(1, a_2), (2, a_1), (3, a_3)]$ and $f(\tilde{P}_N) = [(1, a_2), (2, a_3), (3, a_1)]$, implies f is not non-bossy. It is easy to verify that f is strategy-proof, Pareto efficient, top-envy-proof, and pairwise reallocation-proof.

Example 8.5. In this example, we show that strategy-proofness, Pareto efficiency, top-envy-proofness, and non-bossiness do *not* imply pairwise reallocation-proofness. Consider an allocation problem with three individuals $N = \{1, 2, 3\}$ and three objects $A = \{a_1, a_2, a_3\}$ with a prior order $a_1 \prec a_2 \prec a_3$. Consider the hierarchical exchange rule f^Γ based on the collection of inheritance trees given in Figure 8.1. Consider the assignment rule f such that

$$f = \begin{cases} \text{Serial dictatorship with priority } (2 \succ 1 \succ 3) & \text{if } \tau(P_1) = \tau(P_2) = a_3, \text{ and } \tau(P_3) = a_1; \\ f^\Gamma & \text{otherwise.} \end{cases}$$

Consider the preference profile $P_N = (a_3a_2a_1, a_3a_2a_1, a_1a_2a_3)$ and the preferences $\tilde{P}_1 \in \mathcal{P}_1$, $\tilde{P}_3 \in \mathcal{P}_3$ such that $\tau(\tilde{P}_1) = a_1$ and $\tau(\tilde{P}_3) = a_3$. It follows from the construction of f that $f(P_N) = [(1, a_2), (2, a_3), (3, a_1)]$, $f_1(\tilde{P}_1, P_2, \tilde{P}_3) = f_1(\tilde{P}_1, P_2, P_3) = a_1$, $f_3(\tilde{P}_1, P_2, \tilde{P}_3) = f_3(P_1, P_2, \tilde{P}_3) = a_3$. These facts, along with the fact $a_3P_1a_2$, together imply f is not pairwise reallocation-proof. It can be easily verified that f is strategy-proof, Pareto efficient, top-envy-proof, and non-bossy.



(a) Γ_{a_1} (b) Γ_{a_2} (c) Γ_{a_3}

Figure 8.1: Inheritance trees for Example 8.5

Remark 8.1. The examples in this section also demonstrate that strategy-proofness, Pareto efficiency, top-envy-proofness, non-bossiness, and group-wise reallocation-proofness are all independent for a hierarchical exchange rule. To see this note that except for Example 8.2, all other examples deal with three individuals, and Pareto efficiency and pairwise reallocation-proofness together imply group-wise reallocation-proofness in such cases. The fact that the assignment rule in Example 8.2 satisfies group-wise reallocation-proofness is straightforward, and the assignment rule in Example 8.5 is *not* pairwise reallocation-proof (while being strategy-proof, Pareto efficient, top-envy-proof, and non-bossy), so it will not be group-wise reallocation-proof either.

Appendix A Preliminaries

For $a, b \in A$, let $P^{(a;b)}$ be a single-peaked preference (with respect to the given ordering $\prec$) such that

(i) $\tau(P^{(a;b)}) = a$, and

(ii) $P^{(a;b)}$ is a left (right) single-peaked preference if $b \preceq a$ ($a \prec b$).[17]

Remark A.1. Since $\mathcal{P}_i$ is minimally rich single-peaked domain of preferences (with respect to the given ordering $\prec$) for all $i \in N$, we have $P^{(a;b)} \in \mathcal{P}_i$ for all $i \in N$ and all $a, b \in A$.

Appendix B Proof of Proposition 4.1

Proof of Proposition 4.1. (If part) Assume for contradiction that f is not group strategy-proof. Since f is not group strategy-proof, there exist $P_N \in \mathcal{P}_N$, $S \subseteq N$, and $P'_S \in \prod_{i \in S} \mathcal{P}_i$ such that $f_i(P'_S, P_{-S}) R_i f_i(P_N)$ for all $i \in S$ and $f_j(P'_S, P_{-S}) P_j f_j(P_N)$ for some $j \in S$. Consider the profile of preferences $\tilde{P}_S \in \prod_{i \in S} \mathcal{P}_i$ such that for all $i \in S$,

$$\tilde{P}_i = \begin{cases} P^{(f_i(P'_S, P_{-S}); f_i(P_N))} & \text{if } f_i(P_N) \neq \emptyset; \\ P'_i & \text{if } f_i(P_N) = \emptyset. \end{cases}$$

It follows from the construction of $\tilde{P}_S$ and Remark A.1 that $\tilde{P}_S$ is well-defined.

[17] By $\preceq$ we denote the weak part of $\prec$, that is, for all $a, b \in A$, $a \preceq b$ if and only if $[a \prec b \text{ or } a = b]$.

First, we show that $f(\tilde{P}_S, P_{-S}) = f(P_N)$. Fix $j \in S$.

Claim B.1. $f(\tilde{P}_j, P_{-j}) = f(P_N)$.

Proof of Claim B.1. Suppose $f_j(P_N) = \emptyset$. Then, by strategy-proofness, we have $f_j(\tilde{P}_j, P_{-j}) = \emptyset$. Since $f_j(P_N) = \emptyset$ and $f_j(\tilde{P}_j, P_{-j}) = \emptyset$, by non-bossiness, we have

$$f(\tilde{P}_j, P_{-j}) = f(P_N). \tag{B.1}$$

Now, suppose $f_j(P_N) \neq \emptyset$. Then, by strategy-proofness, we have $f_j(\tilde{P}_j, P_{-j})\tilde{R}_j f_j(P_N)$. Suppose $f_j(\tilde{P}_j, P_{-j})\tilde{P}_j f_j(P_N)$. Since $f_j(\tilde{P}_j, P_{-j})\tilde{P}_j f_j(P_N)$, it follows from the construction of $\tilde{P}_j$ that

$$f_j(P'_S, P_{-S}) \neq f_j(P_N), \text{ and} \tag{B.2a}$$

$$f_j(P'_S, P_{-S}) \preceq f_j(\tilde{P}_j, P_{-j}) \prec f_j(P_N) \quad \text{or} \quad f_j(P_N) \prec f_j(\tilde{P}_j, P_{-j}) \preceq f_j(P'_S, P_{-S}). \tag{B.2b}$$

Since $f_i(P'_S, P_{-S})R_i f_i(P_N)$ for all $i \in S$, by (B.2a) we have $f_j(P'_S, P_{-S})P_j f_j(P_N)$. This, together with (B.2b), implies $f_j(\tilde{P}_j, P_{-j})P_j f_j(P_N)$, a contradiction to strategy-proofness. So, it must be that $f_j(\tilde{P}_j, P_{-j}) = f_j(P_N)$. By non-bossiness, the fact $f_j(\tilde{P}_j, P_{-j}) = f_j(P_N)$ implies

$$f(\tilde{P}_j, P_{-j}) = f(P_N). \tag{B.3}$$

(B.1) and (B.3) together complete the proof of Claim B.1. □

Continuing in this manner, we can move the preferences of all individuals $j \in S$, from the preference P_j to $\tilde{P}_j$ one by one and obtain

$$f(\tilde{P}_S, P_{-S}) = f(P_N). \tag{B.4}$$

Next, we show that $f(\tilde{P}_S, P_{-S}) = f(P'_S, P_{-S})$. Fix $j \in S$. By strategy-proofness, we have $f_j(\tilde{P}_j, P'_{S\setminus\{j\}}, P_{-S})\tilde{R}_j f_j(P'_S, P_{-S})$. Moreover, it follows from the construction of $\tilde{P}_j$ that either $\tau(\tilde{P}_j) = f_j(P'_S, P_{-S})$ or $\tilde{P}_j = P'_j$. This, together with the fact $f_j(\tilde{P}_j, P'_{S\setminus\{j\}}, P_{-S})\tilde{R}_j f_j(P'_S, P_{-S})$, implies $f_j(\tilde{P}_j, P'_{S\setminus\{j\}}, P_{-S}) = f_j(P'_S, P_{-S})$. By non-bossiness, the fact $f_j(\tilde{P}_j, P'_{S\setminus\{j\}}, P_{-S}) = f_j(P'_S, P_{-S})$ implies

$$f(\tilde{P}_j, P'_{S\setminus\{j\}}, P_{-S}) = f(P'_S, P_{-S}).$$

Continuing in this manner, we can move the preferences of all individuals $j \in S$, from the preference P'_j to $\tilde{P}_j$ one by one and obtain

$$f(\tilde{P}_S, P_{-S}) = f(P'_S, P_{-S}). \tag{B.5}$$

However, (B.4) and (B.5) together imply $f(P'_S, P_{-S}) = f(P_N)$, a contradiction to the fact that $f_j(P'_S, P_{-S})P_j f_j(P_N)$ for some $j \in S$. This completes the proof of the "if" part of Proposition 4.1.

(Only-if part) It is obvious that group strategy-proofness implies strategy-proofness and non-bossiness.■

Appendix C Proof of Theorem 5.1

Proof of Theorem 5.1. Suppose $A = \{a_1, a_2, \ldots, a_m\}$ with a prior order $a_1 \prec a_2 \prec \cdots \prec a_m$, where $m \geq 3$. Assume for contradiction that there exists a strategy-proof, non-bossy, Pareto efficient, and strongly pairwise reallocation-proof assignment rule f on $\mathcal{P}_N$. Since $\mathcal{P}_i$ is minimally rich for all $i \in N$, there exists a preference profile $P_N^1 \in \mathcal{P}_N$ such that $P_i^1 = a_2 a_1 a_3 \ldots$ for all $i \in N$. Since $|N| \geq 3$, by Pareto efficiency, we have $\{a_1, a_2, a_3\} \subseteq \bigcup_{i \in N} \{f_i(P_N^1)\}$. Without loss of generality, assume $f_1(P_N^1) = a_1$, $f_2(P_N^1) = a_2$, and $f_3(P_N^1) = a_3$.

Since $\mathcal{P}_i$ is minimally rich for all $i \in N$, we can construct the preference profiles presented in Table C.1. Here, l denotes an individual other than $1, 2, 3$ (if any). Note that such an individual does not change her preference across the mentioned preference profiles.

Preference profiles	Individual 1	Individual 2	Individual 3	...	Individual l
P_N^2	$a_2 \ldots a_m a_1$	$a_1 a_2 a_3 \ldots$	$a_2 a_1 a_3 \ldots$	...	$a_2 a_1 a_3 \ldots$
P_N^3	$a_2 \ldots a_m a_1$	$a_2 a_1 a_3 \ldots$	$a_2 a_1 a_3 \ldots$	...	$a_2 a_1 a_3 \ldots$

Table C.1: Preference profiles for Theorem 5.1

Since $f_1(P_N^1) = a_1$ and $f_2(P_N^1) = a_2$, it follows from strong pairwise reallocation-proofness of f that

$$f_1(P_N^2) = a_2 \text{ and } f_2(P_N^2) = a_1. \tag{C.1}$$

By (C.1) we have $f_2(P_N^2) = a_1$. This, together with strategy-proofness of f, implies $f_2(P_N^3) \in \{a_1, a_2\}$. Suppose $f_2(P_N^3) = a_1$. Since $f_2(P_N^2) = a_1$ and $f_2(P_N^3) = a_1$, by non-bossiness and (C.1), we have $f_1(P_N^3) = a_2$. However, since $a_2 P_1^1 a_1$, the facts $f_1(P_N^1) = a_1$ and $f_1(P_N^3) = a_2$ together contradict strategy-proofness of f. So, it must be that

$$f_2(P_N^3) = a_2. \tag{C.2}$$

Since $f_1(P_N^1) = a_1$ and $f_3(P_N^1) = a_3$, (C.2) together with strong pairwise reallocation-proofness of f, implies that

$$f_1(P_N^3) = a_3, f_2(P_N^3) = a_2, \text{ and } f_3(P_N^3) = a_1. \tag{C.3}$$

By (C.3) we have $f_2(P_N^3) = a_2$ and $f_3(P_N^3) = a_1$. Combining these facts with strong pairwise reallocation-proofness of f, we have $f_2(P_N^2) = a_1$ and $f_3(P_N^2) = a_2$. However, the fact that $f_3(P_N^2) = a_2$ contradicts (C.1). This completes the proof of Theorem 5.1. ■

Appendix D Proof of Theorem 7.1

To prove Theorem 7.1, we use the notations introduced in Section 6. Furthermore, for a preference profile $P_N \in \mathcal{P}_N$ and a hierarchical exchange rule, we assume $F^0(P_N) = \emptyset$ and $W^0(P_N) = \emptyset$.

The following lemma is taken from Pápai (2000). She proves this lemma for the unrestricted domain. Since $\mathcal{P}_N$ is a subset of the unrestricted domain, the result holds for $\mathcal{P}_N$ as well.

Lemma D.1 (Lemma 4 in Pápai (2000))**.** *Let f^Γ be a hierarchical exchange rule, $P_N \in \mathcal{P}_N$, and $i, j \in N$. Suppose $i \in W_s(P_N)$ and $f_j^\Gamma(P_N) \neq f_j^\Gamma(\tilde{P}_i, P_{-i})$ for some $\tilde{P}_i \in \mathcal{P}_i$. Then, either $j \in C_s(i, P_N)$ or $j \notin W^s(P_N)$.*

We obtain the following lemma from Lemma D.1.

Lemma D.2. *Let f^Γ be a hierarchical exchange rule and $P_N \in \mathcal{P}_N$. Suppose $i \in W_{s_i}(P_N)$, $j \in W_{s_j}(P_N)$ and $s_i < s_j$. Then, $f_i^\Gamma(\bar{P}_j, P_{-j}) = f_i^\Gamma(P_N)$ for all $\bar{P}_j \in \mathcal{P}_j$.*

Lemma D.3 establishes a property which says that if an individual j prefers the assignment of another individual i of a hierarchical exchange rule, then it must be that i is assigned before j.

Lemma D.3. *Let f^Γ be a hierarchical exchange rule and $P_N \in \mathcal{P}_N$. Suppose $i \in W_{s_i}(P_N)$ and $j \in W_{s_j}(P_N)$ such that $f_i^\Gamma(P_N) P_j f_j^\Gamma(P_N)$. Then, $s_i < s_j$.*

Proof of Lemma D.3. Assume for contradiction that $s_j \leq s_i$. Since $j \in W_{s_j}(P_N)$, by the definition of f^Γ, we have $f_j^\Gamma(P_N) = \tau(P_j, A \setminus F^{s_j-1}(P_N))$. Furthermore, the fact $i \in W_{s_i}(P_N)$ together with the definition of f^Γ, implies that $f_i^\Gamma(P_N) \in A \setminus F^{s_i-1}(P_N)$. This, together with the fact $s_j \leq s_i$, yields $f_i^\Gamma(P_N) \in A \setminus F^{s_j-1}(P_N)$. However, the facts that $f_j^\Gamma(P_N) = \tau(P_j, A \setminus F^{s_j-1}(P_N))$ and $f_i^\Gamma(P_N) \in A \setminus F^{s_j-1}(P_N)$ together contradict the fact $f_i^\Gamma(P_N) P_j f_j^\Gamma(P_N)$. This completes the proof of Lemma D.3. ■

D.1 Proof of the "if" part of Theorem 7.1

It follows from Pápai (2000) that every hierarchical exchange rule satisfies strategy-proofness, Pareto efficiency, top-envy-proofness, and non-bossiness on the unrestricted domain.[18] Since $\mathcal{P}_N$ is a subset of the unrestricted domain, it follows that every hierarchical exchange rule satisfies strategy-proofness, Pareto efficiency, top-envy-proofness, and non-bossiness on $\mathcal{P}_N$. In what follows, we show that every hierarchical exchange rule satisfies pairwise reallocation-proofness on $\mathcal{P}_N$.

Let f^Γ be a hierarchical exchange rule on $\mathcal{P}_N$. Assume for contradiction that f^Γ does not satisfy pairwise reallocation-proofness. Then, there must exists $P_N \in \mathcal{P}_N$, distinct $i, j \in N$, and $\tilde{P}_i \in \mathcal{P}_i, \tilde{P}_j \in \mathcal{P}_j$ such that

(i) $f_j^\Gamma(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) R_i f_i^\Gamma(P_N)$,

(ii) $f_i^\Gamma(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j^\Gamma(P_N)$, and

[18] For details see Lemma 1, Lemma 7, and the main theorem of Pápai (2000).

(iii) $f_i^\Gamma(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = f_i^\Gamma(\tilde{P}_i, P_{-i})$ and $f_j^\Gamma(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = f_j^\Gamma(\tilde{P}_j, P_{-j})$.

Claim D.1. *$f_i^\Gamma(P_N)$ and $f_j^\Gamma(P_N)$ are distinct objects.*

Proof of Claim D.1. Suppose $f_i^\Gamma(P_N) = \emptyset$. Since f^Γ is strategy-proof, $f_i^\Gamma(P_N) = \emptyset$ implies $f_i^\Gamma(\tilde{P}_i, P_{-i}) = \emptyset$. However, the facts that $f_i^\Gamma(\tilde{P}_i, P_{-i}) = \emptyset$ and $f_i^\Gamma(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = f_i^\Gamma(\tilde{P}_i, P_{-i})$ together imply $f_i^\Gamma(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = \emptyset$, a contradiction to the fact $f_i^\Gamma(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j^\Gamma(P)$. So, it must be that

$$f_i^\Gamma(P_N) \neq \emptyset. \tag{D.1}$$

Since $f_j^\Gamma(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) R_i f_i^\Gamma(P_N)$, (D.1) implies $f_j^\Gamma(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) \neq \emptyset$. This, together with the fact $f_j^\Gamma(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = f_j^\Gamma(\tilde{P}_j, P_{-j})$, implies $f_j^\Gamma(\tilde{P}_j, P_{-j}) \neq \emptyset$. Since f^Γ is strategy-proof, $f_j^\Gamma(\tilde{P}_j, P_{-j}) \neq \emptyset$ implies

$$f_j^\Gamma(P_N) \neq \emptyset. \tag{D.2}$$

(D.1) and (D.2) together complete the proof of Claim D.1. □

It follows from Claim D.1 that there exist stages s_i and s_j of f^Γ at P_N such that $i \in W_{s_i}(P_N)$ and $j \in W_{s_j}(P_N)$. Now, we complete the proof by distinguishing two cases.

CASE 1: Suppose $s_j \leq s_i$.

Since f^Γ is Pareto efficient, $f_i^\Gamma(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j^\Gamma(P_N)$ implies that there exists $k \in N \setminus \{j\}$ such that $f_k^\Gamma(P_N) = f_i^\Gamma(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$. The facts $f_i^\Gamma(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j^\Gamma(P_N)$ and $f_k^\Gamma(P_N) = f_i^\Gamma(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$ together imply $f_k^\Gamma(P_N) P_j f_j^\Gamma(P_N)$ and $f_k^\Gamma(P_N) \in A$. It follows from the fact $f_k^\Gamma(P_N) \in A$ that there exists a stage s_k of f^Γ at P_N such that $k \in W_{s_k}(P_N)$. Since $j \in W_{s_j}(P_N)$, $k \in W_{s_k}(P_N)$, and $f_k^\Gamma(P_N) P_j f_j^\Gamma(P_N)$, by Lemma D.3, we have $s_k < s_j$. This, together with the fact $s_j \leq s_i$, implies $s_k < s_i$. Since $i \in W_{s_i}(P_N)$, $k \in W_{s_k}(P_N)$, and $s_k < s_i$, by Lemma D.2, we have

$$f_k^\Gamma(P_N) = f_k^\Gamma(\tilde{P}_i, P_{-i}). \tag{D.3}$$

Furthermore, the facts $i \in W_{s_i}(P_N)$, $k \in W_{s_k}(P_N)$, and $s_k < s_i$ together imply $i \neq k$. Since $f_k^\Gamma(P_N) \in A$ and $i \neq k$, (D.3) implies

$$f_k^\Gamma(P_N) \neq f_i^\Gamma(\tilde{P}_i, P_{-i}). \tag{D.4}$$

However, the facts $f_i^\Gamma(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = f_i^\Gamma(\tilde{P}_i, P_{-i})$ and $f_k^\Gamma(P_N) = f_i^\Gamma(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$ together contradict (D.4).

CASE 2: Suppose $s_i < s_j$.

If $f_j^\Gamma(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_i f_i^\Gamma(P_N)$, then the proof follows using a similar logic as for Case 1. Since $f_j^\Gamma(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) R_i f_i^\Gamma(P_N)$, let us assume

$$f_j^\Gamma(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = f_i^\Gamma(P_N). \tag{D.5}$$

Since $i \in W_{s_i}(P_N)$, $j \in W_{s_j}(P_N)$, and $s_i < s_j$, by Lemma D.2, we have

$$f_i^\Gamma(\tilde{P}_j, P_{-j}) = f_i^\Gamma(P_N). \tag{D.6}$$

Furthermore, since $f_j^\Gamma(\tilde{P}_j, P_{-j}) = f_j^\Gamma(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$, by (D.5) and (D.6), we have

$$f_i^\Gamma(\tilde{P}_j, P_{-j}) = f_j^\Gamma(\tilde{P}_j, P_{-j}) = f_i^\Gamma(P_N). \tag{D.7}$$

However, by Claim D.1, we have $f_i^\Gamma(P_N) \in A$. Since $f_i^\Gamma(P_N) \in A$ and $i \neq j$, (D.7) implies that $f^\Gamma(\tilde{P}_j, P_{-j})$ is not an allocation, a contradiction.

Since Cases 1 and 2 are exhaustive, it follows that f^Γ satisfies pairwise reallocation-proofness on $\mathcal{P}_N$.

D.2 Proof of the "only-if" part of Theorem 7.1

Let f be a strategy-proof, Pareto efficient, top-envy-proof, non-bossy, and pairwise reallocation-proof assignment rule. We will show that f is a hierarchical exchange rule.

D.2.1 Construction of the inheritance trees based on f

Fix $a \in A$. We proceed to construct an inheritance tree $\Gamma_a = \langle T_a, \zeta_a^{NI}, \zeta_a^{EO} \rangle$ for $a \in A$. Let T_a be a rooted tree that satisfies Condition (i) of Definition 6.1. Let $\zeta_a^{EO} : E(T_a) \to A \setminus \{a\}$ be an edges-to-objects function that satisfies Condition (iii) of Definition 6.1. We will define $\zeta_a^{NI} : V(T_a) \to N$, a nodes-to-individuals function, in accordance with property Condition (ii) of Definition 6.1 based on f.

Let $\mathcal{P}_N^0 \subseteq \mathcal{P}_N$ be the set of all preference profiles P_N such that $\tau(P_i) = a$ for all $i \in N$.

Lemma D.4. *There exists $k \in N$ such that $f_k(P_N) = a$ for all $P_N \in \mathcal{P}_N^0$.*

Proof of Lemma D.4. By Remark 4.1, for every given $P_N \in \mathcal{P}_N^0$, there exists an individual $k \in N$ such that $f_k(P_N) = a$. It remains to show that this individual is unique for all preference profile in $\mathcal{P}_N^0$, that is, $f_k(P_N) = f_k(P'_N) = a$ for all $P_N, P'_N \in \mathcal{P}_N^0$. Assume for contradiction that $f_j(P_N) = f_{j'}(P'_N) = a$ for some $P_N, P'_N \in \mathcal{P}_N^0$ and $j, j' \in N$ such that $j \neq j'$.

Since $f_j(P_N) = a$, $\tau(P_j) = a$, and $a P_k f_k(P_N)$ for all $k \neq j$, by moving the preferences of the individuals $k \neq j$ one by one from P_k to P'_k, and by applying top-envy-proofness condition every time, we obtain $f_j(P_j, P'_{-j}) = a$. Moreover, since $f_{j'}(P'_N) = a$ and $j \neq j'$, we have $f_j(P'_N) \neq a$. This, together with the fact $\tau(P'_j) = a$, implies $a P'_j f_j(P'_N)$. However, the facts $f_j(P_j, P'_{-j}) = a$ and $a P'_j f_j(P'_N)$ together contradict strategy-proofness of f. This completes the proof of Lemma D.4. ■

By Lemma D.4, there exists $i_1 \in N$ such that $f_{i_1}(P_N) = a$ for all $P_N \in \mathcal{P}_N^0$. Define $\zeta_a^{NI}(v_a^1) = i_1$ where v_a^1 is the root-node of T_a. Let $(v_a^1, \ldots, v_a^r)$ with $r \geq 2$ be a path from v_a^1 to v_a^r in T_a. We define ζ_a^{NI} on $\{v_a^s \mid 1 \leq s \leq r\}$ in a recursive manner.

Assume that ζ_a^{NI} is defined on $\{v_a^s \mid 1 \leq s \leq r-1\}$. Let $\zeta_a^{NI}(v_a^s) = i_s$ for all $s = 1, \ldots, r-1$. We proceed to define ζ_a^{NI} on v_a^r. Let $\mathcal{P}_N^{r-1} \subseteq \mathcal{P}_N$ be the set of all preference profiles P_N such that $P_{i_s} = P^{(\zeta_a^{EO}(v_a^s, v_a^{s+1});a)}$ for all $s = 1, \ldots, r-1$, and $\tau(P_i) = a$ otherwise. Note that for all $P_N \in \mathcal{P}_N^{r-1}$ and all $s, s' \in \{1, \ldots, r-1\}$, $\tau(P_{i_s}) \neq \tau(P_{i_{s'}})$ if $s \neq s'$.

Lemma D.5. *There exists $k \in N \setminus \{i_1, \ldots, i_{r-1}\}$ such that $f_k(P_N) = a$ for all $P_N \in \mathcal{P}_N^{r-1}$.*

Proof of Lemma D.5. We first prove two claims that we will use to complete the proof of Lemma D.5.

Claim D.2. *Let $S = \{h_1, \ldots, h_m\} \subsetneq N$ be a set of distinct individuals with $m < |A|$ and let $\{b_1, \ldots, b_m\} \in A \setminus \{a\}$ be a set of distinct objects. Consider the preference profile P_N such that $\tau(P_{h_u}) = b_u$ for all $u = 1, \ldots, m$ and $\tau(P_i) = a$ for all $i \notin S$. Then, there exists $j \in N \setminus S$ such that $f_j(P_N) = a$.*

Proof of Claim D.2. By Remark 4.1, for all $c \in \{a, b_1, \ldots, b_m\}$, there exists $j_c \in N$ such that $f_{j_c}(P_N) = c$. It remains to show $j_a \notin S$. Assume for contradiction that $j_a \in S$. Let $\{j_1, \ldots, j_{t-1}\} \subseteq S$ and $j_t \notin S$ be such that $j_1 = j_a$, $f_{j_{s+1}}(P_N) = \tau(P_{j_s})$ for all $1 \leq s \leq t-1$. Since S is finite, to show such a sequence must exist, it is sufficient to show that $j_1, \ldots, j_{t-1}$ are all distinct. We show this in what follows. Assume for contradiction that l is the first index in the ordering $1, \ldots, t-1$ for which there exists $l < l' \leq t-1$ such that $j_l = j_{l'}$. Suppose $l = 1$. The facts $l = 1$, $j_l = j_{l'}$, $j_1 = j_a$, $f_{j_a}(P_N) = a$ and $f_{j_{l'}}(P_N) = \tau(P_{j_{l'-1}})$ together imply $\tau(P_{j_{l'-1}}) = a$. This is a contradiction since $j_{l'-1} \in S$, which in particular means $\tau(P_{j_{l'-1}}) \in \{b_1, \ldots, b_m\}$. Now, suppose $l > 1$. Then $j_l = j_{l'}$, $f_{j_l}(P_N) = \tau(P_{j_{l-1}})$ and $f_{j_{l'}}(P_N) = \tau(P_{j_{l'-1}})$ together imply

$$\tau(P_{j_{l-1}}) = \tau(P_{j_{l'-1}}). \tag{D.8}$$

However, by our assumption on l, $j_{l-1} \neq j_{l'-1}$. Because $j_{l-1}, j_{l'-1} \in S$ and $j_{l-1} \neq j_{l'-1}$, by the construction of P_N, $\tau(P_{j_{l-1}}) \neq \tau(P_{j_{l'-1}})$, a contradiction to (D.8). This shows that $j_1, \ldots, j_{t-1}$ are all distinct.

By the construction of $\{j_1, \ldots, j_t\}$, $\{f_{j_s}(P_N) \mid s = 1, \ldots, t\} = \{\tau(P_{j_s}) \mid s = 1, \ldots, t\}$. Define the allocation μ such that $\mu(i) = \tau(P_i)$ for all $i \in \{j_1, \ldots, j_t\}$ and $\mu(i) = f_i(P_N)$ for all $i \in N \setminus \{j_1, \ldots, j_t\}$. Clearly μ Pareto dominates $f(P_N)$ at P_N, which violates Pareto efficiency of f at P_N. This completes the proof of Claim D.2. □

Claim D.3. *For all $P_N \in \mathcal{P}_N^{r-1}$ and all $s = 1, \ldots, r-1$, we have $f_{i_s}(P_N) = \tau(P_{i_s})$.*

Proof of Claim D.3. Fix $P_N \in \mathcal{P}_N^{r-1}$. We prove this in two steps.

Step 1. In this step, we show that $f_{i_s}(P_N) P_{i_s} a$ for all $s = 1, \ldots, r-1$. Assume for contradiction that $a R_{i_{s^*}} f_{i_{s^*}}(P_N)$ for some $s^* \in \{1, \ldots, r-1\}$. Consider the preference profile $\tilde{P}_N$ such that $\tilde{P}_{i_t} = P_{i_t}$ for all $t = 1, \ldots, s^*-1$ and $\tau(\tilde{P}_i) = a$, otherwise. By the recursive definition of ζ_a^{NI},

$$f_{i_{s^*}}(\tilde{P}_N) = a. \tag{D.9}$$

Since $\tau(\tilde{P}_i) = a$ for all $i \in N \setminus \{i_1, \dots, i_{s^*-1}\}$, (D.9) implies that $f_{i_{s^*}}(\tilde{P}_N) = \tau(\tilde{P}_{i_{s^*}})$ and $f_{i_{s^*}}(\tilde{P}_N)\tilde{P}_i f_i(\tilde{P}_N)$ for all $i \in N \setminus \{i_1, \dots, i_{s^*}\}$. Therefore, by moving the preferences of all the individuals $i \in N \setminus \{i_1, \dots, i_{s^*}\}$ from $\tilde{P}_i$ to P_i, and by applying top-envy-proofness condition every time, it follows from the construction of $\tilde{P}_N$ that

$$f_{i_{s^*}}(\tilde{P}_{i_{s^*}}, P_{-i_{s^*}}) = a. \tag{D.10}$$

By strategy-proofness, (D.10) implies

$$f_{i_{s^*}}(P_N) R_{i_{s^*}} a. \tag{D.11}$$

By Claim D.2, there exists $j \in N \setminus \{i_1, \dots, i_{r-1}\}$ such that $f_j(P_N) = a$. Since $j \in N \setminus \{i_1, \dots, i_{r-1}\}$ and $f_j(P_N) = a$, (D.11) implies $f_{i_{s^*}}(P_N) P_{i_{s^*}} a$, a contradiction to our assumption. This proves $f_{i_s}(P_N) P_{i_s} a$ for all $s = 1, \dots, r-1$.

Step 2. In this step, we show that $f_{i_s}(P_N) = \tau(P_{i_s})$ for all $s = 1, \dots, r-1$. Assume for contradiction that $f_{i_{s_1}}(P_N) \neq \tau(P_{i_{s_1}})$ for some $s_1 \in \{1, \dots, r-1\}$. Let $s_1, \dots, s_u$ be the *maximal* sequence of distinct elements such that $\{s_1, \dots, s_u\} \subseteq \{1, \dots, r-1\}$ and $f_{i_{s_{t+1}}}(P_N) = \tau(P_{i_{s_t}})$ for all $t = 1, \dots, u-1$. Let $j \in N$ be such that $f_j(P_N) = \tau(P_{i_{s_u}})$. By the maximality assumption of $s_1, \dots, s_u$, either $j \in N \setminus \{i_1, \dots, i_{r-1}\}$ or $j = i_{s_1}$. We distinguish the following two cases.

Case 1: Suppose $j \in N \setminus \{i_1, \dots, i_{r-1}\}$.

By the construction of s_u, we have $f_{i_{s_u}}(P_N) \neq \tau(P_{i_{s_u}})$. Also, since $s_u \in \{1, \dots, r-1\}$, by Step 1, $f_{i_{s_u}}(P_N) P_{i_{s_u}} a$. Combining the facts $f_{i_{s_u}}(P_N) \neq \tau(P_{i_{s_u}})$ and $f_{i_{s_u}}(P_N) P_{i_{s_u}} a$, we have

$$\tau(P_{i_{s_u}}) \; P_{i_{s_u}} \; f_{i_{s_u}}(P_N) \; P_{i_{s_u}} \; a. \tag{D.12}$$

Also, since $s_u \in \{1, \dots, r-1\}$, by the construction of P_N, we have $P_{i_{s_u}} = P^{(\tau(P_{i_{s_u}});a)}$. This, together with (D.12), implies

$$\tau(P_{i_{s_u}}) \prec f_{i_{s_u}}(P_N) \prec a \quad \text{or} \quad a \prec f_{i_{s_u}}(P_N) \prec \tau(P_{i_{s_u}}). \tag{D.13}$$

Since $j \in N \setminus \{i_1, \dots, i_{r-1}\}$, by the construction of P_N, we have $\tau(P_j) = a$. This, together with (D.13), implies

$$a \; P_j \; f_{i_{s_u}}(P_N) \; P_j \; \tau(P_{i_{s_u}}). \tag{D.14}$$

Since $f_j(P_N) = \tau(P_{i_{s_u}})$, (D.12) implies $f_j(P_N) P_{i_{s_u}} f_{i_{s_u}}(P_N)$. Furthermore, since $f_j(P_N) = \tau(P_{i_{s_u}})$, (D.14) implies $f_{i_{s_u}}(P_N) \; P_j \; f_j(P_N)$. However, the facts $f_j(P_N) P_{i_{s_u}} f_{i_{s_u}}(P_N)$ and $f_{i_{s_u}}(P_N) \; P_j \; f_j(P_N)$ together contradict Pareto efficiency of f at P_N.

Case 2: Suppose $j = i_{s_1}$.

By the construction of $\{s_1, \dots, s_u\}$ and j, we have $\{f_{i_{s_t}}(P_N) \mid t = 1, \dots, u\} = \{\tau(P_{i_{s_t}}) \mid t = 1, \dots, u\}$. Let μ be the allocation such that $\mu(i) = \tau(P_i)$ for all $i \in \{i_{s_t} \mid t = 1, \dots, u\}$ and $\mu(i) = f_i(P_N)$ for all

$i \in N \setminus \{i_{s_t} \mid t = 1, \ldots, u\}$. Clearly, μ Pareto dominates $f(P_N)$ at P_N, which violates Pareto efficiency of f at P_N.

Case 1 and Case 2 together complete Step 2, and Step 1 and Step 2 together complete the proof of Claim D.3. □

Now we complete the proof of Lemma D.5. By Claim D.2, for every given $P_N \in \mathcal{P}_N^{r-1}$, there exists an individual $k \in N \setminus \{i_1, \ldots, i_{r-1}\}$ such that $f_k(P_N) = a$. It remains to show that this individual is unique for all preference profile in $\mathcal{P}_N^{r-1}$, that is, $f_k(P_N) = f_k(\tilde{P}_N) = a$ for all $P_N, \tilde{P}_N \in \mathcal{P}_N^{r-1}$. Assume for contradiction that $f_j(P_N) = f_{\tilde{j}}(\tilde{P}_N) = a$ for some $P_N, \tilde{P}_N \in \mathcal{P}_N^{r-1}$ and $j, \tilde{j} \in N \setminus \{i_1, \ldots, i_{r-1}\}$ such that $j \neq \tilde{j}$.

Consider the preference profile $(\tilde{P}_{i_1}, P_{-i_1}) \in \mathcal{P}_N^{r-1}$. Since $P_N, (\tilde{P}_{i_1}, P_{-i_1}) \in \mathcal{P}_N^{r-1}$, by Claim D.3, we have $f_{i_1}(P_N) = f_{i_1}(\tilde{P}_{i_1}, P_{-i_1})$. Using non-bossiness, $f_{i_1}(P_N) = f_{i_1}(\tilde{P}_{i_1}, P_{-i_1})$ implies

$$f(P_N) = f(\tilde{P}_{i_1}, P_{-i_1}).$$

Continuing in this manner, we can move the preferences of all individuals i_s, $s = 0, \ldots, r-1$, from the preference P_{i_s} to $\tilde{P}_{i_s}$ one by one and obtain

$$f(P_N) = f(\tilde{P}_{i_1}, \ldots, \tilde{P}_{i_{r-1}}, P_{-\{i_1,\ldots,i_{r-1}\}}). \tag{D.15}$$

The fact $f_j(P_N) = a$, together with (D.15), implies $f_j(\tilde{P}_{i_1}, \ldots, \tilde{P}_{i_{r-1}}, P_{-\{i_1,\ldots,i_{r-1}\}}) = a$. Since $j \in N \setminus \{i_1, \ldots i_{r-1}\}$ and $\tau(P_i) = a$ for all $i \in N \setminus \{i_1, \ldots i_{r-1}\}$, it follows from the fact $f_j(\tilde{P}_{i_1}, \ldots, \tilde{P}_{i_{r-1}}, P_{-\{i_1,\ldots,i_{r-1}\}}) = a$ that $f_j(\tilde{P}_{i_1}, \ldots, \tilde{P}_{i_{r-1}}, P_{-\{i_1,\ldots,i_{r-1}\}}) = \tau(P_j)$ and $f_j(\tilde{P}_{i_1}, \ldots, \tilde{P}_{i_{r-1}}, P_{-\{i_1,\ldots,i_{r-1}\}}) P_i f_i(\tilde{P}_{i_1}, \ldots, \tilde{P}_{i_{r-1}}, P_{-\{i_1,\ldots,i_{r-1}\}})$ for all $i \in N \setminus \{i_1, \ldots i_{r-1}, j\}$. Therefore, by moving the preferences of all the individuals $i \in N \setminus \{i_1, \ldots i_{r-1}, j\}$ from P_i to $\tilde{P}_i$, and by applying top-envy-proofness condition every time, we obtain

$$f_j(P_j, \tilde{P}_{-j}) = a. \tag{D.16}$$

Since $f_{\tilde{j}}(\tilde{P}_N) = a$ and $j \neq \tilde{j}$, we have $f_j(\tilde{P}_N) \neq a$. Moreover, $j \in N \setminus \{i_1, \ldots i_{r-1}\}$ implies $\tau(\tilde{P}_j) = a$. Combining the facts $f_j(\tilde{P}_N) \neq a$ and $\tau(\tilde{P}_j) = a$, we obtain $a \tilde{P}_j f_j(\tilde{P}_N)$. However, this, together with (D.16), contradicts strategy-proofness of f. This completes the proof of Lemma D.5. ■

By Lemma D.5, there exists $i_r \in N \setminus \{i_1, \ldots i_{r-1}\}$ such that $f_{i_r}(P_N) = a$ for all $P_N \in \mathcal{P}_N^{r-1}$. Define $\zeta_a^{NI}(v_a^r) = i_r$. This completes the recursive definition of ζ_a^{NI}, and thereby completes the construction of Γ_a.

Similarly for each object, an inheritance tree is constructed. Thus, we have constructed a collection of inheritance trees Γ, based on the assignment rule f.

Now, we prove $f(P_N) = f^\Gamma(P_N)$ for all $P_N \in \mathcal{P}_N$, where f^Γ is the hierarchical exchange rule associated with Γ.

D.2.2 $f(P_N) = f^\Gamma(P_N)$ for all $P_N \in \mathcal{P}_N$

Fix $P_N \in \mathcal{P}_N$. We show $f(P_N) = f^\Gamma(P_N)$. We prove this by induction on the stages of f^Γ at P_N.

Base Case: *Assignments in Stage 1.*

(i) $f_i(P_N) = f_i^\Gamma(P_N)$ for all $i \in W^1(P_N)$, and

(ii) $f_i(P'_N) = f_i(P_N)$ for all $i \in W^1(P_N)$, where $P'_N \in \mathcal{P}_N$ is such that for all $i \in W^1(P_N)$ either $\tau(P'_i) = f_i(P_N)$ or $P'_i = P_i$.

Proof of the Base Case. First, we prove a claim that we use in the proof of the Base Case.

Claim D.4. *Let $i \in N$ and let $a \in E_1(i, P_N)$. Suppose $\tilde{P}_N \in \mathcal{P}_N$ is such that $\tau(\tilde{P}_i) = a$. Then $f_i(\tilde{P}_N) = a$.*

Proof of Claim D.4. By the definition of f^Γ, $a \in E_1(i, P_N)$ implies $\zeta_a^{NI}(v_a^1) = i$ where v_a^1 is the root-node of T_a.[19] By the construction of Γ_a, $\zeta_a^{NI}(v_a^1) = i$ implies that

$$f_i(\bar{P}_N) = a \text{ for all } \bar{P}_N \in \mathcal{P}_N \text{ with } \tau(\bar{P}_j) = a \text{ for all } j \in N. \tag{D.17}$$

Now we show $f_i(\tilde{P}_N) = a$ for all $\tilde{P}_N$ with $\tau(\tilde{P}_i) = a$. Consider the preference profile $(\tilde{P}_i, \hat{P}_{-i})$ such that $\tau(\hat{P}_j) = a$ for all $j \neq i$. By (D.17), we have $f_i(\tilde{P}_i, \hat{P}_{-i}) = a$. Since $\tau(\tilde{P}_i) = a$, $f_i(\tilde{P}_i, \hat{P}_{-i}) = a$, and $\tau(\hat{P}_j) = a$ for all $j \neq i$, we have $f_i(\tilde{P}_i, \hat{P}_{-i}) = \tau(\tilde{P}_i)$ and $f_i(\tilde{P}_i, \hat{P}_{-i}) \hat{P}_j f_j(\tilde{P}_i, \hat{P}_{-i})$ for all $j \neq i$. Therefore, by moving the preferences of all the individuals $j \neq i$ from $\hat{P}_j$ to $\tilde{P}_j$, and by applying top-envy-proofness condition every time, we have $f_i(\tilde{P}_N) = a$. This completes the proof of Claim D.4. □

Now, we proceed to prove the Base Case. First we show (i) of the Base Case. Fix $i \in W^1(P_N)$. We complete the proof for (i) of the Base Case by using another level of induction on the number of individuals in $C_1(i, P_N)$.

Base Case (for (i) of the Base Case). Suppose $|C_1(i, P_N)| = 1$. It follows from the definition of f^Γ that $T_1(i, P_N) \in E_1(i, P_N)$ and $T_1(i, P_N) = \tau(P_i)$. Therefore, by Claim D.4, we have

$$f_i(P_N) = T_1(i, P_N). \tag{D.18}$$

By the definition of f^Γ, $|C_1(i, P_N)| = 1$ means

$$f_i^\Gamma(P_N) = T_1(i, P_N). \tag{D.19}$$

By (D.18) and (D.19), we have $f_i(P_N) = f_i^\Gamma(P_N)$. This completes the proof of Base Case (for (i) of the Base Case). Note that since $P_N \in \mathcal{P}_N$ and $i \in W^1(P_N)$ are chosen arbitrarily, using similar logic as above, we have $f_j(\tilde{P}_N) = f_j^\Gamma(\tilde{P}_N)$ for all $\tilde{P}_N \in \mathcal{P}_N$ and all $j \in W^1(\tilde{P}_N)$ with $|C_1(j, \tilde{P}_N)| = 1$.

[19] Recall that $\Gamma_a = \langle T_a, \zeta_a^{NI}, \zeta_a^{EO} \rangle$.

Induction Hypothesis (for (i) of the Base Case). Let $u \geq 2$. Assume that $f_i(P_N) = f_i^\Gamma(P_N)$ for $|C_1(i, P_N)| = u-1$. Assume, furthermore, that for all $\tilde{P}_N \in \mathcal{P}_N$ and all $j \in W^1(\tilde{P}_N)$ such that $|C_1(j, \tilde{P}_N)| = u-1$, we have $f_j(\tilde{P}_N) = f_j^\Gamma(\tilde{P}_N)$.

We show $f_i(P_N) = f_i^\Gamma(P_N)$ for $|C_1(i, P_N)| = u$. Let $C_1(i, P_N) = \{j_1, \ldots, j_u\}$ such that for all $l = 1, \ldots, u$, $T_1(j_l, P_N) \in E_1(j_{l+1}, P_N)$, where $i = j_1$. Assume for contradiction that $f_{j_1}(P_N) \neq f_{j_1}^\Gamma(P_N)$.

Take $\hat{P}_{j_1} = P_{j_u}$ and $\hat{P}_{j_u} = P_{j_1}$. By the construction of $\hat{P}_{j_1}$ and the definition of f^Γ, it follows that $\tau(\hat{P}_{j_1}) \in E_1(j_1, P_N)$. Since $\tau(\hat{P}_{j_1}) \in E_1(j_1, P_N)$, by Claim D.4, we have

$$f_{j_1}(\hat{P}_{j_1}, \hat{P}_{j_u}, P_{-j_1,j_u}) = f_{j_1}(\hat{P}_{j_1}, P_{-j_1}) = \tau(\hat{P}_{j_1}). \tag{D.20}$$

By the definition of $C_1(i, P_N)$ and the construction of $\hat{P}_{j_u}$, it follows that $|C_1(j_u, (\hat{P}_{j_1}, \hat{P}_{j_u}, P_{-j_1,j_u}))| = |C_1(j_u, (\hat{P}_{j_u}, P_{-j_u}))| = u-1$. Therefore, by Induction Hypothesis (for (i) of the Base Case), we have

$$f_{j_u}(\hat{P}_{j_1}, \hat{P}_{j_u}, P_{-j_1,j_u}) = f_{j_u}^\Gamma(\hat{P}_{j_1}, \hat{P}_{j_u}, P_{-j_1,j_u}), \text{ and} \tag{D.21a}$$

$$f_{j_u}(\hat{P}_{j_u}, P_{-j_u}) = f_{j_u}^\Gamma(\hat{P}_{j_u}, P_{-j_u}). \tag{D.21b}$$

By the definition of f^Γ, we have

$$f_{j_1}^\Gamma(P_N) = \tau(P_{j_1}), \text{ and} \tag{D.22a}$$

$$f_{j_u}^\Gamma(\hat{P}_{j_1}, \hat{P}_{j_u}, P_{-j_1,j_u}) = f_{j_u}^\Gamma(\hat{P}_{j_u}, P_{-j_u}) = \tau(\hat{P}_{j_u}). \tag{D.22b}$$

Since $\hat{P}_{j_u} = P_{j_1}$, combining (D.21) and (D.22b), we obtain

$$f_{j_u}(\hat{P}_{j_1}, \hat{P}_{j_u}, P_{-j_1,j_u}) = f_{j_u}(\hat{P}_{j_u}, P_{-j_u}) = \tau(P_{j_1}). \tag{D.23}$$

Since $f_{j_1}(P_N) \neq f_{j_1}^\Gamma(P_N)$ by our assumption, (D.22a) and (D.23) together imply

$$f_{j_u}(\hat{P}_{j_1}, \hat{P}_{j_u}, P_{-j_1,j_u}) P_{j_1} f_{j_1}(P_N). \tag{D.24}$$

By (D.20) and (D.23), we have

$$f_h(\hat{P}_{j_1}, \hat{P}_{j_u}, P_{-j_1,j_u}) = f_h(\hat{P}_h, P_{-h}) \text{ for all } h = j_1, j_u. \tag{D.25}$$

Since $\hat{P}_{j_1} = P_{j_u}$, by (D.20), we have $f_{j_1}(\hat{P}_{j_1}, \hat{P}_{j_u}, P_{-j_1,j_u}) = \tau(P_{j_u})$, which in particular means

$$f_{j_1}(\hat{P}_{j_1}, \hat{P}_{j_u}, P_{-j_1,j_u}) R_{j_u} f_{j_u}(P_N). \tag{D.26}$$

However, (D.24), (D.25) and (D.26) together contradict pairwise reallocation-proofness of f. This completes the proof of (i) of the Base Case. Note, furthermore, that since $P_N \in \mathcal{P}_N$ and $i \in W^1(P_N)$ are chosen arbitrarily, using similar logic as above, we have

$$f_j(\tilde{P}_N) = f_j^\Gamma(\tilde{P}_N) \text{ for all } \tilde{P}_N \in \mathcal{P}_N \text{ and all } j \in W^1(\tilde{P}_N). \tag{D.27}$$

Now we show (ii) of the Base Case. Fix $P'_N \in \mathcal{P}_N$ such that for all $i \in W^1(P_N)$ either $\tau(P'_i) = f_i(P_N)$ or $P'_i = P_i$. From (i) of the Base Case, we have $f_i(P_N) = f_i^\Gamma(P_N)$ for all $i \in W^1(P_N)$. This, together with the definition of f^Γ, implies

$$f_i(P_N) = \tau(P_i) \text{ for all } i \in W^1(P_N). \tag{D.28}$$

It follows from the construction of P'_N and (D.28) that $\tau(P'_i) = \tau(P_i)$ for all $i \in W^1(P_N)$. This, together with the definition of f^Γ, implies

$$W^1(P_N) \subseteq W^1(P'_N), \text{ and} \tag{D.29a}$$

$$f_i^\Gamma(P'_N) = f_i^\Gamma(P_N) \text{ for all } i \in W^1(P_N). \tag{D.29b}$$

(D.29) and (D.27) together complete the proof of (ii) of the Base Case. This completes the proof of the Base Case. □

Now, we proceed to prove the induction step.

Induction Hypothesis: Fix a stage $t \geq 2$. Assume that

(i) $f_i(P_N) = f_i^\Gamma(P_N)$ for all $i \in W^{t-1}(P_N)$, and

(ii) $f_i(P'_N) = f_i(P_N)$ for all $i \in W^{t-1}(P_N)$, where P'_N is such that for all $i \in W^{t-1}(P_N)$ either $\tau(P'_i) = f_i(P_N)$ or $P'_i = P_i$.

We show

(i) $f_i(P_N) = f_i^\Gamma(P_N)$ for all $i \in W^t(P_N)$, and

(ii) $f_i(P'_N) = f_i(P_N)$ for all $i \in W^t(P_N)$, where P'_N is such that for all $i \in W^t(P_N)$ either $\tau(P'_i) = f_i(P_N)$ or $P'_i = P_i$.

First, we prove a claim.

Claim D.5. *Let $i \in N \setminus W^{t-1}(P_N)$ and let $a \in E_t(i, P_N)$. Suppose $\tilde{P}_N \in \mathcal{P}_N$ is such that $\tilde{P}_j = P_j$ for all $j \in W^{t-1}(P_N)$ and $\tau(\tilde{P}_i, A \setminus F^{t-1}(P_N)) = a$. Then, $f_i(\tilde{P}_N) = a$.*

Proof of Claim D.5. Since $i \in N \setminus W^{t-1}(P_N)$ and $a \in E_t(i, P_N)$, it follows from the definition of f^Γ that there exists $r \geq 1$ such that there is a path $(v_a^1, \ldots, v_a^r)$ in T_a from v_a^1 (root-node of T_a) to v_a^r such that $\zeta_a^{NI}(v_a^r) = i$ and for all $s = 1, \ldots, r-1$, we have $\zeta_a^{NI}(v_a^s) \in W^{t-1}(P_N)$ and $f^\Gamma_{\zeta_a^{NI}(v_a^s)}(P_N) = \zeta_a^{EO}(v_a^s, v_a^{s+1})$. Note that for all $s = 1, \ldots, r-1$, by (i) of the Induction Hypothesis, $f_{\zeta_a^{NI}(v_a^s)}(P_N) = f^\Gamma_{\zeta_a^{NI}(v_a^s)}(P_N)$.

First, we show that $f_i(\bar{P}_N) = a$ for all $\bar{P}_N \in \mathcal{P}_N$ such that $\bar{P}_j = P_j$ for all $j \in W^{t-1}(P_N)$ and $\tau(\bar{P}_j) = a$ for all $j \in N \setminus W^{t-1}(P_N)$. Fix $\bar{P}_N \in \mathcal{P}_N$ such that $\bar{P}_j = P_j$ for all $j \in W^{t-1}(P_N)$ and $\tau(\bar{P}_j) = a$ for all $j \in N \setminus W^{t-1}(P_N)$. If $r = 1$, then $a \in E_1(i, P_N)$, and hence by Claim D.4, we have $f_i(\bar{P}_N) = a$. Suppose $r > 1$. Let $S = \{\zeta_a^{NI}(v_a^s) \mid s = 1, \ldots, r-1\}$. By construction, $S \subseteq W^{t-1}(P_N)$. Consider the preference profile $\hat{P}_N$ such that $\hat{P}_j = P^{(f_j(P_N);a)}$ for all $j \in S$, $\tau(\hat{P}_j) = a$ for all $j \in W^{t-1}(P_N) \setminus S$, and $\hat{P}_j = \bar{P}_j$ for all $j \in N \setminus W^{t-1}(P_N)$. Since $f^{\Gamma}_{\zeta_a^{NI}(v_a^s)}(P_N) = \zeta_a^{EO}(v_a^s, v_a^{s+1})$ and $f_{\zeta_a^{NI}(v_a^s)}(P_N) = f^{\Gamma}_{\zeta_a^{NI}(v_a^s)}(P_N)$, by the construction of Γ_a, we have

$$f_i(\hat{P}_N) = a. \tag{D.30}$$

By the construction of $\hat{P}_N$, $\tau(\hat{P}_j) = a$ for all $j \in N \setminus S$. Since $i \in N \setminus W^{t-1}(P_N)$, $S \subseteq W^{t-1}(P_N)$, and $\tau(\hat{P}_j) = a$ for all $j \in N \setminus S$, by (D.30), we have $f_i(\hat{P}_N) = \tau(\hat{P}_i)$ and $f_i(\hat{P}_N)\hat{P}_j f_j(\hat{P}_N)$ for all $j \in W^{t-1}(P_N) \setminus S$. Therefore, by moving the preferences of all the individuals $j \in W^{t-1}(P_N) \setminus S$ from $\hat{P}_j$ to P_j, and by applying top-envy-proofness condition every time, we have

$$f_i(\underline{P}_N) = a, \tag{D.31}$$

where $\underline{P}_j = \hat{P}_j$ for all $j \notin W^{t-1}(P_N) \setminus S$ and $\underline{P}_j = P_j$ for all $j \in W^{t-1}(P_N) \setminus S$. By the construction of $\underline{P}_N$, for all $j \in W^{t-1}(P_N)$, either $\tau(\underline{P}_j) = f_j(P_N)$ or $\underline{P}_j = P_j$. Therefore, by (ii) of the Induction Hypothesis, we obtain

$$f_j(\underline{P}_N) = f_j(P_N) \text{ for all } j \in W^{t-1}(P_N). \tag{D.32}$$

Take $j \in S$. Consider the preference profile P''_N, where $P''_j = P_j$ and $P''_k = \underline{P}_k$ for all $k \neq j$. Since for all $k \in W^{t-1}(P_N)$, either $\tau(P''_k) = f_k(P_N)$ or $\underline{P}_k = P_k$, by (ii) of the Induction Hypothesis, $f_j(P''_N) = f_j(P_N)$. By (D.32), this means $f_j(P''_N) = f_j(\underline{P}_N)$. Since only individual j changes her preference from $\underline{P}_N$ to P''_N and $f_j(P''_N) = f_j(\underline{P}_N)$, by non-bossiness, we have $f(P''_N) = f(\underline{P}_N)$. By moving the preferences of all individuals $j \in S$ from $\underline{P}_j$ to P_j one by one and every time applying a similar logic, we conclude

$$f(\bar{P}_N) = f(\underline{P}_N). \tag{D.33}$$

Combining (D.31) and (D.33), we have

$$f_i(\bar{P}_N) = a. \tag{D.34}$$

Now we complete the proof of Claim D.5. Take $\tilde{P}_N$ such that $\tilde{P}_j = P_j$ for all $j \in W^{t-1}(P_N)$ and $\tau(\tilde{P}_i, A \setminus F^{t-1}(P_N)) = a$. By (D.34) and the construction of $\bar{P}_N$, we have $f_i(\bar{P}_N) = \tau(\bar{P}_i)$ and $f_i(\bar{P}_N)\bar{P}_j f_j(\bar{P}_N)$ for all $j \notin W^{t-1}(P_N) \cup \{i\}$. Therefore, by moving the preferences of all the individuals $j \notin W^{t-1}(P_N) \cup \{i\}$ from $\bar{P}_j$ to $\tilde{P}_j$, and by applying top-envy-proofness condition every time, we obtain

$$f_i(\bar{P}_i, \tilde{P}_{-i}) = a. \tag{D.35}$$

Since f is strategy-proof, (D.35) implies

$$f_i(\tilde{P}_N)\tilde{R}_i a. \tag{D.36}$$

By the choice of $\tilde{P}_N$, we have $\tilde{P}_j = P_j$ for all $j \in W^{t-1}(P_N)$. By (ii) of the Induction Hypothesis

$$f_j(\tilde{P}_N) = f_j(P_N) \text{ for all } j \in W^{t-1}(P_N). \tag{D.37}$$

Since $\tau(\tilde{P}_i, A \setminus F^{t-1}(P_N)) = a$, (D.36) and (D.37) together imply $f_i(\tilde{P}_N) = a$. This completes the proof of Claim D.5. □

Now the proof of the induction step follows by using similar logic as for the proof of the Base Case with Claim D.5 in place of Claim D.4.

Appendix E Proof of Proposition 7.1

Proof of Proposition 7.1. Let f^Γ be a hierarchical exchange rule on $\mathcal{P}_N$. Assume for contradiction that f^Γ does not satisfy group-wise reallocation-proofness. Then, there must exist $P_N \in \mathcal{P}_N$, a set of individuals $S \subseteq N$, a preference profile $\tilde{P}_S$ of the individuals in S, and an allocation $\hat{\mu}$ of $\{f^\Gamma_S(\tilde{P}_S, P_{-S})\}$ over S where $\hat{\mu} \neq f^\Gamma_S(\tilde{P}_S, P_{-S})$ such that

(i) $\hat{\mu}(i)R_i f^\Gamma_i(P_N)$ for all $i \in S$,

(ii) $\hat{\mu}(j)P_j f^\Gamma_j(P_N)$ for some $j \in S$, and

(iii) $f^\Gamma_i(\tilde{P}_i, \tilde{P}_{S\setminus\{i\}}, P_{-S}) = f^\Gamma_i(\tilde{P}_i, P_{S\setminus\{i\}}, P_{-S})$ for all $i \in S$.

Condition (ii) implies that there exists $i^* \in S$ such that $\hat{\mu}(i^*)P_{i^*} f^\Gamma_{i^*}(P_N)$. Moreover, it follows from the definition of $\hat{\mu}$ that there exists a set of individuals $\{i_1 = i^*, \ldots, i_m\} \subseteq S$ such that $\hat{\mu}(i_h) = f^\Gamma_{i_{h+1}}(\tilde{P}_S, P_{-S})$ for all $h = 1, \ldots, m$. Since $\hat{\mu}(i^*)P_{i^*} f^\Gamma_{i^*}(P_N)$, this, together with Condition (iii) and strategy-proofness of f^Γ, implies $m \geq 2$. Combining all these observations with Condition (i), we have

$$f^\Gamma_{i_{h+1}}(\tilde{P}_S, P_{-S})R_{i_h} f^\Gamma_{i_h}(P_N) \text{ for all } h = 2, \ldots, m, \text{ and} \tag{E.1a}$$

$$f^\Gamma_{i_2}(\tilde{P}_S, P_{-S})P_{i_1} f^\Gamma_{i_1}(P_N). \tag{E.1b}$$

Claim E.1. *$f^\Gamma_{i_h}(P_N) \in A$ for all $h = 1, \ldots, m$.*

Proof of Claim E.1. Suppose $f^\Gamma_{i_2}(P_N) = \emptyset$. Since f^Γ is strategy-proof, $f^\Gamma_{i_2}(P_N) = \emptyset$ implies $f^\Gamma_{i_2}(\tilde{P}_{i_2}, P_{-i_2}) = \emptyset$. This, together with Condition (iii), yields $f^\Gamma_{i_2}(\tilde{P}_S, P_{-S}) = \emptyset$, a contradiction to (E.1b). So, it must be that

$$f^\Gamma_{i_2}(P_N) \neq \emptyset. \tag{E.2}$$

Combining (E.1a) and (E.2), we have $f^{\Gamma}_{i_3}(\tilde{P}_S, P_{-S}) \neq \emptyset$. This, together with Condition (iii), yields $f^{\Gamma}_{i_3}(\tilde{P}_{i_3}, P_{-i_3}) \neq \emptyset$. Since f^{Γ} is strategy-proof, $f^{\Gamma}_{i_3}(\tilde{P}_{i_3}, P_{-i_3}) \neq \emptyset$ implies

$$f^{\Gamma}_{i_3}(P_N) \neq \emptyset. \tag{E.3}$$

Continuing in this manner, we obtain

$$f^{\Gamma}_{i_h}(P_N) \neq \emptyset \text{ for all } h = 1, \ldots, m. \tag{E.4}$$

(E.4) completes the proof of Claim E.1. □

It follows from Claim E.1 that for all $h = 1, \ldots, m$, there exists a stage s_h of f^{Γ} at P_N such that $i_h \in W_{s_h}(P_N)$.

Claim E.2. *$s_{h+1} \leq s_h$ for all $h = 2, \ldots, m$.*

Proof of Claim E.2. Assume for contradiction that there exists a $h^* \in \{2, \ldots, m\}$ such that $s_{h^*} < s_{h^*+1}$. By (E.1a), we have $f^{\Gamma}_{i_{h^*+1}}(\tilde{P}_S, P_{-S}) R_{i_{h^*}} f^{\Gamma}_{i_{h^*}}(P_N)$. We complete the proof of Claim E.2 by distinguishing two cases.

CASE 1: Suppose $f^{\Gamma}_{i_{h^*+1}}(\tilde{P}_S, P_{-S}) P_{i_{h^*}} f^{\Gamma}_{i_{h^*}}(P_N)$.

Since f^{Γ} is Pareto efficient, $f^{\Gamma}_{i_{h^*+1}}(\tilde{P}_S, P_{-S}) P_{i_{h^*}} f^{\Gamma}_{i_{h^*}}(P_N)$ implies that there exists $k \in N \setminus \{i_{h^*}\}$ such that $f^{\Gamma}_k(P_N) = f^{\Gamma}_{i_{h^*+1}}(\tilde{P}_S, P_{-S})$. The facts $f^{\Gamma}_{i_{h^*+1}}(\tilde{P}_S, P_{-S}) P_{i_{h^*}} f^{\Gamma}_{i_{h^*}}(P_N)$ and $f^{\Gamma}_k(P_N) = f^{\Gamma}_{i_{h^*+1}}(\tilde{P}_S, P_{-S})$ together imply $f^{\Gamma}_k(P_N) P_{i_{h^*}} f^{\Gamma}_{i_{h^*}}(P_N)$ and $f^{\Gamma}_k(P_N) \in A$. It follows from the fact $f^{\Gamma}_k(P_N) \in A$ that there exists a stage s_k of f^{Γ} at P_N such that $k \in W_{s_k}(P_N)$. Since $i_{h^*} \in W_{s_{h^*}}(P_N)$, $k \in W_{s_k}(P_N)$, and $f^{\Gamma}_k(P_N) P_{i_{h^*}} f^{\Gamma}_{i_{h^*}}(P_N)$, by Lemma D.3, we have $s_k < s_{h^*}$. This, together with the fact that $s_{h^*} < s_{h^*+1}$, implies $s_k < s_{h^*+1}$. Since $i_{h^*+1} \in W_{s_{h^*+1}}(P_N)$, $k \in W_{s_k}(P_N)$, and $s_k < s_{h^*+1}$, by Lemma D.2, we have

$$f^{\Gamma}_k(P_N) = f^{\Gamma}_k(\tilde{P}_{i_{h^*+1}}, P_{-i_{h^*+1}}). \tag{E.5}$$

Furthermore, the facts $i_{h^*+1} \in W_{s_{h^*+1}}(P_N)$, $k \in W_{s_k}(P_N)$, and $s_k < s_{h^*+1}$ together imply $i_{h^*+1} \neq k$. Since $f^{\Gamma}_k(P_N) \in A$ and $i_{h^*+1} \neq k$, (E.5) implies

$$f^{\Gamma}_k(P_N) \neq f^{\Gamma}_{i_{h^*+1}}(\tilde{P}_{i_{h^*+1}}, P_{-i_{h^*+1}}). \tag{E.6}$$

However, the fact $f^{\Gamma}_k(P_N) = f^{\Gamma}_{i_{h^*+1}}(\tilde{P}_S, P_{-S})$ and Condition (iii) together contradict (E.6).

CASE 2: Suppose $f^{\Gamma}_{i_{h^*+1}}(\tilde{P}_S, P_{-S}) = f^{\Gamma}_{i_{h^*}}(P_N)$.

Since $i_{h^*} \in W_{s_{h^*}}(P_N)$, $i_{h^*+1} \in W_{s_{h^*+1}}(P_N)$, and $s_{h^*} < s_{h^*+1}$, by Lemma D.2, we have

$$f^{\Gamma}_{i_{h^*}}(\tilde{P}_{i_{h^*+1}}, P_{-i_{h^*+1}}) = f^{\Gamma}_{i_{h^*}}(P_N). \tag{E.7}$$

Furthermore, since $f^{\Gamma}_{i_{h^*+1}}(\tilde{P}_S, P_{-S}) = f^{\Gamma}_{i_{h^*}}(P_N)$, Condition (iii) and (E.7) together imply

$$f^{\Gamma}_{i_{h^*}}(\tilde{P}_{i_{h^*+1}}, P_{-i_{h^*+1}}) = f^{\Gamma}_{i_{h^*+1}}(\tilde{P}_{i_{h^*+1}}, P_{-i_{h^*+1}}) = f^{\Gamma}_{i_{h^*}}(P_N). \tag{E.8}$$

However, by Claim E.1, we have $f^{\Gamma}_{i_{h^*}}(P_N) \in A$. Since $f^{\Gamma}_{i_{h^*}}(P_N) \in A$ and $i_{h^*} \neq i_{h^*+1}$, (E.8) implies that $f^{\Gamma}(\tilde{P}_{i_{h^*+1}}, P_{-i_{h^*+1}})$ is not an allocation, a contradiction.

Since Cases 1 and 2 are exhaustive, this completes the proof of Claim E.2. □

Now, we complete the proof of Proposition 7.1. By Claim E.2, we have $s_1 \leq s_2$. Moreover, by (E.1b), we have $f^{\Gamma}_{i_2}(\tilde{P}_S, P_{-S}) P_{i_1} f^{\Gamma}_{i_1}(P_N)$. Since $s_1 \leq s_2$ and $f^{\Gamma}_{i_2}(\tilde{P}_S, P_{-S}) P_{i_1} f^{\Gamma}_{i_1}(P_N)$, using a similar logic as for Case 1 in Claim E.2, we get a contradiction. This completes the proof of Proposition 7.1. ■

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