NONZERO-SUM GAMES OF OPTIMAL STOPPING AND GENERALISED NASH EQUILIBRIUM PROBLEMS*

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Abstract. In the nonzero-sum setting, we establish a connection between Nash equilibria in games of optimal stopping (Dynkin games) and generalised Nash equilibrium problems. In the Dynkin game this reveals novel equilibria with complex structures which have not been previously studied. The reward functions need not be differentiable and we also obtain novel results on the existence and uniqueness of threshold-type equilibria, and on their stability under perturbations to the thresholds.

Key words. optimal stopping, nonzero-sum optimal stopping games, Nash equilibrium, Brownian motion, generalised Nash equilibrium problems

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1. Introduction.

In this paper we establish a connection between Nash equilibria in two different types of game. The first type is the two-player, nonzero-sum Dynkin game of optimal stopping (for general background on optimal stopping problems the reader is referred to [24]). Player $i \in \{1, 2\}$ chooses a stopping time τ_i for a strong Markov process $X = (X_t)_{t \geq 0}$ defined on an interval $(x_\ell, x_r) \subseteq \mathbb{R}$. At time $\tau_1 \wedge \tau_2$ the game ends, each player $i \in \{1, 2\}$ receiving a *reward* $\mathcal{J}_i(\tau_1, \tau_2)$ specified by the *reward functions* f_i, g_i, h_i, where

$$\mathcal{J}_i(\tau_1, \tau_2) := f_i(X_{\tau_i})\mathbf{1}_{\{\tau_i < \tau_{-i}\}} + g_i(X_{\tau_{-i}})\mathbf{1}_{\{\tau_{-i} < \tau_i\}} + h_i(X_{\tau_i})\mathbf{1}_{\{\tau_i = \tau_{-i}\}}, \tag{1.1}$$

the subscript $-i$ denoting the other player. In this context equilibrium strategies (τ_1, τ_2) of the form

$$\tau_1 = \inf\{t \geq 0 : X_t \leq \ell\} \quad \text{and} \quad \tau_2 = \inf\{t \geq 0 : X_t \geq r\}, \tag{1.2}$$

for constants $\ell, r \in (x_\ell, x_r)$ with $\ell < r$, are referred to as *threshold-type* equilibria. A recent example is in [11], in which the thresholds ℓ, r are drawn from the disjoint *strategy spaces* $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively where

$$\mathcal{S}_1 := [x_\ell, a], \qquad \mathcal{S}_2 := [b, x_r], \tag{1.3}$$

for some constants a, b with $x_\ell < a < b < x_r$.

The second type of game is a deterministic *generalised game* [13] (or *abstract economy* [1]) with $n \geq 2$ players, where n will depend on the structure of the equilibrium studied in the Dynkin game. Since the examination of all cases $n \geq 2$ is reserved for future work, however, we focus on $n = 2$ and simply provide an example with $n = 3$.

The connection yields novel equilibria in the Dynkin game. This novelty is threefold. Firstly, while threshold-type equilibria correspond to the case $n = 2$, the cases $n > 2$ yield equilibria with more complex structures which, to the best of our knowledge, have not been previously studied. Secondly we obtain novel equilibria of threshold type, since both cases $a < b$ and $a \geq b$ are permitted. Thirdly the reward functions are not required to be differentiable.

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In the threshold-type case, we also obtain the novel result that the equilibria are unique among Markovian strategies, rather than simply in the class of threshold-type strategies. Finally, we obtain sufficient conditions for threshold-type equilibria to be stable under perturbation of the thresholds. More precisely, we show that if the equilibrium threshold of either player is perturbed within appropriate bounds then the equilibrium is restored in the limit through policy iteration. This property is obtained under more general conditions than in previous work: for example, the resulting sequences of thresholds are not necessarily monotone.

1.1. Setting. We will take X to be Brownian motion on $(0,1)$, absorbed at the boundaries $x_\ell = 0$ and $x_r = 1$. That is, let $W = (W_t)_{t\geq 0}$ be a one-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $\mathbb{F}$ is the universally completed filtration [7, p. 27]. We will write the probability measure as $\mathbb{P}^x$ in the case $\mathbb{P}(\{W_0 = x\}) = 1$, and denote the expectation operator with respect to $\mathbb{P}^x$ by $\mathbb{E}^x$. Then set

$$X_t = W_{t\wedge\zeta}, \tag{1.4}$$

where $\zeta = \inf\{t \geq 0\colon W_t \notin (0,1)\}$. We set $\phi(0) = \phi(1) = 0$ for every measurable function ϕ on $[0,1]$. Let $\mathcal{T}$ denote the set of all $\mathbb{F}$-stopping times with values in $[0,\infty]$ and $\mathcal{B}([0,1])$ denote the Borel σ-algebra on $[0,1]$. For each measurable set A, write the associated first entrance (or 'debut') time of X as

$$D_A \coloneqq \inf\{t \geq 0\colon X_t \in A\} = \inf\{t > 0\colon X_t \in A\} \qquad \text{a.s.} \tag{1.5}$$

(The second equality follows since every point is regular for Brownian motion, see for example [21, Remark 8.2].)

The basic assumption in this paper is the following:

Assumption 1.1. For $i = 1,2$ the functions f_i, g_i and h_i are continuous on $[0,1]$, and satisfy $f_i \leq h_i \leq g_i$ and $f_i(x) = g_i(x) = 0$ for $x \in \{0,1\}$.

Although the link which we establish between games is valid in wide generality, obtaining specific results requires specific choices to be made on the geometry of the reward functions in the Dynkin game. We consider two possible choices:

Assumption 1.2 (Section 4). There exist points $a, b \in (0,1)$, not necessarily satisfying $a \leq b$, such that:

$$\begin{aligned} &(i) \quad f_1 \text{ is concave on } [0,a] \text{ and is convex on } [a,1],\\ &(ii) \quad f_2 \text{ is convex on } [0,b] \text{ and is concave on } [b,1],\\ &(iii) \quad \text{If } b \leq a \text{ then } f_i < g_i \text{ on } [b,a] \text{ for } i = 1,2, \end{aligned}$$

or the more complex

Assumption 1.3 (Section 6). There exist points a_1 and a_2 with $0 < a_1 \leq a_2 < b < 1$ such that:

$$\begin{aligned} &(i) \quad f_1 \text{ is convex on } [0,a_1], \text{ concave on } [a_1,a_2] \text{ and convex on } [a_2,1],\\ &(ii) \quad f_2 \text{ is convex on } [0,b] \text{ and concave on } [b,1], \end{aligned}$$

and we leave the construction of further examples to the reader.

The results of Section 5 require more regularity and there we adopt a strengthened version of Assumption 1.2 (Assumption 5.1). Finally we note that the boundary and inequality constraints in Assumption 1.1 can be weakened somewhat (see Section 3.2 and Remark 4.3 respectively).

1.2. Background. For general background on game theory we refer the reader to [14]. Regarding the structure of Nash equilibria, in nonzero-sum Dynkin games this has recently been investigated in [3] and [11]. There, sufficient conditions for the existence of threshold-type equilibria, and their uniqueness in this class, are obtained. A key difference between the case $n = 2$ of the present paper and the latter work is that there, the functions f_i in (1.1) are twice differentiable and have unique points of inflexion a and b respectively with $a < b$, conditions which may all be relaxed in the present approach. Other differences are the inclusion of time discounting and of linear diffusion models for X, and these points are discussed in our setup in Appendix E.

Our results on stability relate to an iterative approximation scheme for Nash equilibria, which has been previously studied outside the Markovian framework in [15] and, in the Markovian framework, in [6], [9], [17] and [22]. In [17] it is assumed that $f_i = g_i$ and in [6], [9] and [22] a condition related to superharmonicity is imposed for the g_i. The latter conditions ensure monotone convergence over the iteration, whereas the approach via stability in Section 5 does not rely on monotonicity.

The remainder of this paper is organised as follows. In Section 2 the two game settings are presented and connected. Useful alternative expressions for the expected rewards in the Dynkin game are developed in Section 3, and our results on existence, uniqueness and stability for threshold-type equilibria follow in Sections 4 and 5. Finally, in Section 6 we discuss Dynkin game equilibria with more complex stopping regions than the threshold type.

2. Two games.

Our first aim in this work is to establish an equivalence between threshold-type equilibrium strategies in Dynkin games and equilibrium strategies in related static, deterministic games. We begin by remarking on the specification $\ell < r$ in (1.2). It is easy to see that both players' stopping times for threshold-type strategies in the Dynkin game are $\mathbb{P}^x-$almost surely positive if and only if $\ell < r$ and $x \in (\ell, r)$. Therefore, when $\ell \geq r$ in (1.2) the Dynkin game is trivial in that it ends immediately, and so we seek to exclude such cases. We will show that the ordering $\ell < r$ in the threshold-type strategy may be induced by generalising the classical deterministic game. Further, in Section 6 the generalised game also provides a convenient way to explicitly establish player 1's stopping structure in a more complex example.

2.1. Generalised Nash equilibrium.

In the n-player generalised game each player's set of available strategies, or *feasible strategy space*, depends on the strategies chosen by the other $n - 1$ players. The case $n = 2$ is as follows. Player $i \in \{1, 2\}$ has a *strategy space* $\mathcal{S}_i$ and a set-valued map $K_i \colon \mathcal{S}_{-i} \rightrightarrows \mathcal{S}_i$ determining their *feasible* strategy space. Denoting a generic strategy for player i by s_i, a strategy pair (s_1, s_2) is then *feasible* if $s_i \in K_i(s_{-i})$ for $i = 1, 2$. Setting $\mathcal{S}_1 = [0, a]$ and $\mathcal{S}_2 = [b, 1]$, the pair of mappings $K_1 \colon [b, 1] \rightrightarrows [0, a]$ and $K_2 \colon [0, a] \rightrightarrows [b, 1]$ will be given by

$$\begin{aligned} K_1(y) &= [0, y \wedge a], \\ K_2(x) &= [x \vee b, 1], \end{aligned} \tag{2.1}$$

where a and b are given constants lying in the interval $(0, 1)$. That is, the feasible strategy pairs are given by the convex, compact set

$$\mathcal{C} = \{(x, y) \in [0, a] \times [b, 1] \colon x \leq y\}. \tag{2.2}$$

This choice of $\mathcal{C}$ will be appropriate for equilibria of the threshold form (1.2) in the Dynkin game. (The set $\mathcal{C}$ will be modified in Section 6 below, where an example of a more complex equilibrium is studied). Letting $\bar{\mathbb{R}} = [-\infty, +\infty]$ denote the extended real line and writing $U_i : \mathcal{C} \to \bar{\mathbb{R}}$ for the utility function of player i, the generalised Nash equilibrium problem is then given by:

Definition 2.1 (GNEP, $n = 2$). *Find $s^* = (s_1^*, s_2^*) \in \mathcal{C}$ which is a Nash equilibrium, that is:*

$$\begin{cases} U_1(s^*) = \sup\limits_{(s_1, s_2^*) \in \mathcal{C}} U_1(s_1, s_2^*), \\ U_2(s^*) = \sup\limits_{(s_1^*, s_2) \in \mathcal{C}} U_2(s_1^*, s_2). \end{cases} \tag{2.3}$$

It is interesting to note that in the case $a < b$, which is analysed in [3] and [11], the generalised problem (2.3) reduces to a classical game (that is, where each player's feasible strategy space does not depend on the other player's chosen strategy). One advantage of the generalised problem (2.3) is therefore in enabling a natural analysis of the case $a \geq b$ as well.

In the proofs below it will be convenient to write $\mathcal{S} := \mathcal{S}_1 \times \mathcal{S}_2$. We will also make use of the following definition:

Definition 2.2. *Let $s = (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$ and $w \in \mathbb{R}$. Then for each $i \in \{1, \ldots, n\}$ we will write (w, s_{-i}) for the vector s modified by replacing its ith entry with w.*

A useful method for establishing the existence of solutions in such nonzero-sum classical games is to appeal to quasi-concavity (see e.g. [14, p. 34]) and we will use this approach as a tool, providing the necessary details in the Appendix.

2.2. Optimal stopping. We also consider a *Dynkin game* with two players which formalises the one in Section 1 with $x_\ell = 0$ and $x_r = 1$. Each player observes the process X and can stop the game to receive a reward (which may be positive or negative) depending on the process value and on who stopped the game first.

Each player $i \in \{1, 2\}$ chooses a stopping time τ_i lying in $\mathcal{T}$ as their strategy. Let f_i, g_i and h_i be real-valued *reward functions* on $[0, 1]$ which respectively determine the reward to player i from stopping first, second, or at the same time as the other player. For convenience we will refer to the f_i as the *leader* reward functions and to the g_i as the *follower* reward functions. Assumption 1.1 (cf. Section 1.1) makes the game similar to a *war of attrition*, and is commonly assumed in stopping games (see for example [6, 9, 12, 22, 23]). Part (iii) of Assumption 1.2 is a mild strengthening of Assumption 1.1 made for technical reasons.

Given a pair of strategies (τ_1, τ_2) and recalling the reward defined in (1.1), we denote the *expected reward* to player i by

$$M_i^x(\tau_1, \tau_2) = \mathbb{E}^x \left[\mathcal{J}_i(\tau_1, \tau_2) \right]. \tag{2.4}$$

The problem of finding a Nash equilibrium for this Dynkin game is then:

Definition 2.3 (DP). *Find a pair $(\tau_1^*, \tau_2^*) \in \mathcal{T} \times \mathcal{T}$ such that for every $x \in (0, 1)$ we have:*

$$\begin{cases} M_1^x(\tau_1^*, \tau_2^*) = \sup\limits_{\tau_1 \in \mathcal{T}} M_1^x(\tau_1, \tau_2^*) \\ M_2^x(\tau_1^*, \tau_2^*) = \sup\limits_{\tau_2 \in \mathcal{T}} M_2^x(\tau_1^*, \tau_2). \end{cases} \tag{2.5}$$

If $\tau_1^* = D_{S_1}$ and $\tau_2^* = D_{S_2}$ with $S_1, S_2 \in \mathcal{B}([0, 1])$, then the Nash equilibrium (D_{S_1}, D_{S_2}) is said to be Markovian.

2.3. Linking the games. We now present the link between the games in the case $n = 2$, which is the setting used until Section 6, where we consider $n = 3$. The idea is that after a suitable transformation of the stopping problems, threshold-type solutions to the DP can be characterised by the slopes $U_1(x, y)$ and $U_2(x, y)$ of certain secant lines. This gives nothing else than a deterministic game, which may be studied in the above generalised setting in order to discover additional novel equilibria. We will close this section by illustrating that this link between the DP and GNEP does not preserve the zero-sum property.

2.3.1. Construction of utility functions for the GNEP. For $(x, y) \in [0, 1]^2$ we define

$$
U_1(x,y) = \begin{cases} \frac{f_1(x)-g_{1,[y,1]}(x)}{y-x}, & x < y, \\ -\infty, & \text{otherwise}, \end{cases}
\qquad
U_2(x,y) = \begin{cases} \frac{f_2(y)-g_{2,[0,x]}(y)}{y-x}, & x < y, \\ -\infty, & \text{otherwise}, \end{cases} \tag{2.6}
$$

where the functions $g_{1,[y,1]}$ and $g_{2,[0,x]}$ are given by:

$$
g_{1,[y,1]}(x) = \begin{cases} g_1(y) \cdot \frac{x}{y}, & \forall x \in [0, y) \\ g_1(x), & \forall x \in [y, 1], \end{cases} \tag{2.7}
$$

$$
g_{2,[0,x]}(y) = \begin{cases} g_2(y), & \forall y \in [0, x] \\ g_2(x) \cdot \frac{1-y}{1-x}, & \forall y \in (x, 1]. \end{cases} \tag{2.8}
$$

Note the utility functions in (2.6) are continuous and bounded above on $\mathcal{C}$ by Assumption 1.1.

Remark 2.4.

(i) The rationale for the form (2.6) of U_1 is as follows (references to the relevant results below are given in parentheses). Suppose that $(D_{[0,\ell]}, D_{[r,1]})$ is a Nash equilibrium in the DP. Then player 1's strategy solves an optimal stopping problem with obstacle $f_1 - g_{1,[r,1]}$ (Lemma 3.4). The function U_1 characterises this solution under our sufficient conditions (Theorem 4.1 and Assumption 1.2). Similar comments of course apply to player 2.

(ii) The so-called *double smooth-fit condition* in the DP holds when in equilibrium the players' expected rewards, considered as functions of the initial point x, are differentiable across the thresholds ℓ and r respectively (see, for example, [3]). The characterisation described in (i) does not assume smooth reward functions. However if the reward functions are differentiable and the equilibrium thresholds lie away from the boundaries (that is, $(\ell, r) \in (0, a) \times (b, 1)$) then the double smooth-fit condition will be seen to hold (Remark 4.3). If either of the equilibrium thresholds lies at a boundary then double smooth fit does not hold in general (Remark 4.3-(iii)).

(iii) In Section 6 we show that more complex equilibria than the threshold type may be obtained by considering GNEPs with three or more players.

2.3.2. Remark on the zero-sum property. It is interesting to note that the zero-sum property in the DP does not imply the same for the GNEP and vice versa.

Suppose that the GNEP (2.6) has zero sum: that is,

$$\sum_{i=1}^{2} U_i(x,y) = 0, \quad \forall\,(x,y) \in \mathcal{S}. \tag{2.9}$$

Recall the definition of the utility functions in (2.6) and that $f_1(0) = g_2(0) = g_1(1) = f_2(1) = 0$. Then considering separately the case $x = 0$, $y \in [b,1]$ in (2.9) and the case $y = 1$, $x \in [0,a]$, we conclude that $f_1(x) = f_2(y) = 0$, $\forall\,(x,y) \in \mathcal{S}$. Then in the DP, any nonzero choice of the reward functions g_i satisfying Assumption 1.1 results in a game with $f_i \neq -g_{-i}$ and hence is nonzero sum.

On the other hand, suppose that $a < b$ and consider the zero-sum DP with reward functions

$$f_1(x) = \begin{cases} x(a-x), & x \in [0,a] \\ (1-x)(a-x), & x \in (a,1], \end{cases}$$

$$g_1(x) = \begin{cases} x(b-x), & x \in [0,b) \\ (1-x)(b-x), & x \in [b,1], \end{cases}$$

$$f_2 = -g_1, \quad g_2 = -f_1, \quad h_1 = -h_2.$$

Then for $(x,y) \in \mathcal{S}$ the sum of the rewards in the GNEP is

$$\sum_{i=1}^{2} U_i(x,y) = x\left(\frac{a-x}{y-x}\right)\left(1 + \frac{1-y}{1-x}\right) - \left(\frac{(1-y)(b-y)}{y-x}\right)\left(\frac{y+x}{y}\right),$$

which is strictly positive for $(x,y) \in \{0,a\} \times (b,1)$, and so the GNEP is not zero sum.

3. Best responses.

In this section we provide three equivalent expressions for best responses in the Dynkin game. These will be used to establish the existence and uniqueness results of Sections 4 and 5.

3.1. Single player problem.

Suppose that in the Dynkin game, the strategy of player $-i$ is specified by a set $A \in \mathcal{B}([0,1])$ on which that player stops.

Definition 3.1. *A measurable function $\phi\colon [0,1] \to \mathbb{R}$ is said to be* superharmonic *on A if for every $x \in [0,1]$ and $\tau \in \mathcal{T}$:*

$$\phi(x) \geq \mathbb{E}^x[\phi(X_{\tau \wedge D_{A^c}})].$$

A measurable function $\phi\colon [0,1] \to \mathbb{R}$ is said to be subharmonic *on A if $-\phi$ is superharmonic on A, and* harmonic *on A if it is both superharmonic and subharmonic on A. If $A = [0,1]$ then ϕ is simply said to be superharmonic, subharmonic, or harmonic as appropriate.*

Taking $A = [0,1]$ and $\tau = \zeta$ in Definition 3.1, the convention $\phi(0) = \phi(1) = 0$ implies that the superharmonic functions ϕ on $[0,1]$ are non-negative. Moreover, since X is a diffusion on its natural scale, superharmonic (respectively subharmonic and harmonic) functions are concave (resp. convex, linear) on convex subsets of $[0,1]$ (see [10, p. 179]).

The following useful result, the proof of which can be found in [4] or [12] for example, states a key property of the resulting optimal stopping value function for player i.

LEMMA 3.2. *For $A \in \mathcal{B}([0,1])$ and functions f, g and h satisfying Assumption 1.1, the map*

$$x \mapsto V^A(x) := \sup_{\tau\in\mathcal{T}} \mathbb{E}^x \left[f(X_\tau)\mathbf{1}_{\{\tau<D_A\}} + g(X_{D_A})\mathbf{1}_{\{D_A<\tau\}} + h(X_{D_A})\mathbf{1}_{\{\tau=D_A\}} \right],$$

is superharmonic on A^c.

DEFINITION 3.3. *Given a bounded measurable function $\phi\colon [0,1] \to \mathbb{R}$, and recalling the first entrance time defined in (1.5), define $\phi_A\colon [0,1] \to \mathbb{R}$ by*

$$\phi_A(x) := \mathbb{E}^x \left[\phi(X_{D_A})\right]. \tag{3.1}$$

It is not difficult to show (using the strong Markov property) that for any measurable function ϕ, the function ϕ_A is harmonic on A^c. Moreover, it is continuous whenever ϕ is continuous [21, Chapter 8]. The next lemma expresses the optimisation problem for player i as equivalent optimal stopping problems.

LEMMA 3.4. *For $x \in (0,1)$ consider the problems*

$$V^A(x) = \sup_{\tau\in\mathcal{T}} M^x(\tau, D_A), \tag{3.2}$$

$$\bar{V}^A(x) = \sup_{\tau\in\mathcal{T}} \bar{M}^x(\tau, D_A), \tag{3.3}$$

$$\tilde{V}^A(x) = \sup_{\tau\in\mathcal{T}} \tilde{M}^x(\tau, D_A), \tag{3.4}$$

where for $\tau \in \mathcal{T}$ we have

$$M^x(\tau, D_A) := \mathbb{E}^x\big[f(X_\tau)\mathbf{1}_{\{\tau<D_A\}} + g(X_{D_A})\mathbf{1}_{\{D_A<\tau\}} + h(X_{D_A})\mathbf{1}_{\{\tau=D_A\}}\big], \tag{3.5}$$

$$\bar{M}^x(\tau, D_A) := \mathbb{E}^x\big[f(X_\tau)\mathbf{1}_{\{\tau<D_A\}} + g(X_{D_A})\mathbf{1}_{\{\tau\ge D_A\}}\big], \tag{3.6}$$

$$\tilde{M}^x(\tau, D_A) := \mathbb{E}^x\big[\big\{f - g_A\big\}(X_\tau)\mathbf{1}_{\{\tau<D_A\}}\big], \tag{3.7}$$

and f, g and h are functions satisfying Assumption 1.1. Then, recalling Definition 3.3, we have

$$V^A(x) = \bar{V}^A(x) = g_A(x) + \tilde{V}^A(x). \tag{3.8}$$

Proof. Let $\tau \in \mathcal{T}$ and $x \in (0,1)$ be arbitrary. We have $\bar{M}^x(\tau, D_A) \ge M^x(\tau, D_A)$ and therefore $\bar{V}^A(x) \ge V^A(x)$. To show the reverse inequality, first recall from Lemma 3.2 that $x \mapsto V^A(x)$ is superharmonic on A^c. By Assumption 1.1 we have $V^A \ge f$ on $(0,1)$, so that $V^A(X_\tau)\mathbf{1}_{\{\tau<D_A\}} \ge f(X_\tau)\mathbf{1}_{\{\tau<D_A\}}$ a.s., while from the strong Markov property we have $V^A(X_{D_A}) = g(X_{D_A})$ a.s. It follows from (3.6) and superharmonicity that

$$\bar{M}^x(\tau, D_A) \le \mathbb{E}^x\big[V^A(X_{\tau\wedge D_A})\big] \le V^A(x),$$

and taking the supremum over τ we have $\bar{V}^A(x) = V^A(x)$. Finally, recalling Definition 3.3 we have

$$\bar{M}(\tau, D_A) - g_A(x) = \mathbb{E}^x\big[\big\{f - g_A\big\}(X_\tau)\mathbf{1}_{\{\tau<D_A\}}\big]. \tag{3.9}$$

∎

Remark 3.5. It follows from (3.8) that

$$V^A(x) = f(x) \iff \tilde{V}^A(x) = f(x) - g_A(x).$$

That is, defining the *stopping region* to be the subset of A^c on which the obstacle equals the value function, the optimal stopping problems $V^A(x)$ and $\tilde{V}^A(x)$ have identical stopping regions. An easy consequence is that if $x \in A^c$ lies in either stopping region then $f(x) \ge g_A(x)$, and that if $f \le g_A$ on A^c then $\tau = D_A$ is optimal in (3.4).

3.2. Rewards at the boundary. We close this section by noting that the results established in the remainder of the paper will remain true if, instead of requiring $f_i(x) = g_i(x) = 0$ for $x \in \{0,1\}$ in Assumption 1.1, the reward functions merely take equal values at the boundary. This slightly more general setting is customary in the literature on optimal stopping games [3, 12]. For this, it suffices to observe that Lemma 3.2 remains true when the same relaxation is made. (An analogous argument outside the Markovian framework can be found in [19, p. 1920].)

COROLLARY 3.6. *The conditions of Lemma 3.4 may be relaxed to allow* $f_i(x) = h_i(x) = g_i(x) =: H_i(x)$, $x \in \{0,1\}$, $i = 1, 2$.

Proof. Consider the expected reward (3.5) with the additional reward $H(x)$ received at the boundary points $x \in \{0,1\}$. Then recalling the definition of ζ from Section 1.1, the new expected reward has the form:

$$\check{M}^x(\tau, D_A) = \mathbb{E}^x\big[\big\{f(X_\tau)\mathbf{1}_{\{\tau<D_A\}} + g(X_{D_A})\mathbf{1}_{\{\tau>D_A\}} + h(X_{D_A})\mathbf{1}_{\{\tau=D_A\}}\big\}\mathbf{1}_{\{(\tau\wedge D_A)<\zeta\}} + H(X_\zeta)\mathbf{1}_{\{(\tau\wedge D_A)\geq\zeta\}}\big]. \tag{3.10}$$

Then defining $H_{\{0,1\}}$ as in Definition 3.3 (with $\phi = H$ and $A = \{0,1\}$) and using the strong Markov property we can show that,

$$\begin{aligned}\check{M}^x(\tau, D_A) - H_{\{0,1\}}(x) &= \mathbb{E}^x\big[\big\{\tilde{f}(X_\tau)\mathbf{1}_{\{\tau<D_A\}} + \tilde{g}(X_{D_A})\mathbf{1}_{\{\tau>D_A\}} \\ &\quad + \tilde{h}(X_{D_A})\mathbf{1}_{\{\tau=D_A\}}\big\}\mathbf{1}_{\{(\tau\wedge D_A)<\zeta\}}\big] \\ &= \mathbb{E}^x\big[\tilde{f}(X_\tau)\mathbf{1}_{\{\tau<D_A\}} + \tilde{g}(X_{D_A})\mathbf{1}_{\{\tau>D_A\}} + \tilde{h}(X_{D_A})\mathbf{1}_{\{\tau=D_A\}}\big],\end{aligned}$$

where $\tilde{f} = f - H_{\{0,1\}}$, $\tilde{g} = g - H_{\{0,1\}}$ and $\tilde{h} = h - H_{\{0,1\}}$, which is nothing more than (3.5) with these new rewards instead of f, g and h respectively. ∎

4. Existence of threshold-type equilibria. In this section we impose Assumption 1.2 and exploit the link between games to establish existence results for the DP. We show, firstly, that there is an equivalence between solutions to the GNEP with utility functions given by (2.6) and threshold-type solutions to the DP (Theorem 4.1). As shown in the Appendix (Lemma A.3), a standard argument using quasi-concavity establishes the existence of solutions to the GNEP under Assumption 1.2. As a corollary we obtain the existence of threshold-type solutions to the DP (Corollary 4.2). This result includes the case $a > b$, which is novel when compared with the existing literature. The case when at least one of the functions f_i is not differentiable is also novel.

Our first main result is the following.

THEOREM 4.1. *Under Assumption 1.2,* $(\ell, r) \in [0, a] \times [b, 1]$ *with* $\ell < r$ *is a solution to the GNEP* (2.3) *if and only if* $(D_{[0,\ell]}, D_{[r,1]})$ *is a Nash equilibrium in the DP* (2.5).

Proof. We first aim to establish that for every $r \in [b, 1]$, a point $\ell_r \in [0, a]$ with $\ell_r < r$ satisfies,

$$U_1(x, r) \leq U_1(\ell_r, r), \quad \forall x \in [0, r), \tag{4.1}$$

if and only if

$$V_1^{[r,1]}(x) := \sup_{\tau_1\in\mathcal{T}} M_1^x(\tau_1, D_{[r,1]}) = M_1^x(D_{[0,\ell_r]}, D_{[r,1]}), \quad \forall x \in [0, 1]. \tag{4.2}$$

Let $r \in [b,1]$ and $\ell_r \in [0,a]$ with $\ell_r < r$ be given. We will make use of the function

$$u_r(x) := M_1^x(D_{[0,\ell_r]}, D_{[r,1]}) - g_{1,[r,1]}(x)$$

$$= \begin{cases} f_1(x) - g_{1,[r,1]}(x), & x \in [0,\ell_r), \\ \left(f_1(\ell_r) - g_{1,[r,1]}(\ell_r)\right) \frac{r-x}{r-\ell_r}, & x \in [\ell_r, r), \\ 0, & x \in [r,1], \end{cases} \tag{4.3}$$

where the middle line is a straightforward consequence of the expected reward for threshold strategies and the fact that, for $x \in [0,r]$, we have

$$g_1(r)\frac{x-\ell_r}{r-\ell_r} - g_{1,[r,1]}(x) = g_1(r)\left(\frac{x-\ell_r}{r-\ell_r} - \frac{x}{r}\right)$$

$$= g_1(r)\left(\frac{r(x-\ell_r) - x(r-\ell_r)}{r(r-\ell_r)}\right)$$

$$= -g_1(r)\left(\frac{\ell_r}{r}\right)\left(\frac{r-x}{r-\ell_r}\right) = -g_{1,[r,1]}(\ell_r)\frac{r-x}{r-\ell_r}. \tag{4.4}$$

Sufficiency ($\Longleftarrow$).

Suppose that (4.2) is satisfied. Substituting this in (4.4), dividing both sides of (4.3) by $r-x$ (when $x<r$), and using the definition (2.6) of U_1, we obtain

$$\frac{V_1^{[r,1]}(x) - g_{1,[r,1]}(x)}{r-x} = \begin{cases} U_1(x,r), & \forall x \le \ell_r \\ U_1(\ell_r, r), & \forall \ell_r < x < r. \end{cases} \tag{4.5}$$

It is easy to see that $V_1^{[r,1]}(r) = g_1(r) = g_{1,[r,1]}(r)$ and $V_1^{[r,1]}(x) \ge f_1(x)$ for all $x \in [0,r]$. Therefore we have,

$$U_1(\ell_r, r) \ge U_1(x,r), \quad \forall x \in (\ell_r, r). \tag{4.6}$$

To treat the case $x \in [0,\ell_r]$, note from Lemma 3.4 and Lemma 3.2 that $x \mapsto V_1^{[r,1]}(x) - g_{1,[r,1]}(x)$ is superharmonic on $[0,r)$ and also non-negative (to see the latter, take $f = f_1$, $g = g_1$, $A = [r,1]$ and $\tau = D_A$ in (3.7)). For $0 \le x < y \le 1$ define $\tau_{x,y} = D_{\{x\}} \wedge D_{\{y\}}$. Using superharmonicity and the fact that X is a positively recurrent diffusion, for every $0 \le x \le \ell_r$ we have,

$$V_1^{[r,1]}(\ell_r) - g_{1,[r,1]}(\ell_r) \ge \mathbb{E}^{\ell_r}\left[V_1^{[r,1]}(X_{\tau_{x,r}}) - g_{1,[r,1]}(X_{\tau_{x,r}})\right]$$

$$= \left(V_1^{[r,1]}(x) - g_{1,[r,1]}(x)\right)\mathbb{E}^{\ell_r}\left[\mathbf{1}_{\{D_{\{x\}} < D_{\{r\}}\}}\right]$$

$$= \left(V_1^{[r,1]}(x) - g_{1,[r,1]}(x)\right)\frac{r-\ell_r}{r-x}. \tag{4.7}$$

Since for all $0 \le x \le \ell_r$ we have $V_1^{[r,1]}(x) = f_1(x)$, (4.7) gives

$$U_1(x,r) \le U_1(\ell_r, r), \quad \forall x \in [0,\ell_r],$$

and together with (4.6) establishes (4.1).

Necessity ($\Longrightarrow$).

Suppose that the pair (ℓ_r, r) satisfies (4.1) with $\ell = \ell_r$. We will establish (4.2) by showing that

$$u_r(x) = V_1^{[r,1]}(x) - g_{1,[r,1]}(x), \quad \forall\, x \in [0,1]. \tag{4.8}$$

By construction (4.8) holds for $x \in [r, 1]$, and so we restrict attention to the domain $[0, r]$. By Lemma 3.4 it is sufficient to show that u_r is the value function of the optimal stopping problem on $[0, r]$ with the obstacle $\vartheta \coloneqq f_1 - g_{1,[r,1]}$. Therefore using Proposition 3.2 in [10], it is enough to show that u_r is the smallest non-negative concave majorant of ϑ on $[0, r]$. The majorant property on $[\ell_r, r)$ follows from (4.1), which gives

$$f_1(x) - g_{1,[r,1]}(x) \leq \left(f_1(\ell_r) - g_{1,[r,1]}(\ell_r)\right)\left(\frac{r - x}{r - \ell_r}\right), \;\; \forall\, x \in [0, r), \tag{4.9}$$

and the majorant property at $x = r$ follows from recalling that $f_1(r) \leq g_1(r)$. For nonnegativity we first recall that the reward functions are null at the boundaries, so taking $x = 0$ in (4.9) gives $0 \leq f_1(\ell_r) - g_{1,[r,1]}(\ell_r) = u_r(\ell_r)$. Combining this with the fact that u_r equals the obstacle on $[0, \ell_r]$, and hence is concave there, establishes nonnegativity. For concavity we note that u_r is a straight line on $[\ell_r, r]$, so it remains only to consider any $x_1 \in [0, \ell_r)$ and $x_2 \in (\ell_r, r]$. Then we have

$$\begin{aligned}
\frac{x_2 - \ell_r}{x_2 - x_1} u_r(x_1) + \frac{\ell_r - x_1}{x_2 - x_1} u_r(x_2) &= \frac{x_2 - \ell_r}{x_2 - x_1}[f_1(x_1) - g_{1,[r,1]}(x_1)] \\
&\quad + \frac{\ell_r - x_1}{x_2 - x_1}\left(f_1(\ell_r) - g_{1,[r,1]}(\ell_r)\right)\left(\frac{r - x_2}{r - \ell_r}\right) \\
&\leq \frac{x_2 - \ell_r}{x_2 - x_1}\left(f_1(\ell_r) - g_{1,[r,1]}(\ell_r)\right)\left(\frac{r - x_1}{r - \ell_r}\right) \\
&\quad + \frac{\ell_r - x_1}{x_2 - x_1}\left(f_1(\ell_r) - g_{1,[r,1]}(\ell_r)\right)\left(\frac{r - x_2}{r - \ell_r}\right) \\
&= f_1(\ell_r) - g_{1,[r,1]}(\ell_r) = u_r(\ell_r),
\end{aligned}$$

where the inequality follows from (4.1). Finally, since u_r equals the obstacle on $[0, \ell_r]$ and is a straight line on $[\ell_r, r]$, it is smaller than any other nonnegative concave majorant on $[0, r]$.

It may be proved similarly that for every $\ell \in [0, a]$, a point $r_\ell \in [b, 1]$ with $\ell < r_\ell$ satisfies,

$$U_2(\ell, y) \leq U_2(\ell, r_\ell), \quad \forall y \in (\ell, 1], \tag{4.10}$$

if and only if

$$V_2^{[0,\ell]}(x) \coloneqq \sup_{\tau_2 \in \mathcal{T}} M_2^x(D_{[0,\ell]}, \tau_2) = M_2^x(D_{[0,\ell]}, D_{[r_\ell,1]}), \quad \forall x \in [0, 1]. \tag{4.11}$$

The proof concludes by noticing that for each $r \in [b, 1]$ and $\ell \in [0, a]$,

$$\sup_{x \in [0,r)} U_1(x, r) = \sup_{x \in [0, a \wedge r]} U_1(x, r), \tag{4.12}$$

$$\sup_{y \in (\ell,1]} U_2(\ell, y) = \sup_{y \in [\ell \vee b, 1]} U_2(\ell, y). \tag{4.13}$$

For $r \in (a, 1]$, eq. (4.12) follows from the convexity of $f_1 - g_{1,[r,1]}$ on $[a, r]$ and the fact that $f_1(r) \le g_1(r) = g_{1,[r,1]}(r)$:

$$\frac{f_1(x) - g_{1,[r,1]}(x)}{r-x} \le \frac{f_1(a) - g_{1,[r,1]}(a)}{r-a} + \left(\frac{f_1(r) - g_{1,[r,1]}(r)}{r-a}\right)\left(\frac{x-a}{r-x}\right)$$

$$\le \frac{f_1(a) - g_{1,[r,1]}(a)}{r-a}, \quad \forall x \in (a,r).$$

The boundary case $x = r$ is excluded since $U_1(r,r) = -\infty$. Similar reasoning establishes (4.13). ◻

COROLLARY 4.2. *Under Assumption 1.2, there exists a pair* $(\ell_*, r_*) \in [0,a] \times [b,1]$ *such that* $(D_{[0,\ell_*]}, D_{[r_*,1]})$ *is a solution to the DP.*

Proof. See Appendix. ◻

Remark 4.3.

(i) Suppose the leader reward functions are differentiable. Then the smooth-fit condition can now easily be obtained for player 1 by differentiating (2.6) at $x = \ell$ and applying (4.1). Smooth fit for player 2, and hence the double smooth-fit condition, follows similarly.

(ii) It follows from the proof of Theorem 4.1 that Assumption 1.1 may be weakened. For example, taking $h_i = g_i$ for simplicity, it is sufficient to assume that $f_i \le g_i$ on $\mathcal{S}_{-i}$.

(iii) Note that thresholds may lie at boundaries: for example, the case $\ell = 0$ is possible. Since the boundaries are absorbing and the rewards are zero there, stopping then becomes irrelevant for player 1. This case is therefore equivalent to player 1 never stopping. Similarly the case $r = 1$ is possible, and is equivalent to player 2 never stopping. In such cases the double smooth-fit condition (Remark 2.4-(ii)) does not hold in general, even when the reward functions are smooth. In Section 5 we provide a condition (Assumption 5.1-4)) which is sufficient to exclude such boundary cases.

5. Stability and uniqueness results. In this section we exploit the above connection to obtain additional novel results for Nash equilibria in the DP. We define a concept of stability and provide a sufficient condition under which it holds locally (Corollary 5.3), showing in Theorem 5.5 that this condition always holds in the particular case of zero-sum Dynkin games. By establishing global stability, Theorem 5.6 provides sufficient conditions for uniqueness of the threshold-type equilibrium of Corollary 4.2 among the Markovian strategies. Theorem 5.9 provides an additional novel uniqueness result for the DP.

5.1. Policy iteration. We will apply the *Gauss-Seidel policy iteration* or *tâtonnement process* [5, 14] to the GNEP. This iteration scheme has previously been used for Dynkin games in [9] and [17] and, outside the Markovian framework, in [15]. Throughout Section 5, for ease of exposition we strengthen Assumption 1.2 to the following:

Assumption 5.1. Assumption 1.2 holds, with:

1) $a < b$,
2) strict convexity and strict concavity,
3) $f_i, g_i \in C^2[0,1]$, and
4) For all $(x,y) \in [0,a] \times [b,1]$ there exists $(\hat{x},\hat{y}) \in (0,a] \times [b,1)$ with $f_1(\hat{x}) > g_1(y) \cdot \frac{\hat{x}}{y}$ and $f_2(\hat{y}) > g_2(x) \cdot \frac{1-\hat{y}}{1-x}$.

Parts 1) and 3) of Assumption 5.1 imply that the GNEP utility functions are finite and smooth on $\mathcal{S}$, which is convenient for the policy iteration. Part 2) says that f_1 is strictly concave on $[0,a]$ and strictly convex on $[a,1]$, and f_2 is strictly convex on $[0,b]$ and strictly concave on $[b,1]$. This ensures that iteration (i) below is well defined. Part 4) removes the need to consider the points 0 and 1 as candidate thresholds during the iteration, which is convenient since the principle of smooth fit (used below) may break down there. Recalling the equality (3.8), this is straightforward to see from (3.4), (3.7) and (2.7)–(2.8). Similarly, Part 4) also ensures that threshold-type equilibria have their thresholds in $(0,1)$ and not at either boundary 0 or 1.

Taking $\ell^{(1)} \in [0,a]$, we consider the following two iteration schemes:

(i) **In the GNEP**: taking $r^{(1)} = \arg\max_{y\in[b,1]} U_2(\ell^{(1)}, y)$, for $n \geq 2$ define

$$\ell^{(n)} = \arg\max_{x\in[0,a]} U_1(x, r^{(n-1)}), \quad r^{(n)} = \arg\max_{y\in[b,1]} U_2(\ell^{(n)}, y). \tag{5.1}$$

(ii) **In the DP**: taking $A_1 = [0, \ell^{(1)}]$, for $n \geq 1$ define

$$\begin{aligned} &(i) && V_{2n}(x) = \sup_\tau \bar{M}_2^x(\tau, D_{A_{2n-1}}),\\ &(ii) && A_{2n} = \{x \in [0,1] \setminus A_{2n-1} \colon V_{2n}(x) = f_2(x)\},\\ &(iii) && V_{2n+1}(x) = \sup_\tau \bar{M}_1^x(\tau, D_{A_{2n}}),\\ &(iv) && A_{2n+1} = \{x \in [0,1] \setminus A_{2n} \colon V_{2n+1}(x) = f_1(x)\}, \end{aligned} \tag{5.2}$$

where $\bar{M}_i^x(\tau, D_A)$, $i \in \{1,2\}$, is given by (3.6) with $f = f_i$ and $g = g_i$.

We will call a solution $s^* = (\ell^*, r^*)$ to the GNEP (2.3) *globally stable* if for any $\ell^{(1)} \in [0,a]$ the iteration (5.1) satisfies $\ell^{(n)} \to \ell^*$ and $r^{(n)} \to r^*$, and *locally stable* if this convergence holds only for $\ell^{(1)}$ in a neighbourhood of ℓ^*. Similarly we call a threshold-type solution $s' = (D_{[0,\ell']}, D_{[r',1]})$ to the DP (2.5) globally stable if for any $\ell^{(1)} \in [0,a]$ the iteration (5.2) satisfies

$$\begin{aligned} \liminf_{n\to\infty} A_{2n-1} &= \limsup_{n\to\infty} A_{2n-1} = [0,\ell'],\\ \liminf_{n\to\infty} A_{2n} &= \limsup_{n\to\infty} A_{2n} = [r',1], \end{aligned}$$

and locally stable if convergence holds only for $\ell^{(1)}$ in a neighbourhood of ℓ'.

5.2. Local stability. We will appeal to the following local stability result for the GNEP:

PROPOSITION 5.2 (Theorem 1.2.3, [18]). *Suppose that Assumption 5.1 holds and that $(\ell_*, r_*) \in (0,a) \times (b,1)$ is a solution to the GNEP. For $w \in \mathcal{S}_1$ set*

$$\begin{aligned} \bar{y} &= \bar{y}(w) = \arg\max_{y\in\mathcal{S}_2} U_2(w,y),\\ \bar{x} &= \bar{x}(w) = \arg\max_{x\in\mathcal{S}_1} U_1(x, \bar{y}(w)), \end{aligned} \tag{5.3}$$

and

$$T(w,\bar{x},\bar{y}) \coloneqq \frac{\partial_{xy} U_1(\bar{x},\bar{y})}{\partial_{xx} U_1(\bar{x},\bar{y})} \frac{\partial_{xy} U_2(w,\bar{y})}{\partial_{yy} U_2(w,\bar{y})}.$$

If it is true that

$$\rho_0 = |T(\ell_*, \ell_*, r_*)| < 1, \tag{5.4}$$

then there exists $\delta > 0$ *such that* $\forall\, \ell^{(1)} \in [0,a]$ *satisfying* $|\ell^{(1)} - \ell_*| < \delta$*, the sequence* $\{\ell^{(n)}\}_{n\geq 1}$ *in* (5.1) *converges to* ℓ_**. The convergence is exponential: for any* $\varepsilon > 0$ *there exists a positive constant* $c(\ell^{(1)};\varepsilon)$ *such that*

$$|\ell^{(n)} - \ell_*| \leq c(\ell^{(1)};\varepsilon)(\rho_0 + \varepsilon)^n. \tag{5.5}$$

Our next result is on local stability for the DP.

Corollary 5.3. *Suppose Assumption* 5.1 *holds. If* $(D_{[0,\ell_*]}, D_{[r_*,1]})$ *is a solution to the DP such that* (5.4) *holds, then it is locally stable.*

Proof. We have from Assumption 5.1 that (ℓ_*, r_*) lies in $(0,a)\times(b,1)$ and, from Theorem 4.1, that it is a solution to the GNEP. Applying Proposition 5.2, take $\ell^{(1)} \in [0,a]$ satisfying $|\ell^{(1)} - \ell_*| < \delta$ and consider the iteration given by (5.1). This yields sequences $(\ell^{(n)}) \to \ell_*$ and $(r^{(n)}) \to r_*$, taking values respectively in $(0,a)$ and $(b,1)$. The proof of Lemma 3.4 and (4.11) then show that the stopping time $D_{[r^{(n)},1]}$ is optimal in (5.2)-i) if $A_{2n-1} = [0,\ell^{(n)}]$. Similarly, the stopping time $D_{[0,\ell^{(n+1)}]}$ is optimal in (5.2)-iii) if $A_{2n} = [r^{(n)},1]$.

Next we establish that the stopping region A_2 is given by $[r^{(1)},1]$. From Remark 3.5, we may study the optimal stopping problem (5.2)-i) in either of its equivalent forms (3.2) or (3.4) (taking $f = f_2$, $g = g_2$ and $A = A_1 = [0,\ell^{(1)}]$). Using (3.2), it is immediate from the strict convexity of the obstacle f_2 on $[\ell^{(1)},b]$ and Dynkin's formula that $A_2 \cap [\ell^{(1)},b] = \emptyset$. On the other hand, considering problem (3.4) it follows from the strict concavity of the obstacle $f_2 - g_{2,A_1}$ on $[b,1]$ and the smooth fit principle that the obstacle lies strictly below the value function on $[b, r^{(1)})$, establishing that $A_2 = [r^{(1)},1]$. Arguing similarly for A_3 and then proceeding inductively we obtain $A_{2n+1} = [0,\ell^{(n+1)}]$ and $A_{2n+2} = [r^{(n+1)},1]$ for all n. ∎

Remark 5.4. The fact that A_1 is an interval plays no role in the above proof, which only uses the inclusion $A_1 \subseteq [0,a]$.

Local stability in the zero-sum DP. We also establish the following result on local stability of equilibria in the zero-sum DP, that is, when $f_i = -g_{-i}$, $i \in \{1,2\}$. The result is novel to the best of our knowledge.

Theorem 5.5. *Under Assumption* 5.1 *every threshold-type solution of the zero-sum DP is locally stable.*

Proof. Let a threshold-type solution $(D_{[0,\ell_*]}, D_{[r_*,1]})$ be given for the DP. We have $V_1^{[r_*,1]} + V_2^{[0,\ell_*]} = 0$. Using the principle of smooth fit we get,

$$\begin{aligned} -g_2'(\ell_*) = f_1'(\ell_*) &= \frac{g_1(r_*) - f_1(\ell_*)}{r_* - \ell_*} \\ &= \frac{[-f_2(r_*) + g_2(\ell_*)]}{r_* - \ell_*} = -f_2'(r_*) = g_1'(r_*). \end{aligned}$$

Using the expressions for U_1 and U_2 in (2.6), the general expressions for the partial derivatives of the utility functions, and the smooth fit principle at $(w,\bar y)$ and $(\bar x,\bar y)$, one can show that

$$T(w,\bar x,\bar y) = \left(\frac{f_1'(\bar x) - g_1'(\bar y)}{f_1''(\bar x)(\bar y - \bar x)}\right)\left(\frac{g_2'(w) - f_2'(\bar y)}{f_2''(\bar y)(\bar y - w)}\right). \tag{5.6}$$

In this zero-sum context we therefore have $T(\ell_*,\ell_*,r_*) = 0$, and the local stability of the equilibrium point now follows from Proposition 5.2. ∎

5.3. Global stability and uniqueness. There is a stronger version of the criterion (5.4) that guarantees the iteration scheme to converge irrespective of player 1's initial strategy $\ell^{(1)} \in [0,a]$. Furthermore, the equilibrium strategy (ℓ_*, r_*) thus obtained is unique.

THEOREM 5.6. *Suppose that Assumption 5.1 holds and that the reward functions f_i and g_i satisfy*

$$\sup_{w \in \mathcal{S}_1} \left| \left(\frac{f_1'(\bar{x}) - g_1'(\bar{y})}{f_1''(\bar{x})(\bar{y} - \bar{x})} \right) \left(\frac{g_2'(w) - f_2'(\bar{y})}{f_2''(\bar{y})(\bar{y} - w)} \right) \right| < 1, \tag{5.7}$$

where $\bar{y} = \bar{y}(w)$ and $\bar{x} = \bar{x}(w)$ are defined by (5.3). Then there exists $(\ell_*, r_*) \in (0,a) \times (b,1)$ such that $(D_{[0,\ell_*]}, D_{[r_*,1]})$ is a solution to the DP. This solution is stable, and is unique in the class of Markovian strategies (D_{S_1}, D_{S_2}) for closed stopping sets $S_1 \subseteq [0,a]$ and $S_2 \subseteq [b,1]$.

Proof. Under Assumption 5.1 every solution (ℓ_*, r_*) to the GNEP lies in $(0,a) \times (b,1)$. A standard contraction argument then shows that under (5.7), there exists a unique solution (ℓ_*, r_*) to the GNEP and, further, that it is globally stable (see for example Theorem 1 in [20] or Proposition 4.1 in [5]; see also Theorem 1.2.1 in [18]).

Thus from Theorem 4.1, $(D_{[0,\ell_*]}, D_{[r_*,1]})$ is a solution to the DP. The fact that it is stable follows from the corresponding property in the GNEP. Suppose that the DP has another solution $(D_{[0,\ell]}, D_{[r,1]})$ with $\ell < r$. Again arguing as in Corollary 5.3, the reward function geometry gives $\ell \in [0,a]$ and $r \in [b,1]$. Therefore (ℓ, r) is a solution to the GNEP and we have $\ell = \ell_*$ and $r = r_*$ by uniqueness.

Suppose that (D_{S_1}, D_{S_2}) is an equilibrium with closed stopping sets $S_1 \subseteq [0,a]$ and $S_2 \subseteq [b,1]$. Recalling Remark 5.4, now consider applying the iteration (ii) above, modified by choosing $A_1 = S_1$, to obtain $A_2 = [r,1]$, say. Then by optimality $S_2 \subseteq A_2$. Finally it is not difficult to see from a standard 'small ball' argument that the strict concavity of f_2 on $[b,1]$ implies that $A_2 \setminus S_2 = \emptyset$. We conclude similarly that A_1 has the form $[0,\ell]$, completing the proof. ∎

Remark 5.7. The sets S_1 and S_2 in Theorem 5.6 are closed in order to avoid trivialities, since every point is regular for standard Brownian motion. Note that the theorem establishes uniqueness among the Markovian strategies, rather than uniqueness among the subset of threshold-type strategies (cf. [11]).

5.4. Examples. We begin this section by constructing an example DP satisfying the global stability condition (5.7). This example is then used to derive a second DP for which local stability, but not global stability, holds. Finally, we discuss local stability of the zero-sum DP.

Global stability. Suppose that $b - a > \frac{1}{2}$ and that F_i, G_i are functions satisfying Assumption 5.1 and furthermore,

$$F_1(x) = x(\tfrac{a}{2} - x), \quad x \in [0, \tfrac{a}{2}].$$

It follows from Assumption 5.1 that F_1 is negative on $[\frac{a}{2}, 1]$. Therefore, for every $w \in \mathcal{S}_1$ the 'best response' $\bar{x}(w)$ to $\bar{y}(w)$ takes values in $[0, \frac{a}{2}]$, where we have the inequality

$$\left| \frac{F_1'(x)}{F_1''(x)} \right| = \left| x - \tfrac{a}{4} \right| \leq \tfrac{1}{4}.$$

Since G_1' is bounded on $[0,a]$ by Assumption 5.1, and recalling that $\bar{y} \in [b,1]$ by definition, for a sufficiently large constant $R_1 > 0$ we have:

$$\left| \frac{F_1'(\bar{x}) - \frac{1}{R_1} G_1'(\bar{y})}{F_1''(\bar{x})(\bar{y} - \bar{x})} \right| \leq 2 \cdot \frac{1}{4} \cdot \frac{1}{b-a} < 1.$$

Therefore if player 1's reward functions in the DP are $f_1 = F_1$ and $g_1 = \frac{1}{R_1} G_1$ (which clearly satisfy Assumption 5.1), then the left hand parenthesis in (5.7) has absolute value less than 1. Similarly if we take $F_2(x) = (x - \frac{b+1}{2})(1-x)$ for all $x \in [\frac{b+1}{2}, 1]$ and let player 2's reward functions be $f_2 = F_2$ and $g_2 = \frac{1}{R_2} G_2$ for a sufficiently large constant R_2, the right hand parenthesis in (5.7) has absolute value less than 1 and so the global stability condition (5.7) holds.

Remark 5.8. Under Assumption 1.1 the reward functions in the DP must satisfy $f_i \leq g_i$ on $[0,1]$. Given the choice of g_i in the example above, $f_i \leq g_i$ implies that the rather strong condition $G_i \geq R_i F_i$ on $[0,1]$ must hold. Although Remark 4.3 shows that $G_i \geq R_i F_i$ is only needed on $\mathcal{S}_{-i}$, there are alternative choices for g_i that satisfy Assumption 1.1 and lead to a conclusion similar to that of the example above. More specifically, in the case $i = 1$, take any $G_1 \geq \max(0, F_1)$ which is in $C^2[0,1]$. We can define a function g_1 which is in $C^2[0,1]$, equal to G_1 on $[0, \frac{a}{2}]$, dominates f_1 on $[0,1]$, and on $[b,1]$ its derivative g_1' is sufficiently small. For example, let $x \mapsto \eta(x)$ be the standard mollifier,

$$\eta(x) = \begin{cases} C \exp\left(\frac{1}{x^2-1}\right), & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

where $C > 0$ is chosen so that $\int_{\mathbb{R}} \eta(x)dx = 1$. For $\epsilon > 0$ define $\eta_\epsilon(x) := \frac{1}{\epsilon}\eta(\frac{x}{\epsilon})$, $H_\epsilon(x) = \int_{-\infty}^{x} \eta_\epsilon(y)dy$ and set $g_1(x;\epsilon) = H_\epsilon(\frac{a}{2} - x + \epsilon)G_1(x)$. For $x \leq \frac{a}{2}$ we have $g_1(x;\epsilon) = G_1(x) \geq F_1(x) = f_1(x)$. For $x \geq \frac{a}{2}$ we have $g_1(x;\epsilon) \geq 0 \geq F_1(x) = f_1(x)$ and, for an appropriate choice of ϵ, $g_1'(x;\epsilon) = 0$ on $[b,1]$.

Local stability only. Global stability implies that the local stability condition (5.4) holds at the unique Nash equilibrium (ℓ_*, r_*) in the DP we have just constructed. Taking the same reward functions in the DP, suppose now that player 1's strategy is $w_0 \in \mathcal{S}_1$ and that player 2's best response is r_*. Then from the smooth fit condition for player 2, the point $(w_0, g_2(w_0))$ must lie on the straight line tangent to f_2 at $(r_*, f_2(r_*))$. We may therefore conclude that if g_2 is not linear on $\mathcal{S}_1$, then there exists a strategy $w_0 \in \mathcal{S}_1 \setminus \{\ell_*\}$ for player 1 to which player 2's best response is $y_0 \in \mathcal{S}_2 \setminus \{r_*\}$. It is also not difficult to see that $y_0 \in (\frac{b+1}{2}, 1)$, and hence smooth fit holds at y_0, provided that g_2 is bounded above by the tangent to f_2 at $(1, f_2(1))$.

Next we remark that the function f_2 may be arbitrarily 'flattened' in a small neighbourhood of y_0 without violating Assumption 5.1. That is, let N_0 be an open neighbourhood of y_0 whose closure does not contain r_* and let $\epsilon \in (f_2''(y_0), 0)$. Then f_2 may be modified on N_0 to produce a new function $\tilde{f}_2$ with

$$\begin{aligned} \tilde{f}_2(y) &= f_2(y), \qquad y \in \{y_0\} \cup N_0^c, \\ \tilde{f}_2'(y_0) &= f_2'(y_0), \\ \tilde{f}_2''(y_0) &= \epsilon, \end{aligned}$$

and such that Assumption 5.1 holds for the reward functions f_1, $\tilde{f}_2$ and g_i. By construction, the smooth fit condition continues to hold at y_0 when f_2 is replaced by $\tilde{f}_2$, so that y_0 remains player 2's best response to w_0. In this way the right hand multiplicand in (5.7) may be made arbitrarily large in absolute value when $w = w_0$ (provided the numerator is non-zero, a mild condition). We thus obtain a DP satisfying Assumption 5.1 which has local, but not global, stability.

5.5. Uniqueness of Nash equilibria. We close this section with a final result on uniqueness of equilibria in the DP by applying a well known condition from [26] for uniqueness of a solution to the GNEP.

THEOREM 5.9. *Suppose that Assumption 5.1 holds,*

$$f_1''(x) \leq -2\frac{f_1(x) + f_1'(x)(y-x) - g_1(y)}{(y-x)^2}, \quad \forall\,(x,y) \in (0,a) \times [b,1], \tag{5.8}$$

$$f_2''(y) \leq -2\frac{f_2(y) - f_2'(y)(y-x) - g_2(x)}{(y-x)^2}, \quad \forall\,(x,y) \in [0,a] \times (b,1), \tag{5.9}$$

and $\exists\,(r_1,r_2) \in [0,\infty) \times [0,\infty)$ *such that* $\forall\,(x,y) \in [0,a] \times [b,1]$,

$$4r_1r_2H_1(x,y)H_2(x,y) - \big(r_1H_3(x,y) + r_2H_4(x,y)\big)^2 > 0, \tag{5.10}$$

where $H_1,\dots,H_4$ *are given by,*

$$\begin{aligned} H_1(x,y) &= f_1''(x)(y-x)^2 + 2\big[f_1(x) + f_1'(x)(y-x) - g_1(y)\big] \\ H_2(x,y) &= f_2''(y)(y-x)^2 + 2\big[f_2(y) - f_2'(y)(y-x) - g_2(x)\big] \\ H_3(x,y) &= 2\big[g_1(y) - f_1(x)\big] - (f_1'(x) + g_1'(y))(y-x) \\ H_4(x,y) &= 2\big[g_2(x) - f_2(y)\big] + (g_2'(x) + f_2'(y))(y-x). \end{aligned} \tag{5.11}$$

Then there exists a unique solution $(\ell_*, r_*) \in [0,a] \times [b,1]$ *to the GNEP* (2.3)*, and therefore* $(D_{[0,\ell_*]}, D_{[r_*,1]})$ *is the unique solution to the DP in the class of Markovian strategies* (D_{S_1}, D_{S_2}) *for closed stopping sets* $S_1 \subseteq [0,a]$ *and* $S_2 \subseteq [b,1]$.

Proof. Conditions (5.8)–(5.9) ensure that each utility function $s_i \mapsto U_i(s_i, s_{-i})$, $i \in \{1,2\}$, is concave on $\mathcal{S}_i$ for each $s_{-i} \in \mathcal{S}_{-i}$. The condition (5.10) is sufficient for *strict diagonal concavity* according to Theorem 6 of [26]. The uniqueness result for the GNEP is an application of Theorem 2 in [26], whereas uniqueness for the DP follows from the proof of Theorem 5.6. ∎

Remark 5.10. Conditions (5.8) and (5.9) are equivalent to concavity of the GNEP utility functions. For possible extensions of Theorem 5.9 to quasi-concave utility functions see, for example, [2]. A comment on the relationship between the sufficient conditions for uniqueness of Nash equilibria used in Theorems 5.6 and 5.9 can be found in Remark 3.3 of [20].

6. Complex equilibria and multiplayer GNEPs. In this section we aim to illustrate that connections may also be made between equilibrium strategies in generalised classical games with $n > 2$ players and more complex equilibria in the two-player Dynkin game of (2.5). Establishing such structures as Dynkin game equilibria is novel to the best of our knowledge. For this, we take Assumption 1.3 from Section 1.1 instead of Assumption 1.2. This means that the reward function f_1 has an additional convex portion, and will correspond to $n = 3$. Since the geometry of Assumption 1.2 suggests an equilibrium strategy for player 1 of the form $D_{[\ell^1,\ell^2]}$ for some $a_1 \leq \ell^1 \leq \ell^2 \leq a_2$, this example illustrates another convenient use of the generalised classical game as it ensures that $\ell^1 \leq \ell^2$ in the arguments below.

Define sets $\hat{\mathcal{S}}_1 = \hat{\mathcal{S}}_2 = [a_1, a_2]$, $\hat{\mathcal{S}}_3 = [b, 1]$ and $\hat{\mathcal{S}} = \prod_{i=1}^3 \hat{\mathcal{S}}_i$. Let the utility functions $\hat{U}_i \colon [0,1]^3 \to \bar{\mathbb{R}}$, $i \in \{1,2,3\}$ be defined by

$$\begin{aligned} \hat{U}_1(x,y,z) &= \frac{f_1(x) - g_{1,[z,1]}(x)}{x}, \\ \hat{U}_2(x,y,z) &= \frac{f_1(y) - g_{1,[z,1]}(y)}{z-y}, \\ \hat{U}_3(x,y,z) &= \frac{f_2(z) - g_{2,[0,y]}(z)}{z-y}, \end{aligned} \tag{6.1}$$

(taking $\hat{U}_2(x,y,z) = \hat{U}_3(x,y,z) = -\infty$ if $y \geq z$). Define the players' feasible strategy spaces by the set-valued maps $\hat{K}_i \colon \hat{\mathcal{S}}_{-i} \rightrightarrows \hat{\mathcal{S}}_i$, where

$$\hat{K}_1(y,z) = [a_1, y \wedge a_2], \;\; \hat{K}_2(x,z) = [x \vee a_1, a_2], \;\; \hat{K}_3(x,y) = [b,1], \tag{6.2}$$

so that the feasible strategy triples belong to the convex, compact set $\hat{\mathcal{C}}$ defined by

$$\hat{\mathcal{C}} = \{(x,y,z) \in [a_1,a_2] \times [a_1,a_2] \times [b,1] \colon x \leq y\}. \tag{6.3}$$

The next result shows that under Assumption 1.3, this more complex equilibrium structure exists in the DP precisely when the corresponding generalised game has a Nash equilibrium satisfying a condition on the sign of its utilities.

THEOREM 6.1. *Suppose that the DP reward functions satisfy Assumption 1.3. Then*

(a) there exists $s^* = (\ell^1, \ell^2, r) \in \hat{\mathcal{C}}$ *with*

$$\hat{U}_i(s^*) = \sup_{(s_i, s^*_{-i}) \in \hat{\mathcal{C}}} \hat{U}_i(s_i, s^*_{-i}), \;\; i \in \{1,2,3\}, \tag{6.4}$$

(b) a solution $s^* = (\ell^1, \ell^2, r) \in \hat{\mathcal{C}}$ *to* (6.4) *satisfies* $\hat{U}_2(s^*) \geq 0$ *if and only if* $(D_{[\ell^1,\ell^2]}, D_{[r,1]})$ *is a Nash equilibrium for the DP.*

Proof. Part (a) follows by a standard argument using quasi-concavity, similar to the proof of Lemma A.3 in the Appendix. For part (b), we claim that the pair (ℓ^1, ℓ^2) solves the following problem:

Problem: Find two points ℓ^1, ℓ^2 satisfying

$$\text{(P)} \qquad \begin{aligned} &i) && a_1 \leq \ell^1 \leq \ell^2 \leq a_2, \\ &ii) && \hat{U}_1(x, \ell^2, r) \leq \hat{U}_1(\ell^1, \ell^2, r), \quad \forall x \in (0, r), \\ &iii) && \hat{U}_2(\ell^1, y, r) \leq \hat{U}_2(\ell^1, \ell^2, r), \quad \forall y \in [0, r). \end{aligned}$$

To establish part iii) note that the function $y \mapsto f_1(y) - g_{1,[r,1]}(y)$ is zero at $y = 0$, convex for $y \in [0, a_1]$, concave for $y \in [a_1, a_2]$, convex for $y \in [a_2, r]$, nonnegative at $y = \ell^2$ and negative at $y = r$. It is then a straightforward exercise in convex analysis, similar to that in the proof of Theorem 4.1, to show that the maximum of the function $y \mapsto \hat{U}_2(\ell^1, y, r)$ on $[0, r)$ must be attained at a point in $[a_1, a_2]$. Taking $i = 2$ in (6.4) then establishes the claim. Part ii) follows similarly.

The necessity and sufficiency claim for the Nash equilibrium in stopping strategies then follows by applying Propositions D.1 and D.2 in the Appendix. ∎

Appendix A. Quasi-concavity and existence of GNEP equilibria. We first recall the definition and some properties of quasi-concave functions (see e.g. [8, Chapter 3.4]).

DEFINITION A.1. *Let $\mathcal{D} \subseteq \mathbb{R}$ be convex. A function $F\colon \mathcal{D} \to \bar{\mathbb{R}}$ is said to be* quasi-concave *if for every $\alpha \in \mathbb{R}$ the superlevel sets L_α^+ defined by*

$$L_\alpha^+ = \{x \in \mathcal{D}\colon F(x) \geq \alpha\}$$

are convex. If the same statement holds but with the sets $\{x \in \mathcal{D}\colon F(x) > \alpha\}$ then F is said to be strictly *quasi-concave. A function F is said to be (strictly) quasi-convex on a convex domain $\mathcal{D}$ if and only if $-F$ is (strictly) quasi-concave.*

All concave functions are quasi-concave. Moreover a function $F\colon \mathcal{D} \to \bar{\mathbb{R}}$ is quasi-concave on a convex domain $\mathcal{D}$ if and only if for any $x_1, x_2 \in \mathcal{D}$ and $0 \leq \theta \leq 1$ we have

$$F(\theta x_1 + (1-\theta)x_2) \geq \min(F(x_1), F(x_2)). \tag{A.1}$$

If (A.1) holds with strict inequality then F is strictly quasi-concave.

LEMMA A.2. *Suppose $\mathcal{D} \subseteq \mathbb{R}$ is convex, $f\colon \mathcal{D} \to \bar{\mathbb{R}}$ is (strictly) concave, and $\varphi\colon \mathcal{D} \to (0, \infty)$ is linear. Then the function $\frac{f}{\varphi}\colon \mathcal{D} \to \bar{\mathbb{R}}$ is (strictly) quasi-concave.*

Proof. In the case of concavity, for each $\alpha \in \mathbb{R}$ define a function $F_\alpha\colon \mathcal{D} \to \bar{\mathbb{R}}$ by $F_\alpha(x) = f(x) - \alpha\varphi(x)$. This function is concave on $\mathcal{D}$, and therefore quasi-concave, which means the superlevel set $\{x \in \mathcal{D}\colon F_\alpha(x) \geq 0\}$ is convex for every $\alpha \in \mathbb{R}$. The function $\frac{f}{\varphi}$ is quasi-concave on $\mathcal{D}$ since for every $\alpha \in \mathbb{R}$,

$$\left\{x \in \mathcal{D}\colon \left(\frac{f}{\varphi}\right)(x) \geq \alpha\right\} = \{x \in \mathcal{D}\colon f(x) \geq \alpha\varphi(x)\} = \{x \in \mathcal{D}\colon F_\alpha(x) \geq 0\}.$$

The proof for strictly concave f follows in the same way. ∎

LEMMA A.3. *Suppose the GNEP (2.3) satisfies for $i = 1, 2$:*

(i) For each fixed $s_{-i} \in \mathcal{S}_{-i}$, the mapping $s_i \mapsto U_i(s_i, s_{-i})$ is quasi-concave on $K_i(s_{-i})$.

(ii) The utility function $s \mapsto U_i(s)$ is continuous in $s = (s_1, s_2)$.

Then there exists a solution $(s_1^*, s_2^*) \in \mathcal{C}$ such that $s_1^* < s_2^*$.

Proof. For $i = 1, 2$ the correspondence K_i is compact and convex valued. Furthermore, using the notion of continuity for set-valued maps in [25], we can confirm that K_1 and K_2 are continuous. Along with the continuity and quasi-concavity properties of the U_i, we conclude by Lemma 2.5 in [1] (or see [16]) that there exists a solution s^* to (2.3). From the construction (2.6), this solution must satisfy $s_1^* < s_2^*$. ∎

A.1. Proof of Corollary 4.2.

Proof. Using Assumption 1.2 and Lemma A.2, we can verify the hypotheses of Lemma A.3 and assert the existence of a pair $(\ell, r) \in [0, a] \times [b, 1]$ with $\ell < r$ that solves the GNEP (2.3),

$$\begin{cases} U_1(x, r) \leq U_1(\ell, r), & \forall x \in [0, r \wedge a], \\ U_2(\ell, y) \leq U_2(\ell, r), & \forall y \in [\ell \vee b, 1], \end{cases}$$

and the result follows from Theorem 4.1. ∎

Appendix B. Expected rewards for threshold strategies. If players 1 and 2 use the strategies $D_{[0,\ell]}$ and $D_{[r,1]}$ respectively, where $0 \leq \ell < r \leq 1$, then the expected reward $M_1^x(D_{[0,\ell]}, D_{[r,1]})$ for player 1 (cf. (2.4)) satisfies,

$$M_1^x(D_{[0,\ell]}, D_{[r,1]}) = \mathbb{E}^x\big[f_1(X_{D_{[0,\ell]}})\mathbf{1}_{\{D_{[0,\ell]}<D_{[r,1]}\}} + g_1(X_{D_{[r,1]}})\mathbf{1}_{\{D_{[r,1]}<D_{[0,\ell]}\}}\big]$$
$$+ \mathbb{E}^x\big[h_1(X_{D_{[0,\ell]}})\mathbf{1}_{\{D_{[0,\ell]}=D_{[r,1]}\}}\big]$$
$$= \begin{cases} f_1(x), & \forall x \in [0,\ell] \\ f_1(\ell)\cdot\frac{r-x}{r-\ell} + g_1(r)\cdot\frac{x-\ell}{r-\ell}, & \forall x \in (\ell, r) \\ g_1(x), & \forall x \in [r,1]. \end{cases}$$

Analogously, the expected reward $M_2^x(D_{[0,\ell]}, D_{[r,1]})$ for player 2 satisfies,

$$M_2^x(D_{[0,\ell]}, D_{[r,1]}) = \begin{cases} g_2(x), & \forall x \in [0,\ell] \\ g_2(\ell)\cdot\frac{r-x}{r-\ell} + f_2(r)\cdot\frac{x-\ell}{r-\ell}, & \forall x \in (\ell, r) \\ f_2(x), & \forall x \in [r,1]. \end{cases}$$

Appendix C. Derivatives of utility functions. Throughout this section we suppose Assumption 5.1 holds. We first provide general formulas for the first and second partial derivatives of a utility function $U(x,y)$ which is of the form $U(x,y) = \frac{F(x,y)}{y-x}$.

$$\partial_x U(x,y) = \frac{\partial_x F(x,y)(y-x) + F(x,y)}{(y-x)^2},$$
$$\partial_y U(x,y) = \frac{\partial_y F(x,y)(y-x) - F(x,y)}{(y-x)^2},$$
$$\partial_{xx} U(x,y) = \frac{\partial_{xx} F(x,y)(y-x)^2 + 2\big[\partial_x F(x,y)(y-x) + F(x,y)\big]}{(y-x)^3},$$
$$\partial_{yy} U(x,y) = \frac{\partial_{yy} F(x,y)(y-x)^2 - 2\big[\partial_y F(x,y)(y-x) - F(x,y)\big]}{(y-x)^3},$$

$$\partial_{xy} U(x,y) = \frac{\partial_{xy} F(x,y)(y-x) + \partial_x F(x,y) + \partial_y F(x,y)}{(y-x)^2}$$
$$- 2\frac{\big[\partial_x F(x,y)(y-x) + F(x,y)\big]}{(y-x)^3}$$
$$= \frac{\partial_{xy} F(x,y)(y-x) - \partial_y F(x,y) - \partial_x F(x,y)}{(y-x)^2}$$
$$+ 2\frac{\big[\partial_y F(x,y)(y-x) - F(x,y)\big]}{(y-x)^3}.$$

Using equation (2.6) for the utility functions gives the following expressions for their partial derivatives,

$$\partial_x U_1(x,y) = \frac{f_1(x) + f_1'(x)(y-x) - g_1(y)}{(y-x)^2},$$

$$\partial_y U_2(x,y) = \frac{g_2(x) + f_2'(y)(y-x) - f_2(y)}{(y-x)^2},$$

$$\partial_{xx} U_1(x,y) = \frac{f_1''(x)(y-x)^2 + 2\big[f_1(x) + f_1'(x)(y-x) - g_1(y)\big]}{(y-x)^3},$$

$$\partial_{yy} U_2(x,y) = \frac{f_2''(y)(y-x)^2 + 2\big[f_2(y) - f_2'(y)(y-x) - g_2(x)\big]}{(y-x)^3},$$

$$\partial_{xy} U_1(x,y) = \frac{2\big[g_1(y) - f_1(x)\big] - (f_1'(x) + g_1'(y))(y-x)}{(y-x)^3},$$

$$\partial_{xy} U_2(x,y) = \frac{2\big[g_2(x) - f_2(y)\big] + (g_2'(x) + f_2'(y))(y-x)}{(y-x)^3}.$$

Appendix D. A verification theorem using multiplayer GNEPs.

PROPOSITION D.1. *Under Assumption 1.3 and given $r \in (a_2, 1]$, (ℓ^1, ℓ^2) is a solution to Problem (P) if and only if*

$$V_1^{[r,1]}(x) := \sup_{\tau_1 \in \mathcal{T}} M_1^x(\tau_1, D_{[r,1]}) = M_1^x(D_{[\ell^1,\ell^2]}, D_{[r,1]}), \quad \forall x \in [0,1]. \tag{D.1}$$

Proof. The arguments are more or less the same as those establishing Theorem 4.1. For the sake of brevity we therefore only show the proof of necessity (Problem (P) $\implies$ (D.1)).

Define u_r on $[0,1]$ by,

$$\begin{aligned} u_r(x) &= M_1^x(D_{[\ell^1,\ell^2]}, D_{[r,1]}) - g_{1,[r,1]}(x) \\ &= \begin{cases} \big(f_1(\ell^1) - g_{1,[r,1]}(\ell^1)\big)\,\frac{x}{\ell^1}, & x \in [0,\ell^1), \\ f_1(x) - g_{1,[r,1]}(x), & x \in [\ell^1,\ell^2), \\ \big(f_1(\ell^2) - g_{1,[r,1]}(\ell^2)\big)\,\frac{r-x}{r-\ell^2}, & x \in [\ell^2, r), \\ 0, & x \in [r,1]. \end{cases} \end{aligned} \tag{D.2}$$

Suppose (ℓ^1, ℓ^2) is a solution to Problem (P). Similarly to Theorem 4.1, we will prove (D.1) by showing that u_r is the smallest non-negative concave majorant of $f_1 - g_{1,[r,1]}$ on $[0,r]$. Initially we will analyse u_r separately on $[0,\ell^1]$ and $[\ell^1,\ell^2]$.

Observe firstly that the function $f_1 - g_{1,[r,1]}$ is nonnegative when evaluated at the points ℓ^1 and ℓ^2 and hence, by concavity, on $[\ell^1,\ell^2]$. Recalling (6.1), this follows from (P), since $f_1(0) = g_{1,[r,1]}(0)$ and so $f_1(\ell^2) - g_{1,[r,1]}(\ell^2) \geq 0$. Also

$$f_1(x) - g_{1,[r,1]}(x) \leq \big(f_1(\ell^1) - g_{1,[r,1]}(\ell^1)\big)\,\frac{x}{\ell^1}, \;\; \forall x \in (0,r),$$

and taking $x = \ell^2$ shows that $f_1(\ell^1) - g_{1,[r,1]}(\ell^1) \geq 0$. Therefore u_r is a non-negative majorant of $f_1 - g_{1,[r,1]}$ on $[0,\ell^1]$. This is also true on $[\ell^1, r]$, since $f_1(r) \leq g_1(r)$ and so

$$f_1(x) - g_{1,[r,1]}(x) \leq \big(f_1(\ell^2) - g_{1,[r,1]}(\ell^2)\big)\left(\frac{r-x}{r-\ell^2}\right), \;\; \forall x \in [0,r]. \tag{D.3}$$

Concavity holds for u_r on the three intervals $[0,\ell^1]$, $[\ell^1,\ell^2]$ and $[\ell^2,r]$ separately and, arguing as in the proof of Theorem 4.1, we can show that u_r is continuous and concave on the entire interval $[0,r]$, completing the proof. ∎

PROPOSITION D.2. *Under Assumption 1.3, for every ℓ^1,ℓ^2 satisfying $0<\ell^1\le\ell^2<b$, a point $r\in[b,1]$ satisfies (4.10) with $\ell=\ell^2$ and $U_2=\hat U_3$ if and only if*

$$V_2^{[\ell^1,\ell^2]}(x) := \sup_{\tau_2\in\mathcal{T}} M_2^x(D_{[\ell^1,\ell^2]},\tau_2) = M_2^x(D_{[\ell^1,\ell^2]},D_{[r,1]}),\quad \forall x\in[0,1]. \tag{D.4}$$

Proof. By Lemma 3.4 it is sufficient merely to consider the optimal stopping problem on the set $[0,\ell^1]\cup[\ell^2,1]$ with obstacle $f_2-g_{2,[\ell^1,\ell^2]}$, and we will only sketch the solution. Note that since $f_2\le g_2$ it is clearly suboptimal to stop in $[\ell^1,\ell^2]$. From Dynkin's formula it is also suboptimal to stop on $[0,\ell^1]$, since $f_2-g_{2,[\ell^1,\ell^2]}$ is convex there and $f_2(x)-g_{2,[\ell^1,\ell^2]}(x)\le 0$ for $x\in\{0,\ell^1\}$. The solution is nontrivial only on $(\ell^2,1]$, where the arguments used for Theorem 4.1 are sufficient to complete the proof. ∎

Appendix E. Other Markov processes and discounting. Let $X=(X_t)_{t\ge0}$ be a continuous strong Markov process absorbed at the endpoints of an interval $E=(\ell,r)\subseteq\mathbb{R}$. Suppose that the rewards in the DP are discounted by a factor $\lambda\ge0$, so that (1.1) becomes

$$\begin{aligned}\mathcal{J}_i(\tau_1,\tau_2) := e^{-\lambda(\tau_i\wedge\tau_{-i})}\{&f_i(X_{\tau_i})\mathbf{1}_{\{\tau_i<\tau_{-i}\}}+g_i(X_{\tau_{-i}})\mathbf{1}_{\{\tau_{-i}<\tau_i\}}\\ &+h_i(X_{\tau_i})\mathbf{1}_{\{\tau_i=\tau_{-i}\}}\},\qquad i\in\{1,2\}.\end{aligned} \tag{1.1'}$$

Lemma 3.4 has a straightforward extension to the case $\lambda>0$. Extending the concept of superharmonic functions in Definition 3.1, we say that a measurable function $\phi\colon\bar E\to\mathbb{R}$ is *λ-superharmonic* on a set $A\in\mathcal{B}(\bar E)$ if for every $x\in\bar E$ and $\tau\in\mathcal{T}$,

$$\phi(x)\ge\mathbb{E}^x[e^{-\lambda(\tau\wedge D_{A^c})}\phi(X_{\tau\wedge D_{A^c}})].$$

The function ϕ_A introduced in Definition 3.3 is given more generally by,

$$\phi_A(x) := \mathbb{E}^x\left[e^{-\lambda D_A}\phi(X_{D_A})\right].$$

It was noted in Section 3.1 that ϕ_A is continuous when $\lambda=0$ and ϕ is continuous. This same property, which is important for ensuring that the obstacle in problem (3.4) is continuous, also holds for $\lambda\ge0$ when X is a more general regular diffusion with strictly positive diffusion coefficient [27]. Furthermore, when $X_t=Z_{t\wedge\zeta}$ for $t\ge0$, where $Z=(Z_t)_{t\ge0}$ is a regular diffusion on E and $\zeta=\inf\{t\ge0\colon Z_t\notin E\}$, the results in Sections 4–5 hold with obvious modifications. We now briefly discuss this extension when Z satisfies the stochastic differential equation,

$$dZ_t=\mu(Z_t)dt+\sigma(Z_t)dW_t, \tag{E.1}$$

where $W=(W_t)_{t\ge0}$ is a standard Brownian motion and $\mu\colon\bar E\to\mathbb{R}$, $\sigma\colon\bar E\to\mathbb{R}$ are Borel-measurable functions such that for every $x\in E$,

$$\begin{aligned} &i)\quad \sigma^2(x)>0,\\ &ii)\quad \int_{x-\varepsilon}^{x+\varepsilon}\frac{1+|\mu(y)|}{\sigma^2(y)}dy<\infty\ \text{ for some }\varepsilon>0.\end{aligned}$$

Let $\mathcal{G}=\frac12\sigma^2(\cdot)\frac{d^2}{dx}+\mu(\cdot)\frac{d}{dx}$ denote the infinitesimal generator corresponding to Z.

E.1. Undiscounted rewards. For the case $\lambda = 0$, we first recall from [10] that there is a continuous increasing function S on E, the *scale function*, which satisfies $\mathcal{G}S(\cdot) \equiv 0$. Let $\tilde{\ell} = S(\ell)$, $\tilde{r} = S(r)$, $\tilde{X} = (\tilde{X}_t)_{t\geq 0}$ with $\tilde{X}_t = S(X_t)$, and $\tilde{E} = (\tilde{\ell}, \tilde{r})$. The process $\tilde{X}$ is a diffusion on its natural scale on $\tilde{E}$. It follows from Proposition 3.3 of [10] that the DP corresponding to the process X and rewards f_i, g_i and h_i on E can be studied by an equivalent DP corresponding to $\tilde{X}$ with reward functions $\tilde{f}_i(\cdot) = f_i(S^{-1}(\cdot))$, $\tilde{g}_i(\cdot) = g_i(S^{-1}(\cdot))$, $\tilde{h}_i(\cdot) = h_i(S^{-1}(\cdot))$ on $\tilde{E}$.

E.2. Discounted rewards. For the case $\lambda > 0$, we first let ψ^λ and ϕ^λ denote the fundamental solutions to the diffusion generator equation $\mathcal{G}w = \lambda w$, where ψ^λ is strictly increasing and ϕ^λ is strictly decreasing [10, p. 177]. Let $F(\cdot) = \frac{\psi^\lambda(\cdot)}{\phi^\lambda(\cdot)}$, $\tilde{\ell} = F(\ell)$, $\tilde{r} = F(r)$, $\tilde{X} = (\tilde{X}_t)_{t\geq 0}$ with $\tilde{X}_t = F(X_t)$, and $\tilde{E} = (\tilde{\ell}, \tilde{r})$. The process $\tilde{X}$ is a diffusion on its natural scale on $\tilde{E}$. It follows from Proposition 4.3 of [10] that the DP corresponding to the process X and rewards f_i, g_i and h_i on E discounted by $\lambda > 0$ can be studied by an equivalent DP corresponding to $\tilde{X}$ with reward functions $\tilde{f}_i(\cdot) = \frac{f_i}{\phi^\lambda}(F^{-1}(\cdot))$, $\tilde{g}_i(\cdot) = \frac{g_i}{\phi^\lambda}(F^{-1}(\cdot))$, $\tilde{h}_i(\cdot) = \frac{h_i}{\phi^\lambda}(F^{-1}(\cdot))$ on $\tilde{E}$ *without discounting*.

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