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# A $(p,\nu)$-EXTENSION OF SRIVASTAVA'S TRIPLE HYPERGEOMETRIC FUNCTION $H_C$

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ABSTRACT. We obtain a $(p,\nu)$-extension of Srivastava's triple hypergeometric function $H_C(\cdot)$ by employing the extended Beta function $B_{p,\nu}(x,y)$ introduced in Parmar et al. [J. Class. Anal. **11** (2017), 91–106]. We give some of the main properties of this extended function, which include several integral representations, the Mellin transform, a differential formula, recursion formulas and a bounded inequality.

## 1. Introduction and preliminaries

In the present paper, we employ the following notations:

$$\mathbf{N} := \{1,2,\dots\}, \quad \mathbf{N}_0 := \mathbf{N}\cup\{0\}, \quad \mathbf{Z}_0^- := \mathbf{Z}^-\cup\{0\},$$

where the symbols $\mathbf{N}$ and $\mathbf{Z}$ denote the set of integer and natural numbers; as usual, the symbols $\mathbf{R}$ and $\mathbf{C}$ denote the set of real and complex numbers, respectively.

Hypergeometric functions of a single variable have a long history and arise in numerous branches of mathematics and physics. The Gauss hypergeometric function is defined for $b_1, b_2 \in \mathbf{C}$, $c_1 \in \mathbf{C}\backslash\mathbf{Z}_0^-$ by

$$(1.1)\qquad {}_2F_1\left(\begin{matrix} b_1, b_2 \\ c_1 \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(b_1)_n (b_2)_n}{(c_1)_n} \frac{z^n}{n!} \quad (|z|<1),$$

where $(a)_n$ denotes the Pochhammer symbol (or the shifted factorial) defined by $(a)_n = \Gamma(a+n)/\Gamma(a) = a(a+1)\dots(a+n-1)$. Extensions of this function to include $p$ numerator parameters $b_j$ $(1 \leqslant j \leqslant p)$ and $q$ denominator parameters $c_j$ $(1 \leqslant j \leqslant q)$ also find wide application; see [**17**]. Triple hypergeometric functions (that is functions of three variables $x$, $y$ and $z$) have been introduced and investigated by Srivastava and Karlsson [**21**, Chapter 3] who provide a table of 205 distinct such functions. In [**18**,**19**], Srivastava introduced the triple hypergeometric functions $H_A, H_B$ and $H_C$ of the second order. It is known that $H_C$ and $H_B$ are

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generalizations of Appell's hypergeometric functions $F_1$ and $F_2$, while $H_A$ is the generalization of both $F_1$ and $F_2$.

In the present study, we shall confine our attention to Srivastava's triple hypergeometric function $H_C$ given by [**21**, p. 43, 1.5(11) to 1.5(13)] (see also [**18**] and [**20**, p. 68])

$$(1.2)\quad H_C(b_1,b_2,b_3;c_1;x,y,z) := \sum_{m,n,k=0}^{\infty} \frac{(b_1)_{m+k}(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_{m+n+k}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}$$
$$= \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n} \frac{B(b_1+m+k, c_1+n-b_1)}{B(b_1, c_1+n-b_1)} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}.$$

Here $B(\alpha,\beta)$ denotes the classical Beta function defined by [**13**, (5.12.1)]

$$(1.3)\qquad B(\alpha,\beta) = \begin{cases} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt, & (\mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0) \\ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, & ((\alpha,\beta)\in \mathbf{C}\backslash\mathbf{Z}_0^-). \end{cases}$$

The convergence region for the hypergeometric series $H_C(\cdot)$ is given in [**11**, p. 243] as $|x|<\alpha$, $|y|<\beta$, $|z|<\gamma$, where $\alpha$, $\beta$, $\gamma$ satisfy the relation

$$(1.4)\qquad \alpha+\beta+\gamma-2\sqrt{(1-\alpha)(1-\beta)(1-\gamma)} < 2.$$

We shall also find it convenient to introduce an additional parameter $a$ into $H_C(\cdot)$ in the form

$$(1.5)\quad H_C^{(a)}(b_1,b_2,b_3;c_1;x,y,z) :=$$
$$\sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n} \frac{B(b_1+a+m+k, c_1+a+n-b_1)}{B(b_1, c_1+n-b_1)} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}.$$

which reduces to (1.2) when $a=0$.

In 1997, Chaudhry et al. [**1**, Eq. (1.7)] introduced a $p$-extension of the Beta function $B(x,y)$ given by

$$B(x,y;p) = \int_0^1 t^{x-1}(1-t)^{y-1}\exp\Big[\frac{-p}{t(1-t)}\Big]dt, \quad (\mathrm{Re}(p)>0).$$

Also, Chaudhry et al. [**2**] employed this function to extend the Gauss hypergeometric series ${}_2F_1(\cdot)$ and its integral representations. A further extension of the Beta function has been given by Choi et al. [**8**] in the form

$$B_{p,q}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\exp\Big[-\frac{p}{t}-\frac{q}{1-t}\Big]dt \quad (\mathrm{Re}(p)>0,\ \mathrm{Re}(q)>0),$$

which reduces to $B(x,y;p)$ when $p=q$. Recently, Parmar et al. [**15**] have given a different extension of the Beta function in the form

$$(1.6)\qquad B_{p,\nu}(x,y) = \sqrt{\frac{2p}{\pi}}\int_0^1 t^{x-\frac{3}{2}}(1-t)^{y-\frac{3}{2}} K_{\nu+\frac{1}{2}}\Big(\frac{p}{t(1-t)}\Big)dt,$$

where $\mathrm{Re}(p) > 0$, $\nu \geqslant 0$ and $K_\nu(z)$ is the modified Bessel function (sometimes known as the Macdonald function) of order $\nu$. When $\nu = 0$, (1.6) reduces to $B(x, y; p)$, since $K_{\frac{1}{2}}(z) = \sqrt{\pi/(2z)}e^{-z}$.

Many authors have studied integral representations of Srivastava's triple hypergeometric function $H_C(\cdot)$ defined in (1.2); see [**3**–**7**]. Our aim in this paper is to introduce a $(p, \nu)$-extension of this function, which we denote by $H_{C,p,\nu}(\cdot)$, based on the extended Beta function in (1.6). The Appell hypergeometric function of two variables defined by

$$F_1(b_1, b_2, b_3; c_1; x, y) = \sum_{m,n=0}^{\infty} \frac{(b_2)_n (b_3)_m B(b_1 + m + n, c_1 - b_1)}{B(b_1, c_1 - b_1)} \frac{x^m}{m!} \frac{y^n}{n!}$$

where $(|x| < 1, |y| < 1)$ has been extended by replacement of the numerator Beta function (of the same arguments) with $B(x, y; p)$ in [**14**] and with $B_{p,\nu}(x, y)$ in [**9**]. Similar extensions of $H_A$ and $H_B$ have been carried out in [**10**, **16**].

The plan of this paper is as follows. The extended function $H_{C,p,\nu}(\cdot)$ is defined in Section 2 and some integral representations are presented involving the modified Bessel function and the Gauss hypergeometric function ${}_2F_1$. The main properties of $H_{C,p,\nu}(\cdot)$ namely, its Mellin transform, a differential formula, a bounded inequality and recursion formulas are established in Sections 3 to 6. Some concluding remarks are made in Section 7.

## 2. The $(p, v)$-extended Srivastava triple hypergeometric function $H_{C,p,\nu}(\cdot)$

Srivastava introduced the triple hypergeometric function $H_C(\cdot)$, together with its integral representations, in [**18**] and [**20**]. Here we consider the following $(p, \nu)$-extension of this function, which we denote by $H_{C,p,\nu}(\cdot)$, based on the extended Beta function $B_{p,\nu}(x, y)$ defined in (1.6). This is given by

$$\begin{aligned} (2.1)\quad & H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ &= \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1{+}m{+}k, c_1{+}n{-}b_1)}{B(b_1, c_1 + n - b_1)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!}, \end{aligned}$$

where the parameters $b_1, b_2, b_3 \in \mathbf{C}$ and $c_1 \in \mathbf{C}\backslash\mathbf{Z}_0^-$. The region of convergence is $|x| < \alpha$, $|y| < \beta$, $|z| < \gamma$, where $\alpha$, $\beta$, $\gamma$ satisfy (1.4). This definition clearly reduces to the original function when $\nu = 0$.

An integral representation for $H_{C,p,\nu}(\cdot)$ involving the Gauss hypergeometric function ${}_2F_1$ defined in (1.1) can be given. We have

THEOREM 2.1. *The following integral representation of the function* $H_{C,p,\nu}(\cdot)$ *holds for* $\mathrm{Re}(p) > 0$, $\mathrm{Re}(b_j) > 0$ $(j = 1, 2, 3)$ *and* $\mathrm{Re}(c_1 - b_1) > 0$*:*

$$\begin{aligned} (2.2)\quad & H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ &= \frac{\Gamma(c_1)}{\Gamma(b_1)\Gamma(c_1 - b_1)} \sqrt{\frac{2p}{\pi}} \int_0^1 t^{b_1-\frac{3}{2}}(1-t)^{c_1-b_1-\frac{3}{2}} K_{\nu+\frac{1}{2}}\Big(\frac{p}{t(1-t)}\Big) \end{aligned}$$

$$\times (1-xt)^{-b_2}(1-zt)^{-b_3}\,{}_2F_1\Big(\begin{matrix} b_2, b_3 \\ c_1-b_1 \end{matrix}; \frac{(1-t)y}{(1-xt)(1-zt)}\Big)dt, \tag{2.3}$$

*where* $|x|<1$, $|y|<1$ *and* $|z|<1$.

PROOF. The proof of integral representation (2.3) follows by use of the extended beta function (1.6) in (2.1), a change in the order of integration and summation (with uniform convergence of the integral) to find

$$\begin{aligned} &H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ &\quad = \frac{\Gamma(c_1)}{\Gamma(b_1)\Gamma(c_1-b_1)}\sqrt{\frac{2p}{\pi}}\int_0^1 t^{b_1-\frac{3}{2}}(1-t)^{c_1-b_1-\frac{3}{2}} K_{\nu+\frac{1}{2}}\Big(\frac{p}{t(1-t)}\Big) \\ &\qquad\qquad \times \sum_{m,nk=0}^{\infty} \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1-b_1)_n}\,\frac{(xt)^m}{m!}\,\frac{(y(1-t))^n}{n!}\,\frac{(zt)^k}{k!}\,dt. \end{aligned}$$

Making use of the result $(a)_{m+n} = (a)_n(a+n)_m$ we can express the treble sum as

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(b_2)_n(b_3)_n}{(c_1-b_1)_n}\,\frac{(y(1-t))^n}{n!}\sum_{m=0}^{\infty}(b_2+n)_m\frac{(xt)^m}{m!}\sum_{k=0}^{\infty}(b_3+n)_k\frac{(zt)^k}{k!} \\ &\qquad = (1-xt)^{-b_2}(1-zt)^{b_3}\sum_{n=0}^{\infty}\frac{(b_2)_n(b_3)_n}{(c_1-b_1)_n}\,\frac{X^n}{n!}, \quad X = \frac{y(1-t)}{(1-xt)(1-zt)}, \end{aligned}$$

where we have employed the binomial theorem

$$\sum_{n=0}^{\infty}\frac{(a)_n w^n}{n!} = (1-w)^{-a} \quad (a \in \mathbf{C},\ |w|<1)$$

to evaluate the sums over $m$ and $k$. Identification of the sum over $n$ as a Gauss hypergeometric function by (1.1), then yields (2.3). □

The following variants of (2.3) can be obtained by making appropriate transformations of the integration variable. We have

$$\begin{aligned} &\frac{\Gamma(b_1)\Gamma(c_1-b_1)}{\Gamma(c_1)}\sqrt{\frac{\pi}{2p}}\,H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ &\quad = \int_0^{\infty} \xi^{b_1-\frac{3}{2}}(1+\xi)^{b_2+b_3-c_1+1}\,\Omega_1^{-b_2}\Omega_2^{-b_3}\,K_{\nu+\frac{1}{2}}\Big(\frac{p}{\sigma_1\sigma_2}\Big)\,{}_2F_1\Big(\begin{matrix} b_2, b_3 \\ c_1-b_1 \end{matrix}; \sigma_3 y\Big)d\xi, \end{aligned} \tag{2.4}$$

where

$$\sigma_1 = \frac{\xi}{1+\xi}, \quad \sigma_2 = \frac{1}{1+\xi}, \quad \sigma_3 = \frac{1+\xi}{\sigma_1\sigma_2}, \quad \Omega_1 = 1+(1-x)\xi, \quad \Omega_2 = 1+(1-z)\xi;$$

$$\begin{aligned} &\frac{\Gamma(b_1)\Gamma(c_1-b_1)}{\Gamma(c_1)}\sqrt{\frac{\pi}{2p}}\,H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ &\quad = \frac{(\beta-\gamma)^{b_1-\frac{1}{2}}(\alpha-\gamma)^{c_1-b_1-\frac{1}{2}}}{(\beta-\alpha)^{c_1-b_2-b_3-2}} \times \int_{\alpha}^{\beta}\frac{(\xi-\alpha)^{b_1-\frac{3}{2}}(\beta-\xi)^{c_1-b_1-\frac{3}{2}}}{(\xi-\gamma)^{c_1-b_2-b_3-1}}\,\Omega_1^{-b_2}\Omega_2^{-b_3} \end{aligned} \tag{2.5}$$

$$\times K_{\nu+\frac{1}{2}}\Big(\frac{p}{\sigma_1\sigma_2}\Big)\, {}_2F_1\Big(\begin{matrix} b_2, b_3 \\ c_1-b_1 \end{matrix}; \sigma_3 y\Big) d\xi,$$

where, with $\gamma < \alpha < \beta$,

$$\sigma_1 = \frac{(\beta-\gamma)(\xi-\alpha)}{(\beta-\alpha)(\xi-\gamma)}, \quad \sigma_2 = \frac{(\alpha-\gamma)(\beta-\xi)}{(\beta-\alpha)(\xi-\gamma)}, \quad \sigma_3 = \frac{(\alpha-\gamma)(\beta-\alpha)(\beta-\xi)(\xi-\gamma)}{\Omega_1\Omega_2},$$
$$\Omega_1 = (\beta-\alpha)(\xi-\gamma) - x(\beta-\gamma)(\xi-\alpha), \quad \Omega_2 = (\beta-\alpha)(\xi-\gamma) - z(\beta-\gamma)(\xi-\alpha);$$

$$\text{(2.6)} \quad \frac{\Gamma(b_1)\Gamma(c_1-b_1)}{\Gamma(c_1)}\sqrt{\frac{\pi}{2p}}\, H_{C,p,\nu}(b_1,b_2,b_3;c_1;x,y,z)$$
$$= 2\int_0^{\pi/2} (\sin^2\xi)^{b_1-1}(\cos^2\xi)^{c_1-b_1-1}\,\Omega_1^{-b_2}\Omega_2^{-b_3}$$
$$\times K_{\nu+\frac{1}{2}}\Big(\frac{p}{\sigma_1\sigma_2}\Big)\, {}_2F_1\Big(\begin{matrix} b_2, b_3 \\ c_1-b_1 \end{matrix}; \sigma_3 y\Big) d\xi,$$

where

$$\sigma_1 = \sin^2\xi, \quad \sigma_2 = \cos^2\xi, \quad \sigma_3 = \frac{\cos^2\xi}{\Omega_1\Omega_2}, \quad \Omega_1 = 1 - x\sin^2\xi, \quad \Omega_2 = 1 - z\sin^2\xi;$$

and

$$\text{(2.7)} \quad \frac{\Gamma(b_1)\Gamma(c_1-b_1)}{\Gamma(c_1)}\sqrt{\frac{\pi}{2p}}\, H_{C,p,\nu}(b_1,b_2,b_3;c_1;x,y,z)$$
$$= (1+\lambda)^{b_1-\frac{1}{2}}\int_0^1 \frac{\xi^{b_1-\frac{3}{2}}(1-\xi)^{c_1-b_1-\frac{3}{2}}}{(1+\lambda\xi)^{c_1-b_2-b_3-1}}\,\Omega_1^{-b_2}\Omega_2^{-b_3}$$
$$\times K_{\nu+\frac{1}{2}}\Big(\frac{p}{\sigma_1\sigma_2}\Big)\, {}_2F_1\Big(\begin{matrix} b_2, b_3 \\ c_1-b_1 \end{matrix}; \sigma_3 y\Big) d\xi,$$

where, with $\lambda > -1$,

$$\sigma_1 = \frac{(1+\lambda)\xi}{1+\lambda\xi}, \quad \sigma_2 = \frac{1-\xi}{1+\lambda\xi}, \quad \sigma_3 = \frac{(1-\xi)(1+\lambda\xi)}{\Omega_1\Omega_2},$$
$$\Omega_1 = 1+\lambda\xi-(1+\lambda)x\xi, \quad \Omega_2 = 1+\lambda\xi-(1+\lambda)z\xi.$$

Integral representations (2.4)–(2.7) can be proved directly by using the following transformations

$$\text{(2.4)}: \qquad t = \frac{\xi}{1+\xi}, \quad \frac{dt}{d\xi} = \frac{1}{(1+\xi)^2}$$
$$\text{(2.5)}: \qquad t = \frac{(\beta-\gamma)(\xi-\alpha)}{(\beta-\alpha)(\xi-\gamma)}, \quad \frac{dt}{d\xi} = \frac{(\beta-\gamma)(\alpha-\gamma)}{(\beta-\alpha)(\xi-\gamma)^2},$$
$$\text{(2.6)}: \qquad t = \sin^2\xi, \quad \frac{dt}{d\xi} = 2\sin\xi\cos\xi$$
$$\text{(2.7)}: \qquad t = \frac{(1+\lambda)\xi}{1+\lambda\xi}, \quad \frac{dt}{d\xi} = \frac{(1+\lambda)}{(1+\lambda\xi)^2},$$

in turn in (2.3) to obtain the right-hand side of each result.

Finally, use of the integral representation [**13**, p. 388]

$$ {}_2F_1\left(\begin{matrix} b_1, b_2 \\ c_1 \end{matrix}; z\right) = \frac{1}{B(b_2, c_1 - b_2)} \int_0^1 \frac{t^{b_2-1}(1-t)^{c_1-b_2-1}}{(1-zt)^{b_1}} dt, \quad (|\arg(1-z)| < \pi) $$

for $\mathrm{Re}(c_1) > \mathrm{Re}(b_2) > 0$, shows that

$$ \begin{aligned} &H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ &= \frac{\Gamma(c_1)}{\Gamma(b_1)\Gamma(b_2)\Gamma(c_1-b_1-b_2)} \sqrt{\frac{2p}{\pi}} \int_0^1 \int_0^1 s^{b_2-1} t^{b_1-\frac{3}{2}} (1-s)^{c_1-b_1-b_2-1} (1-t)^{c_1-b_1-\frac{3}{2}} \\ &\qquad \times \frac{(1-xt)^{b_3-b_2}}{\{(1-xt)(1-zt) - ys(1-t)\}^{b_3}} K_{\nu+\frac{1}{2}}\left(\frac{p}{t(1-t)}\right) ds\, dt \end{aligned} $$

provided, in addition, $\mathrm{Re}(c_1 - b_1 - b_2) > 0$.

## 3. The Mellin transform for $H_{C,p,\nu}(\cdot)$

The Mellin transform of a locally integrable function $f(x)$ on $(0, \infty)$ is given by (see, for example, [**12**, p. 193, Section 2.1])

$$ \Phi(s) = \mathcal{M}\{f(x)\}(s) = \int_0^\infty x^{s-1} f(x) dx, \tag{3.1} $$

which defines an analytic function in its strip of analyticity $a < \mathrm{Re}(s) < b$. The inverse Mellin transform of the above function (3.1) is defined by

$$ f(x) = \mathcal{M}^{-1}\{\Phi(s)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \Phi(s) ds \quad (a < c < b). $$

Theorem 3.1. *The following Mellin transform of the extended Srivastava triple hypergeometric function $H_{C,p,\nu}(\cdot)$ holds true:*

$$ \begin{aligned} \mathcal{M}\{H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)\}(s) &= \int_0^\infty p^{s-1} H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) dp, \\ &= \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s-\nu}{2}\right) \Gamma\left(\frac{s+\nu+1}{2}\right) H_C^{(s)}(b_1, b_2, b_3; c_1; x, y, z), \end{aligned} \tag{3.2} $$

*where* $\mathrm{Re}(s) > \nu > 0$*,* $c_1 \in \mathbf{C} \setminus \mathbf{Z}_0^-$ *and* $H_C^{(s)}(\cdot)$ *is defined in* (1.5).

Proof. Substituting the extended Srivastava function (2.1) into the integral on the left-hand side of (3.2) and changing the order of integration (by the uniform convergence of the integral), we obtain

$$ \begin{aligned} \mathcal{M}\{H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)\}(s) &= \sum_{m,n,k=0}^\infty \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n\ B(b_1, c_1+n-b_1)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!} \\ &\qquad \times \left\{ \int_0^\infty p^{s-1} B_{p,\nu}(b_1+m+k, c_1+n-b_1) dp \right\}. \end{aligned} $$

Use of the extended Beta function (1.6) then shows that

$$ \mathcal{M}\{H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)\}(s) $$

$$= \sqrt{\frac{2}{\pi}} \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n\ B(b_1, c_1+n-b_1)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!}$$

$$\times \int_0^1 t^{b_1+m+k-\frac{3}{2}}(1-t)^{c_1+n-b_1-\frac{3}{2}} \Big\{ \int_0^\infty p^{s-\frac{1}{2}} K_{\nu+\frac{1}{2}}\Big(\frac{p}{t(1-t)}\Big) dp \Big\} dt.$$

If we apply the result [**13**, (10.43.19)]

$$\int_0^\infty w^{s-\frac{1}{2}} K_{\alpha+\frac{1}{2}}(w) dw = 2^{s-\frac{3}{2}} \Gamma\Big(\frac{s-\alpha}{2}\Big)\Gamma\Big(\frac{s+\alpha+1}{2}\Big) \quad (|\operatorname{Re}(\alpha)| < \operatorname{Re}(s))$$

followed by the substitution $w = p/(t(1-t))$, we obtain

$$\mathcal{M}\{H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)\}(s) = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\Big(\frac{s-\nu}{2}\Big)\Gamma\Big(\frac{s+\nu+1}{2}\Big)$$

$$\times \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n B(b_1, c_1+n-b_1)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!}$$

$$\times \Big\{ \int_0^1 t^{b_1+m+k+s-1}(1-t)^{c_1+n+s-b_1-1} dt \Big\}.$$

Evaluation of the integral in terms of the classical Beta function, then finally yields

$$\Phi(s) = \mathcal{M}\{H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)\}(s) = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\Big(\frac{s-\nu}{2}\Big)\Gamma\Big(\frac{s+\nu+1}{2}\Big)$$

$$\times \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n} \frac{B(b_1+s+m+k, c_1+s+n-b_1)}{B(b_1, c_1+n-b_1)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!}.$$

Identifying the above sum as $H_C^{(s)}(b_1, b_2, b_3; c_1; x, y, z)$ defined in (1.5), we obtain the right-hand side of (3.2). □

COROLLARY 3.1. *The following inverse Mellin formula for $H_{C,p,\nu}(\cdot)$ holds:*

$$H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) = \mathcal{M}^{-1}\{\Phi(s)\} =$$

$$\frac{\pi^{-3/2}}{4i} \int_{c-i\infty}^{c+i\infty} \Big(\frac{2}{p}\Big)^s \Gamma\Big(\frac{s-\nu}{2}\Big)\Gamma\Big(\frac{s+\nu+1}{2}\Big) H_C^{(s)}(b_1, b_2, b_3; c_1; x, y, z) ds,$$

*where $c > \nu$.*

## 4. A differentiation formula for $H_{C,p,\nu}(\cdot)$

THEOREM 4.1. *The following derivative formula for $H_{C,p,\nu}(\cdot)$ holds:*

$$\frac{\partial^{M+N+K}}{\partial x^M \partial y^N \partial z^K} H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) = \frac{(b_1)_{M+K}(b_2)_{M+N}(b_3)_{N+K}}{(c_1)_{M+N+K}} \tag{4.1}$$

$$\times H_{C,p,\nu}(b_1+M+K, b_2+M+N, b_3+N+K; c_1+M+N+K; x, y, z),$$

*where $M$, $N$, $K \in \mathbf{N}_0$.*

PROOF. If we differentiate partially the series for $\mathcal{H} \equiv H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)$ in (2.1) with respect to $x$ we obtain

$$\frac{\partial \mathcal{H}}{\partial x} = \sum_{m=1}^{\infty} \sum_{n,k=0}^{\infty} \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1 + m + k, c_1 + n - b_1)}{B(b_1, c_1 + n - b_1)} \frac{x^{m-1}}{(m-1)!} \frac{y^n}{n!} \frac{z^k}{k!}.$$

Making use of the fact that

$$B(b_1, c_1 + n - b_1) = \frac{(c_1 + n)}{b_1} B(b_1 + 1, c_1 + n - b_1) \tag{4.2}$$

and $(\lambda)_{m+n} = (\lambda)_m(\lambda + m)_n$, we have upon setting $m \to m + 1$

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial x} &= \frac{b_1 b_2}{c_1} \sum_{m,n,k=0}^{\infty} \frac{(b_2 + 1)_{m+n}(b_3)_{n+k}}{(c_1 + 1)_n} \frac{B_{p,\nu}(b_1{+}1{+}m{+}k, c_1{+}n{-}b_1)}{B(b_1 + 1, c_1 + n - b_1)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!} \\ &= \frac{b_1 b_2}{c_1} H_{C,p,\nu}(b_1 + 1, b_2 + 1, b_3; c_1 + 1; x, y, z). \end{aligned} \tag{4.3}$$

Repeated application of (4.3) then yields for $M = 1, 2, \ldots$

$$\frac{\partial^M}{\partial x^M} \mathcal{H} = \frac{(b_1)_M (b_2)_M}{(c_1)_M} H_{C,p,\nu}(b_1 + M, b_2 + M, b_3; c_1 + M; x, y, z).$$

A similar reasoning shows that

$$\begin{aligned} \frac{\partial^{M+1}}{\partial x^M \partial y} \mathcal{H} &= \frac{(b_1)_M (b_2)_M}{(c_1)_M} \sum_{m,k=0}^{\infty} \sum_{n=1}^{\infty} \frac{(b_2{+}M)_{m+n}(b_3)_{n+k}}{(c_1{+}M)_n} \\ &\quad \times \frac{B_{p,\nu}(b_1{+}M{+}m{+}k, c_1{+}M{+}n{+}k)}{B(b_1{+}M, c_1{+}M)} \frac{x^m}{m!} \frac{y^{n-1}}{(n-1)!} \frac{z^k}{k!} \\ &= \frac{(b_1)_M (b_2)_{M+1} b_3}{(c_1)_{M+1}} H_{C,p,\nu}(b_1 + M, b_2 + M + 1, b_3 + 1; c_1 + M + 1; x, y, z) \end{aligned} \tag{4.4}$$

upon putting $n \to n + 1$ and using the property of the Beta function in (1.3). Repeated differentiation of (4.4) $N$ times with respect to $y$ then produces

$$\frac{\partial^{M+N}}{\partial x^M \partial y^N} \mathcal{H} = \frac{(b_1)_M (b_2)_{M+N} (b_3)_N}{(c_1)_{M+N}} H_{C,p,\nu}(b_1{+}M, b_2{+}M{+}N, b_3{+}N; c_1{+}M{+}N; x, y, z).$$

Application of the same procedure (making use of (4.2)) to deal with differentiation with respect to $z$ then yields the result stated in (4.1) □

## 5. An upper bound for $H_{C,p,\nu}(\cdot)$

THEOREM 5.1. *Let the parameters $c_1 > 0$, $b_j > 0$ $(1 \leqslant j \leqslant 3)$ with $c_1 - b_1 > 0$ and the variables $x$, $y$, $z$ be complex. Then the following bound for $H_{C,p,\nu}(\cdot)$ holds:*

$$\begin{aligned} &|H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z)| \\ &\qquad < \frac{2^\nu |p|^{\nu+1}}{\sqrt{\pi}(\mathrm{Re}(p))^{2\nu+1}} \Gamma(\nu + 1/2) H_C^{(\nu)}(b_1, b_2, b_3; c_1; |x|, |y|, |z|), \end{aligned} \tag{5.1}$$

*where* $\mathrm{Re}(p) > 0$, $\nu > 0$ *and* $H_C^{(\nu)}(\cdot)$ *is defined in* (1.5)

The integral representation of the extension $H_{B,p,\nu}(\cdot)$ in (2.3) is associated with the modified Bessel function of the second kind, for which we have the following expression [**13**, (10.32.8)]

$$K_{\nu+\frac{1}{2}}(z)=\frac{\sqrt{\pi}\left(\frac{1}{2}z\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)}\int_1^\infty e^{-zt}(t^2-1)^\nu dt,\quad (\nu>-1,\ \mathrm{Re}(z)>0).$$

In our problem we have $\nu>0$, $\mathrm{Re}(z)>0$. Further, we let $x=\mathrm{Re}(z)$, so that

$$|K_{\nu+\frac{1}{2}}(z)|\leqslant\frac{\sqrt{\pi}\left(\frac{1}{2}|z|\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)}\left|\int_1^\infty e^{-zt}(t^2-1)^\nu dt\right|<\frac{\sqrt{\pi}\left(\frac{1}{2}|z|\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)}\int_0^1 t^{2\nu}e^{-xt}dt$$
$$=\frac{\sqrt{\pi}\left(\frac{1}{2}|z|\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)}\frac{\Gamma(2\nu+1,x)}{x^{2\nu+1}},\tag{5.2}$$

where $\Gamma(a,z)$ is the upper incomplete gamma function [**13**, (8.2.2)]. We can simplify (5.2) by making use of the simple inequality $\Gamma(2\nu+1,x)<\Gamma(2\nu+1)$ to find

$$|K_{\nu+\frac{1}{2}}(z)|<\frac{\sqrt{\pi}\left(\frac{1}{2}|z|\right)^{\nu+\frac{1}{2}}}{\Gamma(\nu+1)}\frac{\Gamma(2\nu+1)}{x^{2\nu+1}}=\frac{1}{2}\Big(\frac{2|z|}{x^2}\Big)^{\nu+\frac{1}{2}}\Gamma\Big(\nu+\frac{1}{2}\Big),\tag{5.3}$$

upon use of the duplication formula for the gamma function.

Proof. : Setting $z=p/(t(1-t))$, where $t\in(0,1)$ and $\mathrm{Re}(p)>0$, in (5.3) we obtain

$$\left|K_{\nu+\frac{1}{2}}\Big(\frac{p}{t(1-t)}\Big)\right|<\frac{1}{2}\Big(\frac{2|p|t(1-t)}{(\mathrm{Re}(p))^2}\Big)^{\nu+\frac{1}{2}}\Gamma\Big(\nu+\frac{1}{2}\Big).$$

We shall assume that the parameters $c_1>0$, $b_j>0$ $(1\leqslant j\leqslant 3)$, with $c_1-b_1>0$. Then, from (2.3),

$$|H_{C,p,\nu}(b_1,b_2,b_3;c_1;x,y,z)|\leqslant\frac{2^\nu|p|^{\nu+1}\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}(\mathrm{Re}(p))^{2\nu+1}}\sum_{m,n,k=0}^\infty\frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_nB(b_1,c_1-b_1+n)}$$
$$\times\frac{|x|^m}{m!}\frac{|y|^n}{n!}\frac{|z|^k}{k!}\int_0^1 t^{b_1+\nu+m+k-1}(1-t)^{c_1-b_1+\nu+n-1}dt$$
$$<\frac{2^\nu|p|^{\nu+1}\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}(\mathrm{Re}(p))^{2\nu+1}}\sum_{m,n,k=0}^\infty\frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n}$$
$$\times\frac{B(b_1+\nu+m+k,c_1-b_1+\nu+n)}{B(b_1,c_1-b_1+n)}\frac{|x|^m}{m!}\frac{|y|^n}{n!}\frac{|z|^k}{k!}$$

which is the result stated in (5.1). □

## 6. Recursion formulas for $H_{C,p,\nu}(\cdot)$

In this section, we obtain two recursion formulas for the extended Srivastava function $H_{C,p,\nu}(\cdot)$. The first formula gives recursions with respect to the numerator parameters $b_2$ and $b_3$, and the second a recursion with respect to the denominator parameter $c_1$.

THEOREM 6.1. *The following recursions for $H_{C,p,\nu}(\cdot)$ with respect to the numerator parameters $b_2$ and $b_3$ hold:*

$$(6.1)\quad H_{C,p,\nu}(b_1,b_2+1,b_3;c_1;x,y,z) = H_{C,p,\nu}(b_1,b_2,b_3;c_1;x,y,z) + \frac{xb_1}{c_1}H_{C,p,\nu}(b_1+1,b_2+1,b_3;c_1+1;x,y,z) + \frac{yb_3}{c_1}H_{C,p,\nu}(b_1,b_2+1,b_3+1;c_1+1;x,y,z),$$

$$(6.2)\quad H_{C,p,\nu}(b_1,b_2,b_3+1;c_1;x,y,z) = H_{C,p,\nu}(b_1,b_2,b_3;c_1;x,y,z) + \frac{yb_2}{c_1}H_{C,p,\nu}(b_1,b_2+1,b_3+1;c_1+1;x,y,z) + \frac{zb_1}{c_1}H_{C,p,\nu}(b_1+1,b_2,b_3+1;c_1+1;x,y,z).$$

PROOF. From (2.1) and the result $(b_2+1)_{m+n} = (b_2)_{m+n}(1+m/b_2+n/b_2)$, we obtain

$$(6.3)\quad H_{C,p,\nu}(b_1,b_2+1,b_3;c_1;x,y,z) = \sum_{m,n,k=0}^{\infty}\frac{(b_2+1)_{m+n}(b_3)_{n+k}}{(c_1)_n}\frac{B_{p,\nu}(b_1+m+k,c_1+n-b_1)}{B(b_1,c_1+n-b_1)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}$$
$$= H_{C,p,\nu}(b_1,b_2,b_3;c_1;x,y,z) + \frac{x}{b_2}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n}\frac{B_{p,\nu}(b_1+m+k,c_1+n-b_1)}{B(b_1,c_1+n-b_1)}\frac{x^{m-1}}{(m-1)!}\frac{y^n}{n!}\frac{z^k}{k!}$$
$$+ \frac{y}{b_2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n}\frac{B_{p,\nu}(b_1+m+k,c_1+n-b_1)}{B(b_1,c_1+n-b_1)}\frac{x^m}{m!}\frac{y^{n-1}}{(n-1)!}\frac{z^k}{k!}.$$

Consider the first sum in (6.3) which we denote by $S$. Put $m\to m+1$ and use the identity $(a)_{n+1} = a(a+1)_n$ to find

$$S = \frac{x}{b_2}\sum_{m,n,k=0}^{\infty}\frac{(b_2)_{m+n+1}(b_3)_{n+k}}{(c_1)_n}\frac{B_{p,\nu}(b_1+1+m+k,c_1+n-b_1)}{B(b_1,c_1+n-b_1)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}$$
$$= x\sum_{m,n,k=0}^{\infty}\frac{(b_2+1)_{m+n}(b_3)_{n+k}}{(c_1)_n}\frac{B_{p,\nu}(b_1+1+m+k,c_1+n-b_1)}{B(b_1,c_1+n-b_1)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}.$$

Using (4.2), we then obtain

$$(6.4)\quad S = \frac{xb_1}{c_1}\sum_{m,n,k=0}^{\infty}\frac{(b_2+1)_{m+n}(b_3)_{n+k}}{(c_1+1)_n}\frac{B_{p,\nu}(b_1+1+m+k,c_1+n-b_1)}{B(b_1+1,c_1+n-b_1)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}$$
$$= \frac{xb_1}{c_1}H_{C,p,\nu}(b_1+1,b_2+1,b_3;c_1+1;x,y,z).$$

Proceeding in a similar manner for the second series in (6.3) with $n \to n+1$, we find that this sum can be expressed as

$$\frac{yb_3}{c_1} H_{C,p,\nu}(b_1, b_2+1, b_3+1; c_1+1; x, y, z). \tag{6.5}$$

Combination of (6.4) and (6.5) with (6.3) then produces the result stated in (6.1). The proof of (6.2) can be established in a similar manner. □

Corollary 6.1. *From* (6.1) *and* (6.2) *the following recursions hold:*

$$\begin{aligned} H_{C,p,\nu}(b_1, b_2+N, b_3; c_1; x, y, z) &= H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ &+ \frac{xb_1}{c_1} \sum_{\ell=1}^{N} H_{C,p,\nu}(b_1+1, b_2+\ell, b_3; c_1+1; x, y, z) \\ &+ \frac{yb_3}{c_1} \sum_{\ell=1}^{N} H_{C,p,\nu}(b_1, b_2+\ell, b_3+1; c_1+1; x, y, z), \end{aligned}$$

$$\begin{aligned} H_{C,p,\nu}(b_1, b_2, b_3+N; c_1; x, y, z) &= H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \\ &+ \frac{yb_2}{c_1} \sum_{\ell=1}^{N} H_{C,p,\nu}(b_1, b_2+1, b_3+\ell; c_1+1; x, y, z) \\ &+ \frac{zb_1}{c_1} \sum_{\ell=1}^{N} H_{C,p,\nu}(b_1+1, b_2, b_3+\ell; c_1+1; x, y, z) \end{aligned}$$

*for positive integer* $N$.

Theorem 6.2. *The following* 3*-term recursion for* $H_{C,p,\nu}(\cdot)$ *with respect to the denominator parameter* $c_1$ *holds:*

$$\begin{aligned} H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) &= H_{C,p,\nu}(b_1, b_2, b_3; c_1+1; x, y, z) \\ &+ \frac{yb_2b_3}{c_1(c_1+1)} H_{C,p,\nu}(b_1, b_2+1, b_3+1; c_1+2; x, y, z). \end{aligned} \tag{6.6}$$

Proof. Consider the case when $c_1$ is reduced by 1, namely

$$H \equiv H_{C,p,\nu}(b_1, b_2, b_3; c_1-1; x, y, z)$$

and use $(c_1-1)_n = (c_1)_n / \{1 + \frac{n}{c_1-1}\}$. Then

$$\begin{aligned} H &= \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1-1)_n} \frac{B_{p,\nu}(b_1+m+k, c_1+n-b_1)}{B(b_1, c_1+n-b_1)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!} \\ &= \sum_{m,n,k=0}^{\infty} \frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n} \frac{B_{p,\nu}(b_1+m+k, c_1+n-b_1)}{B(b_1, c_1+n-b_1)} \left(1 + \frac{n}{c_1-1}\right) \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^k}{k!} \\ &= H_{C,p,\nu}(b_1, b_2, b_3; c_1; x, y, z) \end{aligned}$$

$$+\frac{y}{c_1-1}\sum_{m=1}^{\infty}\sum_{n,k=0}^{\infty}\frac{(b_2)_{m+n}(b_3)_{n+k}}{(c_1)_n}\frac{B_{p,\nu}(b_1+m+k,c_1+n-b_1)}{B(b_1,c_1+n-b_1)}\frac{x^m}{m!}\frac{y^{n-1}}{(n-1)!}\frac{z^k}{k!}.$$

Putting $n \to n+1$ in the above sum, we obtain

$$H_{C,p,\nu}(b_1,b_2,b_3;c_1-1;x,y,z) = H_{C,p,\nu}(b_1,b_2,b_3;c_1;x,y,z)$$
$$+\frac{yb_2b_3}{c_1(c_1-1)}\sum_{m,n,k=0}^{\infty}\frac{(b_2+1)_{m+n}(b_3+1)_{n+k}}{(c_1+1)_n}\frac{B_{p,\nu}(b_1+m+k,c_1+1+n-b_1)}{B(b_1,c_1+1+n-b_1)}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}$$
$$= H_{C,p,\nu}(b_1,b_2,b_3;c_1;x,y,z)+\frac{yb_2b_3}{c_1(c_1-1)}H_{C,p,\nu}(b_1,b_2+1,b_3+1;c_1+1;x,y,z).$$

Replacement of $c_1$ by $c_1+1$ then yields the result stated in (6.6). □

## 7. Concluding remarks

We have introduced the $(p,\nu)$-extension of Srivastava's triple hypergeometric function given by $H_{C,p,\nu}(\cdot)$ in (2.1). We have given some integral representations of this function that involve the modified Bessel function of the second kind and a Gauss hypergeometric function. We have also established some properties of the function $H_{C,p,\nu}(\cdot)$, namely the Mellin transform, a differential formula, a bounded inequality and some recursion relations.

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