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On Strategy-proofness and the Salience of Single-peakedness in a Private Goods Economy*

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Abstract: We consider strategy-proof rules operating on a rich domain of preference profiles in a set up where multiple private goods have to be assigned to a set of agents with entitlements where preferences display satiation. We show that if the rule is in addition "desirable", in that it is tops-only, continuous, same-sided and individually rational with respect to the entitlements, then the preferences in the domain have to satisfy a variant of single-peakedness (referred to as semilattice single-peakedness). We also provide a converse of this main finding. It turns out that this domain coincides with the one already identified in a general set up with a public good. Finally, we relate semilattice single-peaked domains to well-known restricted domains under which strategy-proof and desirable rules do exist.

Keywords: Strategy-proofness; Semilattice Single-peakedness.
JEL Classification Number: D71.

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1 Introduction

1.1 Goal and main result

The notion of single-peakedness has played a fundamental role in the design of rules with appealing incentive properties in various economic and political models with public or private components. We seek to identify the underlying fundamental property of a domain of preferences that admits a non-trivial strategy-proof rule. In particular, we enquire whether single-peakedness is indeed indispensable to the design of such rules in a set up where multiple private goods are assigned to a set of agents with entitlements.[1]

Our methodology postulates preference domains that admit the design of a strategy-proof rule that satisfies some additional axioms,[2] and investigates the implications of this postulate on the structure of preferences in the domain. Our main result is that if the domain satisfies some "richness" conditions, then the existence of a rule satisfying these axioms implies that the domain must satisfy a particular weakening of single-peakedness, called semilattice single-peakedness. We also show that any domain of semilattice single-peaked preferences admits a strategy-proof rule satisfying the same axioms.

The restriction of semilattice single-peakedness assumes a semilattice[3] on the set of alternatives and requires that for any triple x, y, z of alternatives, a preference ordering that has x as its peak must rank the supremum of the pair (x, y) above the supremum of the pair (z, y). Chatterji and Massó (2018) show that this is equivalent to requiring that (i) moving "closer" to the top in the order of the semilattice is improving and (ii) the supremum of the peak of the preference ordering x and any other alternative w not above x according to the order of the semilattice is at least as preferred as w.

The domain of semilattice single-peaked preferences is known to be salient for the design of non-trivial and "simple" strategy-proof rules in the voting model (see Chatterji and Massó (2018)). In spite of the significant differences between the voting model and the private goods setting, the domain implications of the existence of strategy-proof rules (satisfying other appealing properties) turns out to be identical. This may be seen as evidence to support the view that some form of single-peakedness lies at the heart of possibility results in the literature.

In the next subsection, we provide a heuristic description of our main result. The rest

[1] Three well studied partially predecessors of this formulation are Sprumont (1991), Mas-Colell (1992) and Barberà and Jackson (1995). See also Barberà, Jackson and Neme (1997) and Klaus, Peters and Storcken (1998). We describe the relation of those papers, and others, with our contribution in Section 4.

[2] Specifically, tops-onlyness, continuity, same-sidedness and individual rationality with respect to the entitlements.

[3] A semilattice on a set is a binary relation satisfying reflexivity, antisymmetry and transitivity with the property that every pair of elements in the set has supremum.

of the paper is organized as follows. Section 2 introduces basic definitions and notation, the desirable properties of rules and properties of preferences and domains. Sections 3 contains the results of the paper. In Section 4 we relate our results to existing results in the literature on domain restrictions for strategy-proof rules for private goods, present some corollaries of our main result, and discuss our axioms and richness conditions. An Appendix collects the proofs of complementary results omitted in the main text.

1.2 A heuristic description of the main result

Consider a set of individual allotments given by the interval $[0, W]$, where $W > 0$. Given agent i and $q^i \in [0, W]$, we define a semilattice $\succeq^{q^i}$ on $[0, W]$ as follows: for any pair $x^i, y^i \in [0, W]$ on the same side of q^i (either to the left or to the right of q^i with respect to the natural order of the reals), $x^i \succeq^{q^i} y^i$ whenever x^i is closer to q^i than y^i. If x^i and y^i are on opposite sides of q^i, they are unrelated under $\succeq^{q^i}$. Indeed, every pair $x^i, y^i \in [0, W]$ has a supremum (or least upper bound or join): if $x^i \succeq^{q^i} y^i$ then $\sup_{\succeq^{q^i}}\{x^i, y^i\} = x^i$ and if x^i and y^i are unrelated under $\succeq^{q^i}$, then $\sup_{\succeq^{q^i}}\{x^i, y^i\} = q^i$.

Consider a preference R^i of agent i with top $t(R^i)$, assumed unique, and with entitlement q^i. Semilattice single-peakedness with respect to $\succeq^{q^i}$ requires (i) preferences between $t(R^i)$ and q^i are weakly declining in the usual sense of single-peakedness, and (ii) alternatives "beyond" q^i are dispreferred to q^i. Figure 1.a (where arrows indicate the directions in which $\succeq^{q^i}$ is increasing) depicts a semilattice single-peaked preference with respect to q^i for the case where $t(R^i) < q^i$. Figure 1.b (where arrows indicate the increasing direction of the reals) depicts a single-peaked preference with the same top and makes clear that semilattice single-peakedness is a significant weakening of single-peakedness. Notice in particular that allotments to the left of $t(R^i)$ in Figure 1.a are ranked arbitrarily, and that q^i does not play any role in Figure 1.b.

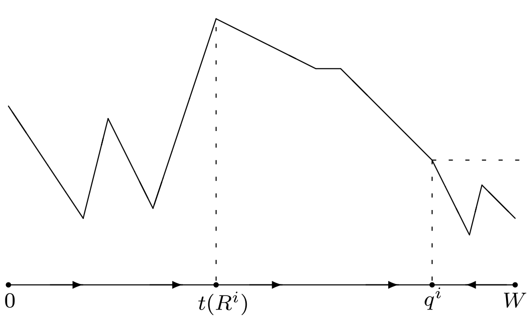

Figure 1.a

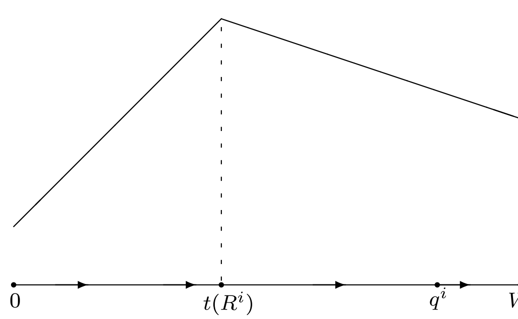

Figure 1.b

The main intuition of why a domain that admits a strategy-proof and desirable rule f has to be semilattice single-peaked can be obtained by looking at the set of allotments that f can assign to i at R^i, together with some profile of the other agents' preferences. This set in Figure 1.a is $[t(R^i), q^i]$, provided that f is strategy-proof and individually rational with respect to q^i, and the domain of f is sufficiently rich. We sketch out why the shape of R^i has to be as in Figure 1.a. First, assume that $t(R^i) < x^i < y^i \leq q^i$. Then, there exists a profile of the other agents' preferences at which, together with R^i, f selects x^i while f selects y^i if i submits any preference with top at y^i. Strategy-proofness implies that $x^i R^i y^i$, and this argument applies also to single-peaked domains. Second, assume that $q^i < x^i \leq W$. Then, there exists a profile of the other agents' preferences (each with allotment $\frac{W-q^i}{n-1}$ as top, where n is the number of agents) at which, together with R^i, f selects q^i while f selects x^i if i submits any preference with top at x^i. Strategy-proofness implies that $q^i R^i x^i$, and no restriction can be obtained on the preference ordering between any pair of i's allotments above q^i. Third, assume $x^i < t(R^i)$. Then, $t(R^i) P^i x^i$ holds trivially, and no restriction can be obtained on the preference ordering between any pair of i's allotments below $t(R^i)$. Our Theorem 1 provides a precise formulation of this intuition in a setting with potentially many private goods.

We now consider three formulations of private goods allocation models studied in the literature. Barberà, Jackson and Neme (1997) consider the problem of allocating W units of a perfectly divisible private good among n agents, each i with entitlement q^i.[4] This requires that if i asks for q^i she must receive it. Individual rationality with respect to q^i would therefore require each agent to always receive an allotment at least as preferred as q^i. Sprumont (1991) studies the model without entitlements, and without explicitly requiring individual rationality.[5] He characterizes the uniform rule as the unique one satisfying strategy-proofness, efficiency and anonymity. His axioms, in particular anonymity, effectively guaranties $q^i = \frac{W}{n}$ to each agent i. All these models assume agents have single-peaked preferences on the set of individual allotments $[0, W]$ and characterize specific families of rules that are strategy-proof.[6]

In contrast, rather than assuming single-peakedness from the outset, we postulate in a multi-dimensional version of these models[7] that the domain satisfies some "richness" conditions and admits a strategy-proof rule that is tops-only, continuous, same-sided and individually rational with respect $q = (q^1, \ldots, q^n)$. We assume richness because in

[4] This is a particular version of their general model without entitlements. Klaus, Peters and Storcken (1998) conforms precisely to this description.

[5] This is also the general set up in Barberà, Jackson and Neme (1998).

[6] For instance, Barberà, Jackson and Neme (1997) are also interested in situations where anonymity is not a reasonable requirement, and consequently the uniform allocation rule is not appropriate. Agents may have a wide range of priorities, seniorities or rights (different to entitlements) that the rule ought to respect, at least partially. They characterize the class of all strategy-proof, efficient and replacement monotonic rules as the family of sequential allotment rules (in Subsection 3.3 we describe with detail one rule within this family).

[7] Considered first by Amorós (2002) and fully studied by Morimoto, Serizawa and Ching (2013). These contributions are discussed in Subsection 4.3.

its absence strategy-proofness may be ineffectual. We show that the domain has the semilattice single-peaked structure and observe that this structure suffices for the design of a strategy-proof rule with these properties. The semilattice single-peakedness is a generalization of Figure 1.a to $\mathbb{R}^m_+$ using the L_1 norm, where m is the number of goods to be allotted (see Section 2 for details).

Moulin (1980) considers a public good model where the level of the public good belongs to $[0, W]$. A two-agent, anonymous, efficient and strategy-proof rule can be defined by selecting a fixed ballot at some level $q \in [0, W]$; the rule selects at every profile of preferences the median of the set of two tops and q. The preference restriction that is implied by the strategy-proofness of this rule is exactly the semilattice single-peakedness displayed in Figure 1.a. Chatterji and Massó (2018) consider the general version of the public good problem with no structure on the set of alternatives and show that the same notion of semilattice single-peakedness emerges as a consequence of strategy-proofness along with tops-only, unanimity and anonymity, provided that the domain is rich. It is known that only for $n = 2$, the private good case can be formulated as a public good case: this is not the case for $n \geq 3$ and multiple goods.

The main contribution of this paper is to highlight the role of semilattice single-peakedness as the fundamental underlying structure of preferences in a domain that permits the design of strategy-proof rules in the disparate private and public goods models.

2 Preliminaries

2.1 Basic definitions and notation

Our general setup closely follows Morimoto, Serizawa and Ching (2013). Let $N = \{1, \ldots, n\}$ be the finite set of *agents*, with $n \geq 2$, and let $M = \{1, \ldots, m\}$ be the set of perfectly divisible goods, with $m \geq 1$. For each $\ell \in M$, let $W_\ell \in \mathbb{R}_{++}$ be the strictly positive amount of good ℓ that has to be allotted among agents in N and let $W = (W_1, \ldots, W_m) \in \mathbb{R}^M_{++}$. For each $\ell \in M$, let $X_\ell = [0, W_\ell]$ and $X_{-\ell} = \prod_{\ell' \in M \setminus \{\ell\}} X_{\ell'}$. For each agent $i \in N$, let

$$X = \prod_{\ell \in M} X_\ell = \{x^i = (x^i_1, \ldots, x^i_m) \in \mathbb{R}^M_+ \mid 0 \leq x^i_\ell \leq W_\ell \text{ for each } \ell \in M\}$$

be agent i's set of possible allotments, which is the same for everyone. To emphasize agent i's allotment, we often write $x = (x^i, x^{-i}) \in X^N$ and given $\bar{r} \in \mathbb{R}$ we write $(\bar{r})^{j \neq i}$ as the $n-1$ dimensional vector with all components different to i equal to $\bar{r}$.

Let

$$Z = \{x = (x^1, \ldots, x^n) \in X^N \mid \sum_{i \in N} x^i = W\}$$

be the set of feasible allotments.

Given $x^i, y^i \in X$ define the *minimal box* $MB(x^i, y^i)$ of x^i and y^i as the set of possible allotments for agent i that lie between x^i and y^i in the L_1-norm, denoted by $\|\cdot\|_{L_1}$, where for any $z \in \mathbb{R}^M$, $\|z\|_{L_1} = \sum_{\ell \in M} |z_\ell|$; namely,

$$MB(x^i, y^i) = \{z^i \in X \mid \|x^i - y^i\|_{L_1} = \|x^i - z^i\|_{L_1} + \|z^i - y^i\|_{L_1}\}.$$

Remark 1. The minimal box between any pair $x^i, y^i \in X$ can be written as a Cartesian product of intervals; namely,

$$MB(x^i, y^i) = \prod_{\ell \in M} [\min\{x^i_\ell, y^i_\ell\}, \max\{x^i_\ell, y^i_\ell\}]. \tag{1}$$

■

Each agent $i \in N$ has a preference $R^i \in \mathcal{D}^i \subseteq \mathcal{R}$ over X, where $\mathcal{D}^i$ is a subset of $\mathcal{R}$, the set of all complete and transitive binary relations over X. We do not impose the continuity of preferences. Note that different agents may have different sets of preferences. For any $x^i, y^i \in X$, $x^i R^i y^i$ means that agent i considers allotment x^i to be at least as preferred as allotment y^i. Let P^i and I^i denote the strict and indifference relations induced by R^i over X, respectively. We assume that for each $R^i \in \mathcal{D}^i$ there exists $t(R^i) \in X$, the *top* of R^i, such that $t(R^i) P^i y^i$ for all $y^i \in X \backslash \{t(R^i)\}$. When R^i is obvious from the context we write t^i instead of $t(R^i)$. For $x^i \in X$, let $R^i_{x^i}$ denote any preference $R^i \in \mathcal{D}^i$ with $t(R^i) = x^i$. We assume throughout the paper that for each $i \in N$ and each $x^i \in X$, the set $\mathcal{D}^i$ contains at least one preference $R^i_{x^i}$.

We refer to the set $\mathcal{D}^1 \times \cdots \times \mathcal{D}^n$ as a *domain* of preferences, and often denote it as $\mathcal{D}$. A *profile* $R = (R^1, \ldots, R^n) \in \mathcal{D}$ is a n-tuple of preferences, one for each agent. To emphasize R^i in profile R we often write $R = (R^i, R^{-i})$.

2.2 Properties of rules

Since individual preferences are private information, they must be elicited through a rule. A *rule* is a mapping $f : \mathcal{D} \to Z$ that assigns to every profile $R \in \mathcal{D}$, a feasible allotment $f(R) \in Z$.

We are interested in rules that induce agents to tell the truth. A rule $f : \mathcal{D} \to Z$ is *strategy-proof* if for all $R \in \mathcal{D}$, all $i \in N$, and all $\widehat{R}^i \in \mathcal{D}^i$,

$$f^i(R) R^i f^i(\widehat{R}^i, R^{-i}).$$

A rule $f : \mathcal{D} \to Z$ is *tops-only* if for all $R, \widehat{R} \in \mathcal{D}$ such that $t(R^i) = t(\widehat{R}^i)$ for all $i \in N$, $f(R) = f(\widehat{R})$. Hence, a tops-only rule $f : \mathcal{D} \to Z$ can be written as a function $f : X^N \to Z$. Accordingly, we will often use the notation $f(t^1, \ldots, t^n)$ interchangeably with $f(R^1, \ldots, R^n)$ since all the rules we study are tops-only.

A tops-only rule $f : \mathcal{D} \to Z$ is *component-wise unanimous* if for all $\ell \in M$ and all $(x^1_\ell, \ldots, x^n_\ell) \in X^N_\ell$ such that $\sum_{i \in N} x^i_\ell = W_\ell$, $f^i_\ell((x^1_\ell, x^1_{-\ell}), \ldots, (x^n_\ell, x^n_{-\ell})) = x^i_\ell$ holds for all

$i \in N$ and all $(x^1_{-\ell}, \ldots, x^n_{-\ell}) \in X^N_{-\ell}$. When $m = 1$ we refer to a component-wise unanimous rule as just being *unanimous*.

A tops-only rule $f : \mathcal{D} \to Z$ is *continuous* if its associated function $f : X^N \to Z$ is continuous.

Let $q \in Z$ be a feasible allotment. A rule $f : \mathcal{D} \to Z$ satisfies *individual rationality with respect to* q if for all $R \in \mathcal{D}$ and all $i \in N$, $f^i(R)R^iq^i$.

A rule $f : \mathcal{D} \to Z$ is *same-sided* if for all $R \in \mathcal{D}$ and all $\ell \in M$,
(i) if $\sum_{j\in N} t_\ell(R^j) \geq W_\ell$, then $f^i_\ell(R) \leq t_\ell(R^i)$ for each $i \in N$, and
(ii) if $\sum_{j\in N} t_\ell(R^j) \leq W_\ell$, then $f^i_\ell(R) \geq t_\ell(R^i)$ for each $i \in N$.

We refer to a rule $f : \mathcal{D} \to Z$ satisfying strategy-proofness, tops-onlyness, continuity, same-sidedness and individual rationality with respect to q as a *desirable* rule.

A rule $f : \mathcal{D} \to Z$ is *efficient* if for all $R \in \mathcal{D}$, the allotment $f(R)$ is Pareto efficient; namely, there exists no $y \in Z$ such that $y^iR^if^i(R)$ for all $i \in N$ and $y^jP^jf^j(R)$ for at least one $j \in N$.

Observe that same-sidedness implies component-wise unanimity. For general domains, efficiency and same-sidedness are two independent properties. We justify the property of same-sidedness as a fairness axiom: if at a profile there is scarcity (or abundance) of a good, then agents should be rationed by all receiving an allotment smaller (or larger) than their tops. Under certain domains (for instance, the domain of single-peaked preferences in the one-dimensional case), same-sidedness and efficiency are equivalent.

2.3 Properties of preferences and domains

A preference $R^i \in \mathcal{D}^i$ is *top-separable* if for each each $\ell \in M$ and $x^i \in X$ we have that $(t^i_\ell, x^i_{-\ell})R^i(x^i_\ell, x^i_{-\ell})$ for all $x^i_{-\ell} \in X_{-\ell}$.[8]

A preference $R^i \in \mathcal{D}^i$ is *separable* if for each pair $x^i, y^i \in X$ and each $\ell \in M$ we have that, for all $x^i_{-\ell}, y^i_{-\ell} \in X_{-\ell}$, $(x^i_\ell, x^i_{-\ell})R^i(y^i_\ell, x^i_{-\ell})$ if and only if $(x^i_\ell, y^i_{-\ell})R^i(y^i_\ell, y^i_{-\ell})$.

Figure 2.a illustrates the allotment t^i and the two allotments $(t^i_\ell, x^i_{-\ell})$ and $(x^i_\ell, x^i_{-\ell})$ involved in the top-separability condition and Figure 2.b illustrates the four allotments $(x^i_\ell, x^i_{-\ell})$, $(y^i_\ell, x^i_{-\ell})$, $(x^i_\ell, y^i_{-\ell})$ and $(y^i_\ell, y^i_{-\ell})$ involved in the separability condition.

A preference $R^i \in \mathcal{D}^i$ is *Euclidean* if, for each $x^i, y^i \in X$, $x^iR^iy^i$ if and only if $\|t(R^i) - x^i\| \leq \|t(R^i) - y^i\|$, where $\|\cdot\|$ is the Euclidean norm. Let $\mathcal{E}$ be the set of all Euclidean preferences. It is immediate to see that all Euclidean preferences are separable and top-separable.

We will assume that each $\mathcal{D}^i$ satisfies the following conditions.

[8] Top-separability was first introduced by Le Breton and Weymark (1999).

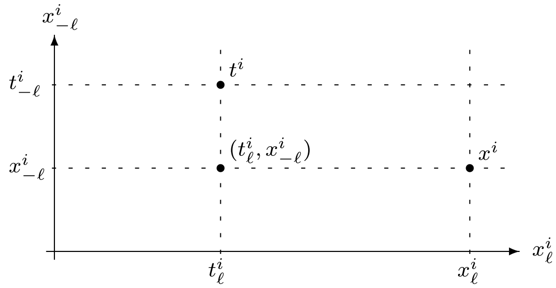

Figure 2.a

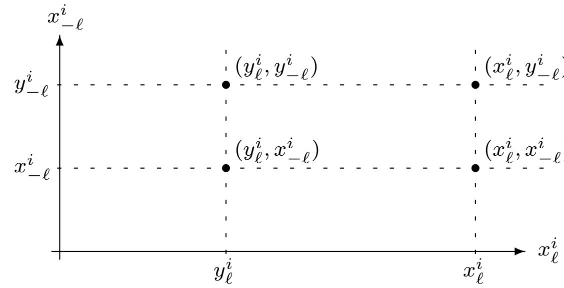

Figure 2.b

Separable Richness: A set of preferences $\mathcal{D}^i$ is *separably rich* if for every non-separable preference $R^i \in \mathcal{D}^i$ and every pair $x^i, y^i \in X$ such that $x^i P^i y^i$ there exists a separable preference $\hat{R}^i \in \mathcal{D}^i$ such that $t(R^i) = t(\hat{R}^i)$ and $x^i \hat{P}^i y^i$. A domain $\mathcal{D}^1 \times \cdots \times \mathcal{D}^n$ is *separably rich* if, for each $i \in N$, $\mathcal{D}^i$ is separably rich.

Richness relative to q^i: A set of preferences $\mathcal{D}^i$ is *rich relative to* $q^i \in X$ if for all $x^i, y^i, z^i \in X$ such that $x^i \neq y^i$, $q^i \notin intMB(x^i, y^i)$ and $z^i \notin MB(x^i, y^i)$ there exist $R^i_{x^i}, R^i_{y^i} \in \mathcal{D}^i$ such that $y^i P^i_{x^i} z^i$ and $x^i P^i_{y^i} z^i$. A domain $\mathcal{D}^1 \times \cdots \times \mathcal{D}^n$ is *rich relative to* $q = (q^1, \ldots, q^n) \in Z$ if, for each $i \in N$, $\mathcal{D}^i$ is rich relative to q^i.

The property of richness relative to q^i translates to this Euclidean setting the richness condition used in Chatterji and Massó (2018) for semilattices in the case of a public good. However, the present version of richness relative to $q^i \in X$ is weaker because the existence of the two preferences $R^i_{x^i}, R^i_{y^i} \in \mathcal{D}^i$ is only required if $q^i \notin intMB(x^i, y^i)$.[9]

Figure 3.a depicts allotments $x^i, y^i, z^i, q^i \in X$ satisfying the hypothesis of the property of richness relative to q^i. Figure 3.b illustrates the hypothesis of the property when $y^i = q^i$.

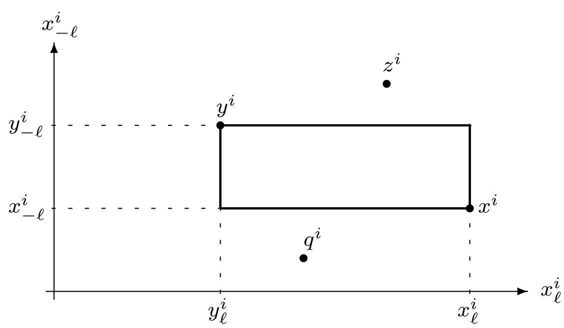

Figure 3.a

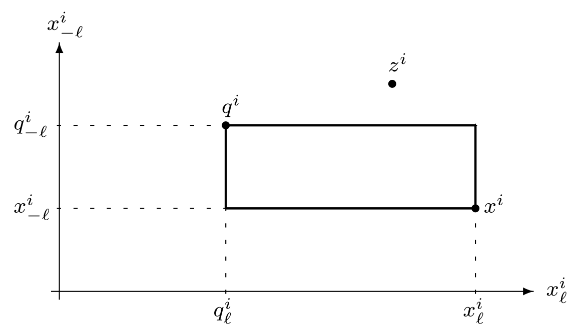

Figure 3.b

Separable richness requires that the domain $\mathcal{D}$ exclude certain non-separable preferences. For instance, assume $m = 2$ and consider $x^i, y^i \in X$ with $x^i_\ell < y^i_\ell$ for $\ell = 1, 2$. Consider the non-separable preference $R^i \in \mathcal{R}^i$ with $t(R^i) = x^i$, where

$$(x^i_1, x^i_2) P^i (y^i_1, y^i_2) P^i (y^i_1, x^i_2) P^i (x^i_1, y^i_2).$$

[9] See Subsection 4.1 for a general discussion of the connection between the two approaches. It is easy to see that the domain of Euclidean preferences does not satisfy this richness property since for each allotment $x^i \in X$ there exists a uniqye Euclidean preference whose peak is x^i.

Then, any separable preference $\hat{R}^i$ with $t(\hat{R}^i) = (x_1^i, x_2^i)$ has to have $(y_1^i, x_2^i)\hat{P}^i(y_1^i, y_2^i)$ and $(x_1^i, y_2^i)\hat{P}^i(y_1^i, y_2^i)$. Accordingly, separability would require that $R^i \notin \mathcal{D}^i$. However, note that R^i is not top-separable. In Appendix 1 (proof of Statement 1) we show that, for any non-separable but top-separable preference $R^i \in \mathcal{D}^i$ and for any pair $x^i, y^i \in X$ such that $x^i P^i y^i$, there exists a separable preference $\hat{R}^i \in \mathcal{R}^i$ with the property that $t(R^i) = t(\hat{R}^i)$ and $x^i \hat{P}^i y^i$. Separable richness requires that at least one such preference $\hat{R}^i$ belongs to $\mathcal{D}^i$.

A preference $R^i \in \mathcal{D}^i$ is *multidimensional single-peaked* if for all $x^i, y^i \in X$ such that $x^i \in MB(y^i, t^i)$, $x^i R^i y^i$. Denote the set of all multidimensional single-peaked preferences by $\mathcal{MSP}$. For $x^i, y^i, z^i \in X$ such that $x^i \notin MB(y^i, z^i)$, denote by $\widetilde{x}^i$ as the allotment in $MB(y^i, z^i)$ which is closest to x^i; namely, $\widetilde{x}^i = \arg\min_{r^i \in MB(y^i, z^i)} \left\| x^i - r^i \right\|_{L_1}$. Observe that $\widetilde{x}^i$ is unique. Moreover, the following holds for every $\ell \in M$.

$$\text{If } \min\{y_\ell^i, z_\ell^i\} \leq x_\ell^i \leq \max\{y_\ell^i, z_\ell^i\}, \text{ then } \widetilde{x}_\ell^i = x_\ell^i.$$
$$\text{If } x_\ell^i < \min\{y_\ell^i, z_\ell^i\}, \text{ then } \widetilde{x}_\ell^i = \min\{y_\ell^i, z_\ell^i\}.$$
$$\text{If } \max\{y_\ell^i, z_\ell^i\} < x_\ell^i, \text{ then } \widetilde{x}_\ell^i = \max\{y_\ell^i, z_\ell^i\}.$$

Figure 4.a depicts a situation where $x^i R^i y^i$ according to multidimensional single-peakedness and Figure 4.b illustrates geometrically for x^i and w^i the corresponding $\widetilde{x}^i$ and $\widetilde{w}^i$.

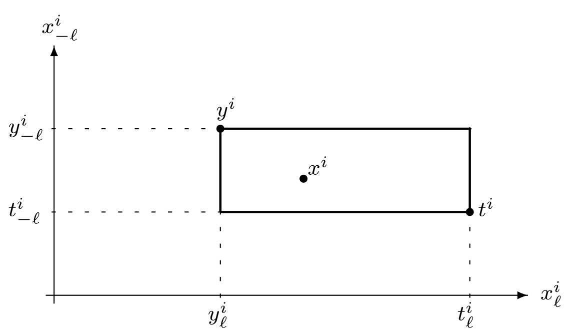

Figure 4.a

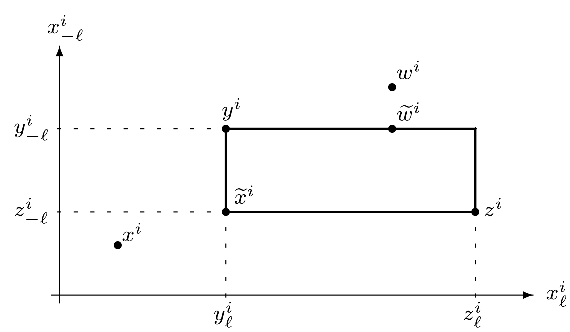

Figure 4.b

We now present the definition of semilattice single-peakedness.[10]

Semilattice Single-peakedness: A preference $R^i \in \mathcal{D}^i$ is *semilattice single-peaked* with respect to $q^i \in X$ if

(SSP.1) for all $x^i, y^i \in X$ such that $x^i \in MB(q^i, t^i)$ and $y^i \in MB(x^i, t^i)$, we have $y^i \ R^i \ x^i$, and

[10]The notion of semilattice single-peakedness was first introduced in Chatterji and Massó (2018) for the public good case by using a semilattice obtained from a given rule satisfying strategy-proofness, unanimity, anonymity and tops-onlyness. Since the condition identified here for the private goods case corresponds to the one identified for the public good case, we have decided to retain the same name, even though our multidimensional Euclidean setting does not require the explicit reference to a semilattice. In Subsection 4.1 we will describe how to obtain in our private goods case a semilattice on X inducing the same domain of semilattice single-peaked preferences.

(SSP.2) for all $x^i \notin MB(q^i, t^i)$, $\widetilde{x}^i \; R^i \; x^i$.

Given $q^i \in X$, denote the set of all semilattice single-peaked preferences by $\mathcal{SSP}(q^i)$, and given $q \in Z$, define $\mathcal{SSP}(q) = \mathcal{SSP}(q^1) \times \cdots \times \mathcal{SSP}(q^n)$.

Figures 5.a illustrates the hypotheses of the property (SSP.1), under which $y^i R^i x^i$ has to hold. Figure 5.b illustrates the hypothesis of the property (SSP.2), under which $\widetilde{x}^i R^i x^i$ has to hold, and that $\mathcal{SSP}(q^i)$ is larger than the domain $\mathcal{MSP}$ because there are $R^i \in \mathcal{SSP}(q^i)$ and $y^i, z^i \in X$ such that $t(R^i) = t^i$ but $z^i \in MB(y^i, t^i)$ and $y^i P_i z^i$.

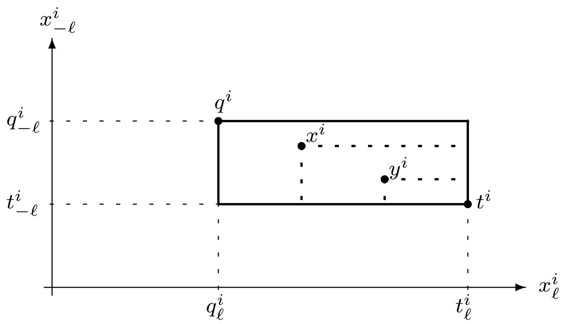

Figure 5.a

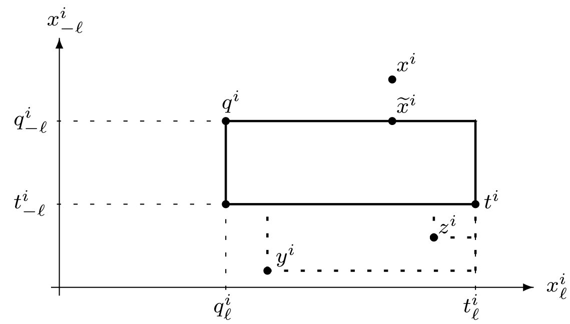

Figure 5.b

Note that $\mathcal{MSP}$ and $\mathcal{SSP}(q^i)$ contain non-separable preferences. It is easy to check that the following properties hold for each $q^i \in X$.

(1) $\mathcal{MSP}$ and $\mathcal{SSP}(q^i)$ are top-separable, separably rich and rich relative to q^i,

(2) $\mathcal{E} \subseteq \mathcal{MSP} \subsetneq \mathcal{SSP}(q^i)$ and $\bigcap_{q^i \in X} \mathcal{SSP}(q^i) = \mathcal{MSP}$, and

(3) $\mathcal{E}$ is top-separable, separable, separably rich (trivially) but it is not rich relative to any q^i. Figure 6 illustrates this last sentence by depicting two circles, with centers at x^i and y^i, representing indifference classes for the two unique Euclidean preferences $R^i_{x^i}$ and $R^i_{y^i}$ with peaks at x^i and y^i, respectively. Then, $z^i P^i_{x^i} y^i$ and $z^i P^i_{y^i} x^i$ necessarily hold, since z^i is closer to x^i than y^i is to x^i and z^i is closer to y^i than x^i is to y^i.

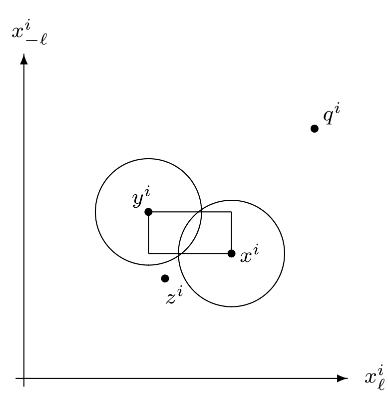

Figure 6

3 Results

3.1 Preliminaries

Remark 2. Let $\mathcal{D}^1 \times \cdots \times \mathcal{D}^n$ be a rich domain relative to $q \in Z$ and let $f : \mathcal{D} \to Z$ be a tops-only rule satisfying individual rationality with respect to q. Then, for all $t = (t^1, \ldots, t^n) \in X^N$ such that either (a) $t^i \in MB(0, q^i)$ for all $i \in N$ or (b) $t^i \in MB(q^i, W)$ for all $i \in N$, we have that $f(t) = q$. To see that, assume $t = (t^1, \ldots, t^n) \in X^N$ is such that $t^i \in MB(0, q^i)$ for all $i \in N$ and $f(t) \neq q$. Then, by feasibility of $f(t)$ and q, there exists $i \in N$ such that $f^i(t) \notin MB(t^i, q^i)$. Since $q^i \notin intMB(t^i, q^i)$, by richness relative to q^i and tops-onlyness, there exists $\widehat{R}^i_{t^i}$ such that $q^i \widehat{P}^i_{t^i} f^i(t)$, which together with tops-onlyness contradicts individual rationality with respect to q^i. The argument for case (b) is analogous. ■

Remark 3. Let $\mathcal{D}^1 \times \cdots \times \mathcal{D}^n$ be a rich domain relative to $q \in Z$ and let $f : \mathcal{D} \to Z$ be a tops-only rule satisfying same-sidedness and individual rationality with respect to q. Fix $\ell \in M$ and let $w_{-\ell}$ be such that $\sum_{j \in N} w^j_k = W_k$ for all $k \neq \ell$. Assume that $t_\ell = (t^1_\ell, \ldots, t^n_\ell) \in X^N_\ell$ is such that $t^i = (t^i_\ell, w^i_{-\ell})$ for all $i \in N$ and either (a) $t^j_\ell \geq q^j_\ell$ for all $j \in N$ or (b) $t^j_\ell \leq q^j_\ell$ for all $j \in N$. Then, for all $i \in N$,

$$f^i_\ell(t_\ell, w_{-\ell}) = q^i_\ell.$$

To see that, assume first that (a) holds. Hence, $\sum_{j \in N} t^j_\ell \geq W_\ell$. By same-sidedness, $f^i_\ell(t_\ell, w_{-\ell}) \leq t^i_\ell$ for all $i \in N$. To obtain a contradiction, assume $f^j_\ell(t_\ell, w_{-\ell}) \neq q^j_\ell$ holds for some $j \in N$. By feasibility of q, there exists $i \in N$ such that $f^i_\ell(t_\ell, w_{-\ell}) < q^i_\ell$. Accordingly $f^i(t_\ell, w_{-\ell}) \notin MB(q^i, (t^i_\ell, w^i_{-\ell}))$. Since $q^i \notin intMB(q^i, (t^i_\ell, w^i_{-\ell}))$, by richness relative to q^i, there exists $\widehat{P}^i_{(t^i_\ell, w^i_{-\ell})}$ such that

$$q^i \widehat{P}^i_{(t^i_\ell, w^i_{-\ell})} f^i((t^i_\ell, w^i_{-\ell}), (t^j_\ell, w^j_{-\ell})^{j \neq i}),$$

which, together with tops-onlyness, contradicts that f is individually rational with respect to q^i. The argument for case (b) is analogous. ■

3.2 The main result

Theorem 1. *Let $\mathcal{D}$ be a domain that is top-separable, separably rich and rich relative to $q \in Z$. Let $f : \mathcal{D} \to Z$ be a strategy-proof, tops-only and continuous rule satisfying same-sidedness and individual rationality with respect to q. Then, for each $i \in N$, $\mathcal{D}^i$ is a set of semilattice single-peaked preferences with respect to q^i.*

Proof: Let $x^i, y^i \in X$ be such that $x^i \in MB(q^i, t^i)$ and $y^i \in MB(x^i, t^i)$. For (SSP.1) we are required to show $y^i R^i_{t^i} x^i$.

For $\ell \in M$, let $z^j_{-\ell} = \frac{1}{n-1}(W_{-\ell} - t^i_{-\ell})$ for all $j \in N, j \neq i$. By same-sidedness, f is component-wise unanimous and consequently, for all $j \neq i$, all $p^i_\ell \in [0, X_\ell]$ and all $(p^j_\ell)^{j \neq i} \in [0, X_\ell]^{N \setminus \{i\}}$,

$$f^i_{-\ell}((p^i_\ell, t^i_{-\ell}), (p^j_\ell, z^j_{-\ell})^{j \neq i}) = t^i_{-\ell}. \tag{2}$$

We only consider the case that i's allotments $(x^i_\ell, t^i_{-\ell}), (y^i_\ell, t^i_{-\ell}) \in X$ are such that $q^i_\ell \leq x^i_\ell \leq y^i_\ell \leq t^i_\ell$. An analogous argument applies to the excluded case where $q^i_\ell \geq x^i_\ell \geq y^i_\ell \geq t^i_\ell$.

Claim 1 $(y^i_\ell, x^i_{-\ell}) R^i_{t^i} (x^i_\ell, x^i_{-\ell})$.

Proof of Claim 1 We note first that if $y^i_\ell = t^i_\ell$, the required conclusion follows from the assumption that preferences are top-separable. If $x^i_\ell = y^i_\ell$, the claim follows from reflexivity of $R^i_{t^i}$. So we henceforth assume $q^i_\ell \leq x^i_\ell < y^i_\ell < t^i_\ell$.

To obtain a contradiction to Claim 1, suppose that $(x^i_\ell, x^i_{-\ell}) P^i_{t^i} (y^i_\ell, x^i_{-\ell})$. By same-sidedness, f is component-wise unanimous, and therefore

$$f^i_\ell((W_\ell, t^i_{-\ell}), (0, z^j_{-\ell})^{j \neq i}) = W_\ell. \tag{3}$$

By Remark 3,

$$f^i_\ell((W_\ell, t^i_{-\ell}), (q^j_\ell, z^j_{-\ell})^{j \neq i}) = q^i_\ell. \tag{4}$$

By continuity, there exists r^{-i}_ℓ such that for all $j \neq i$, $0 \leq r^j_\ell \leq q^j_\ell$ and

$$f^i_\ell((W_\ell, t^i_{-\ell}), (r^j_\ell, z^j_{-\ell})^{j \neq i}) = y^i_\ell. \tag{5}$$

We now show that

$$f^i_\ell((t^i_\ell, t^i_{-\ell}), (r^j_\ell, z^j_{-\ell})^{j \neq i}) \leq y^i_\ell \tag{6}$$

holds as well. To obtain a contradiction, assume $f^i_\ell((t^i_\ell, t^i_{-\ell}), (r^j_\ell, z^j_{-\ell})^{j \neq i}) > y^i_\ell$.

If $W_\ell = f^i_\ell((t^i_\ell, t^i_{-\ell}), (r^j_\ell, z^j_{-\ell})^{j \neq i})$, then

$$f^i((t^i_\ell, t^i_{-\ell}), (r^j_\ell, z^j_{-\ell})^{j \neq i}) = (W_\ell, t^i_{-\ell}) P^i_{(W_\ell, t^i_{-\ell})} (f^i((W_\ell, t^i_{-\ell}), (r^j_\ell, z^j_{-\ell})^{j \neq i}), t^i_{-\ell}) = (y^i_\ell, t^i_{-\ell}),$$

which contradicts strategy-proofness.

Hence, $W_\ell > f^i_\ell((t^i_\ell, t^i_{-\ell}), (r^j_\ell, z^j_{-\ell})^{j \neq i}) > y^i_\ell > q^i_\ell$ and so (see Figure 7)

$$(y^i_\ell, t^i_{-\ell}) \notin MB((f^i_\ell((t^i_\ell, t^i_{-\ell}), (r^j_\ell, z^j_{-\ell})^{j \neq i}), t^i_{-\ell}), (W_\ell, t^i_{-\ell})). \tag{7}$$

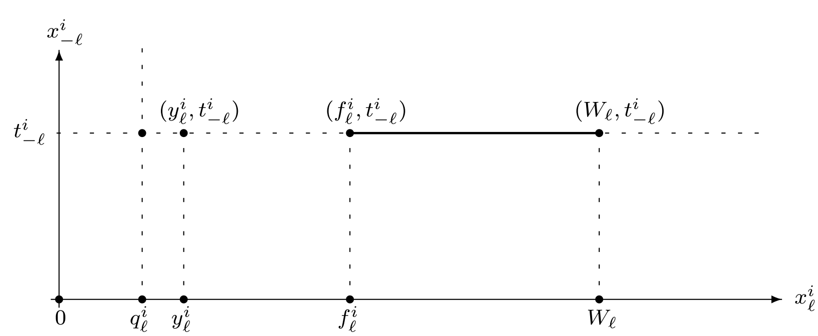

Figure 7

Because $q^i_\ell < y^i_\ell$ and (5) hold, we have that $q^i \notin intMB(f^i((t^i_\ell, t^i_{-\ell}), (r^j_\ell, z^j_{-\ell})^{j\neq i}), (W_\ell, t^i_{-\ell}))$ and, since (7) holds as well, we can apply richness relative to q^i. Therefore, there exists $\widehat{R}^i_{(W_\ell, t^i_{-\ell})} \in \mathcal{D}^i$ such that $f^i((t^i_\ell, t^i_{-\ell}), (r^j_\ell, z^j_{-\ell})^{j\neq i})\widehat{P}^i_{(W_\ell, t^i_{-\ell})}(y^i_\ell, t^i_{-\ell})$, which contradicts strategy-proofness since, by (5),

$$f^i((t^i_\ell, t^i_{-\ell}), (r^j_\ell, z^j_{-\ell})^{j\neq i})\widehat{P}^i_{(W_\ell, t^i_{-\ell})}(f^i((W_\ell, t^i_{-\ell}), (r^j_\ell, z^j_{-\ell})^{j\neq i})) = (y^i_\ell, t^i_{-\ell}).$$

Hence, (6) holds. Since for all $j \neq i$, $0 \leq r^j_\ell \leq q^j_\ell$ and $0 \leq q^i_\ell$, Remark 3 implies that

$$f^i_\ell((0, t^i_{-\ell}), (r^j_\ell, z^j_{-\ell})^{j\neq i}) = q^i_\ell. \tag{8}$$

By continuity, (5) and (8) there exists $\widehat{r}^i_\ell \in (0, W_\ell)$ such that (if $x^i_\ell = q^i_\ell$ then $\widehat{r}^i_\ell = q^i_\ell$)

$$f^i_\ell((\widehat{r}^i_\ell, t^{-i}_{-\ell}), (r^j_\ell, z^j_{-\ell})^{j\neq i}) = x^i_\ell. \tag{9}$$

Next note that our contradiction hypothesis $(x^i_\ell, x^i_{-\ell})P^i_{t^i}(y^i_\ell, x^i_{-\ell})$ and that the domain $\mathcal{D}$ is separably rich ensures that there exists a separable preference $\hat{R}^i_{t^i} \in \mathcal{D}^i$ such that $(x^i_\ell, t^i_{-\ell})\hat{P}^i_{t^i}(y^i_\ell, t^i_{-\ell})$. By strategy-proofness, tops-onlyness, top-separability, (9) and the contradiction hypothesis, we have that

$$\begin{aligned} f^i((t^i_\ell, t^i_{-\ell}), (r^i_\ell, z^j_{-\ell})^{j\neq i}) &= (f^i_\ell((t^i_\ell, t^i_{-\ell}), (r^i_\ell, z^j_{-\ell})^{j\neq i}), t^i_{-\ell}) \\ &\quad \hat{R}^i_{t^i}(f^i_\ell((\widehat{r}^i_1, t^i_{-1}), (r^j_1, z^j_{-1})^{j\neq i}), t^i_{-\ell}) \\ &= (x^i_\ell, t^i_{-\ell}) \\ &\quad \hat{P}^i_{t^i}(y^i_\ell, t^i_{-\ell}). \end{aligned}$$

By (6), and the strict preference in the expression above, $f^i_\ell((t^i_\ell, t^i_{-\ell}), (r^j_\ell, z^j_{-\ell})^{j\neq i}) < y^i_\ell < t^i_\ell$, where the last strict inequality follows from our general assumption that $q^i_\ell \leq x^i_\ell < y^i_\ell < t^i_\ell$. Hence, $(f^i_\ell((t^i_\ell, t^i_{-\ell}), (r^j_\ell, z^j_{-\ell})^{j\neq i})), t^i_{-\ell}) \notin MB((y^i_\ell, t^i_{-\ell}), (t^i_\ell, t^i_{-\ell}))$ and $(q^i_\ell, t^i_{-\ell}) \notin intMB((y^i_\ell, t^i_{-\ell}), (t^i_\ell, t^i_{-\ell}))$. By richness relative to q^i, there exists a preference $\tilde{R}^i_{t^i} \in \mathcal{D}^i$ such that $(y^i_\ell, t^i_{-\ell})\tilde{P}^i_{t^i}(f^i_\ell((t^i_\ell, t^i_{-\ell}), (r^j_\ell, z^j_{-\ell})^{j\neq i})), t^i_{-\ell})$. By (5),

$$(f^i_\ell((W_\ell, t^i_{-\ell}), (r^j_\ell, z^j_{-\ell})^{j\neq i}), t^i_{-\ell})\tilde{P}^i_{t^i}(f^i_\ell((t^i_\ell, t^i_{-\ell}), (r^j_\ell, z^j_{-\ell})^{j\neq i})), t^i_{-\ell}),$$

which contradicts strategy-proofness. This completes the proof of Claim 1. □

Next, we repeat the argument used in the verification of Claim 1 to conclude that, for any pair $\ell, \ell' \in M$, $(y^i_\ell, y^i_{\ell'}, x^i_{-\{\ell,\ell'\}})R^i_{t^i}(y^i_\ell, x^i_{\ell'}, x^i_{-\{\ell,\ell'\}})$ and use transitivity to conclude that $(y^i_\ell, y^i_{\ell'}, x^i_{-\{\ell,\ell'\}})R^i_{t^i}(x^i_\ell, x^i_{\ell'}, x^i_{-\{\ell,\ell'\}})$. Repeating the argument for $\ell'' \in M\backslash\{\ell, \ell'\}$ gives the desired conclusion $y^i R^i_{t^i} x^i$.

To prove (SSP.2), assume $x^i \notin MB(t^i, q^i)$. We want to show that $\widetilde{x}^i R^i x^i$. Define

$$\begin{aligned} L_1 &= \{\ell \in M \mid x^i_\ell < q^i_\ell \leq t^i_\ell\} \\ L_2 &= \{\ell \in M \mid q^i_\ell \leq t^i_\ell < x^i_\ell\} \\ L_3 &= \{\ell \in M \mid x^i_\ell < t^i_\ell \leq q^i_\ell\} \\ L_4 &= \{\ell \in M \mid t^i_\ell \leq q^i_\ell < x^i_\ell\}. \end{aligned}$$

By hypotheses, $L_1 \cup L_2 \cup L_3 \cup L_4 \neq \emptyset$. Suppose $L_1 \neq \emptyset$ and let $\ell \in L_1$. By definition of $\widetilde{x}^i$, $\widetilde{x}^i_\ell = q^i_\ell$. Accordingly, $x^i_\ell < \widetilde{x}^i_\ell = q^i_\ell \leq t^i_\ell$.

Claim 2.1 $(\widetilde{x}^i_\ell, x^i_{-\ell}) R^i (x^i_\ell, x^i_{-\ell})$.

Proof of Claim 2.1 We note first that if $\widetilde{x}^i_\ell = t^i_\ell$, the required conclusion follows from the assumption that preferences are top-separable. So we henceforth assume

$$x^i_\ell < \widetilde{x}^i_\ell = q^i_\ell < t^i_\ell.$$

Let $z^j_{-\ell} = \frac{1}{n-1}(W_{-\ell} - t^i_{-\ell})$ for all $j \in N \backslash \{i\}$. To obtain a contradiction to Claim 2.1, suppose that $(x^i_\ell, x^i_{-\ell}) P^i_{t^i} (q^i_\ell, x^i_{-\ell})$. By separable richness, there is a separable preference $\hat{P}^i_{t^i} \in \mathcal{D}^i$ such that $(x^i_\ell, x^i_{-\ell}) \hat{P}^i_{t^i} (q^i_\ell, x^i_{-\ell})$. By separability,

$$(x^i_\ell, t^i_{-\ell}) \hat{P}^i_{t^i} (q^i_\ell, t^i_{-\ell}). \tag{10}$$

By Remark 3, and component-wise unanimity that follows from same-sidedness,

$$f^i((t^i_\ell, t^i_{-\ell}), (W_\ell, z^j_{-\ell})^{j \neq i}) = (q^i_\ell, t^i_{-\ell}). \tag{11}$$

By same-sidedness,

$$f^i_\ell((x^i_\ell, t^i_{-\ell}), (W_\ell, z^j_{-\ell})^{j \neq i}) \leq x^i_\ell.$$

We next claim that

$$f^i_\ell((x^i_\ell, t^i_{-\ell}), (W_\ell, z^j_{-\ell})^{j \neq i}) = x^i_\ell.$$

If this is not true,

$$f^i_\ell((x^i_\ell, t^i_{-\ell}), (W_\ell, z^j_{-\ell})^{j \neq i}) = y^i_\ell < x^i_\ell \tag{12}$$

holds. Then, and since $(y^i_\ell, q^i_{-\ell}) \notin MB((x^i_\ell, t^i_{-\ell}), q^i)$ and $(q^i_\ell, q^i_{-\ell}) \notin intMB((x^i_\ell, t^i_{-\ell}), q^i)$, by richness relative to q^i, there exists a preference $P^i_{(x^i_\ell, t^i_{-\ell})} \in \mathcal{D}^i$ such that

$$(q^i_\ell, q^i_{-\ell}) P^i_{(x^i_\ell, t^i_{-\ell})} (y^i_\ell, q^i_{-\ell}).$$

By separable richness, there is a separable preference $\hat{P}^i_{t^i} \in \mathcal{D}^i$ such that $t(\hat{R}^i) = (x^i_\ell, t^i_{-\ell}) = t(R^i)$ and $(q^i_\ell, q^i_{-\ell}) \hat{P}^i_{(x^i_\ell, t^i_{-\ell})} (y^i_\ell, q^i_{-\ell})$. By separability, $(q^i_\ell, t^i_{-\ell}) \hat{P}^i_{(x^i_\ell, t^i_{-\ell})} (y^i_\ell, t^i_{-\ell})$. Hence, by (11) and (12),

$$f^i((t^i_\ell, t^i_{-\ell}), (W_\ell, z^j_{-\ell})^{j \neq i}) = (q^i_\ell, t^i_{-\ell}) \hat{P}^i_{(x^i_\ell, t^i_{-\ell})} (y^i_\ell, t^i_{-\ell}) = f^i_\ell((x^i_\ell, t^i_{-\ell}), (W_\ell, z^j_{-\ell})^{j \neq i}),$$

a contradiction with strategy-proofness. Hence, $f^i_\ell((x^i_\ell, t^i_{-\ell}), (W_\ell, z^j_{-\ell})^{j \neq i}) = x^i_\ell$. Thus, by (11), (10), $t(\hat{R}^i) = (x^i_\ell, t^i_{-\ell})$, tops-onlyness and component-wise unanimity that follows from same-sidedness,

$$f^i((x^i_\ell, t^i_{-\ell}), (W_\ell, z^j_{-\ell})^{j \neq i}) = (x^i_\ell, t^i_{-\ell}) \hat{P}^i_{t^i} (q^i_\ell, t^i_{-\ell}) = f^i((t^i_\ell, t^i_{-\ell}), (W_\ell, z^j_{-\ell})^{j \neq i}),$$

a contradiction with strategy-proofness. This finishes the proof of Claim 2.1. □

Suppose $L_4 \neq \emptyset$ and let $\ell \in L_4$. By definition of $\widetilde{x}^i$, $\widetilde{x}^i_\ell = q^i_\ell$. Accordingly, $t^i_\ell \leq q^i_\ell = \widetilde{x}^i_\ell < x^i_\ell$.

Claim 2.2 $(\widetilde{x}^i_\ell, x^i_{-\ell}) R^i (x^i_\ell, x^i_{-\ell})$.

Proof of Claim 2.2 We note first that if $\widetilde{x}^i_\ell = t^i_\ell$, the required conclusion follows from the assumption that preferences are top-separable. So we henceforth assume

$$t^i_\ell < q^i_\ell = \widetilde{x}^i_\ell < x^i_\ell.$$

Let $z^j_{-\ell} = \frac{1}{n-1}(W_{-\ell} - t^i_{-\ell})$ for all $j \in N \backslash \{i\}$. To obtain a contradiction to Claim 2.2, suppose that $(x^i_\ell, x^i_{-\ell}) P^i_{t^i} (q^i_\ell, x^i_{-\ell})$. By separable richness, there is a separable preference $\hat{P}^i_{t^i} \in \mathcal{D}^i$ such that $(x^i_\ell, x^i_{-\ell}) \hat{P}^i_{t^i} (q^i_\ell, x^i_{-\ell})$. By separability,

$$(x^i_\ell, t^i_{-\ell}) \hat{P}^i_{t^i} (q^i_\ell, t^i_{-\ell}). \tag{13}$$

By Remark 3, and component-wise unanimity that follows from same-sidedness,

$$f^i((t^i_\ell, t^i_{-\ell}), (W_\ell, z^j_{-\ell})^{j \neq i}) = (q^i_\ell, t^i_{-\ell}). \tag{14}$$

By same-sidedness,

$$f^i_\ell((x^i_\ell, t^i_{-\ell}), (W_\ell, z^j_{-\ell})^{j \neq i}) \leq x^i_\ell.$$

We next claim that

$$f^i_\ell((x^i_\ell, t^i_{-\ell}), (W_\ell, z^j_{-\ell})^{j \neq i}) = x^i_\ell.$$

If this is not true

$$f^i_\ell((x^i_\ell, t^i_{-\ell}), (W_\ell, z^j_{-\ell})^{j \neq i}) = y^i_\ell < x^i_\ell \tag{15}$$

holds. Then, and since $(y^i_\ell, q^i_{-\ell}) \notin MB((x^i_\ell, t^i_{-\ell}), q^i)$ and $(q^i_\ell, q^i_{-\ell}) \notin intMB((x^i_\ell, t^i_{-\ell}), q^i)$, by richness relative to q^i, there exists a preference $P^i_{(x^i_\ell, t^i_{-\ell})} \in \mathcal{D}^i$ such that

$$(q^i_\ell, q^i_{-\ell}) P^i_{(x^i_\ell, t^i_{-\ell})} (y^i_\ell, q^i_{-\ell}).$$

By separable richness, there is a separable preference $\hat{P}^i_{t^i} \in \mathcal{D}^i$ such that $t(\hat{R}^i) = (x^i_\ell, t^i_{-\ell}) = t(R^i)$ and $(q^i_\ell, q^i_{-\ell}) \hat{P}^i_{(x^i_\ell, t^i_{-\ell})} (y^i_\ell, q^i_{-\ell})$. By separability, $(q^i_\ell, t^i_{-\ell}) \hat{P}^i_{(x^i_\ell, t^i_{-\ell})} (y^i_\ell, t^i_{-\ell})$. Hence, by (14), (15) and

$$f^i((t^i_\ell, t^i_{-\ell}), (W_\ell, z^j_{-\ell})^{j \neq i}) = (q^i_\ell, t^i_{-\ell}) P^i_{(x^i_\ell, t^i_{-\ell})} (y^i_\ell, t^i_{-\ell}) = f^i_\ell((x^i_\ell, t^i_{-\ell}), (W_\ell, z^j_{-\ell})^{j \neq i}),$$

a contradiction with strategy-proofness. Hence, $f^i_\ell((x^i_\ell, t^i_{-\ell}), (W_\ell, z^j_{-\ell})^{j \neq i}) = x^i_\ell$. Thus, by (14), (15), $t(\hat{R}^i) = (x^i_\ell, t^i_{-\ell})$, tops-onlyness and component-wise unanimity that follows from same-sidedness,

$$f^i((x^i_\ell, t^i_{-\ell}), (W_\ell, z^j_{-\ell})^{j \neq i}) = (x^i_\ell, t^i_{-\ell}) \hat{P}^i_{t^i} (q^i_\ell, t^i_{-\ell}) = f^i((t^i_\ell, t^i_{-\ell}), (W_\ell, z^j_{-\ell})^{j \neq i}),$$

a contradiction with strategy-proofness. This finishes the proof of Claim 2.2. □

Suppose $\ell \in L_2 \cup L_3$. By definition of $\widetilde{x}^i$, $\widetilde{x}^i_\ell = t^i_\ell$. Then, $(\widetilde{x}^i_\ell, x^i_{-\ell}) R^i (x^i_\ell, x^i_{-\ell})$ follows from the assumption that preferences are top-separable.

Moving from x^i to $\widetilde{x}^i$ coordinate-by-coordinate and applying Claim 2.1 or Claim 2.2 or this last observation together with transitivity of R^i we establish that $x^i R^i \widetilde{x}^i$. This finishes the proof of Theorem 1. ■

3.3 Semilattice single-peaked domains admit desirable rules

In this subsection we show that, for each $q \in Z$, the domain $\mathcal{SSP}(q)$ admits a strategy-proof, tops-only and continuous rule that satisfies same-sidedness and individual rationality with respect to q. The rule that we will exhibit is the $m-$dimensional sequential allotment rule $f^q : \mathcal{SSP}(q) \to Z$ where, for each $R \in \mathcal{SSP}(q)$, $f^q(R) = (f^{q_\ell}_\ell(t_\ell(R^1), \ldots, t_\ell(R^n)))_{\ell \in M}$ and, for each $\ell \in M$, (i) $f^{q_\ell}_\ell : \mathcal{SSP}(q_\ell) \to [0, W_\ell]$ is the sequential allotment rule that satisfies individual rationality with respect to q_ℓ (as defined in Barberà, Jackson and Neme (1997) on the domain of single-peaked preferences on $[0, W_\ell]$) and (ii) its sequential adjustment function is uniform (up to feasibility). Observe that sequential allotment rules are tops-only, and since the domain of single-peaked preferences on $[0, W_\ell]$ is a subset of $\mathcal{SSP}(q_\ell)$, the rule $f^{q_\ell}_\ell : \mathcal{SSP}(q_\ell) \to [0, W_\ell]$ can be identified with $f^{q_\ell}_\ell : X^N_\ell \to [0, W_\ell]$, and accordingly it can be extended to operate on the larger domain $\mathcal{SSP}(q_\ell)$. Moreover, as established in Barberà, Jackson and Neme (1997), $f^{q_\ell}_\ell$ has the property that (i) $f^{q_\ell}_\ell(0, \ldots, 0) = f^{q_\ell}_\ell(W_\ell, \ldots, W_\ell) = q_\ell$ and (ii) q_ℓ can be seen as a vector of guaranteed endowments since, for each $i \in N$ and each $t^{-i}_\ell \in X^{N \setminus \{i\}}_\ell$, $f^{q_\ell, i}_\ell(q^i_\ell, t^{-i}_\ell) = q^i_\ell$.

For each $q \in Z$ and $\ell \in M$, the definition of $f^{q_\ell}_\ell : X^N_\ell \to [0, W_\ell]$ is sequential and as follows. Let $t_\ell = (t^1_\ell, \ldots, t^n_\ell) \in X^N_\ell$ be arbitrary.

Suppose $\sum_{j \in N} t^j_\ell = W_\ell$. Then, $f^{q_\ell}_\ell(t_\ell) = t_\ell$.

Suppose $\sum_{j \in N} t^j_\ell > W_\ell$. If $t^i_\ell \geq q^i_\ell$ for all $i \in N$, then $f^{q_\ell}_\ell(t_\ell) = q_\ell$. Otherwise, each i with $t^i_\ell \leq q^i_\ell$ receives t^i_ℓ and leaves the process with t^i_ℓ, while the other agents remain. The guaranteed endowments of the remaining agents are weakly increased by distributing among them the not yet allotted amount uniformly. Agents with a top smaller (larger) than or equal to the new guaranteed endowment receive the top and leave the process, while the others remain. The process proceeds this way until there is no agent with a top smaller (larger) than the current guaranteed endowment, and the rule assigns to those remaining agents their last guaranteed endowment.

Suppose $\sum_{j \in N} t^j_\ell < W_\ell$. If $t^i_\ell \leq q^i_\ell$ for all $i \in N$, then $f^{q_\ell}_\ell(t_\ell) = q_\ell$. Otherwise, each i with $t^i_\ell \geq q^i_\ell$ receives t^i_ℓ and leaves the process with t^i_ℓ, while the other agents remain. The guaranteed endowments of the remaining agents are weakly decreased uniformly, keeping them feasible and non-negative. Agents with a top larger than or equal to the new guaranteed endowment receive the top and leave the process, while the others remain. The process proceeds this way until there is no agent with a top larger than the current guaranteed endowment, and the rule assigns to those remaining agents their last guaranteed endowment.

For further reference, we state as Remark 4 a property of any of these sequential allotment rules.

Remark 4. Let q_ℓ be such that $\sum_{i \in N} q_\ell^i = W_\ell$ and $f_\ell^{q_\ell} : \mathcal{SSP}(q_\ell) \to [0, W_\ell]$ be the sequential allotment rule that satisfies individual rationality with respect to q_ℓ and its sequential adjustment function is uniform (up to feasibility). Then, the followings hold:
(i) For each $t_\ell = (t_\ell^1, \ldots, t_\ell^n) \in X_\ell^N$, at the end of the process, each agent i receives either t_ℓ^i or i's final guaranteed endowment which has been moving monotonically towards t_ℓ^i along the process.
(ii) *Uncompromisingness*: For each $R \in \mathcal{SSP}(q)$, each $\ell \in M$, each $i \in N$ and each $\widehat{R}^i \in \mathcal{SSP}(q^i)$, if $f_\ell^{q_\ell,i}(R) < t_\ell(R^i)$ and $f_\ell^{q_\ell,i}(R) \leq t_\ell(\widehat{R}^i)$ ($f_\ell^{q_\ell,i}(R) > t_\ell(R^i)$ and $f_\ell^{q_\ell,i}(R) \geq t_\ell(\widehat{R}^i)$), then $f_\ell^{q_\ell,i}(\widehat{R}^i, R^{-i}) = f_\ell^{q_\ell,i}(R)$.
(iii) *Peak-monotonicity*: For each $R \in \mathcal{SSP}(q)$, each $\ell \in M$, each $i \in N$ and each $\widehat{R}^i \in \mathcal{SSP}(q^i)$, if $t_\ell(\widehat{R}^i) \geq t_\ell(R^i)$ ($t_\ell(\widehat{R}^i) \leq t_\ell(R^i)$), then $f_\ell^{q_\ell,i}(\widehat{R}^i, R^{-i}) \geq f_\ell^{q_\ell,i}(R)$ ($f_\ell^{q_\ell,i}(\widehat{R}^i, R^{-i}) \leq f_\ell^{q_\ell,i}(R)$). ■

The next example illustrates one of these sequential allotment rules $f_\ell^{q_\ell} : \mathcal{SSP}(q_\ell) \to [0, W_\ell]$ by evaluating it at two different profiles of tops.

Example 1. Let $N = \{1, 2, 3, 4\}$ and $W_\ell = 7$. Assume $q_\ell = (0, 1, 2, 4) \in Z$ is the initial vector of guaranteed endowments, represented in Figure 8.a by the four circles. In Figures 8 and 9 the horizontal axes represent the assignments of the good while the vertical axes represent the agents. To simplify notation, we omit the reference to q_ℓ and write f_ℓ^i instead of $f_\ell^{q_\ell,i}$.

Consider the vector of tops $\bar{t}_\ell = (5, 1, 2.5, 2) \in X_\ell^N$. Since $\sum_{j \in N} \bar{t}_\ell^j > 7$, $\bar{t}_\ell^2 = 1 = q_\ell^2$ and $\bar{t}_\ell^4 = 2 < 4 = q_\ell^4$, $f_\ell^2(\bar{t}_\ell) = 1$ and $f_\ell^4(\bar{t}_\ell) = 2$, and agents 2 and 4 leave with their tops. The amount not allotted yet is $\overline{W}_\ell = 4$. The new updated guaranteed endowments for agents 1 and 3 that remain are $\bar{q}_\ell^1 = q_\ell^1 + x = x$ and $\bar{q}_\ell^3 = q_\ell^3 + x = 2 + x$, where x is such that $\bar{q}_\ell^1 + \bar{q}_\ell^3 = 4$. Hence, $x = 1$ and so $\bar{q}_\ell^1 = 1$ and $\bar{q}_\ell^3 = 3$, represented in Figure 8.b by the two circles.

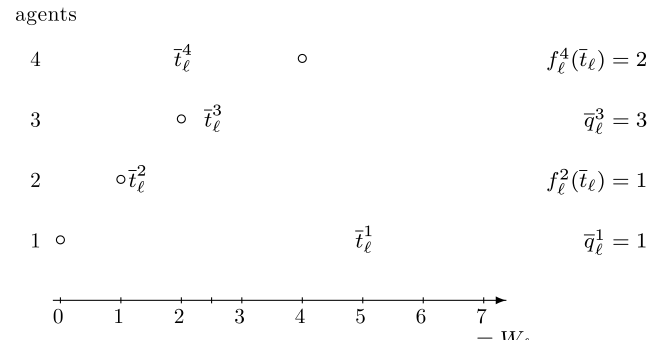

Figure 8.a

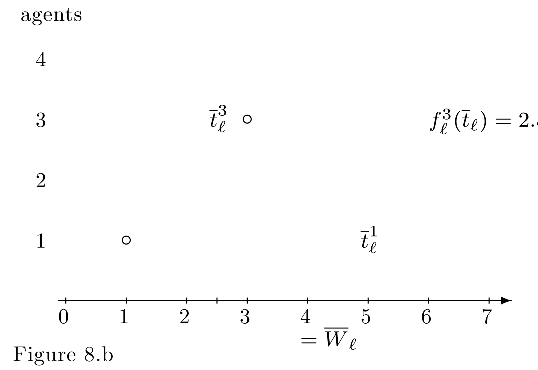

Figure 8.b

Since $\overline{t}_\ell^3 = 2.5 < 3 = \overline{q}_\ell^3$, $f_\ell^3(\overline{t}_\ell) = 2.5$. Since only agent 1 remains and one a half units have not been allotted yet, the new guaranteed endowment for agent 1 is equal to $\overline{\overline{q}}_\ell^1 = 1.5$, strictly smaller than $\overline{t}_\ell^1 = 5$. Hence, $f_\ell^1(\overline{t}_\ell) = 1.5$. Therefore, $f_\ell^{q_\ell}(5, 1, 2.5, 2) = (1.5, 1, 2.5, 2)$.

Consider the vector of tops $\underline{t}_\ell = (4, 0, 0.5, 1) \in X_\ell^N$. Since $\sum_{j \in N} \underline{t}_\ell^j < 7$ and $\underline{t}_\ell^1 = 4 > 0 = q_\ell^1$ agent 1 leaves with her top.

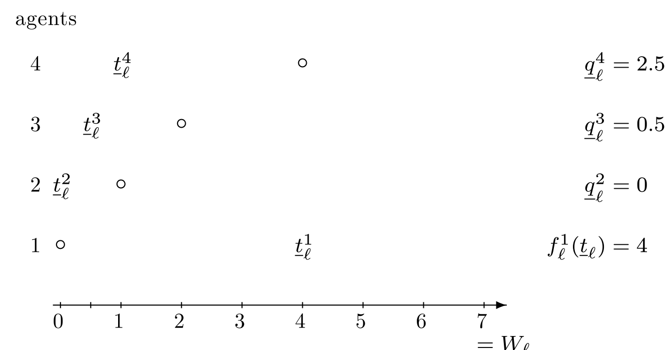

Figure 9.a

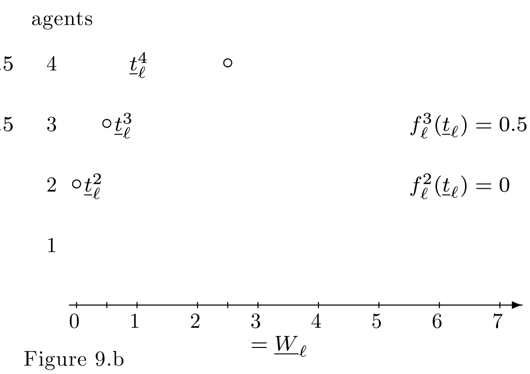

Figure 9.b

The amount not allotted yet is $\underline{W}_\ell = 3$, and the updated guaranteed endowments for agents 2, 3 and 4 that remain are $\underline{q}_\ell^2 = \max\{q_\ell^2 - x, 0\} = \max\{1 - x, 0\}$, $\underline{q}_\ell^3 = \max\{q_\ell^3 - x, 0\} = \max\{2 - x, 0\}$ and $\underline{q}_\ell^4 = \max\{q_\ell^4 - x, 0\} = \max\{4 - x, 0\}$, where x is such that $\underline{q}_\ell^2 + \underline{q}_\ell^3 + \underline{q}_\ell^4 = 3$. Hence, $x = 1.5$ and so $\underline{q}_\ell^2 = 0$, $\underline{q}_\ell^3 = 0.5$ and $\underline{q}_\ell^4 = 2.5$, represented in Figure 9.b by three circles. Since $\underline{t}_\ell^2 = \underline{q}_\ell^2 = 0$ and $\underline{t}_\ell^3 = \underline{q}_\ell^3 = 0.5$ agents 2 and 3 leave with their tops and agent 4 receives her guaranteed endowment 2.5. Therefore, $f_\ell^{q_\ell}(4, 0, 0.5, 1) = (4, 0, 0.5, 2.5)$. ■

Proposition 1. *For each $q \in Z$, the rule $f^q : \mathcal{SSP}(q) \to Z$ whose sequential adjustment function is uniform (up to feasibility) is strategy-proof, tops-only, continuous and satisfies same-sidedness and individual rationality with respect to q.*

Proof: Tops-onlyness, continuity and same-sidedness follow directly from the definition of f^q. It remains to verify strategy-proofness and individual rationality with respect to q.

Fix arbitrary agent $i \in N$ and profile $R \in \mathcal{SSP}(q)$. By Remark 4(i), for all $\ell \in M$, $f_\ell^{q,i}(t_\ell(R^1), \ldots, t_\ell(R^n))$ is either $t_\ell(R^i)$ or i's last updated guaranteed endowment of good ℓ used in the sequential process to define $f^q_\ell(t_\ell(R^1), \ldots, t_\ell(R^n))$. In both cases, $f^{q,i}_\ell(t_\ell(R^1), \ldots, t_\ell(R^n)) \in MB(t^i_\ell, q^i_\ell)$. Hence,

$$f^{q,i}(R) \in MB(t^i, q^i).$$

By (SSP.1), $f^{q,i}(R)\ R^i\ q^i$ which means that f^q satisfies individual rationality with respect to q^i.

To show that f^q is strategy-proof, let $\widehat{R}^i \in \mathcal{SSP}(q^i)$ be arbitrary. We have to verify that

$$x^i \equiv f^{q,i}(R)\ R^i\ f^{q,i}(\widehat{R}^i, R^{-i}) \equiv \widehat{x}^i. \tag{16}$$

By Remark 4(i), $x^i \in MB(t^i, q^i)$. Consider the case of $\widehat{x}^i \in MB(x^i, q^i)$. Then by $x^i \in MB(t^i, q^i)$, $\widehat{x}^i \in MB(t^i, q^i)$ and $x^i \in MB(t^i, \widehat{x}^i)$. Thus by (SSP.1), (16) holds.

Next consider the case of $\widehat{x}^i \notin MB(x^i, q^i)$. By Remark 4(ii) and (iii),

$$\widehat{x}^i_\ell \in [0, x_\ell] \text{ if } x^i_\ell < t^i_\ell; \quad \widehat{x}^i_\ell \in [0, W_\ell] \text{ if } x^i_\ell = t^i_\ell; \quad \widehat{x}^i_\ell \in [x_\ell, W_\ell] \text{ if } x^i_\ell > t^i_\ell.$$

Thus by $x^i \in MB(t^i, q^i)$, there are the following cases;

Case A:	$\widehat{x}^i_\ell \leq q^i_\ell \leq x^i_\ell < t^i_\ell$,	Case B: $q^i_\ell \leq \widehat{x}^i_\ell \leq x^i_\ell < t^i_\ell$,
Case C:	$\widehat{x}^i_\ell \leq q^i_\ell \leq x^i_\ell = t^i_\ell$,	Case D: $q^i_\ell \leq \widehat{x}^i_\ell \leq x^i_\ell = t^i_\ell$,
Case E:	$q^i_\ell \leq x^i_\ell = t^i_\ell \leq \widehat{x}^i_\ell$,	Case F: $\widehat{x}^i_\ell \leq x^i_\ell = t^i_\ell \leq q^i_\ell$,
Case G:	$x^i_\ell = t^i_\ell \leq q^i_\ell \leq \widehat{x}^i_\ell$,	Case H: $x^i_\ell = t^i_\ell \leq \widehat{x}^i_\ell \leq q^i_\ell$,
Case I:	$\widehat{x}^i_\ell \geq q^i_\ell \geq x^i_\ell > t^i_\ell$,	Case J: $q^i_\ell \geq \widehat{x}^i_\ell \geq x^i_\ell > t^i_\ell$.

Thus by $\widehat{x}^i \notin MB(x^i, q^i)$, $\widehat{x}^i \notin MB(q^i, t^i)$. Let $\widetilde{x}^i$ be such that $\widetilde{x}^i_\ell = q^i_\ell$ for Cases A, C, G and I; $\widetilde{x}^i_\ell = \widehat{x}^i_\ell$ for Cases B, D, H and J; and $\widetilde{x}^i_\ell = x^i_\ell$ for Cases E and F. Then, $x^i \in MB(\widetilde{x}^i, t^i)$, $\widetilde{x}^i \in MB(x^i, q^i) \subseteq MB(q^i, t^i)$, and $\widetilde{x}^i$ is the closet point to $\widehat{x}^i$ in $MB(q^i, t^i)$. By (SSP.1), $\widetilde{x}^i \in MB(t^i, q^i)$ and $x^i \in MB(\widetilde{x}^i, t^i)$, $x^i R^i \widetilde{x}^i$. Since $\widetilde{x}^i$ is the closet point to $\widehat{x}^i$ in $MB(q^i, t^i)$, by (SSP.2) and $\widehat{x}^i \notin MB(q^i, t^i)$, $\widetilde{x}^i R^i \widehat{x}^i$. Thus, by transitivity, $x^i R^i \widehat{x}^i$ and (16) holds. ■

Observe that, given $q \in Z$, there are many sequential allotment rules; in particular, those that use sequential adjustment functions that are not necessarily uniform (see Barberà, Jackson and Neme (1997)).

4 Discussion and related literature

4.1 Semilattice single-peakedness for public and private goods

The notion of semilattice single-peakedness used in this paper corresponds to the definition of semilattice single-peakedness relative to a semilattice over the set of alternatives used in

Chatterji and Massó (2018) in a public good context. We now show how to obtain in our private goods case a semilattice on X inducing the same set of semilattice single-peaked preferences.

Let $q^i \in X$ be given. Define the binary relation $\succeq^{q^i}$ over X as follows. For each pair $x^i, y^i \in X$, set

$$x^i \succeq^{q^i} y^i \Leftrightarrow x \in MB(y^i, q^i).$$

It is immediate to check that the binary relation $\succeq^{q^i}$ is reflexive (*i.e.*, for all $x \in X$, $x^i \succeq^{q^i} x^i$) and antisymmetric (for all $x^i, y^i \in X$, $x^i \succeq^{q^i} y^i$ and $y^i \succeq^{q^i} x^i$ imply $x^i = y^i$). To check that $\succeq^{q^i}$ is transitive, let $x^i, y^i, z^i \in X$ be such that $x^i \succeq^{q^i} y^i$ and $y^i \succeq^{q^i} z^i$. Equivalently, assume that $x^i \in MB(y^i, q^i)$ and $y^i \in MB(z^i, q^i)$ hold. Fix $\ell \in M$. The fact that $y^i \in MB(z^i, q^i)$ means that $\min\{z^i_\ell, q^i_\ell\} \leq y^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}$. We distinguish between two cases. First, $z^i_\ell \leq q^i_\ell$, and so $z^i_\ell \leq y^i_\ell \leq q^i_\ell$. The fact that $x^i \in MB(y^i, q^i)$ means that $\min\{y^i_\ell, q^i_\ell\} = y^i_\ell \leq x^i_\ell \leq q^i_\ell = \max\{y^i_\ell, q^i_\ell\}$, which implies that $\min\{z^i_\ell, q^i_\ell\} \leq x^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}$. Second, $q^i_\ell \leq z^i_\ell$, and so $q^i_\ell \leq y^i_\ell \leq z^i_\ell$. The fact that $x^i \in MB(y^i, q^i)$ means that $\min\{y^i_\ell, q^i_\ell\} = q^i_\ell \leq x^i_\ell \leq y^i_\ell = \max\{y^i_\ell, q^i_\ell\}$, which implies that $\min\{z^i_\ell, q^i_\ell\} \leq x^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}$. Hence, in both cases, $x^i_\ell \in [\min\{z^i_\ell, q^i_\ell\}, \max\{z^i_\ell, q^i_\ell\}]$, and so $x^i_\ell \in MB(z^i_\ell, q^i_\ell)$. Since this holds for all $\ell \in M$, we have that $x^i \in MB(z^i, q^i)$. Thus, $x^i \succeq^{q^i} z^i$, and $\succeq^{q^i}$ is transitive.

Therefore, $\succeq^{q^i}$ is a partial order over X. Statement 2 in the Appendix says that $\succeq^{q^i}$ is a semilattice; namely, for any pair $x^i, y^i \in X$, $\sup_{\succeq^{q^i}}\{x^i, y^i\}$ does exist. Moreover, our definition of semilattice single-peakedness with respect to q^i here corresponds to the notion of semilattice single-peakedness on $(X, \succeq^{q^i})$ given by Chatterji and Massó (2018) for the case of a public good (*i.e.*, if X were the set of social alternatives). Indeed given any triple t^i, x^i, y^i of alternatives, a preference ordering that has t^i as its top must rank the supremum of the pair (t^i, y^i) above the supremum of the pair (x^i, y^i) (see Figures 5.a and 5.b).

4.2 The case of one private good

We consider now the special case of our model where W units of one perfectly divisible private good has to be distributed among the set of agents N. This is a special case of our model with $m = 1$. We therefore set $W_1 = W$ and $X = [0, W]$.

In this case, the domain requirements of top-separability and separable richness are vacuously satisfied. We only require that the domain be rich relative to q. Sprumont (1991) studied this problem assuming that preferences are continuous and single-peaked. He characterized the uniform rule as the unique rule that is strategy-proof, efficient and anonymous on the single-peaked domain.[11] The statement in Corollary 1 below corre-

[11] A rule $f : \mathcal{D} \to Z$ is *efficient* if for all $R \in \mathcal{D}$, there is no $x \in Z$ such that $x^i R^i f^i(R)$ for all $i \in N$

sponds to our Theorem 1 for this one-dimensional case.

Corollary 1. *Assume $m = 1$ and let $\mathcal{D} = \mathcal{D}^1 \times \cdots \times \mathcal{D}^n$ be a rich domain relative to $q \in Z$ and let $f : \mathcal{D} \to Z$ be a strategy-proof, tops-only and continuous rule satisfying same-sidedness and individual rationality with respect to q. Then, for each $i \in N$, $\mathcal{D}^i$ is a set of semilattice single-peaked preferences with respect to q^i.*

A simple adaptation of the proof of Theorem 1 for the case $m = 1$ shows that Corollary 1 remains true after replacing same-sidedness by efficiency. In the one-dimensional case with single-peaked preferences, same-sidedness is indeed equivalent to efficiency. However, Morimoto, Serizawa and Ching (2012) showed that in the multiple-dimensional case with continuous, strictly convex, and separable preferences, efficiency implies same-sidedness, with the converse not being true.

Massó and Neme (2001) identified *the* maximal domain of preferences that admits a strategy-proof, efficient and strong symmetric rule.[12] Massó and Neme (2004) identified *a* maximal domain of preferences that admits a strategy-proof, efficient, tops-only and continuous rule. These two domains are similar to the one described here in Figure 1.a, with $q^i = \frac{W}{n}$. The main differences between the two domains and our are that (i) the unique top condition is not imposed from the outset and (ii) preferences with some (and very specific) indifference intervals at the same side of the peak have to be excluded. The reasons underlying the second difference is that Massó and Neme (2001 and 2004) required the rule to be efficient but not individually rational (because agents do not have entitlements). For the case of a variable amount of the good, Ching and Serizawa (1998) showed that the single-plateaued domain is *the* unique maximal domain containing the domain of single-peaked preferences while admitting a strategy-proof, efficient and symmetric rule.

4.3 The maximal domain property

A domain $\mathcal{D}$ is *maximal for a list of properties* if (i) there is a rule on $\mathcal{D}$ satisfying the properties and (ii) there is no domain $\mathcal{D}' \supsetneq \mathcal{D}$ such there is a rule on $\mathcal{D}'$ satisfying the properties. Note that by Theorem 1 and Proposition 1, we have the following result.

Corollary 2. *The set of all semilattice single-peaked preferences with respect to q is the unique maximal domain for strategy-proofness, tops-onlyness, continuity, same-sidedness and individual rationality with respect to q that is top-separable, separably rich and rich relative to $q \in Z$.*

and $x^j P^j f^j(R)$ for some $j \in N$.

[12] A rule $f : \mathcal{D} \to Z$ is *strong symmetric* if for all $R \in \mathcal{D}$ and all $i, j \in N$ such that $R_i = R_j$, $f^i(R) = f^j(R)$.

Proof: By Proposition 1, there is a rule on $\mathcal{SSP}(q)$ satisfying strategy-proofness, tops-onlyness, continuity, same-sidedness and individual rationality with respect to q. Thus, $\mathcal{SSP}(q)$ satisfies (i) of domain maximality.

Let $\mathcal{D}$ be a top-separable, separably rich and rich relative to $q \in Z$. Theorem 1 states $\mathcal{D} \subseteq \mathcal{SSP}(q)$. Thus, $\mathcal{SSP}(q)$ also satisfies (ii) of domain maximality. Since $\mathcal{D}$ is an arbitrary top-separable, separably rich and rich relative to $q \in Z$, $\mathcal{SSP}(q)$ is the unique maximal domain for these properties. ■

Note that a maximal domain for a list of properties may not be unique. Thus, the uniqueness claim of Corollary 2 demonstrates that semilattice single-peakedness is essential for our desirable properties (strategy-proofness, tops-onlyness, continuity, same-sidedness and individual rationality with respect to q).

4.4 The model with variable endowments

We consider now a variant of our model where the entitlements are a variable as in earlier work (Ching and Serizawa, 1998; Mizobuchi and Serizawa, 2006).

Let

$$Q = \{q = (q^1, \ldots, q^n) \mid \sum_{i \in N} q^i = W\}$$

be the set of entitlement profiles. A *rule* is a mapping $f : \mathcal{D} \times Q \to Z$ that assigns to every profile $(R, q) \in \mathcal{D} \times Q$ of preferences and entitlements a feasible allotment $f(R, q) \in Z$.

The following result is a direct corollary of Theorem 1.

Corollary 3. *Let $\mathcal{D}$ be a domain that is top-separable, separably rich and rich relative to all $q \in Q$. Let $f : \mathcal{D} \times Q \to Z$ be a strategy-proof, tops-only and continuous rule satisfying same-sidedness and individual rationality with respect to all $q \in Q$. Then, for each $i \in N$, $\mathcal{D}^i$ is a set of multidimensional single-peaked preferences.*

Proof: Let $i \in N$, $R^i \in \mathcal{D}^i$, $x^i \in X$ and $y^i \in X$ be such that $y^i \in MB(x^i, t^i)$. We need to show $y^i \, R^i \, x^i$. Let $q^i = x^i$. Then, $x^i \in MB(q^i, t^i)$. By Theorem 1, R^i is semilattice single-peaked with respect to q^i. Thus, (SSP.1) of semilattice single-peakedness implies $y^i \, R^i \, x^i$. ■

4.5 Other related literature: the case of many private goods

There is a rich literature studying rules in economies with more than one private good. In contrast to our approach, they assume a given domain of preferences and identify rules satisfying desirable properties on the domain.

Morimoto, Serizawa and Ching (2012) studied the multi-dimensional extension of

Sprumont (1991).[13] They show that on the class of continuous, strictly convex, and separable preferences a rule satisfies strategy-proofness, unanimity, weak symmetry and nonbossiness if and only if it is the uniform rule.[14] This result extends to the class of continuous, strictly convex, and multidimensional single-peaked preferences. Adachi (2010) provides a similar characterization of the uniform rule using strategy-proofness, same-sidedness and envy-freeness.

Mas-Colell (1992) considers an economy with private goods and production. Agents have continuous and convex preferences (*i.e.*, they might or might not be satiated). He defines the notion of Walrasian equilibrium with slacks (an extension of the notion of competitive equilibrium in this more general setting with potentially satiated agents). The main contribution of Mas-Colell (1992) is to identify sufficient conditions on the economy under which a Walrasian equilibrium with slacks does exist and it is efficient; the issue of truthful revelation is not addressed.[15]

Barberà and Jackson (1995) considers an exchange economy with private goods. Agents have initial entitlements and continuous, strictly quasi-concave and increasing preferences, which are represented by utility functions. For the case of two agents they characterize fixed-proportion trading as the class of all strategy-proof and individually rational rules. For the case of any number of agents, they characterize fixed-proportion anonymous trading as the class of all strategy-proof, non-bossy, anonymous and tie-free rules. In contrast with the rule that we exhibit in Proposition 1, those rules are not tops-only (although they are tops-only on the range). Observe that their assumption that agents have increasing preferences and our assumption that agents are satiated (*i.e.*, for each $x^i \in X$ there exists at least one $R^i \in \mathcal{D}^i$ such that $t(R^i) = x^i$) imply that the two domains are different and they reflect two very distant economic settings.

Moulin (2017) studies a family of collective decision problems where each agent i's preferences are single-peaked over a set in which agent i is interested, and this set is one-dimensional. The model includes among others the voting model (Moulin (1980)), when the set is a common subset of real numbers, and Sprumont's (1991) division problem, when the set of alternatives has n private components, and each agent i cares only about his/her one-dimansional private component. Moulin (2017) shows the existence of strategy-proof

[13] Amorós (2002) previously studied this extension. However, he only considered the case of two agents, which can easily be seen as being equivalent to a public good case: by feasibility, once the allotment of an agent is determined (the "public good"), the allotment of the other agent is determined as well.

[14] A rule $f : \mathcal{D} \to Z$ is *weak symmetric* if for all $R \in \mathcal{D}$ and all $i, j \in N$ such that $R_i = R_j$, $f^i(R) I^i f^j(R)$. A rule $f : \mathcal{D} \to Z$ is *non-bossy* if the change of one agent's preference does not alter allocations unless it alters her own assignment.

[15] Hurwicz (1972) and Zhou (1991) address this issue for two-agent pure exchange economies. Later Serizawa (2002) and Serizawa and Weymark (2003) extend the analysis to many-agent pure exchange economies.

rules satisfying additional desirable properties like efficiency and fairness. The existence result in Proposition 1 here can be recast as a multi-dimensional version of Moulin (2017) where we are able to weaken the requirement that the domain of preferences be single-peaked over a one-dimensional set to the requirement that the domain of preferences be semilattice single-peaked over a multi-dimensional set.

4.6 Our axioms

The key ingredients in our analysis (and indeed in the entire literature on the division problem) are the features that (i) the rules we consider satisfy the tops-only property and continuity and (ii) the preference domain is rich relative to the entitlements. We conjecture that the tops-only property and continuity of rules can be deduced from the hypotheses of strategy-proofness and (a possibly strengthened version of) our richness condition. We leave the search for a proof, or alternatively, for examples of rich domains that violate semilattice single-peakedness with respect to entitlements but admit strategy-proof rules that are individual rationality with respect to q_i for each i and that violate tops-onlyness/continuity, for future work. We instead provide two examples of non-rich domains which violate semilattice single-peakedness with respect to entitlements but admit rules that violate respectively continuity and tops-onlyness while satisfying the other axioms (unanimity instead of same-sidedness in the first case), and conclude with two examples that show the indispensability of strategy-proofness and individual rationality respectively for Theorem 1.

Example 2a. Let $M = 1$ and the total amount of the single good available W_1 be one. Let $N = 2$. In this simple case, the division problem can be reformulated as a pure public good problem as follows. Let $A = [0, 1]$ and $a \in A$ denotes the division whereby agent 1 receives a and agent 2 receives $1 - a$ and where agents' preferences are directly formulated on A. That is, saying that agent 2 strictly prefers a to b means now that agent 2 strictly prefers $1 - a$ to $1 - b$. Assume that $q := q^1 = q^2 = 1/2$.

The domain of preferences $\hat{\mathcal{D}}$ is identical across the two agents and is specified as follows. First, there is exactly one preference $R^i_{1/2}$ that has $1/2$ as its top-ranked alternative and this preference is single peaked. Second, for every $x^i \in A\backslash\{1/2\}$ there is exactly one preference $R^i_{x^i}$ that has x^i as its top-ranked alternative. In this preference ordering, $1/2$ is ranked second and the preferences over the remaining alternatives are single-peaked over $A\backslash\{x^i\}$ with peak at $1/2$. Figures 10.a depicts the utility representation of a preference $R^i_{1/2}$, the unique one in $\hat{\mathcal{D}}$ with peak at $1/2$, and Figure 10.b depicts the utility representation of a preference $R^i_{3/4}$, the unique one in $\hat{\mathcal{D}}$ with peak at $3/4$.

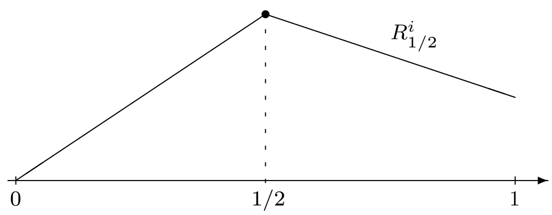

Figure 10.a

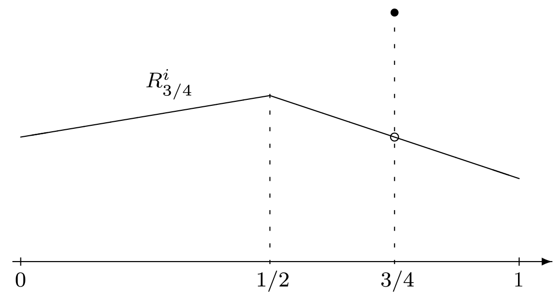

Figure 10.b

It is straightforward to verify that this preference domain *violates* both richness relative to q^i and semilattice single-peakedness with respect to entitlements.

Consider the tops-only rule which picks the agents common peak whenever they announce the same peak and q otherwise. This rule is unanimous and respects individual rationality with respect to q. It is easily verified that this rule is strategy-proof and violates continuity.[16] ■

The following is an example (adapted from Chatterji and Massó (2018)) that shows that one may construct a non-tops-only rule (and therefore a non continuous rule, as continuity of rules is defined only for tops-only rules in our set up) that satisfies our remaining axioms on a domain and that violates semilattice single-peakedness with respect to entitlements and richness relative to q.

Example 2b. The setting is identical to the one in the previous example. The preference domain is different from the previous example (but identical across agents). To describe the preferences of agents on A, we partition it into the following four intervals: $X \equiv [0, 0.25), Y \equiv [0.25, 0.5), q \equiv [0.5, 0.5], Z \equiv (0.5, 1]$. We will postulate that there are 5 categories of preferences R^i in the domain. Each preference in the category R^i_a will be assumed to have a common structure as shown in the table below, where for instance, R^i_x ranks the block X above the block Y, Y above q and finally block Z is ranked last. Analogous restrictions apply to R^i_y, etc. Furthermore, the ranking of alternatives *within* each of the blocks X, Y, Z will be assumed to be single peaked. Given a preference R^i drawn from this domain, we will we let $\tau_k(R^i)$ denote the peak of the preference R^i restricted to the block $k \in \{X, Y, Z\}$.

R^i_x	R^i_y	R^i_z	R'^i_z	R^i_q
X	Y	Z	Z	q
Y	X	Y	q	Z
q	q	q	X	X
Z	Z	X	Y	Y

.

[16] This rule however does not satisfy same-sidedness. We also leave for future work to determine whether an example can be constructed where the exhibited rule satisfies in addition same-sidedness.

To see that the domain does not satisfy richness, consider the allotments $z^i = 0.1$, $x^i = 0.2$ and $y^i = 0.3$ and note that the hypothesis of the property are satisfied but the domain does not contain a preference R^i_x for which $y^i P^i_x z^i$. To define a two-agent rule on this domain, we proceed as follows. Given a profile $R = (R^1, R^2)$ of preferences, let $\sigma_k(R) = \max(\tau_k(R^1), \tau_k(R^2))$ for $k \in \{X, Y, Z\}$.[17] Finally, consider the non tops-only, and unanimous rule $f : \mathcal{D}^1 \times \mathcal{D}^2 \to A$ defined by the following table where we suppress, for $k \in \{X, Y, Z\}$, the dependence of $\sigma_k(R)$ on R for notational convenience:

f	R^2_x	R^2_y	R^2_z	R'^2_z	R^2_q
R^1_x	σ_X	σ_Y	σ_Y	q	q
R^1_y	σ_Y	σ_Y	σ_Y	q	q
R^1_z	σ_Y	σ_Y	σ_Z	σ_Z	q
R'^1_z	q	q	σ_Z	σ_Z	q
R^1_q	q	q	q	q	q

.

By construction, f satisfies individual rationality with respect to q. To verify that f is strategy-proof, we proceed in two steps. The configuration of preferences in the table is used to argue that no agent can manipulate to a preferred block. Within a block, strategy-proofness is guaranteed by the assumption that preferences within the block are single-peaked and a particular phantom voter rule is used. Even though preferences within a block are single-peaked, the overall preferences are not semilattice single-peaked with respect to entitlements. ■

It is straightforward to exhibit the indispensability of two of our axioms, respectively strategy-proofness and individual rationality with respect to q_i for each i, for Theorem 1.

Example 2c. Here too the setting is identical to the one in Example2a. The preference domain is different (but identical across agents). (i) Assume the universal domain of preferences. This domain is rich relative to q^i for each i, violates semilattice single-peakedness with respect to entitlements, but admits a dictatorial rule which satisfies all our axioms other than individual rationality with respect to q^i for each i.

(ii) Consider now the case where the domain $\mathcal{D} = \hat{\mathcal{D}} \cup \mathcal{D}^{SP}$ where $\hat{\mathcal{D}}$ is the set of all complete, transitive preference that are represented by continuous utility functions that attain a unique maximum and attain a minimum at q whenever the top is different from q, and $\mathcal{D}^{SP}$ is the set of all single-peaked preferences on A with respect to the natural order. Since $\mathcal{D}^{SP}$ is rich relative to q^i for each i, so is $\mathcal{D}$. The rule that selects the median of the two tops and $q^1 = 0.5$ on this domain (which corresponds to the uniform rule in

[17] Recall that a preference R^i induces a single-peaked preference on each $k \in \{X, Y, Z\}$: The alternative $\sigma_k(R)$ is the one chosen by a phantom voter rule applied to the interval $k \in \{X, Y, Z\}$ where the phantom is located at the upper end of the interval.

the original private good setting) satisfies all our axioms except strategy-proofness and the domain is evidently not semilattice single-peaked with respect to entitlements. ■

5 Appendix

5.1 Separable richness

We show that the following statement about separable richness holds.

Statement 1. *Assume $m \geq 2$ and let $R^i \in \mathcal{R}$ be a top-separable but non-separable preference with the properties that $t(R^i) = x^i$ and, for a pair $y^i, z^i \in X$, $y^i P^i z^i$. Then, there exists a separable preference $\hat{R}^i \in \mathcal{R}$ with the properties that $t(\hat{R}^i) = x^i$ and $y^i \hat{P}^i z^i$.*

Separable richness of a domain $\mathcal{D}^i$ requires that, for any top-separable preference $R^i \in \mathcal{D}^i$ and any pair $y^i, z^i \in X$ with $y^i P^i_{x^i} z^i$, there exists a separable preference $\hat{R}^i \in \mathcal{D}^i$ with $y^i \hat{P}^i_{x^i} z^i$ (the statement above) *and* that $\hat{R}^i \in \mathcal{D}^i$ (the separable richness requirement imposed on the domain $\mathcal{D}^i$).

The set of additive preferences will be useful in the proof of Statement 1. A preference R^i is additively representable if for each $\ell \in M$ there exists $u^i_\ell : X_\ell \to \mathbb{R}$ such that, for each pair $x^i, y^i \in X$,

$$x^i R^i y^i \Leftrightarrow \textstyle\sum_{\ell \in M} u^i_\ell(x^i_\ell) \geq \sum_{\ell \in M} u^i_\ell(y^i_\ell).$$

For every $z^i \in X$, define $u^i(z^i) = \sum_{\ell \in M} u^i_\ell(z^i_\ell)$. It is immediate to see that any additively representable preference is separable and top-separable.

Proof of Statement 1: Let $R^i \in \mathcal{R}$ be a top-separable but non-separable preference and let $y^i, z^i \in X$ be any pair such that $y^i P^i_{x^i} z^i$. If $x^i = y^i$ the statement follows trivially since $x^i \hat{P}^i z^i$ holds for any separable preference $\hat{R}^i$ with $t(\hat{R}^i) = x_i$.

Assume $x^i \neq y^i$, and define the non-empty set

$$M_{x \neq y} = \{\ell \in M \mid x^i_\ell \neq y^i_\ell\}.$$

We proceed by distinguishing between two cases.

Case 1: There exists $\widehat{\ell} \in M_{x \neq y}$ such that $x^i_{\widehat{\ell}} \neq z^i_{\widehat{\ell}}$ and $y^i_{\widehat{\ell}} \neq z^i_{\widehat{\ell}}$.
Define the separable preference $\hat{R}^i$ by choosing a family of functions $u^i_\ell : X_\ell \to \mathbb{R}$, one for each $\ell \in M$, and an $\varepsilon > 0$ such that (i) for all $w^i_{\widehat{\ell}} \notin \{x^i_{\widehat{\ell}}, y^i_{\widehat{\ell}}, z^i_{\widehat{\ell}}\}$,

$$u^i_{\widehat{\ell}}(x^i_{\widehat{\ell}}) > u^i_{\widehat{\ell}}(y^i_{\widehat{\ell}}) > u^i_{\widehat{\ell}}(z^i_{\widehat{\ell}}) > u^i_{\widehat{\ell}}(w^i_{\widehat{\ell}}) = \varepsilon \tag{17}$$

and (ii) for all $\ell \in M \backslash \{\widehat{\ell}\}$ and $w^i \in X$, $u^i_\ell(w^i_\ell) = \varepsilon$. Then,

$$\begin{aligned} u^i(x^i) &= u^i_{\widehat{\ell}}(x^i_{\widehat{\ell}}) + \varepsilon(m-1) \\ u^i(y^i) &= u^i_{\widehat{\ell}}(y^i_{\widehat{\ell}}) + \varepsilon(m-1) \\ u^i(z^i) &= u^i_{\widehat{\ell}}(z^i_{\widehat{\ell}}) + \varepsilon(m-1). \end{aligned}$$

Therefore, by (17), $u^i(x^i) > u^i(y^i) > u^i(z^i)$, $t(\hat{R}^i) = x^i$ and $y^i \hat{P}^i_{x^i} z^i$.

Case 2: For all $\ell \in M_{x \neq y}$ either $x^i_\ell = z^i_\ell$ (and $y^i_\ell \neq z^i_\ell$) or $y^i_\ell = z^i_\ell$ (and $x^i_\ell \neq z^i_\ell$).

Define the sets

$$M_{x=z} = \{\ell \in M_{x\neq y} \mid x^i_\ell = z^i_\ell\}$$
$$M_{y=z} = \{\ell \in M_{x\neq y} \mid y^i_\ell = z^i_\ell\},$$

and note that $M_{x=z} \cup M_{y=z} = M_{x\neq y}$ and $M_{x=z} \cap M_{y=z} = \emptyset$. Since $M_{x\neq y} \neq \emptyset$, we have that $M_{x=z} \neq \emptyset$ or $M_{y=z} \neq \emptyset$. We proceed by distinguishing between Subcase 2.1 when $M_{x=z} \neq \emptyset$ and Subcase 2.2 when $M_{x=z} = \emptyset$ and $M_{y=z} \neq \emptyset$.

Subcase 2.1: $M_{x=z} \neq \emptyset$.

This means that there exists $\widehat{\ell} \in M_{x=z}$ such that $x^i_{\widehat{\ell}} = z^i_{\widehat{\ell}} \neq y^i_{\widehat{\ell}}$. But, since z^i can be obtained from y^i by substituting one by one y^i_ℓ by x^i_ℓ in each $\ell \in M_{x=z}$, applying top-separability at each step, and by transitivity of R^i, we obtain that $z^i P^i y^i$, a contradiction with the initial hypothesis. Observe that $\ell^* \notin M_{y=z}$ and accordingly, $\ell^* \notin M_{x\neq y}$ (*i.e.*, $x^i_{\ell^*} = y^i_{\ell^*}$). Define the separable preference $\hat{R}^i$ by choosing a family of functions $u^i_\ell : X_\ell \to \mathbb{R}$, one for each $\ell \in M$, and an $\varepsilon > 0$ such that (i) $u^i_{\widehat{\ell}}(x^i_{\widehat{\ell}}) = u^i_{\widehat{\ell}}(z^i_{\widehat{\ell}}) = u^i_{\widehat{\ell}}(y^i_{\widehat{\ell}}) + \varepsilon > 2\varepsilon$, (ii) for all $w^i_{\widehat{\ell}} \notin \{x^i_{\widehat{\ell}}, y^i_{\widehat{\ell}}, z^i_{\widehat{\ell}}\}$, $u^i_{\widehat{\ell}}(w^i_{\widehat{\ell}}) = \varepsilon$, (iii) $u^i_{\ell^*}(x^i_{\ell^*}) = u^i_{\ell^*}(y^i_{\ell^*}) = u^i_{\ell^*}(z^i_{\ell^*}) + 2\varepsilon > 4\varepsilon$, (iv) for all $w^i_{\ell^*} \notin \{x^i_{\ell^*}, y^i_{\ell^*}, z^i_{\ell^*}\}$, $u^i_{\ell^*}(w^i_{\ell^*}) = \varepsilon$, (v) for all $\ell \in M\backslash\{\widehat{\ell}, \ell^*\}$, $u^i_\ell(x^i_\ell) = u^i_\ell(y^i_\ell) = 4\varepsilon$, $u^i_\ell(z^i_\ell) = \varepsilon$, and (vi) for all $w^i \notin \{x^i, y^i, z^i\}$ and $\ell \in M\backslash\{\widehat{\ell}, \ell^*\}$, $u^i_\ell(w^i_\ell) = \varepsilon$. Then,

$$u^i(x^i) = u^i_{\widehat{\ell}}(y^i_{\widehat{\ell}}) + \varepsilon + u^i_{\ell^*}(z^i_{\ell^*}) + 2\varepsilon + 4\varepsilon(m-2)$$
$$u^i(y^i) = u^i_{\widehat{\ell}}(y^i_{\widehat{\ell}}) + u^i_{\ell^*}(z^i_{\ell^*}) + 2\varepsilon + 4\varepsilon(m-2)$$
$$u^i(z^i) = u^i_{\widehat{\ell}}(y^i_{\widehat{\ell}}) + \varepsilon + u^i_{\ell^*}(z^i_{\ell^*}) + \varepsilon(m-2)$$

and, for all $w^i \notin \{x^i, y^i, z^i\}$, $u^i(w^i) = \varepsilon m$. Therefore, since $\varepsilon > 0$ and $m > \frac{5}{3}$, $u^i(x^i) > u^i(y^i) > u^i(z^i) > u^i(w^i) = \varepsilon m$. Hence, $t(\hat{R}^i) = x^i$ and $y^i \hat{P}^i_{x^i} z^i$.

Subcase 2.2: $M_{x=z} = \emptyset$ and $M_{y=z} \neq \emptyset$.

This means that there exists $\widehat{\ell} \in M_{y=z}$ such that $y^i_{\widehat{\ell}} = z^i_{\widehat{\ell}} \neq x^i_{\widehat{\ell}}$. Moreover, there exists $\ell^* \notin M_{x\neq y}$ such that $x^i_{\ell^*} = y^i_{\ell^*} \neq z^i_{\ell^*}$, otherwise $y^i = z^i$ would hold, which contradicts the hypothesis that $y^i P^i z^i$. Define the separable preference $\hat{R}^i$ by choosing a family of functions $u^i_\ell : X_\ell \to \mathbb{R}$, one for each $\ell \in M$, and an $\varepsilon > 0$ such that (i) $u^i_{\widehat{\ell}}(x^i_{\widehat{\ell}}) = u^i_{\widehat{\ell}}(z^i_{\widehat{\ell}}) + 2\varepsilon = u^i_{\widehat{\ell}}(y^i_{\widehat{\ell}}) + 2\varepsilon > 3\varepsilon$, (ii) for all $w^i_{\widehat{\ell}} \notin \{x^i_{\widehat{\ell}}, y^i_{\widehat{\ell}}, z^i_{\widehat{\ell}}\}$, $u^i_{\widehat{\ell}}(w^i_{\widehat{\ell}}) = \varepsilon$, (iii) $u^i_{\ell^*}(x^i_{\ell^*}) = u^i_{\ell^*}(y^i_{\ell^*}) = u^i_{\ell^*}(z^i_{\ell^*}) + \varepsilon > 2\varepsilon$, (iv) for all $w^i_{\ell^*} \notin \{x^i_{\ell^*}, y^i_{\ell^*}, z^i_{\ell^*}\}$, $u^i_{\ell^*}(w^i_{\ell^*}) = \varepsilon$, and (v) for all $w^i \in X$ and $\ell \in M\backslash\{\widehat{\ell}, \ell^*\}$, $u^i_\ell(w^i_\ell) = \varepsilon$. Then,

$$u^i(x^i) = u^i_{\widehat{\ell}}(z^i_{\widehat{\ell}}) + 2\varepsilon + u^i_{\ell^*}(z^i_{\ell^*}) + \varepsilon + \varepsilon(m-2)$$
$$u^i(y^i) = u^i_{\widehat{\ell}}(z^i_{\widehat{\ell}}) + u^i_{\ell^*}(z^i_{\ell^*}) + \varepsilon + \varepsilon(m-2)$$
$$u^i(z^i) = u^i_{\widehat{\ell}}(z^i_{\widehat{\ell}}) + u^i_{\ell^*}(z^i_{\ell^*}) + \varepsilon(m-2)$$

and, for all $w^i \notin \{x^i, y^i, z^i\}$, $u^i(w^i) = \varepsilon m$. Therefore, since $\varepsilon > 0$ and $u^i_{\widehat{\ell}}(z^i_{\widehat{\ell}}), u^i_{\ell^*}(z^i_{\ell^*}) > \varepsilon$, $u^i(x^i) > u^i(y^i) > u^i(z^i) > u^i(w^i) = \varepsilon m$. Hence, $t(\hat{R}^i) = x^i$ and $y^i \hat{P}^i_{x^i} z^i$. ■

5.2 Semilattice

We show that the partial order $\succeq^{q^i}$ over X defined in Subsection 4.1 is a semilattice.

Statement 2. *The partial order $\succeq^{q^i}$ over X is a semilattice.*

Proof of Statement 2: Let $x^i, y^i \in X$ be arbitrary. We prove that there exists $\sup_{\succeq^{q^i}}\{x^i, y^i\}$. Note that the set of upper bounds of x^i and y^i can be written as

$$I_{x^iy^i} = MB(x^i, q^i) \cap MB(y^i, q^i).$$

Note that this set is non-empty since $q^i \in I_{x^iy^i}$. Define z^i as follows; for each $\ell \in M$,

(a) if $\min\{x^i_\ell, y^i_\ell\} \leq q^i_\ell \leq \max\{x^i_\ell, y^i_\ell\}$, set $z^i_\ell = q^i_\ell$,

(b) if $q^i_\ell \leq \min\{x^i_\ell, y^i_\ell\}$, set $z^i_\ell = \min\{x^i_\ell, y^i_\ell\}$,

(c) if $\max\{x^i_\ell, y^i_\ell\} \leq q^i_\ell$, set $z^i_\ell = \max\{x^i_\ell, y^i_\ell\}$.

Figure 11 illustrates the definition of z^i as the projection of q^i to $MB(x^i, y^i)$; that is, z^i is the closest allotment to q^i in $MB(x^i, y^i)$ according to the L_1-norm.

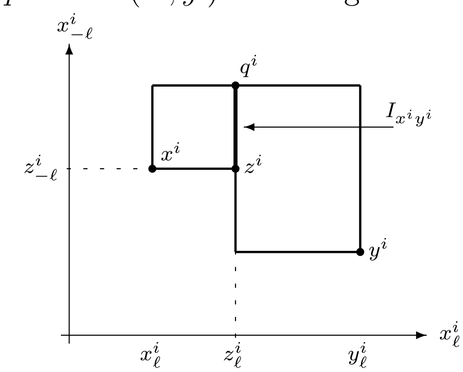

Figure 11

Claim 1 $\sup_{\succeq^{q^i}}\{x^i, y^i\} = z^i$.

Proof of Claim 1 We verify Claim 1 in two steps.

Step 1 The allotment z^i is an upper bound of x^i and y^i. Namely, $z^i \succeq^{q^i} x^i$ and $z^i \succeq^{q^i} y^i$ or, equivalently, $z^i \in MB(x^i, q^i) \cap MB(y^i, q^i)$. We show that $z^i \in MB(y^i, q^i)$. To show that $z^i \in MB(x^i, q^i)$ is similar, and therefore omitted.

To obtain a contradiction, assume $z^i \notin MB(y^i, q^i)$. By (1), there exists $\ell \in M$ such that either $z^i_\ell < \min\{y^i_\ell, q^i_\ell\}$ or $\max\{y^i_\ell, q^i_\ell\} < z^i_\ell$.

Assume $z^i_\ell < \min\{y^i_\ell, q^i_\ell\}$ holds. We proceed by distinguishing between two cases.

Case 1: $z^i_\ell < q^i_\ell \leq y^i_\ell$. Since $z^i_\ell \neq q^i_\ell$, Case (a) does not hold. Case (b) does not hold either, because otherwise $q^i_\ell = z^i_\ell$. In order that Case (c) holds, $\max\{y^i_\ell, q^i_\ell\} = z^i_\ell$, but then, $y^i_\ell \leq z^i_\ell$ which is a contradiction.

Case 2: $z^i_\ell < y^i_\ell \leq q^i_\ell$. Since $z^i_\ell \neq q^i_\ell$ Case (a) does not hold. Case (b) does not hold either, because otherwise $q^i_\ell = z^i_\ell$. In order that Case (c) holds, $\max\{y^i_\ell, q^i_\ell\} = z^i_\ell$, but then, $y^i_\ell \leq z^i_\ell$ which is a contradiction.

Assume $\max\{y^i_\ell, q^i_\ell\} < z^i_\ell$ holds. We proceed by distinguishing between two cases.

Case 1: $q^i_\ell \leq y^i_\ell < z^i_\ell$. Since $z^i_\ell \neq q^i_\ell$, Case (a) does not hold. Case (b) does not hold either, because otherwise $q^i_\ell = z^i_\ell$. In order that Case (c) holds, $\max\{y^i_\ell, q^i_\ell\} = z^i_\ell$, but then, $y^i_\ell = z^i_\ell$ which is a contradiction.

Case 2: $z^i_\ell < y^i_\ell \leq q^i_\ell$. Since $z^i_\ell \neq q^i_\ell$, Case (a) does not hold. Case (b) does not hold either, because otherwise $z^i_\ell = y^i_\ell$. In order that Case (c) holds, $\max\{y^i_\ell, q^i_\ell\} = z^i_\ell$, but then, $y^i_\ell \leq z^i_\ell$ which is a contradiction.

Hence, $z^i \in MB(y^i, q^i)$ as well as $z^i \in MB(x^i, q^i)$. Accordingly, z^i is an upper bound of x^i and y^i.

Step 2 The allotment z^i is the smallest upper bound of x^i and y^i. Namely, if $\widehat{z}^i \in I_{x^i y^i}$, then $\widehat{z}^i \succeq^{q^i} z^i$ (or equivalently, $\widehat{z}^i \in MB(z^i, q^i)$), which is the same that ,for all $\ell \in M$, $\min\{z^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}$. Fix $\ell \in M$. We distinguish among the three cases (a), (b) or (c), depending under which one of them z^i_ℓ is defined.

Assume it is case (a). Then, $\min\{z^i_\ell, q^i_\ell\} \leq q^i_\ell = z^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}$. First, suppose $x^i_\ell \leq y^i_\ell$. Then, $x^i_\ell \leq q^i_\ell = z^i_\ell \leq y^i_\ell$. Because, $\widehat{z}^i \in MB(x^i, q^i)$, $\min\{x^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{x^i_\ell, q^i_\ell\}$, which means that $\max\{y^i_\ell, q^i_\ell\} = q^i_\ell = z^i_\ell \geq \widehat{z}^i_\ell$ and so

$$\widehat{z}^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}. \tag{18}$$

Because $\widehat{z}^i \in MB(y^i, q^i)$, $\min\{y^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{y^i_\ell, q^i_\ell\}$, which means that $\min\{y^i_\ell, q^i_\ell\} = q^i_\ell = z^i_\ell \leq \widehat{z}^i_\ell$, and so

$$\min\{q^i_\ell, z^i_\ell\} \leq \widehat{z}^i_\ell. \tag{19}$$

By (18) and (19), we have $\min\{z^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}$. Suppose now that $y^i_\ell \leq x^i_\ell$. Then, $y^i_\ell \leq q^i_\ell = z^i_\ell \leq x^i_\ell$. Because, $\widehat{z}^i \in MB(x^i, q^i)$, $\min\{x^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{x^i_\ell, q^i_\ell\}$, which means that $\min\{x^i_\ell, q^i_\ell\} = q^i_\ell = z^i_\ell \leq \widehat{z}^i_\ell$ and so

$$\min\{q^i_\ell, z^i_\ell\} \leq \widehat{z}^i_\ell. \tag{20}$$

Because $\widehat{z}^i \in MB(y^i, q^i)$, $\min\{y^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{y^i_\ell, q^i_\ell\}$, which means that $\widehat{z}^i_\ell \leq \max\{y^i_\ell, q^i_\ell\} = q^i_\ell = z^i_\ell$, and so

$$\widehat{z}^i_\ell \leq \max\{q^i_\ell, z^i_\ell\}. \tag{21}$$

By (20) and (21), we have $\min\{z^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}$.

Assume it is case (b). Then, $q^i_\ell \leq z^i_\ell = \min\{x^i_\ell, y^i_\ell\}$. First, suppose $x^i_\ell \leq y^i_\ell$. Then, $q^i_\ell \leq z^i_\ell = x^i_\ell \leq y^i_\ell$. Because, $\widehat{z}^i \in MB(x^i, q^i)$, $\min\{x^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{x^i_\ell, q^i_\ell\}$, which means that $q^i_\ell \leq \widehat{z}^i_\ell \leq z^i_\ell = x^i_\ell$ and so

$$\min\{q^i_\ell, z^i_\ell\} \leq \widehat{z}^i_\ell. \tag{22}$$

Because $\widehat{z}^i \in MB(x^i, q^i)$, $\min\{x^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{x^i_\ell, q^i_\ell\}$, which means that $\min\{x^i_\ell, q^i_\ell\} = q^i_\ell \leq \widehat{z}^i_\ell \leq z^i_\ell = x^i_\ell$ and so

$$\widehat{z}^i_\ell \leq \max\{q^i_\ell, z^i_\ell\}. \tag{23}$$

By (22) and (23), we have $\min\{z^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}$. Suppose now that $y^i_\ell \leq x^i_\ell$. Then, $q^i_\ell \leq z^i_\ell = y^i_\ell \leq x^i_\ell$. Because, $\widehat{z}^i \in MB(y^i, q^i)$, $\min\{y^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{y^i_\ell, q^i_\ell\}$, which means that $\min\{y^i_\ell, q^i_\ell\} = q^i_\ell \leq \widehat{z}^i_\ell$ and so

$$\min\{z^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell. \tag{24}$$

Moreover, $q^i_\ell \leq \widehat{z}^i_\ell \leq \max\{y^i_\ell, q^i_\ell\} = y^i_\ell = z^i_\ell$, and so

$$\widehat{z}^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}. \tag{25}$$

By (24) and (25), we have $\min\{z^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}$.

Assume it is case (c). Then, $\max\{x^i_\ell, y^i_\ell\} = z^i_\ell \leq q^i_\ell$. First, suppose $x^i_\ell \leq y^i_\ell$. Then, $x^i_\ell \leq y^i_\ell = z^i_\ell \leq q^i_\ell$. Because $\widehat{z}^i \in MB(y^i, q^i)$, $\min\{y^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{y^i_\ell, q^i_\ell\}$, which means that $\min\{y^i_\ell, q^i_\ell\} = y^i_\ell = z^i_\ell \leq \widehat{z}^i_\ell$ and so

$$\min\{q^i_\ell, z^i_\ell\} \leq \widehat{z}^i_\ell. \tag{26}$$

Moreover, $\widehat{z}^i_\ell \leq \max\{y^i_\ell, q^i_\ell\} = q^i_\ell$, and so

$$\widehat{z}^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}. \tag{27}$$

By (26) and (27), we have $\min\{z^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}$. Suppose now that $y^i_\ell \leq x^i_\ell$. Then, $y^i_\ell \leq x^i_\ell = z^i_\ell \leq q^i_\ell$. Because $\widehat{z}^i \in MB(x^i, q^i)$, $\min\{x^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{x^i_\ell, q^i_\ell\}$, which means that $\min\{x^i_\ell, q^i_\ell\} = x^i_\ell = z^i_\ell \leq \widehat{z}^i_\ell$ and so

$$\min\{z^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell. \tag{28}$$

Moreover, $\widehat{z}^i_\ell \leq \max\{x^i_\ell, q^i_\ell\} = q^i_\ell$, and so

$$\widehat{z}^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}. \tag{29}$$

This shows that $\widehat{z}^i \succeq^{q^i} z^i$, and that z^i is the smallest upper bound of x^i and y^i. Hence $\sup_{\succeq^{q^i}}\{x^i, y^i\} = z^i$.[18] ■

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