Universitat de Barcelona
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BASICS ON THE THEORY OF INTEREST

Applications to the Spanish financial market

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PROLOGUE

This manuscript is intended for readers who are interested in knowing the basic concepts and mathematical tools that are currently used in financial markets. It is aimed at both university students who will require this knowledge when they start working in these markets and professionals who already work in them.

The book comprises four units. Unit 1, Financial transactions and financial regimes, deals with the time value of money as a main idea, as well as other related concepts such as financial capital, financial equivalence, simple and compound interest, annual effective interest rate, etc. Unit 2 is devoted to Annuities. In particular, level or constant annuities and varying annuities, both in geometric and in arithmetic progression are explained. A sound grounding of the first two units is essential to understand Unit 3 and Unit 4 which analyze Loans and Bonds, respectively. The manuscript concludes with some information sources.

Practically all the concepts introduced in the book are followed by illustrative examples. On the one hand, academic examples addressed to understand the theoretical foundations are stated. Additionally, the examples are supplemented with real examples taken from the financial market so that the reader can understand their practical and professional application. In this regards, all real applications considered are based on the Spanish financial market.

Furthermore, in order to assimilate properly the concepts in each unit, a set of problems to be solved is proposed, alongside a list of the correct answers.

The recommended approach for using this book is to read each unit, work on the embedded examples, and then to practice by solving the set of problems.

Please note that all content, as well as any mistakes that might be found, are the sole and exclusive responsibility of the authors. Should you find any errors, we would appreciate if you could let us know at lgonzalezv@ub.edu.

The authors
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UNIT 1.

FINANCIAL TRANSACTIONS AND FINANCIAL REGIMES

1.1. Financial capital

Prior to defining the concept of a financial transaction, we should highlight that in a financial context it is not enough to just talk of a specific amount of money. It is also necessary to state when this amount will be available. This latter information is related to what is called the **time value of money**.

One of the core principles of finance states that money that is available at present is worth more than the same amount in the future, due to its potential earning capacity. That is to say, the sooner any amount of money is received, the more valuable it will be (**time preference theory**).

Bearing this idea in mind, we now define **financial capital** as any monetary quantity at a specific moment in time:

$$(C, T)\ C \geq 0, T \geq 0$$

where C is the amount of money, which we will express in €, and T is the moment in the future when the money will be available, which we will measure in years.

Financial capital is usually called **payment**(s) or **repayment**(s) depending on the economic agent who hands it over.

Example 1.

€500 available today is indicated as $(500, 0)$

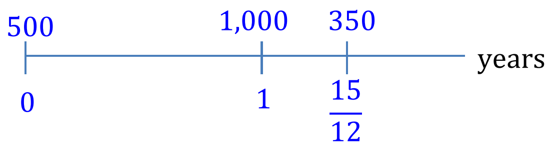
€1,000 available in one year is indicated as $(1{,}000, 1)$

€350 available in fifteen months is indicated as $\left(350, \frac{15}{12}\right)$

Unit 1. Financial transactions and financial regimes
1. Financial transaction: Definition, elements and classification

Any financial capital can be represented graphically on the real line, from the origin $T = 0$.

Example 2. The first, second and third financial capital contained in Example 1 can be represented on the real line, usually called a **timeline**, in the following way:



1.2. Definition of financial transaction

A financial transaction can be defined as the exchange, between economic agents, of payment(s) and repayment(s) at different moments in time.

All financial transactions have the same elements: Personal Elements, Material Elements and Formal Elements.

1.3. Elements of a financial transaction

1) Personal elements

These are economic agents who partake in the financial transaction.

- **Active party or lender**: this is the party that transfers payment(s) for a specific term. In exchange for this financial service, the active party or lender receives a remuneration or price.
- **Passive party or borrower**: this is the party that is committed to return payment(s) received and to pay a remuneration or price.

Unit 1. Financial transactions and financial regimes
1. Financial transaction: Definition, elements and classification

2) Material elements

Payment(s) handed over by the active party. When returned by the passive party, including the price, the material element is called **repayment**(s).

3) Formal elements

Deals or conditions that are agreed on by lender and borrower. These conditions govern the financial transaction and specify, among other things, how the price is calculated, when this price is paid and the way in which the payment(s) that has been lent will be repaid by the borrower.

Example 3. Today, Juan deposits €10,000 in a fixed-term deposit, offered by Bank Z, in exchange for receiving €10,500 in 13 months from now. The elements of this financial transaction are:

Personal elements

- **Active party or lender**: Juan.
- **Passive party or borrower**: Bank Z.

For lending €10,000 today, in 13 months Juan will receive a remuneration, paid by Bank Z, of €500 in addition to repayment of the €10,000.

Material elements

Payment: €10,000 today or $(10{,}000, 0)$

Repayment: €10,500 in 13 months or $\left(10{,}500, \frac{13}{12}\right)$

Formal element

The fixed term deposit agreement, signed by both parties, which lays out the conditions of this financial transaction.

Example 4. A company discounts today, at Bank Z, a trade bill with a face value of €2,500 which matures in 90 days, receiving €2,450. Now the elements are:

Personal elements

Active party or lender: Bank Z.

Passive party or borrower: The company.

For receiving €2,450 today, in exchange for €2,500 that will be available in 90 days, the company pays a price of €50. When the trade bill matures, Bank Z will receive €2,500.

* This kind of transaction will be described in detail in Section 4.2 of this unit.

Material elements

Payment: €2,450 today or $(2{,}450, 0)$

Repayment: €2,500 in 90 days or $\left(2{,}500, \frac{90}{365}\right)$

Formal element

The discount agreement, signed by both parties, that includes the conditions of this financial transaction.

1.4. Classification of financial transactions

Financial transactions can be classified by considering different aspects.

Regarding the number of payment(s) and repayment(s) that the lender and the borrower exchange:

1) **Simple financial transaction**

   Both parties hand over only one payment and one repayment to the other.

2) **Partially complex financial transaction**

   The active or the passive party gives the other party only one payment or repayment while the other party transfers a set of payments or repayments to the former.

3) **Complex financial transaction**

   Both parties transfer several payments or repayments to the other.

Example 5.

The financial transactions included in Examples 3 and 4 are simple.

A loan to be repaid in monthly instalments over the next 3 years is a partially complex financial transaction.

A pension scheme in which the active party will deposit different amounts in exchange for a guaranteed lifelong income, starting from the moment of retirement, is a complex financial transaction.

Regarding the party that is the focus of the study of the financial transaction:

1) **Interest (or accumulation) financial transaction**

   Financial capital is lent with the aim of receiving a greater amount of capital at the end of the term. In this case, the remuneration that is received by the active party is called **interest**.

2) **Discount financial transaction**

   Financial capital that will be available in the future is anticipated at a prior moment in time. The price of such anticipation, which is paid by the passive party, is now called a **discount.**

Example 6.

The financial transaction included in Example 3 is an interest financial transaction. The interest paid by the active party is equal to €500.

The financial transaction of Example 4 is a discount financial transaction. The active party pays a discount of €50.

2. Equilibrium in a financial transaction

2.1. Financial equivalence

The economic agents involved in a financial transaction (lender and borrower) agree, in accordance with certain conditions, the payment(s) and repayment(s) to be exchanged.

Therefore, both parties consider them to be financially equivalent, that is to say, **financial equivalence** has been agreed on payment(s) and repayment(s). We will symbolize financial equivalence by the symbol $\sim$.

2.2. Financial equivalence representation

Financial equivalence is represented in different ways, depending on the kind of financial transaction considered.

1) Simple financial transaction

Payment: (C,T)

Repayment: (C',T')

Financial equivalence:
$(C,T)\sim(C',T')$

2) Partially complex financial transaction

Payment: (C,T)

Repayments: $\{(C_1',T_1'),(C_2',T_2'),\cdots,(C_n',T_n')\}$

Financial equivalence:
$(C,T)\sim\{(C_r',T_r')\}_{r=1,2,\cdots,n}$

Or alternatively:

Payments: $\{(C_1,T_1),(C_2,T_2),\cdots,(C_n,T_n)\}$

Repayment: (C',T')

Financial equivalence:
$\{(C_r,T_r)\}_{r=1,2,\cdots,n}\sim(C',T')$

3) Complex financial transaction

Payments: $\{(C_1,T_1),(C_2,T_2),\cdots,(C_n,T_n)\}$

Repayments: $\{(C_1',T_1'),(C_2',T_2'),\cdots,(C_m',T_m')\}$

Financial equivalence:
$\{(C_r,T_r)\}_{r=1,2,\cdots,n}\sim\{(C_s',T_s')\}_{s=1,2,\cdots,m}$

2.3. Accumulation and discount functions

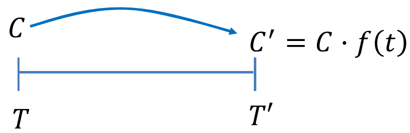
Let us consider a simple financial transaction of interest. The active party gives the passive party the financial capital (C, T) with the aim of obtaining the financial capital (C', T'), where $C' > C$ and $T' > T$. Thus, it could be said that over the term of the financial transaction, $t = T' - T$, the amount C grows and becomes C'.

We define the **accumulation function (or factor)**, denoted by $f(t)$, as the **function that represents the way in which money grows as time goes by**.

In this way, the accumulation factor $f(t)$ can be understood as the **final amount obtained after *t* years from an initial amount of €1**. So, we write:

$$C' = C \cdot f(t)$$

Graphically:



Unit 1. Financial transactions and financial regimes
2. Equilibrium in a financial transaction

Since $C' > C$, it follows that in an interest financial transaction, **the accumulation function is greater than 1**, i.e. $f(t) > 1$.

That is:

$$f(t) = \frac{C'}{C}$$

Note that the accumulation factor is a **function that depends on *t*** in such a way that the greater the term of the financial transaction, the greater the accumulated value, C'.

Finally, since (C, T) and (C', T') are financially equivalent, the financial accumulation factor can also be defined as the **function that allows us to obtain the amount of money at T', C', which is equivalent to C at T**.

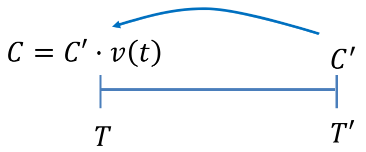
To define the **discount factor**, we consider a simple financial transaction of discount. In this operation, a financial capital available at a future moment, (C', T'), is anticipated at a previous moment in time, T. So, the amount at T, C, satisfies the relationship $C < C'$.

We define the **discount function (or factor)** and denote it by $v(t)$, as the **<u>function that allows us to obtain the amount of money at time T , C, which is equivalent to C' at time T'</u>**.

So, we write:

$$C = C' \cdot v(t)$$

Graphically:



From the definition of the discount factor, **<u>it follows that</u>**

$$v(t) = \frac{C}{C'}$$

and so it is straightforward to see that $v(t) = \frac{1}{f(t)}$

Note also that **<u>the discount function is less than 1</u>**: $v(t) < 1$.

Following similar reasoning to that in the case of the accumulation factor, it is also possible to affirm the following:

- The discount function represents **<u>the way in which money decreases as it is anticipated from a future moment in time to a previous one</u>**.
- The discount factor $v(t)$ can be understood as the **<u>initial amount obtained by anticipating €1 available in t years</u>**.
- The discount factor is a **<u>decreasing function with respect to t</u>**.

Accumulation function	Discount function
Function that represents the way in which money grows as time goes by	Function that represents the way in which money decreases as it is anticipated from a future moment in time to a previous one
Final amount obtained after $\boldsymbol{t}$ years from an initial amount of €**1**	Initial amount obtained by anticipating €**1** available in $\boldsymbol{t}$ years
$f(t) > 1$	$v(t) < 1$
$f(t) = \frac{C'}{C}$	$v(t) = \frac{C}{C'}$
Function that increases with respect to $\boldsymbol{t}$	Function that decreases with respect to $\boldsymbol{t}$
Function that allows us to obtain the amount of money at time T', C', which is equivalent to C at the previous time T	Function that allows us to obtain the amount of money at time T, C, which is equivalent to C' at the later time T'

Unit 1. Financial transactions and financial regimes
2. Equilibrium in a financial transaction

2.4. Interest prices

Given the simple financial transaction $(C, T) \sim (C', T')$, with $T' > T$, we can define the following interest prices.

1) Total interest. This is the total benefit obtained by the lender through the financial transaction, expressed in monetary units.

$$\Delta C = C' - C$$

Example 7. In the financial transaction $(2{,}000, 0) \sim (2{,}320, 2)$, the total interest is:

$$\Delta C = 2{,}320 - 2{,}000 = €320$$

This means that in exchange for €2,000 given today, the lender receives a total remuneration of €320 in 2 years.

2) Unit price of interest or effective interest rate. This is the benefit obtained for each monetary unit invested in the financial transaction.

$$I = \frac{\Delta C}{C} = \frac{C' - C}{C}$$

Example 8. In the financial transaction $(2{,}000, 0) \sim (2{,}320, 2)$, the unit price is:

$$I = \frac{\Delta C}{C} = \frac{C' - C}{C} = \frac{2{,}320 - 2{,}000}{2{,}000} = 0.16 \equiv 16\%$$

This means that for each euro given by the active party, they receive €0.16 in 2 years. Therefore, this interest could also be called 16% biennial*. Alternatively, we can also say that for each €100 given, €16 are received.

* Biennial means once every 2 years. Biannual means twice every year.

3) Unit and mean price of interest or nominal interest rate. This is the benefit obtained for each monetary unit and year.

$$i = \frac{I}{T' - T} = \frac{C' - C}{C \cdot (T' - T)} = \frac{C' - C}{C \cdot t}$$

Example 9. In the financial transaction $(2{,}000, 0) \sim (2{,}320, 2)$, the unit and mean price is:

$$i = \frac{I}{T' - T} = \frac{C' - C}{C \cdot (T' - T)} = \frac{0.16}{2} = 0.08 \equiv 8\%$$

This means that for each €100 given and for each year that the financial transaction lasts, the lender receives €8.

NOTE: When the term of a financial transaction is expressed in days or by calendar dates, the value of t depends on the criterion used. In general, there are 4 criteria:

- If $t \leq 1$,

Criterion	Meaning
$\frac{30}{360}$	Numerator: All months have 30 days Denominator: Always 360
$\frac{Act}{360}$	Numerator: Actual days of the transaction Denominator: Always 360
$\frac{Act}{365}$	Numerator: Actual days of the transaction Denominator: Always 365
$\frac{Act}{Act}$	Numerator: Actual days of the transaction Denominator: 365 or 366 (if it is a leap year)

- If $t > 1$, it is necessary to add, to the number of whole years, the remaining fraction of the year that will be calculated according to one of the previous criteria.

Example 10. Obtain the term (in years) that exists between January 18 and April 18 of a leap year.

Criterion	t
$\frac{30}{360}$	$t = \frac{12+30+30+18}{360} = \frac{90}{360} = 0.25$
$\frac{Act}{360}$	$t = \frac{13+29+31+18}{360} = \frac{91}{360} = 0.252\dot{7}$
$\frac{Act}{365}$	$t = \frac{13+29+31+18}{365} = \frac{91}{365} = 0.249315$
$\frac{Act}{Act}$	$t = \frac{13+29+31+18}{366} = \frac{91}{366} = 0.248634$

3. Financial regime: Definition and classification

3.1. Definition of financial regimes

3.2. Classification of financial regimes

3.1. Definition of financial regimes

A financial regime is the formal expression of the set of deals or conditions that both parties agree on in a financial transaction. These refer to how the price of the transaction is calculated and when it is paid.

3.2. Classification of financial regimes

Depending on how the price is calculated:

1) **Simple financial regimes.** These are those in which the term of the financial transaction is considered as a single period and, therefore, the price is calculated only once. Simple financial regimes are used in short-term financial transactions.
2) **Compound financial regimes.** These are those in which the term of the financial transaction is split into periods and, therefore, the price is calculated for each of them.

Regarding the economic agent:

1) **Interest financial regime.** The financial transaction is studied from the perspective of the active party.
2) **Discount financial regime.** The financial transaction is analyzed from the point of view of the passive party.

As a consequence of both classifications, we will study the following financial regimes:

	INTEREST	DISCOUNT
SIMPLE	**SIMPLE INTEREST**	**SIMPLE DISCOUNT**
COMPOUND	**COMPOUND INTEREST WITH A CONSTANT INTEREST RATE**	
	COMPOUND INTEREST WITH VARIABLE INTEREST RATES	

4. Simple financial regimes

4.1. Simple interest regime

4.2. Simple discount regime

4.1. Simple interest (payable in arrears) regime

If a financial transaction is ruled by a simple interest regime, it means that both the active and the passive party have agreed on the following:

1) The total price of the transaction, ΔC, is proportional to the initial amount, C, and to the term of the operation, $t = T' - T$, through a rate, measured as per 1: $i > 0$.

$$\Delta C = C \cdot i \cdot (T' - T) = C \cdot i \cdot t$$

2) The total price is received at the end of the financial transaction T', with the repayment of the initial amount, so the final amount repaid is C'.

$$C' = C + \Delta C$$

Thus, the formal expression of both conditions is:

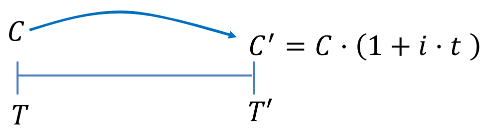
$$C' = C + C \cdot i \cdot t$$

which is equivalent to:

$$\boxed{C' = C \cdot (1 + i \cdot t\,)}$$

The simple interest regime is commonly used for short-term financial transactions such as current accounts, savings accounts, fixed-term deposits, loads repaid through a simple repayment, etc.

Since:



It turns out that the accumulation function of the simple interest regime is:

$$f(t) = \frac{C'}{C} = 1 + i \cdot t$$

Furthermore, we can obtain the following interest prices.

Total price of interest:

$$\Delta C = C' - C = C \cdot i \cdot t$$

Unit price of interest or effective interest rate:

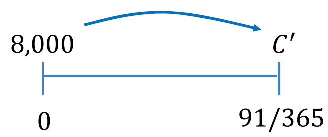
$$I = \frac{\Delta C}{C} = \frac{C \cdot i \cdot t}{C} = i \cdot t$$

Unit and mean price of interest or nominal interest rate:

$$i = \frac{I}{t} = \frac{i \cdot t}{t} = i$$

Unit 1. Financial transactions and financial regimes
4. Simple financial regimes

Example 11. Determine the final capital and interest prices obtained by investing €8,000 in a fixed-term deposit, which matures in 91 days (use the Act/365 criterion), at an annual simple interest rate of 1.75%.



The final amount is:

$$C' = C \cdot (1 + i \cdot t) = 8{,}000 \cdot \left(1 + 0.0175 \cdot \frac{91}{365}\right) = €8{,}034.90$$

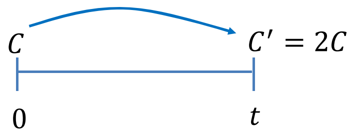
The interest prices are:

$$\Delta C = 8{,}034.90 - 8{,}000 = €34.90$$ Total price

$$I = \frac{\Delta C}{C} = i \cdot t = 0.0175 \cdot \frac{91}{365} = 0.004363 \equiv 0.4363\%$$ Effective rate earned in 91 days

$$i = \frac{I}{t} = \frac{0.004363}{\frac{91}{365}} = 0.0175 \equiv 1.75\%$$ Nominal rate

Example 12. How many years will it take to double capital invested today at an annual simple interest rate equal to 5%?



$$C' = C \cdot (1 + i \cdot t)$$

$$2C = C \cdot (1 + 0.05 \cdot t) \Leftrightarrow 2 = 1 + 0.05 \cdot t \Leftrightarrow 1 = 0.05 \cdot t$$

$$t = 20 \text{ years}$$

4.2. Simple (commercial) discount regime

In a financial transaction ruled by a simple discount regime, the active and passive parties agree on the following:

1) A financial capital available at a future moment, (C', T'), is anticipated at a previous moment, T. The total price of the transaction, ΔC, is proportional to the final amount, C', and to the term of the operation, $t = T' - T$, through a rate, measured as per 1: $d > 0$.

$$\Delta C = C' \cdot d \cdot (T' - T) = C' \cdot d \cdot t$$

2) The total price is paid at the beginning of the financial transaction, T, and is deducted from the final amount, so the amount C is received.

$$C = C' - \Delta C$$

Thus, the formal expression of both conditions is:

$$C = C' - C' \cdot d \cdot t$$

$$\boxed{C = C' \cdot (1 - d \cdot t)}$$

Face value

$t = T' - T$

Maturity or Due date

The simple discount regime is commonly used to discount bills of exchange, trade bills or commercial bills. A bill of exchange is a document issued by the **seller** (drawer) and accepted/signed by the **buyer** (drawee) for the value of goods delivered by the former.

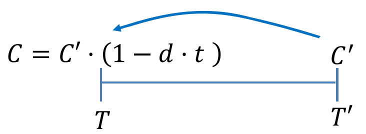
This document obliges the **buyer** to pay a particular amount of money (face value of the bill of exchange) at a particular time (maturity or due date) in the short-term.

Once the bill of exchange has been issued and accepted, its owner (the **seller** of the goods) can either keep it until its maturity date and receive its face value at that moment from the **buyer**, or discount (and give) it at a bank and receive a lower amount than the face value. This is why we say that in the simple discount regime, a financial capital available at a future moment is anticipated at a previous moment in time.

If the bill of exchange is discounted, the bank (its new owner) will have the right to receive its face value, from the **buyer**, on its maturity date.

Unit 1. Financial transactions and financial regimes
4. Simple financial regimes

Graphically:



In this case, since we are studying the financial transaction from the perspective of the passive party, it makes sense to obtain the discount function, which is:

$$v(t) = \frac{C}{C'} = 1 - d \cdot t$$

Moreover, it is possible to obtain the following discount prices.

Total price of discount:

$$\Delta C = C' - C = C' \cdot d \cdot t$$

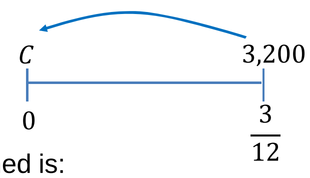
Unit price of discount or effective discount rate:

$$D = \frac{\Delta C}{C'} = \frac{C' \cdot d \cdot t}{C'} = d \cdot t$$

Unit and mean price of discount or nominal discount rate:

$$d = \frac{D}{t} = \frac{d \cdot t}{t} = d$$

Example 13. Determine the amount obtained by the discount of a trade bill with a face value of €3,200, if it matures in 3 months and a simple discount regime at an annual discount rate of 4.50% is applied. Determine also the discount prices.



So, the amount obtained is:

$$C = C' \cdot (1 - d \cdot t) = 3{,}200 \cdot \left(1 - 0.0450 \cdot \frac{3}{12}\right) = €3{,}164$$

Discount prices:

$$\Delta C = 3{,}200 - 3{,}164 = €36$$ Total price

$$D = \frac{\Delta C}{C'} = \frac{36}{3{,}200} = 0.01125 \equiv 1.125\%$$ Effective rate payed within 3 months

$$d = \frac{D}{t} = \frac{0.01125}{3/12} = 0.045 \equiv 4.50\%$$ Nominal rate

When applying the simple interest regime, the price is paid at the end of the financial transaction, at an interest rate i; but in the simple discount regime, the price is paid at the beginning, at a discount rate d.

So, from the perspective of the borrower, a question may arise:

> What is the maximum annual simple interest rate I would have to pay in a simple interest regime to borrow the same initial amount that I obtained in the simple discount regime, and then repay this amount with the face value of the trade bill when it matures?

In order to answer this question, let us find the equivalence between the rates, i and d.

Simple interest regime: $C' = C \cdot (1 + i \cdot t)$

Simple discount regime: $C = C' \cdot (1 - d \cdot t)$

By replacing the second expression into the first one, we get:

$$C' = C' \cdot (1 - d \cdot t) \cdot (1 + i \cdot t) \Leftrightarrow 1 = (1 - d \cdot t) \cdot (1 + i \cdot t)$$

$$1 + i \cdot t = \frac{1}{1 - d \cdot t} \Leftrightarrow i \cdot t = \frac{d \cdot t}{1 - d \cdot t} \qquad \boxed{i = \frac{d}{1 - d \cdot t}}$$

Example 14. With the information contained in Example 13, find the equivalent annual simple interest rate.

By using the previous expression:

$$i = \frac{d}{1 - d \cdot t} = \frac{0.045}{1 - 0.045 \cdot 3/12} = 0.045512 \equiv 4.5512\%$$

This result could also be obtained by using the formal expression of the simple interest regime, $C' = C \cdot (1 + i \cdot t)$, and substituting all the known magnitudes except i:

$$3{,}200 = 3{,}164 \cdot \left(1 + i \cdot \frac{3}{12}\right) \Leftrightarrow i = 0.045512 \equiv 4.5512\%$$

This means that, for the borrower, discounting a trade bill with a face value of €3,200 and maturity in 3 months at an annual discount rate of 4,50%, is equivalent to borrowing €3,200 for 3 months at an annual simple interest rate of 4.55%, and then repaying it with the face value of the trade bill when it matures.

It can also be understood that **this financial transaction results in a profitability of a 4.55% annual simple interest rate for the bank**.

Financial impact of the initial commissions on the commercial discount

Sometimes, when the simple discount regime is used, not only is the nominal discount rate, d, charged but so is an initial fee. This fee, measured as per 1, g, is applied to the face value of the commercial bill in such a way that we calculate the amount C that the borrower will finally receive as follows:

$$C = C' \cdot (1 - d \cdot t) - g \cdot C'$$

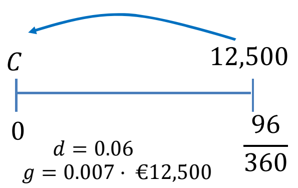
$$\boxed{C = C' \cdot (1 - d \cdot t - g)}$$

If a simple discount transaction has an initial fee, in order to obtain the equivalent annual simple interest rate, it is necessary to use the known expression of the simple interest regime.

NOTE: If the term of the financial transaction, t, is expressed in **days** or by calendar dates, **the criterion Act/360 will be used** for the determination of the term, unless stated otherwise.

Example 15. A company has a portfolio of bills of exchange with a face value of €12,500 and a due date of 96 days. Today, it discounts the portfolio at its bank and is charged with an annual simple discount rate of 6%. The financial transaction involves an initial fee of 0.7% of the face value. Calculate:

1) The amount obtained by the company after discounting the portfolio.
2) The equivalent annual simple interest rate of this financial transaction. Use the criterion Act/365.



So, the amount obtained by the company is:

$$C = C' \cdot (1 - d \cdot t - g) = 12{,}500 \cdot \left(1 - 0.06 \cdot \frac{96}{360} - 0.007\right) = €12{,}212.50$$

where $12{,}500 \cdot 0.06 \cdot \frac{96}{360} = €200$ is the total discount paid and $12{,}500 \cdot 0.007 = €87.5$ is the initial fee.

To calculate the equivalent annual simple interest rate, it is necessary to use the simple interest regime expression:

$$C' = C \cdot (1 + i \cdot t)$$

$$12{,}500 = 12{,}212.5 \cdot \left(1 + i \cdot \frac{96}{365}\right) \Leftrightarrow 1 + i \cdot \frac{96}{365} = 1.023541$$

$$i = 0.089507 \equiv 8.9507\%$$

NOTE: The expression $i = \frac{d}{1-d \cdot t}$ obtained in the previous pages cannot be used in this case because it does not consider situations in which there exists an initial fee. Hence, it is necessary to use the general expression of the simple interest rate.

5. Compound financial regimes

5.1. Compound interest with a constant interest rate regime

A compound interest regime arises when the term of the financial transaction is split into periods, and the interest is calculated at the end of these periods and is simultaneously reinvested in order to earn additional interest in the following periods.

For this regime, we consider the following variables:

p : length (in years) of the period into which the term of the transaction is split, it will be called **capitalization period**.

m: **capitalization frequency**, number of periods of length p in 1 year. Thus, $m = \frac{1}{p}$.

n: **number of capitalization periods** in the term of the transaction. So, $n = m \cdot t$.

i_m: **nominal interest rate**, annual interest rate applied each period.

I_m: **effective interest rate per period**. $I_m = i_m \cdot p$, and since $p = \frac{1}{m}$, it turns out that:

$$I_m = \frac{i_m}{m}$$

Example 16. The nominal interest rate is 6% and the term of the financial transaction is 5 years, $t = 5$:

- If it is split into 6-month periods: $p = \frac{1}{2}$ years, $m = 2$, $n = 2 \cdot 5 = 10$ periods, $i_2 = 0.06$ and $I_2 = \frac{i_2}{2} = 0.03 \equiv 3\%$.

- If it is split into quarters (or 3-month periods): $p = \frac{1}{4}$ years, $m = 4$, $n = 4 \cdot 5 = 20$ quarters, $i_4 = 0.06$ and $I_4 = \frac{i_4}{4} = 0.015 \equiv 1.5\%$.

It is worth noting how banks advertise both nominal and effective interest rates.

For nominal interest rates:

$$i_m\% \begin{bmatrix} \text{nominal} \\ \text{annual} \end{bmatrix} \begin{bmatrix} \text{convertible} \\ \text{compounded} \\ \text{payable} \\ \text{accumulated} \\ \dots \end{bmatrix} \begin{bmatrix} \text{annually} \\ \text{quarterly} \\ \text{semi-annually} \\ \text{monthly} \\ \dots \end{bmatrix}$$

And for effective interest rates:

$$I_m\% \begin{bmatrix} \text{annual} \\ \text{quarterly} \\ \text{semi-} \\ \text{annual} \\ \text{monthly} \\ \dots \end{bmatrix} \text{(effective) interest rate}$$

Example 17. For the following statements, write down the corresponding interest rate:

- 1.5% annual interest rate compounded monthly.

$$i_{12} = 0.015 \equiv 1.5\%$$

- 0.8% semi-annual interest rate.

$$I_2 = 0.008 \equiv 0.8\%$$

- Biennial interest rate of 16%.

$$I_{0.5} = 0.16 \equiv 16\%$$

- Nominal interest rate convertible quarterly equal to 4%.

$$i_4 = 0.04 \equiv 4\%$$

- 2.1% annual interest rate accumulated every two months.

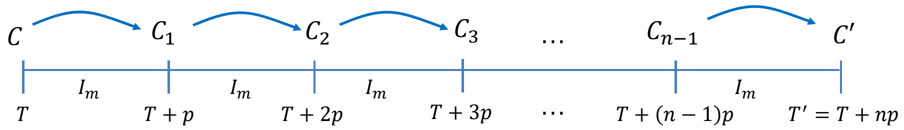
$$i_6 = 0.021 \equiv 2.1\%$$

Now, we obtain the formal expression corresponding to the compound interest regime by first considering its conditions.

If a financial transaction is governed by a regime of compound interest with a constant interest rate, it means that both the active and the passive party have agreed on the following:

1) The term of the financial transaction, $t = T' - T$, is split into n periods of length p (in years). So, $t = n \cdot p$. The interest is calculated at the end of each period by an effective interest rate, measured as per 1, $I_m > 0$, applied on the **amount accumulated at the beginning of the period**. The interest for each period is added to the aforementioned amount in such a way that the accumulated amount at the end of a period is the same as the initial amount of the following period.
2) Although the price is calculated in each period, the total price is only received once, at the end of the financial transaction, T'.

Graphically:



Hence, the accumulated amount at the end of each period will be:

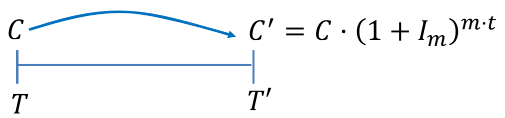
Time	Accumulated amount
T	C
$T+p$	$C_1 = C + C \cdot I_m = C \cdot (1+I_m)$
$T+2p$	$C_2 = C_1 + C_1 \cdot I_m = C_1 \cdot (1+I_m) = C \cdot (1+I_m)^2$
$T+3p$	$C_3 = C_2 + C_2 \cdot I_m = C_2 \cdot (1+I_m) = C \cdot (1+I_m)^3$
…	…
$T' = T+np$	$C' = C \cdot (1+I_m)^n$

As we saw, $n = m \cdot t$, so we can rewrite the above expression as:

$$\boxed{C' = C \cdot (1+I_m)^{m \cdot t}}$$

Many financial transactions are governed by a compound interest regime, both at a constant interest rate and at variable interest rates. Examples include investment funds, pension schemes and loans; all of which share a common characteristic, that of being arranged in medium or long-term operations.

Since:



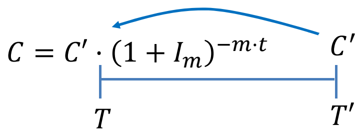
It turns out that the accumulation function of the compound interest regime would be:

$$f(t) = \frac{C'}{C} = (1+I_m)^{m \cdot t}$$

As we will see later, this regime can also be used in discount transactions. So, the discount function associated with the compound interest rate is:

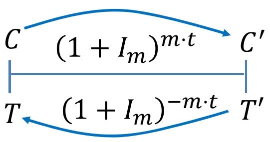
$$v(t) = \frac{C}{C'} = (1 + I_m)^{-m \cdot t}$$

which, on the timeline, can be represented graphically as:



Summarizing:

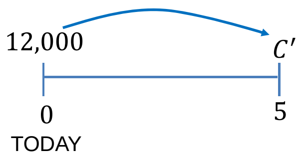
Accumulation: $C' = C \cdot (1 + I_m)^{m \cdot t}$



$$I_m = \frac{i_m}{m} \quad n = m \cdot t$$

Discount: $C = C' \cdot (1 + I_m)^{-m \cdot t}$

Example 18. An investor deposits €12,000 today in a savings account that pays a nominal interest rate compounded quarterly of 3%. He makes no further deposits or withdrawals until he closes the account, 5 years after he opened it. How much money does the investor receive when he cancels the account?

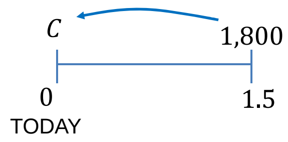


$$m = 4$$
$$i_4 = 0.03$$
$$I_4 = \frac{0.03}{4} = 0.0075$$

The amount obtained is:

$$C' = C \cdot (1 + I_m)^{m \cdot t} = 12{,}000 \cdot (1 + 0.0075)^{4 \cdot 5} = €13{,}934.21$$

Example 19. What deposit made today will provide a payment of €1,800 in 1 and a half years if the financial transaction is governed by a 0.2% semi-annual effective interest rate?

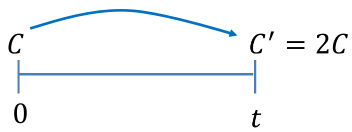


$$m = 2$$
$$I_2 = 0.002$$

The deposit is:

$$C = C' \cdot (1 + I_m)^{-m \cdot t} = 1{,}800 \cdot (1 + 0.002\,)^{-2 \cdot 1.5} = €1{,}789.24$$

Example 20. How many years will it take to double capital invested today at a nominal interest rate of 5% compounded annually?



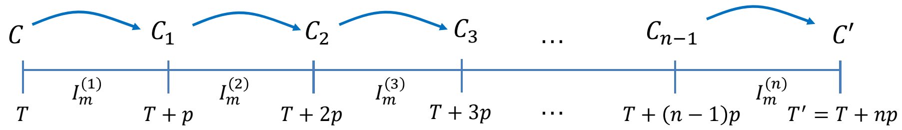
$$m = 1$$
$$i_1 = 0.05$$
$$I_1 = \frac{0.05}{1} = 0.05$$

$$C' = C \cdot (1 + I_m)^{m \cdot t}$$

$$2C = C \cdot (1 + 0.05\,)^{1 \cdot t} \Leftrightarrow 2 = 1.05^t \Leftrightarrow t = \frac{\ln 2}{\ln 1.05} = 14.21 \text{ years}$$

5.2. Compound interest with variable interest rates regime

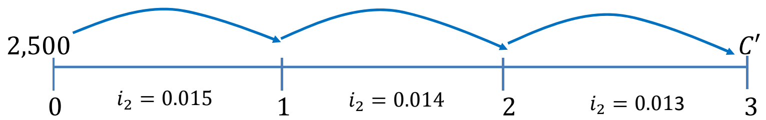
All the conditions in this regime are the same as in the regime above. However, the interest rate varies over the term of the financial transaction.



In such a situation, it is easy to check that:

$$C' = C \cdot \prod_{s=1}^{n} \left(1 + I_m^{(s)}\right)$$

Example 21. Bank Z offers a fixed-term deposit for 3 years at an interest rate convertible semi-annually. The nominal interest rate changes every year. If it is 1.5%, 1.4% and 1.3%, for the first, second and third year, respectively, and €2,500 is deposited today, what will the accumulated amount be?



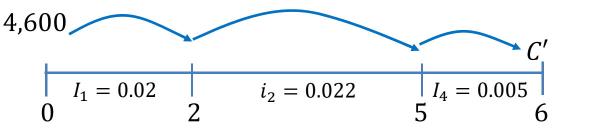
$I_2 = \frac{0.015}{2} = 0.0075$ $\quad I_2 = \frac{0.014}{2} = 0.007$ $\quad I_2 = \frac{0.013}{2} = 0.0065$

So, the accumulated amount is:

$$C' = \underbrace{\underbrace{2{,}500 \cdot (1+0.0075)^{2\cdot 1}}_{2{,}537.64} \cdot (1+0.007)^{2\cdot 1}}_{2{,}573.29} \cdot (1+0.0065)^{2\cdot 1} = €2{,}606.85$$

Example 22. An account is governed by compound interest. The annual interest rate for the first 2 years is 2%, for the next 3 years the nominal interest rate is 2.2% compounded semi-annually and for the last year the quarterly effective interest rate is 0.5%.

- Find the accumulated value at the end of 6 years if the initial amount is €4,600.



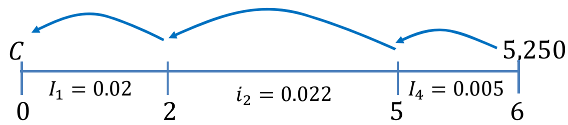
$$I_2 = \frac{0.022}{2} = 0.011$$

$$C' = 4{,}600 \cdot (1 + 0.02\,)^{1 \cdot 2} \cdot (1 + 0.011\,)^{2 \cdot 3} \cdot (1 + 0.005\,)^{4 \cdot 1}$$

$$C' = €5{,}213.50$$

- Find the initial amount that should have been deposited to have an accumulated value of €5,250 at the end of 6 years.

  In this case, by discounting, we will have:

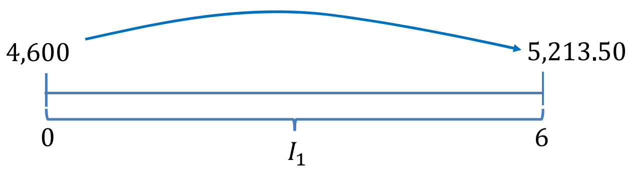


$$I_2 = \frac{0.022}{2} = 0.011$$

$$C = 5{,}250 \cdot (1 + 0.005\,)^{-4 \cdot 1} \cdot (1 + 0.011\,)^{-2 \cdot 3} \cdot (1 + 0.02\,)^{-1 \cdot 2}$$

$$C = €4{,}632.20$$

- Find the annual effective interest rate that would have been paid over the 6 years in order to get €5,213.50 by depositing €4,600.



$$5{,}213.50 = 4{,}600 \cdot (1 + I_1)^6$$

$$I_1 = \left(\frac{5{,}213.50}{4{,}600}\right)^{\frac{1}{6}} - 1 = 0.021085 \equiv 2.1085\%$$

We have the same initial and accumulated amounts by using several interest rates over the term of the financial transaction as well as by using a single annual effective rate for the whole term. So, it could be said that this single rate, I_1, is equivalent to all the other interest rates applied over the 6 years.

Unit 1. Financial transactions and financial regimes
5. Compound financial regimes

5.3. Equivalent effective interest rates

Let us consider a financial transaction governed by an effective interest rate of frequency m, I_m. The initial amount deposited is C and the accumulated value after t years is C'. That is to say:

$$C' = C \cdot (1 + I_m)^{m \cdot t}$$

We want to find the effective interest rate with capitalization frequency k, I_k, that should be applied to the same initial amount, C, over the same term of t years, in order to arrive at the same final amount, C'. Thus:

$$C' = C \cdot (1 + I_k)^{k \cdot t}$$

If we equate the 2 expressions, we obtain the relationship between effective interest rates of different capitalization frequencies, m and k, also called equivalent effective interest rates.

$$(1 + I_k)^k = (1 + I_m)^m$$

or, equivalently:

$$\boxed{I_k = (1 + I_m)^{m/k} - 1}$$

This means that a monetary amount invested at either of the two effective interest rates will generate the same final capital, regardless of the transaction term and of the capital invested.

That is the reason why it is said that I_k and I_m are equivalent and it is symbolized as:

$$I_k \sim I_k$$

Example 23. Calculate the semi-annual effective interest rate equivalent to a nominal interest rate compounded monthly that equals 4%.

$$m = 12, \qquad i_{12} = 0.04 \Leftrightarrow I_{12} = \frac{0.04}{12} = 0.00\dot{3}, \qquad k = 2$$

$$\boxed{I_k = (1 + I_m)^{m/k} - 1}$$

$$I_2 = \left(1 + 0.00\dot{3}\right)^{12/2} - 1 = 0.020167 \equiv 2.0167\%$$

Example 24. A fund has earned an effective interest rate for a two-year period of 14%. Find the nominal interest rate convertible biennially and the annual effective interest rate that has governed the fund. In this case we have:

$$m = 0.5, \qquad I_{0.5} = 0.14 \Leftrightarrow i_{0.5} = 0.5 \cdot 0.14 = 0.07, \qquad k = 1$$

$$I_1 = (1 + 0.14)^{0.5/1} - 1 = 0.067708 \equiv 6.7708\%$$

5.4. Annual effective interest rate and *TAE*

The most commonly used effective interest rate in the market is the annual one, i.e. I_1. By using the expression obtained previously, it is straightforward to see that the annual effective interest rate equivalent to the effective interest rate with capitalization frequency m is:

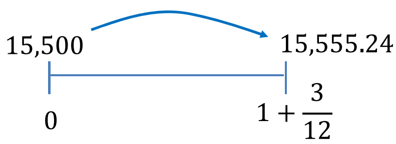
$$\boxed{I_1 = (1 + I_m)^m - 1}$$

It is also possible to obtain the annual effective interest rate of any operation through the expression of the compound interest regime if the initial amount, the accumulated value and the term of the financial transaction are known.

$$C' = C \cdot (1 + I_1)^t \Rightarrow \boxed{I_1 = \left(\frac{C'}{C}\right)^{1/t} - 1}$$

Unit 1. Financial transactions and financial regimes
5. Compound financial regimes

Example 25. By investing €15,500 in a financial product for 1 year and 3 months an investor receives €15,555.24. Find the annual effective interest rate associated with this financial transaction.



$$I_1 = \left(\frac{15{,}555.24}{15{,}500}\right)^{\frac{1}{1+\frac{3}{12}}} - 1 = 0.0028501 \equiv 0.28501\%$$

Since 1990, the Bank of Spain has obliged banks and other financial institutions to inform about the ***TAE*** (***tasa anual equivalente***) of any financial transaction.

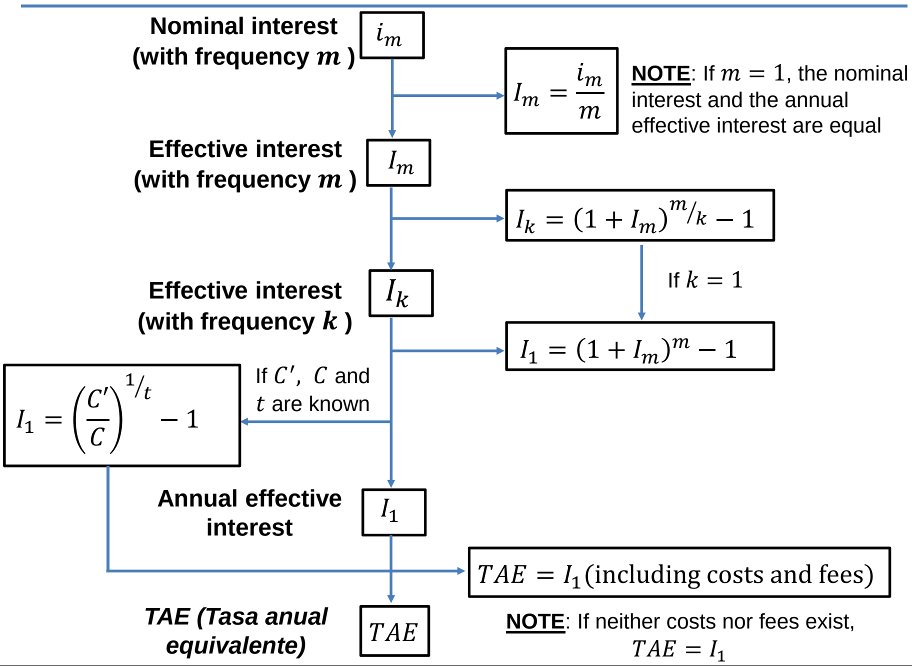
The *TAE,* which is more a legal than a financial concept, is the annual effective interest rate associated with a financial transaction once the costs and fees indicated by the Bank of Spain have been taken into account.

The aim of this rule is to facilitate comparisons among different financial products, offered by several financial institutions, by obliging these institutions to inform about the same information, the *TAE*.

The TAE was first defined by the Ministerial Order of 12/12/1989. The costs and fees that have to be considered in order to calculate the *TAE* of a financial transaction have changed over time.

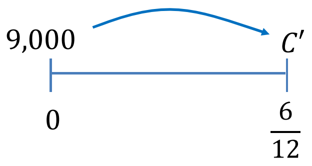
If a financial transaction does not charge fees or costs, both the *TAE* and the annual effective interest rate, I_1, are exactly the same.

Most developed countries have laws in force that use similar concepts. For example, in the UK, the concept of **effective APR** (annual percentage rate) is used.



Example 26. Brian borrows €9,000 from Bank X. The loan will be repaid by a single repayment made 6 months from now. The nominal interest rate of the loan is 6% payable monthly and the application fee is €100, payable when the loan is given.

- Find the amount to be paid by Brian in 6 months.



$$i_{12} = 0.06$$

$$I_{12} = \frac{0.06}{12} = 0.005$$

$$C' = 9{,}000 \cdot (1 + 0.005\,)^{6} = €9{,}273.40$$

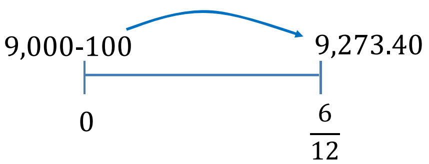
- Find the annual effective rate of the loan.

$$I_1 = (1 + 0.005\,)^{12} - 1 = 0.061678 \equiv 6.1678\%$$

Unit 1. Financial transactions and financial regimes
5. Compound financial regimes

- Obtain the *TAE* of the loan.

  Since the actual amount borrowed by Brian is €9,000 - €100 = €8,900, the *TAE* of the loan is:



$$TAE = \left(\frac{9{,}273.40}{8{,}900}\right)^{\frac{1}{\frac{6}{12}}} - 1 = 0.085670 \equiv 8.5670\%$$

Example 27. Today, a person deposits €3,000 in a fixed-term deposit for 1 year. The deposit earns an annual interest rate of 0.90% and has a maintenance fee of €20, which is paid at the end of the year.

- Find the amount accumulated after 1 year, before and after paying the fee.

$$\underset{0}{3{,}000} \longrightarrow \underset{1}{C'} \qquad I_1 = 0.0090$$

Before paying the maintenance fee:

$$C' = 3{,}000 \cdot (1 + 0.0090\,)^1 = €3{,}027$$

And after paying it:

$$C' = 3{,}000 \cdot (1 + 0.0090)^1 - 20 = €3{,}007$$

- What is the annual effective interest rate of the deposit?

$$I_1 = 0.0090$$

- Obtain the *TAE* of the deposit.

  In order to calculate the *TAE*, the maintenance fee has to be considered.

$$\underset{0}{3{,}000} \longrightarrow \underset{1}{3{,}007} \qquad I_1 = \left(\frac{3{,}007}{3{,}000}\right)^1 - 1 = 0.002333 \equiv 0.2333\%$$

Example 28. Obtain the *TAE* of a pension scheme which, for a payment of €6,000 on 20th December of year A, will provide a guaranteed income of €9,609.91 on 17th May of year A+12 (use the criterion Act/365).

Since no fees or other costs exist, the *TAE* and the annual effective rate of the pension scheme are equal.

$$\underset{\substack{20.12.\text{A} \\ 0}}{6{,}000} \longrightarrow \underset{\substack{17.05.\text{A}+12 \\ 12+\frac{148}{365}}}{9{,}609.91}$$

$$I_1 = TAE = \left(\frac{9{,}609.91}{6{,}000}\right)^{\frac{1}{12+\frac{148}{365}}} - 1$$

$$= 0.0387 \equiv 3.87\%$$

Example 29. Let us suppose you want to invest €30,000 in this fixed-term deposit offered by bank Z.

1-YEAR TERM DEPOSIT

2.50% TAE [1]	✓ **Receive interest every 3 months**
	✓ **No fees or other costs exist**

Do I have to have any other product besides the bank deposit?
If you are a new customer, you will only have to open an AA account that does not have any kind of costs or fees.

Is interest reinvested in the same deposit?
No, interest will be paid into your AA account quarterly.

[1] Nominal interest rate of 2.48% (*TAE* 2.50%) payable for 12 months.
Example: By investing €30,000, upon maturity of the deposit the client will have received €744.00.

- Check the calculation of the *TAE*.

$$m = 4, \qquad i_4 = 0.0248 \iff I_4 = \frac{0.0248}{4} = 0.0062, \qquad k = 1$$

$$I_1 = (1 + 0.0062)^4 - 1 = 0.02503 \equiv 2.50\%$$

- Calculate the total interest you are supposed to have at the maturity of the deposit.

$$C' = 30{,}000 \cdot (1 + 0.0062)^4 = 30{,}000 \cdot (1 + 0.025\,)^1 = €30{,}750$$

$$Y = 30{,}750. - 30{,}000 = €750$$

- Explain why the amount calculated in the previous section is not equal to the one that the advert says.

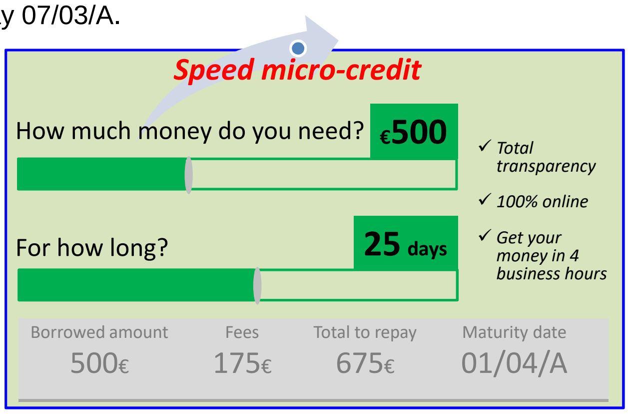
  Since the interest received every three months is not reinvested in the fixed-term deposit, it turns out that the total interest is:

  1st quarter: $30{,}000 \cdot \frac{0.0248}{4} = €186$

  2nd quarter: $30{,}000 \cdot 0.0062 = €186 \ldots$

  At the maturity of the fixed-term deposit: $186 \cdot 4 = €744$

Example 30. A relative of yours has applied for the following online fast loan today 07/03/A.



Obtain the *TAE* (use the criterion Act/365):

$$TAE = \left(\frac{675}{500}\right)^{\frac{365}{25}} - 1 = 78.96 = 7{,}896\%$$

6. Financial value

6.1. Financial value at any moment in time

6.2. Present value and final value

6.1. Financial value at any moment in time

So far simple financial transactions, i.e., those ones with a single payment and a single repayment, have been analyzed.

Now we will study complex financial transactions (either partially or totally complex), in which we will have several payments or repayments.

In this latter situation, it may be useful to **financially value the set of payments or repayments at a specific moment in time**.

When financially valuing this set, it is replaced by a single amount of financial capital that, somehow, represents the set. This single amount is called the **financial value** (or **financial sum**) associated with the set of payments or repayments (depending on what we are considering) and it could be said that both, the set and its financial value, are **financially equivalent**.

In order to perform a financial valuation, the idea of the time value of money has to be taken into account and, consequently, a financial regime will have to be considered.

NOTE: From now on, unless otherwise indicated, to determine financial values (or financial sums), **we will use the compound interest regime**.

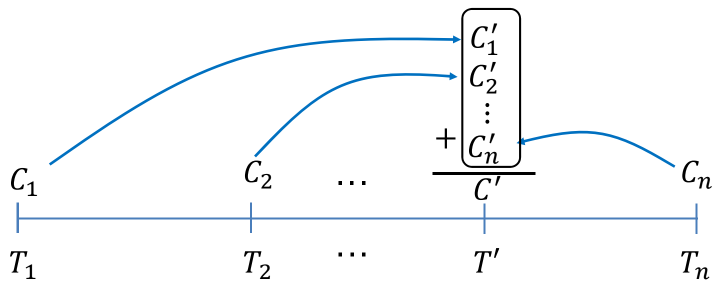
Given the set of payments or repayments $\{(C_r, T_r)\}_{r=1,2,\cdots,n}$ and the effective interest rate I_m, its **financial value** (or **financial sum**) at moment T' is defined as the financial capital (C', T') where:

$$C' = \sum_{r=1}^{n} C_r \cdot (1 + I_m)^{m \cdot (T' - T_r)}$$

It can be written as:

$$\{(C_r, T_r)\}_{r=1,2,\cdots,n} \underset{\displaystyle I_m}{\underset{\downarrow}{\sim}} (C', T')$$

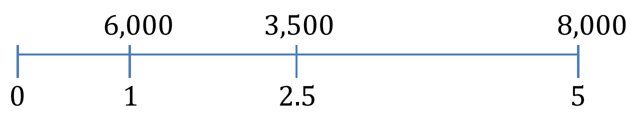
Graphically,



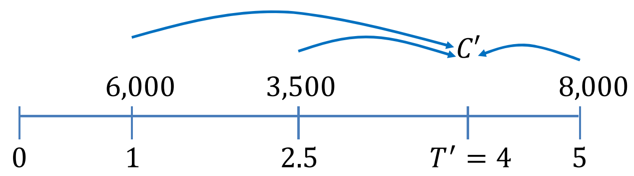
Example 31. Find the financial sum at $T' = 4$ of the following set of repayments if an annual effective interest rate of 5% is considered:

$$\{(6{,}000, 1), (3{,}500, 2.5), (8{,}000, 5)\}$$

On the timeline, we have:



So:



$I_1 = 0.05$

$$C' = 6{,}000 \cdot (1 + 0.05)^{3} + 3{,}500 \cdot (1 + 0.05)^{1.5} +$$
$$+8{,}000 \cdot (1 + 0.05)^{-1}$$
$$C' = €18{,}330.55$$

The financial value at $T' = 4$ is the financial capital:

$$(C', T') = (18{,}330.55, 4)$$

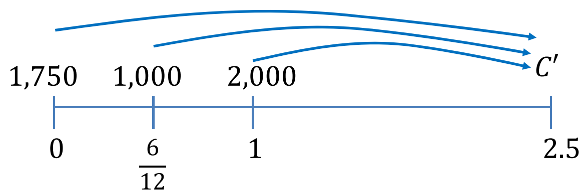
Furthermore, for the interest rate given:

$$\{(6{,}000, 1), (3{,}500, 2.5), (8{,}000, 5)\} \sim (18{,}330.55, 4)$$

Example 32. Today, Mary deposits €1,750 in a bank account. In 6 months' time, Mary will deposit an additional amount of €1,000; and in 1 year, further €2,000.

- Find the account balance in 2 and a half years if the account pays a nominal interest rate compounded semi-annually equal to 4.92%.

  First, we draw the payments on the timeline.

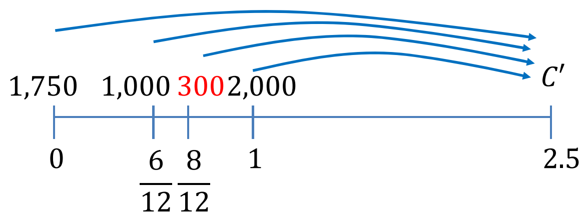


$$i_2 = 0.0492$$
$$I_2 = \frac{0.0492}{2} = 0.0246$$

- Then, we calculate the financial value at $T' = 2.5$

$$C' = 1{,}750 \cdot (1 + 0.0246)^{2 \cdot 2.5} + 1{,}000 \cdot (1 + 0.0246)^{2 \cdot 2} +$$
$$+2{,}000 \cdot (1 + 0.0246)^{2 \cdot 1.5} = €5{,}229.46$$

- If Mary withdraws €300 in 8 months, what will the account balance be in 2 and a half years?

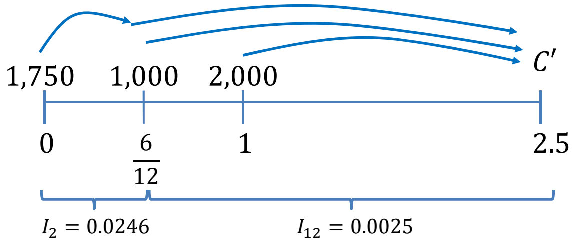


$$i_2 = 0.0492$$

$$I_2 = \frac{0.0492}{2} = 0.0246$$

$$C' = 1{,}750 \cdot (1 + 0.0246)^{2 \cdot 2.5} + 1{,}000 \cdot (1 + 0.0246)^{2 \cdot 2} -$$
$$-300(1 + 0.0246)^{2 \cdot \left(1 + \frac{10}{12}\right)} + 2{,}000 \cdot (1 + 0.0246)^{2 \cdot 1.5} = €4{,}901.50$$

- Find the account balance in 2 and a half years if the account pays a nominal interest rate equal to 4.92% compounded semi-annually for the first 6 months and a 0.25% monthly interest rate for the rest of the term.



$$C' = 1{,}750 \cdot (1 + 0.0246)^{2 \cdot 0.5} \cdot (1 + 0.0025)^{12 \cdot 2}$$
$$+1{,}000 \cdot (1 + 0.0025)^{12 \cdot 2} +$$
$$+2{,}000 \cdot (1 + 0.0025)^{12 \cdot 1.5} = €5{,}057.48$$

- Write the equation that allows us to calculate the annual effective interest rate which is equivalent to those that are paid over the 2 and a half years in the previous example.

  As this annual effective interest rate will give the same accumulated value:

$$1{,}750 \cdot (1 + I_1)^{2.5} + 1{,}000 \cdot (1 + I_1)^{2} + 2{,}000 \cdot (1 + I_1)^{1.5} = 5{,}057.48$$

Example 33. A person plans to invest in the following payment-in-kind deposit offered by Bank Z. Find the interest of this deposit (value of the laptop).

https://www.pexels.com/

Fixed-term deposit

- Amount: €90,000
- Term: 6 months
- *TAE*: 1,40%

For the purposes of personal income tax, the laptop is considered remuneration in kind. There is no cash remuneration.

We have to consider that the laptop is received at the beginning of the financial transaction.

90,000 | 9,000
Interest
0 | $\frac{6}{12}$

The person and the bank consider that payments and repayments involved in the financial transaction are equivalent:

$$(90{,}000,\, 0\,) \sim \{(\text{Interest}, 0), (90{,}000, 1)\}$$

It means that their financial value must be equal. So, by valuing at 0:

$$90{,}000 = \text{Interest} + 90{,}000 \cdot (1 + 0.014)^{-\frac{6}{12}}$$

$$\text{Interest} = €623.43$$

You should only make this deposit if you really want to have the laptop and if it is worth more than €623.43 on the market.

Unit 1. Financial transactions and financial regimes
6. Financial value

6.2. Present value and final value

If the financial value of the set of payments or repayments is made at a **fixed origin**, which is usually denoted as 0, the value obtained is called the **present value of the set** and we represent it by V_0.

That is to say, given the set $\{(C_r, T_r)\}_{r=1,2,\cdots,n}$ and the effective interest rate I_m, its present value is:

- **Present value:**

$$V_0 = \sum_{r=1}^{n} C_r \cdot (1 + I_m)^{-m \cdot T_r}$$

Graphically:

V_0

$C_1 \quad C_2 \quad \cdots \quad C_{n-1} \quad C_n$

$\mathbf{0} \quad T_1 \quad T_2 \quad \cdots \quad T_{n-1} \quad T_n$

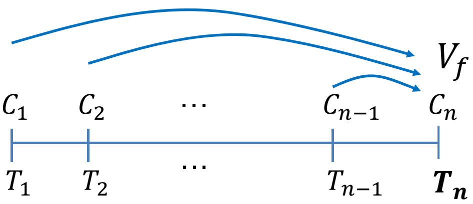
$$\{(C_r, T_r)\}_{r=1,2,\cdots,n} \underset{\substack{\downarrow \\ I_m}}{\sim} (V_0, 0)$$

Similarly, if the financial value of the set of payments or repayments is calculated at a **final moment**, T_n, the amount obtained is called the **final (or accumulated) value of the set** and we represent it by V_f.

- **Final (or accumulated) value:**

$$V_f = \sum_{r=1}^{n} C_r \cdot (1 + I_m)^{m \cdot (T_n - T_r)}$$

Graphically:



$$\{(C_r, T_r)\}_{r=1,2,\cdots,n} \underset{I_m}{\overset{}{\sim}} (V_f, T_n)$$

It is worth remarking that since, for the given interest rate I_m:

$$\{(C_r, T_r)\}_{r=1,2,\cdots,n} \sim (V_0, 0)$$

and:

$$\{(C_r, T_r)\}_{r=1,2,\cdots,n} \sim (V_f, T_n)$$

it turns out that:

$$(V_0, 0) \sim (V_f, T_n)$$

The same equivalence holds for the financial valuation at any moment, T'.

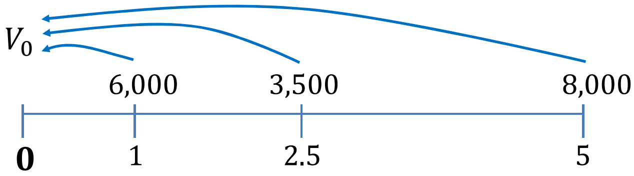
Therefore, **once the present value** (or the final value) **of a set of payments or repayments has been obtained, we can find the financial value** (or financial sum) **of the set at any moment in time** by simply using an accumulation or discount factor, depending on when the financial value has to be calculated.

Example 34. For the set of payments:

$$\{(6{,}000, 1), (3{,}500, 2.5), (8{,}000, 5)\}$$

with an annual effective interest rate of 5%:

- Find the present value.



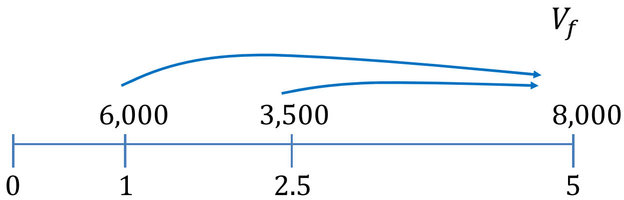
$$I_1 = 0.05$$

$$V_0 = 6{,}000 \cdot (1 + 0.05\,)^{-1} + 3{,}500 \cdot (1 + 0.05\,)^{-2.5} + $$
$$+ 8{,}000 \cdot (1 + 0.05\,)^{-5}$$

$$V_0 = €15{,}080.59$$

Unit 1. Financial transactions and financial regimes
6. Financial value

- Find the final value (at $T_n = 5$)



$$I_1 = 0.05$$

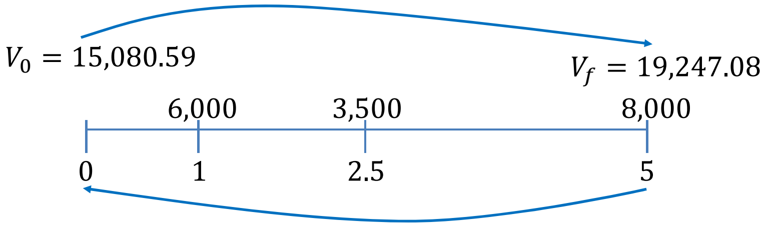
$$V_f = 6{,}000 \cdot (1 + 0.05\,)^{4} + 3{,}500 \cdot (1 + 0.05\,)^{2.5} + 8{,}000$$

$$V_f = €19{,}247.08$$

- Check that, for the given interest, $I_1 = 0.05$, the present and the final values of the set of payments are equivalent:

$$(15{,}080.59\,,0) \sim (19{,}247.08\,,5)$$

$$V_f = 15{,}080.59 \cdot (1+0.05)^{5} = €19{,}247.08$$



$$V_0 = 19{,}247.08 \cdot (1+0.05)^{-5} = €15{,}080.59$$

UNIT 1 PROBLEMS

1. A person deposits €2,500 in an account at SA Bank today. He will get €2,750 in 2 years. Find the total price, the unit price and the unit and mean price. Explain the results obtained.

2. A financial transaction began on February 17th of a leap year and ended on July 10th of the same year. Find the term of the transaction by using the 30/360, Actual/360, Actual/365 and the Actual/Actual methods.

3. You invest €3,000 at an annual simple interest rate of 2.5%. Find the accumulated value after 91 days by using the Actual/365 criterion. Calculate the total interest, the nominal interest rate and the effective interest rate.

4. Mary borrows €2,000 from Mark at a 5% annual simple interest rate and agrees to repay the loan in 9 months. What is the amount that Mary is required to repay to Mark?

5. How many years will it take for €500 to increase to €600 at a 4.5% annual simple interest rate?

6. At what annual rate of simple interest will €8,000 increase to €8,128.05 in 182 days? Consider the Actual/365 criterion.

7. What initial amount will earn interest of €100 in 6 months at an annual simple interest rate of 3.20%?

8. A trade bill with a face value of €9,000 that matures in 2 months is discounted. The bank applies a simple discount regime at an annual rate of 6%.

a) Calculate the amount received by the passive party.
b) Find the total price, the unit price and the unit and mean price of the financial transaction.
c) How much would be received if there was an initial fee of 0.5% of the face value?

9. The amount obtained by discounting a commercial bill under the simple discount regime is €1,000. Find the face value of the commercial bill if its due date is in 62 days and the annual discount rate applied is 5.95%. What would the face value be if the financial transaction involved initial expenses equal to 0.7% of the face value?

10. A company has a bill of exchange with a face value of €3,000 and a maturity of 3 months from now. It trades the bill at its bank and obtains €2,963.25.

a) Find the simple annual discount rate applied by the bank.
b) Find the equivalent simple interest rate of this financial transaction.

11. A company discounts a portfolio of bills of exchange today. The portfolio has a face value of €12,500 and matures in 31 days. The bank charges a simple annual discount rate equal to 6% and an initial fee of 0.7% and it uses the Act/360 criterion. Calculate the profitability obtained by the bank if simple interest regime is considered (use the Act/365 criterion).

12. A company wants to buy a car in 3 months. The car will have a price of €29,700. In order to make this future payment, the company performs the following financial transactions today:

- It cancels a bank account that was opened with €10,000 9 months ago. The account has paid an annual simple interest rate of 3.50%.
- It discounts a portfolio of bills of exchange at an annual simple discount rate of 6% and an initial fee of 0.4%. The face value of the portfolio is €20,000. 40% of the portfolio matures in 6 months and the rest in 9 months.

a) Find the amount obtained by the company today when cancelling the bank account.
b) Find the amount obtained by the company today when discounting the portfolio.
c) Does the company have enough money today to buy the car? If it does not, the company will invest the amount obtained with the above financial transactions in a fixed-term deposit, from today until the moment of purchasing the car. What is the minimum annual simple interest rate that the fixed-term deposit must pay in order for the company to be able to buy the car?

13. A woman invests €6,320 today. Her investment grows according to a nominal interest rate compounded quarterly that is equal to 3.20%.

a) Find her balance after 3 and a half years.
b) How much interest does she earn?
c) How long will €6,320 have to be invested to obtain a balance of €7,471.16?

14. On his fifth birthday, a boy received an inheritance from his aunt. The inheritance grew to €32,168 by the time he was 18 years old. If the money grew by compound interest at a semi-annual effective interest rate of 1.15%, find the amount of the inheritance.

15. Mr. Smith deposits €1,750 in a bank account for 2 years and 3 months. How much money will he receive when he closes the bank account if it is governed by:

a) A 4.5% nominal interest rate compounded annually?
b) A semi-annual (biannual) effective rate equalling 2.15%?
c) An 8% biennially effective interest rate?
d) An annual interest rate of 4% convertible quarterly (every three months)?
e) A monthly interest rate of 0.35%?
f) A 3.60% nominal interest rate accumulated every four months?
g) A 0.8% effective interest rate every two months?

16. A financial asset has a face value of €1,000 and matures in 2 years. Today, you buy it for €945.73. Find the annual effective interest rate associated with this financial transaction.

17. Suppose that the growth of money is ruled by compound interest at an annual rate of 5% payable semi-annually. How much money must you invest today in order to have a balance of €23,000 3 and a half years from now?

18. An investor deposits €9,000 today in a savings account that accumulates interest at varying rates. For the first 2 years, the nominal interest rate is 2% compounded quarterly. For the next 1 and a half years, a 0.3% monthly interest rate is applied. For the rest of the term, the interest rate is equal to an annual 4%. What is the investor’s balance at the end of 6 years?

19. Bank Z offers a savings account with varying interest rates for 3 years. For the first 8 months the monthly effective interest rate is 0.2%, for the next 9 months the nominal interest rate compounded semi-annually is 2% and for the rest of the term the annual effective interest rate is 3%.

a) What deposit made into this account today will provide a balance of €10,000 in 3 years?
b) If you deposit €5,500 today:
   b.1) Find the accumulated value in 3 years.
   b.2) Find the annual effective interest rate that should have been paid over the 3 years in order to achieve the same accumulated value.

20. Find the following equivalent interest rates:

a) Quarterly effective interest rate equivalent to 6% nominal interest compounded quarterly?
b) Annual effective rate equivalent to an annual interest rate of 7.2% convertible monthly?
c) Nominal interest rate compounded semi-annually equivalent to a 5.9% biennially effective interest rate.
d) Interest rate compounded quarterly equivalent to an interest rate of 8% payable semi-annually.

21. Find the effective interest rate over a three-year period (triennially) that is equivalent to an annual effective rate of 3% the first year, 2% the second year and 1% the third year.

22. Calculate the annual effective interest rate and the *TAE* of a loan of €3,000, to be repaid by a single repayment in 3 months, if a nominal interest rate of 6.75% accumulated monthly has been agreed and there is an application fee that equals 2% on the principal, which is paid by the borrower when the loan is given.

23. Find the annual effective interest rate and the *TAE* of a financial transaction in which, by investing €8,250 on October 8th of a given year, €9,156 is obtained on October 21st two years later. Consider that at the moment of investing, a fee of 3% of the initial amount is paid by the investor. (Use the Actual/365 criterion).

24. If you invest €10,000 today in a financial product offered by Bank Triple Z, you will obtain an accumulated profitability of 15% in 4 and a half years.

a) Find the annual effective interest rate and the *TAE* of this financial product.
b) What would the effective annual interest and the *TAE* of this financial product be if at its maturity you had to pay a management fee of 1% on the invested amount?

25. After saving for some years, an investor has €20,000 and he is offered 3 different alternatives to invest this amount.

- Alternative 1: open a fixed-term deposit that pays a 4% nominal interest rate compounded monthly.
- Alternative 2: buy a bill of exchange with a face value of €20,500 and maturity within 8 months.
- Alternative 3: invest in a bank account that offers an annual simple interest rate of 7%, in which no interest is earned on the first €8,000.

a) Calculate the amount accumulated after 8 months according to the 3 alternatives.
b) Find the *TAE* of the 3 alternatives if none of them has fees or costs.

26. In order to buy a small apartment, you are given two different alternatives today.

- Alternative A: A €24,000-payment in 2 and a half years and €100,000 in 5 years.
- Alternative B: A single payment of €120,000 in 3 years.

If the current interest rate is equal to 6% annual, which alternative do you prefer?

27. A person must pay €6,000 in 1 year and €5,000 in 4 and a half years. Obtain the capital that must be paid within 2 years at an annual interest rate of 5% in order to cancel both debts.

28. Susan deposited €9,800 in a bank account 18 months ago. 6 months later, she deposited €3,300 more. Finally, 2 months ago, she made an additional deposit of €4,500.

a) If the bank account was governed by an interest rate of 2.5% compounded semi-annually, what is the balance of the account today (accumulated value)?

b) What would the balance of the account be today if it had been governed by the following interest rates: an annual effective interest rate of 2% for the first 3 months, a 2.4% annual interest rate convertible monthly for the next year, and a 1.3% semi-annual interest rate for the rest of the term?

29. A Bank offers a savings account with varying interest rates for 3 years: for the first 8 months the monthly effective interest rate is 0.2%, for the next 9 months the nominal interest rate compounded quarterly is 1% and for the rest of the term the annual interest rate is 3%.

a) If you deposit €5,500 today:
   - a.1) Find the accumulated value in 3 years.
   - a.2) Find the annual effective interest rate that should have been paid over the 3 years in order to get the same accumulated value.

b) If you deposit today €5,500, in 6 months you withdraw €2,000 and in 1 year you deposit €1,000:
   - b.1) Find the balance in 3 years.
   - b.2) Write the equation that allows you to calculate the annual effective interest rate that should have been paid over the 3 years in order to get the same accumulated value.

c) What deposit made today will provide for a payment of €1,000 in 10 months and another of €2,000 in 1 and a half years?

Answers

1. $\Delta C = €250$. This means that in exchange for €2,000 given today, the person receives a total remuneration of €250 in 2 years.
$I = 0.10 \equiv 10\%$. This means that for each euro given by the person, he/she receives €0.10 in 2 years.
$i = 0.05 \equiv 5\%$. That is to say, for each €100 given and for each year that the financial transaction lasts, the person receives €5.

2.

Criterion	t
$\frac{30}{360}$	$t = 0.397\dot{2}$
$\frac{Act}{360}$	$t = 0.4$
$\frac{Act}{365}$	$t = 0.394521$

$\frac{Act}{Act}$	$t = 0.393443$

3. $C' = €3{,}018.70 \quad \Delta C = €18.70 \quad I = 0.006233 \equiv 0.6233\% \quad i = 0.025 \equiv 2.5\%$.

4. $C' = €2{,}075$.

5. $t = 4.\dot{4}$.

6. $i = 0.0321 \equiv 3.21\%$.

7. $C = €6{,}250$.

8. a) $C = €8{,}910$.
b) $\Delta C = €90 \quad D = 0.001 \equiv 1\% \quad d = 0.06 \equiv 6\%$.
c) $C = €8{,}865$.

9. If there are no expenses $C' = €1{,}010.36$. When expenses exist, $C' = €1{,}017.55$.

10. a) $d = 0.0490 \equiv 4.90\%$.
b) $i = 0.049608 \equiv 4.9608\%$.

11. $i = 0.145014 \equiv 14.5014\%$.

12. a) $C' = €10{,}262.50$.
b) $C = €19{,}140$.
c) No, there is not enough money today. $i = 0.040473 \equiv 4.0473\%$.

13. a) $C' = €7{,}065.85$.
b) $\Delta C = €745.85$.
c) $t = 5.25$.

14. $C = €23{,}895.22$.

15. a) $C' = €1{,}932.19$.
b) $C' = €1{,}925.80$.
c) $C' = €1{,}908.27$.
d) $C' = €1{,}913.95$.
e) $C' = €1{,}923.12$.
f) $C' = €1{,}896.73$.
g) $C' = €1{,}948.75$.

16. $I_1 = 0.028292 \equiv 2.8292\%$.

17. $C = €19{,}349.10$.

18. $C' = €10{,}903.63$.

19. a) $C = €9{,}252.32$.
b) b.1) $C' = €5{,}944.46$.
b.2) $I_1 = 0.026242 \equiv 2.6242\%$.

20. a) $I_4 = 0.015 \equiv 1.5\%$.
b) $I_1 = 0.074424 \equiv 7.4424\%$.
c) $i_2 = 0.028869 \equiv 2.8869\%$.
d) $i_4 = 0.079216 \equiv 7.9216\%$.

21. $I_{1/3} = 0.061106 \equiv 6.1106\%$.

22. $I_1 = 0.069628 \equiv 6.9628\%$ $\quad TAE = 0.159654 \equiv 15.9654\%$.

23. $I_1 = 0.052519 \equiv 5.2519\%$ $\quad TAE = 0.037346 \equiv 3.7346\%$.

24. a) $I_1 = 0.031546 \equiv 3.1546\%$ $\quad TAE = 0.031546 \equiv 3.1546\%$.
b) $I_1 = 0.031546 \equiv 3.1546\%$ $\quad TAE = 0.029545 \equiv 2.9545\%$.

25.

Alternative	a) Accumulated amount	b) *TAE*
1	€20,539.60	$0.040742 \equiv 4.0742\%$
2	€20,500.00	$0.037733 \equiv 3.7733\%$
3	€20,560.00	$0.042293 \equiv 4.2293\%$

26. Since the present values of alternatives A and B are, respectively, €95,472.40 and €100,754.31, as a buyer you should choose alternative A.

27. €10,725.85.

28. a) $C' = €18{,}073.80$.
b) $C' = €18{,}054.05$.

29. a) a.1) $C' = €5{,}900.42$.
a.2) $I_1 = 0.023701 \equiv 2.3701\%$.
b) b.1) $C' = €4{,}832.66$.
b.2) $5{,}500(1 + I_1)^3 - 2{,}000(1 + I_1)^{2.5} + 1{,}000\,(1 + I_1)^2 = 4{,}832.66$.
c) $C = €2{,}931.30$

UNIT 2.

ANNUITIES

1. Definition and classification

2. Level or constant annuities

3. Annuities with payments in geometric progression

4. Annuities with payments in arithmetic progression

1. Definition and classification

1.1. Definition

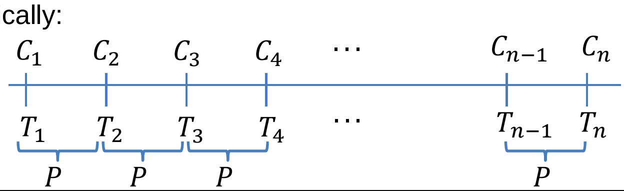
1.2. Classification

1.1. Definition

An **annuity** is a set of payments (or repayments), $\{(C_r, T_r)\}_{r=1,2,\cdots,n}$, made at equal intervals in time. Common examples of annuities include house rents, mortgage payments on homes, salaries earned by workers or instalments paid for cars.

The interval between annuity payments is called the **annuity period**, P, and the number of annuity periods in 1 year is named the **annuity frequency**, M, with the relationship $M = 1/P$. Furthermore, n is the **number of payments** of the annuity.

Graphically:



Example 1.

Repayments made by a borrower, at the end of each quarter, to repay a loan over the next 10 years are an annuity with $P = 1/4$, $M = 4$ and $n = 40$.

The rent paid for a flat at the beginning of each month over 5 years is an annuity with $P = 1/12$, $M = 12$ and $n = 60$.

1.2. Classification

Annuities can be classified by considering different aspects.

1) Depending on the number of payments, n :
- **Temporary annuity**: If n is a finite number.
- **Perpetuity**: If payments last forever, i.e. n is infinite.

2) Depending on the structure of the amounts C_r:
- **Constant or level annuity**: If $C_r = C, \forall r = 1,2, \dots, n$
- **Non-level annuity**: If the amounts are not constant.

3) Depending on the **fixed origin**, 0:
- **Non-deferred annuity**: The fixed origin and the annuity origin are both the same.
- **Deferred annuity**: The fixed origin is earlier than the annuity origin.

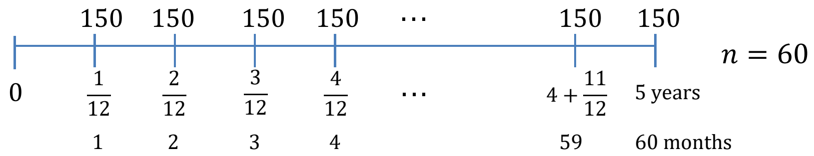
4) Depending on the annuity period:
- **Monthly annuity**: If $M = 12$
- **Quarterly annuity**: If $M = 4$
- **Semi-annual annuity**: If $M = 2$
- **Etc.**

5) Depending on when the payments are made each annuity period:
- **Immediate annuity**: If payments are made at the end of each annuity period.
- **Annuity-due**: If payments are made at the beginning of each period.

Example 2.

Draw the following annuities on the timeline:

- A monthly non-deferred immediate annuity, consisting of level payments over the next 5 years, equal to €150.



- A 2-year-deferred annuity composed of semi-annual immediate payments to be made for 12 years. The first payment is equal to €300 whereas the rest will increase in such a way that each payment is 5% greater than the previous one.

  The first payment is: $C_1 = 300$

  The second payment is:

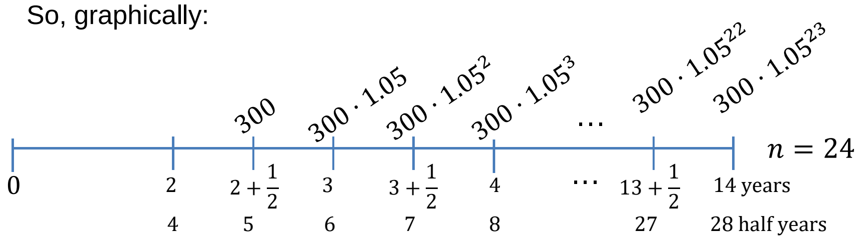
  $$C_2 = 300 + 300 \cdot 0.05 = 300 \cdot (1 + 0.05) = 300 \cdot 1.05$$

  The third payment is:

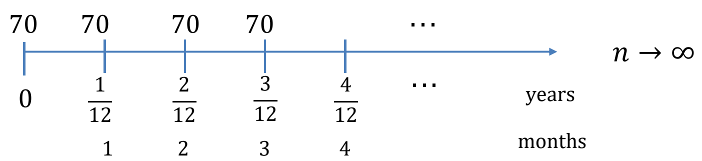
  $$C_3 = (300 \cdot 1.05) \cdot 1.05 = 300 \cdot 1.05^2$$

And so on.

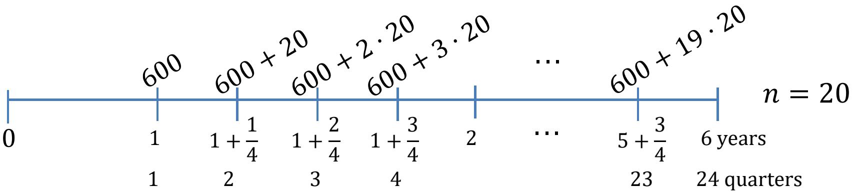
So, graphically:



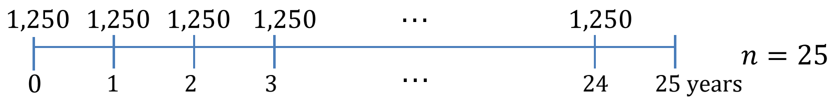
- A non-deferred perpetuity-due that will pay €70 a month from now on.



- A twenty-payment quarterly annuity-due that is deferred 1 year with the first payment equal to €600 and the payments increasing by €20 each quarter.



- A constant non-deferred annuity-due, to be paid annually over the next 25 years, with payments of €1,250.



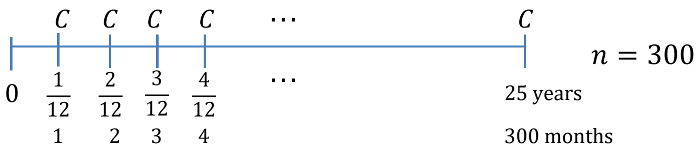
<u>NOTE</u>

Annuities are commonly not named as in the previous example. This means that, from the information given in each financial transaction which involves annuities, we will have to be able to determine all of the annuity characteristics. More particularly:

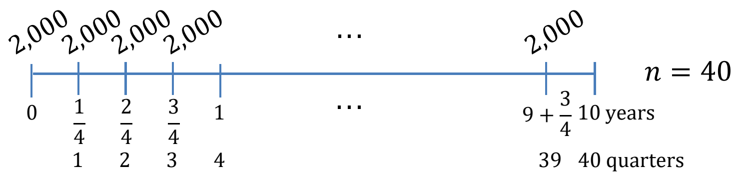
- If an annuity is deferred, it will be explicitly stated (using the corresponding term or a synonym). If nothing is said, it must be assumed that the annuity is non-deferred.
- If the annuity is due, this is stated explicitly (using this term or a synonym) So, if nothing is stated, it must be understood that the annuity is immediate.

Example 3.
Draw on the timeline the annuities involved in the following financial transactions:

- A loan of €250,000 to be repaid with constant payments at the end of each month over the next 25 years.



- A person wants to save for retirement by depositing €2,000 at the beginning of each quarter, for 10 years.



2. Level or constant annuities

2.1. Definition

An annuity is level or constant if all the payments or amounts have the same value, i.e.:

$$C_r = C, \qquad \forall r$$

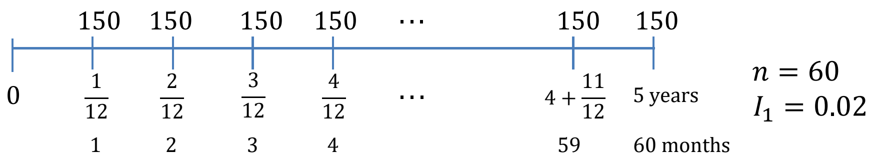
2.2. Financial value of the Non-deferred immediate annuity with n payments and period P

We begin by calculating the **present value**. From it, the final or accumulated value of the annuity will be obtained.

First, let us consider an example.

Example 4. Find the present value of the first annuity drawn in example 2 if an annual effective interest rate equal to 2% is considered.

We had a monthly non-deferred immediate annuity, with level payments over the next 5 years equal to €150. Graphically:



Unit 2. Annuities
2. Level or constant annuities

Although it is not necessary, to have an easier expression, let us calculate the **monthly effective interest rate** that it is equivalent to $I_1 = 0.02$ (note that the annuity frequency is $M = 12$).

$$I_{12} = (1 + 0.02)^{1/12} - 1 = 0.001652$$

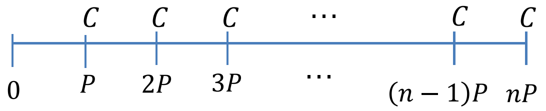
So, the present value is:

$$V_0 = 150 \cdot (1 + 0.001652)^{-1} + 150 \cdot (1 + 0.001652)^{-2} + \\ + \cdots + 150 \cdot (1 + 0.001652)^{-60}$$

$$V_0 = €8{,}561.72$$

Calculating the present value in this way is an annoying task. So, we are going to look for a **formula** which will allow us to evaluate annuities more easily.

In general, the graph of a non-deferred immediate annuity with n payments and period P is:



<u>NOTE</u>: When evaluating annuities, we must bear in mind that **<u>the effective interest rate that appears in the formulas must have the same frequency as the annuity,</u>** that is, M.

The present value of the annuity, V_0, with the effective interest rate I_M is:

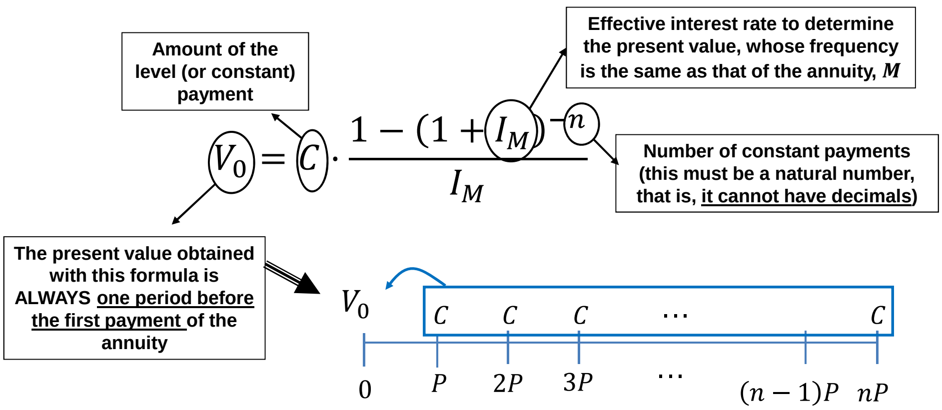
$$V_0 = C \cdot (1+I_M)^{-1} + C \cdot (1+I_M)^{-2} + \cdots + C \cdot (1+I_M)^{-n}$$

$$V_0 = C \cdot \sum_{r=1}^{n} (1+I_M)^{-r}$$

It can be demonstrated that this sum is equal to:

$$V_0 = C \cdot \frac{1-(1+I_M)^{-n}}{I_M}$$

Explanation of the **formula** of the present value of a level non-deferred immediate annuity with n payments and period P (frequency M)

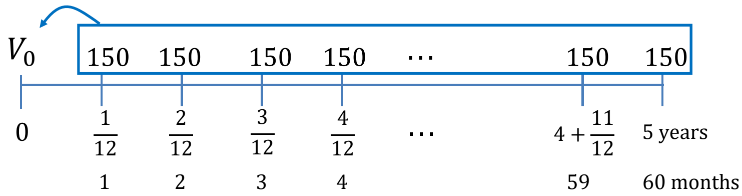


The expression $\frac{1-(1+I_M)^{-n}}{I_M}$ is usually written as $a_{\overline{n|}I_M}$ and, as a result, in general:

$$\boxed{V_0 = C \cdot a_{\overline{n|}I_M}}$$

Example 5. For the first annuity illustrated in Example 2, find its present value if an annual effective interest rate equal to 2% is considered.

The graph of the annuity is:



with $C = 150,\ M = 12$ and $n = 60$.

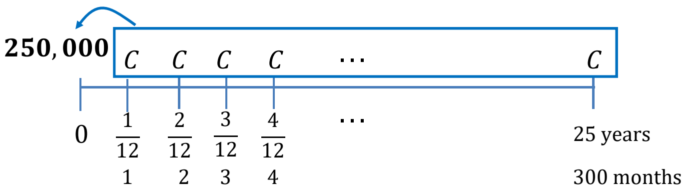
The **monthly effective interest rate** is $I_{12} = 0.001652$.

So, by using the formula we just saw:

$$V_0 = 150 \cdot a_{\overline{60}|0.001652} = 150 \cdot \frac{1-(1+0.001652)^{-60}}{0.001652} = €8{,}561.72$$

Example 6. For the loan in Example 3, find the amount of constant payments if the loan is governed by an annual interest rate of 4.80% accumulated monthly.

The loan has a principal of €250,000 and it will be repaid by constant payments at the end of each month over the next 25 years.



with $M = 12$ and $n = 300$.

In this case, the interest rate we use to value is $i_{12} = 0.0480$. So, the **monthly effective interest rate** is:

$$I_{12} = \frac{0.0480}{12} = 0.004$$

Now, by using the formula for the present value:

$$250{,}000 = C \cdot a_{\overline{300}|0.004}$$

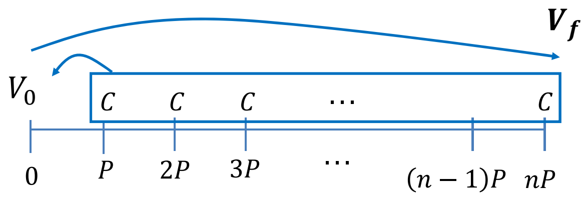
$$250{,}000 = C \cdot \frac{1-(1+0.004)^{-300}}{0.004}$$

And so, by isolating C, we get:

$$C = €1{,}432.49$$

This means that to pay off the amount borrowed at the interest rate agreed, the borrower has to pay €1,432.49 at the end of each month, for the next 25 years.

We can also calculate the **<u>accumulated (or final) value</u>** of the annuity from its present value.

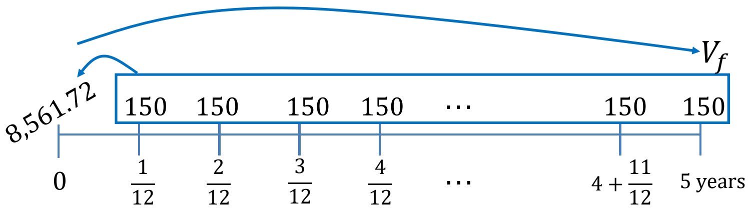


$$\boxed{V_f = V_0 \cdot (1+I_M)^{M\cdot t}}$$

being t the term (in years) between 0 and nP.

Example 7. For the annuity in Example 4, find its accumulated value if an annual effective interest rate equal to 2% is considered.

We already saw that the annuity graph on the timeline is:



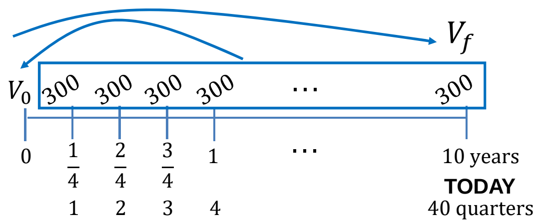
and the **monthly effective interest rate** is $I_{12} = 0.001652$.

Therefore, from its present value, the accumulated value will be:

$$V_f = 8{,}561.72 \cdot (1 + 0.001652)^{12 \cdot 5} = €9{,}452.83$$

Unit 2. Annuities
2. Level or constant annuities

Example 8. Dr Collins has been depositing €300 at the end of each trimester for the last 10 years in a savings account that is governed by a 1.20% nominal interest rate paid quarterly. What is the savings account balance today, just after making the corresponding deposit?



$C = 300$
$M = 4$
$n = 40$

From the given interest rate $i_4 = 0.0120$, we determine the **quarterly effective interest rate**:

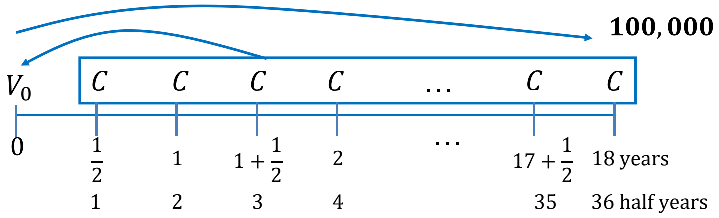
$$I_4 = \frac{0.0120}{4} = 0.003$$

Now, from the formula we already know:

$$V_f = \underbrace{300 \cdot a_{\overline{40|}0.003}}_{11{,}292.02} \cdot (1 + 0.003)^{4 \cdot 10} = €12{,}729.43$$

Example 9. An investor wishes to accumulate €100,000 at the time of her last deposit in a pension scheme by contributing, from now on, with an amount C at the end of each 6-month period for 18 years. If the annual effective interest rate of the pension scheme is 1.5%, what amount is necessary for her to reach her goal?

Bearing in mind that $M = 2$ and $n = 36$, we first draw the annuity on the timeline.



Then, from the given interest rate $I_1 = 0.0150$, we determine the <u>**semi-annual effective interest rate**</u>:

$$I_2 = (1 + 0.0150)^{1/2} - 1 = 0.007472$$

Now, from the formula:

$$100{,}000 = \underbrace{C \cdot a_{\overline{36|}0.007472}}_{V_0} \cdot (1 + 0.007472\,)^{2\cdot 18}$$

$$100{,}000 = C \cdot \frac{1 - (1 + 0.007472\,)^{-36}}{0.007472} \cdot (1 + 0.007472\,)^{2\cdot 18}$$

And, by isolating C, we get:

$$C = €2{,}431.21$$

NOTE:

So far, for **level annuities**, we have obtained a formula from which it is easy to calculate the present value and the accumulated value of a **temporary non-deferred immediate** annuity.

In order to obtain the present value and the accumulated value of a:

- **temporary deferred immediate annuity**
- **temporary non-deferred annuity-due**
- **temporary deferred annuity-due**

we will use **exactly the same formula** and the proper **accumulation or discount factor**, depending on the moment in time for which we need the value of the annuity.

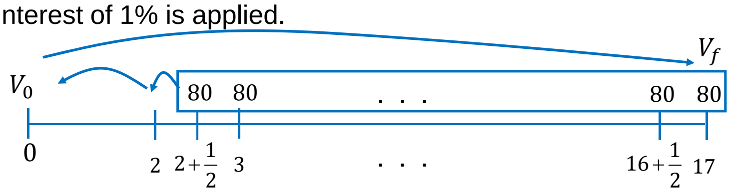
Unit 2. Annuities
2. Level or constant annuities

Summarizing:

In order to properly calculate the financial value (present, accumulated or at any other moment) of any level annuity, the following steps should be followed:

1. Draw the annuity on the timeline.
2. Think of where you want the annuity to be financially valued.
3. Check if the capitalization frequency of the given interest rate is equal to the annuity frequency. If it is not, then obtain the equivalent interest rate.
4. Use the formula $a_{\overline{n}|I_M}$.
5. Reflect on where the financial value obtained with the above formula is placed.
6. If needed, correct this financial value with an accumulation factor or a discount factor, so as to arrive to the financial value at the desired moment.

Example 10. Find the present value and the accumulated value of a semi-annual immediate annuity to be paid for 15 years, after a 2-year deferment. The amount of each payment is €80 and a monthly effective interest of 1% is applied.



$C = 80$
$M = 2$
$n = 30$

Semi-annual effective rate equivalent to $I_{12} = 0.01$:

$$I_2 = (1+0.01)^{12/2} - 1 = 0.06152$$

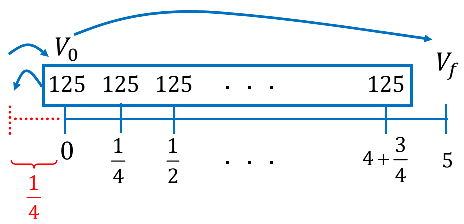
Present value:

$$V_0 = \underbrace{80 \cdot a_{\overline{30}|0.06152}}_{\text{Value at } T=2:\ 1,083.50} \cdot (1+0.06152)^{-2\cdot 2} = €853.33$$

Accumulated (final) value:

$$V_f = 853.33 \cdot (1+0.06152)^{2\cdot 17} = €6,496.47$$

Example 11. Find the present value and the final value, under an annual interest rate of 4% payable quarterly, of a non-deferred quarterly annuity-due to be paid for 5 years, with payments of €125.



$C = 125$
$M = 4$
$n = 20$

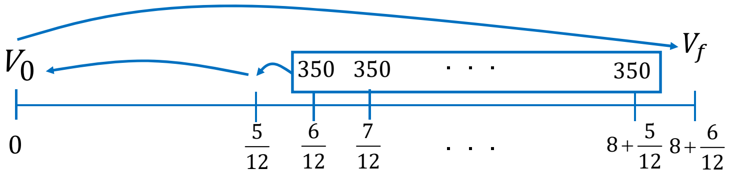
Quarterly effective interest rate:

$$i_4 = 0.04 \rightarrow I_4 = \frac{0.04}{4} = 0.01$$

Present value: $V_0 = \underbrace{125 \cdot a_{\overline{20}|0.01}}_{\text{Value at } T=-\frac{1}{4}:\ 2,255.69} (1+0,01)^{1} = €2.278,25$

Accumulated value: $V_f = 2.278,25 \cdot (1+0,01)^{20} = €2,779.90$

Example 12. Let us consider an annuity-due payable for 8 years after a 6-month deferment. It has level payments of €350 a month. Obtain its present value and its accumulated value at an annual interest rate of 4% compounded monthly.



$C = 350$
$M = 12$
$n = 96$

Monthly effective interest rate:

$$i_{12} = 0.04 \rightarrow I_{12} = \frac{0.04}{12} = 0.00\dot{3}$$

Present value:

$$V_0 = \underbrace{350 \cdot a_{\overline{96}|0.00\dot{3}}}_{\text{Value at } T=\frac{5}{12}:\ 28{,}713.77} \cdot \left(1+0.00\dot{3}\right)^{-5} = €28{,}239.96$$

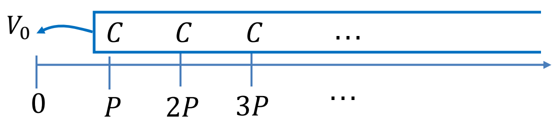
Accumulated value:

$$V_f = 28{,}239.96 \cdot \left(1+0.00\dot{3}\right)^{12 \cdot 8.5} = €39{,}653.24$$

2.3. Financial value of the Non-deferred immediate perpetuity with period *P*

A perpetuity is an annuity with an infinite term. That is to say, a perpetuity has payments that last forever. Therefore, it makes no sense to calculate accumulated values for perpetuities.

Graphically:



The present value of the annuity, V_0, with the effective interest rate I_M is:

$$V_0 = C \cdot (1+I_M)^{-1} + C \cdot (1+I_M)^{-2} + \cdots =$$
$$= C \cdot \sum_{r=1}^{\infty} (1+I_M)^{-r}$$

Alternatively, by using the formula we already know, it is also possible to write:

$$V_0 = \lim_{n\to\infty} C \cdot a_{\overline{n|}I_M} = C \cdot \lim_{n\to\infty} \frac{1-(1+I_M)^{-n}}{I_M}$$

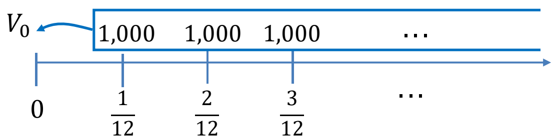
So:

$$\boxed{V_0 = C \cdot \frac{1}{I_M}}$$

The present value obtained with this formula is ALWAYS given <u>one period before the first payment</u> of the annuity.

Example 13. What would you be willing to pay today for an infinite stream of constant end-of-month payments of €1,000, if the annual interest rate compounded monthly was 1.5%?

Graphically:



$$M = 12$$
$$i_{12} = 0.015$$

From the given interest rate, we obtain the **<u>monthly effective interest rate</u>**:

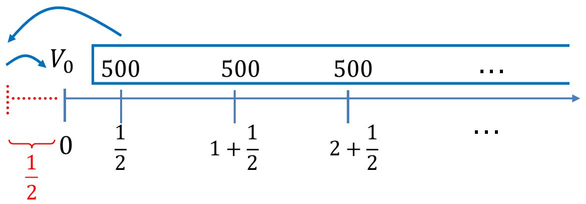
$$I_{12} = \frac{0.015}{12} = 0.00125$$

So, the present value of the infinite stream of payments is:

$$V_0 = 1{,}000 \cdot \frac{1}{0.00125} = €800{,}000$$

Example 14. An investment made today will provide for €500 a year forever, with the first payment being made 6 months from today. If a semi-annual effective interest rate equal to 1% is used, find the amount invested.

First, we draw the perpetuity on the timeline.



$$M = 1$$
$$I_2 = 0.01$$

The **annual effective interest rate** is:

$$I_1 = (1+0.01)^2 - 1 = 0.0201$$

So, the present value is:

$$V_0 = 500 \cdot \frac{1}{0.0201} \cdot (1+0.0201)^{0.5} = €25{,}124.38$$

3. Annuities with payments in geometric progression

3.1. Definition

3.2. Financial value of the Non-deferred immediate annuity with n payments and period P

3.3. Financial value of the Non-deferred immediate perpetuity with period P

3.1. Definition

We now consider annuities with payments that vary in a geometric progression. In this case, the payments of the annuity are:

$$C_r = C_1 \cdot q^{r-1}, q > 0, \qquad \forall r$$

This means that, from a first payment, C_1, any payment is obtained by multiplying the preceding one by a constant, q, called the **common ratio**.

If $q > 1$, the payments of the annuity increase; if $0 < q < 1$, they decrease.

Example 15. It is forecast that the turnover of an SME will be, at the end of this year, €700,000. Then, it will rise by 2% annually and cumulatively for the next 9 years. Find the general expression of the turnover for these 10 years.

The first year's turnover is:

$$C_1 = 700{,}000$$

The second year's is:

$$C_2 = 700{,}000 + 700{,}000 \cdot 0.02 = 700{,}000 \cdot 1.02$$

The third is:

$$C_3 = (700{,}000 \cdot 1.02\ ) \cdot 1.02 = 700{,}000 \cdot 1.02^{\ 2}$$

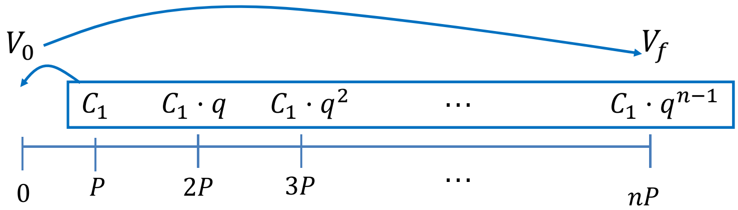
So, $C_1 = 700{,}000$, $q = 1.02$ and, in general:

$$C_r = 700{,}000 \cdot 1.02^{r-1}, \qquad \forall r = 1,2, \dots, 10$$

3.2. Financial value of the Non-deferred immediate annuity with n payments and period P

As we did for level annuities, we begin by calculating the **present value**. From it, the final or accumulated value of the annuity will be obtained.

Graphically:



The present value of the annuity, V_0, with the effective interest rate I_M, is:

$$V_0 = C_1 \cdot (1+I_M)^{-1} + C_2 \cdot (1+I_M)^{-2} + \cdots + C_n \cdot (1+I_M)^{-n} =$$

$$= \sum_{r=1}^{n} C_r \cdot (1+I_M)^{-r} = \sum_{r=1}^{n} C_1 \cdot q^{r-1} \cdot (1+I_M)^{-r}$$

It can be demonstrated that this expression turns into:

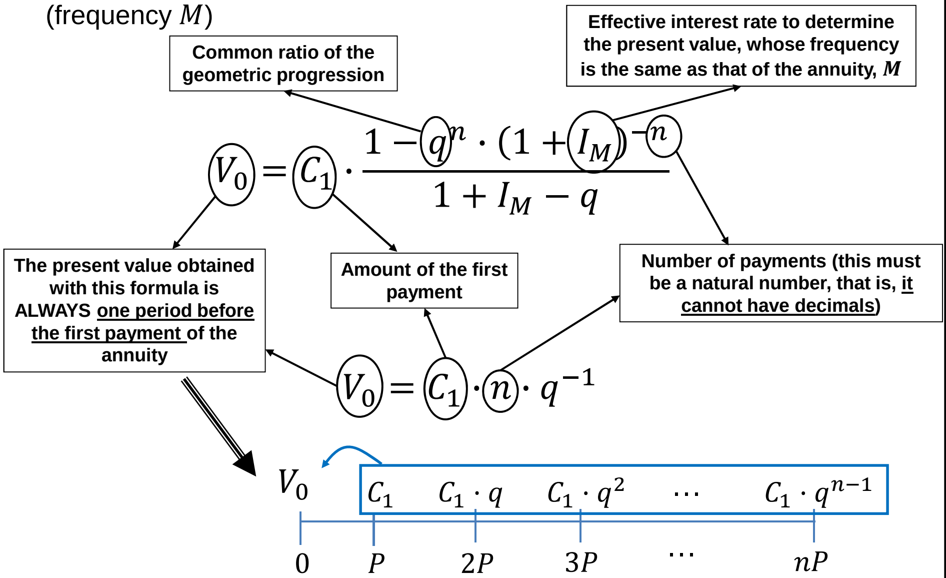
$$\boxed{V_0 = \begin{cases} C_1 \cdot \dfrac{1 - q^n \cdot (1+I_M)^{-n}}{1 + I_M - q} & \text{if } q \neq 1 + I_M \\[2ex] C_1 \cdot n \cdot q^{-1} & \text{if } q = 1 + I_M \end{cases}}$$

And for the accumulated (final) value of the annuity:

$$\boxed{V_f = V_0 \cdot (1+I_M)^{M \cdot t}}$$

Unit 2. Annuities
3. Annuities with payments in geometric progression

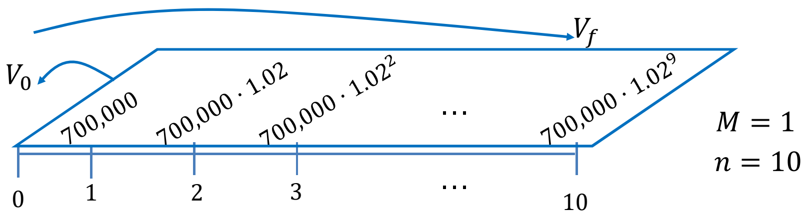
Explanation of the **formula** for the present value of a non-deferred immediate annuity with n payments in geometric progression and period P (frequency M)



Example 16. For the annuity in Example 15, find:

- Its present value (today) if an annual interest rate equal to 1.5% is applied.

  The graph of the annuity on the timeline is:



  The given interest rate is $I_1 = 0.015$. That is, it has the same frequency as that of the annuity. Moreover, $q = 1.02 \neq 1 + I_1 = 1.015$. Thus, we can write:

$$V_0 = 700{,}000 \cdot \frac{1 - 1.02^{10} \cdot (1 + 0.015)^{-10}}{1 + 0.015 - 1.02} = €7{,}051{,}456.63$$

- Its present value (today) if an annual interest rate equal to 2% is applied.

  In this case, we have $I_1 = 0.02$. Therefore, $q{=}1 + I_1{=}1.02$ and the present value would be:

$$V_0 = 700{,}000 \cdot 10 \cdot 1.02^{-1} = €6{,}862{,}745.10$$

- Its accumulated value, in 10 years, with both the interest rates given above.

  For $I_1 = 0.015$:

$$V_f = 7{,}051{,}456.63 \cdot (1 + 0.015)^{10} = €8{,}183{,}503.30$$

  And for $I_1 = 0.02$:

$$V_f = 6{,}862{,}745.10 \cdot (1 + 0.02)^{10} = €8{,}365{,}647.98$$

NOTE:

For **annuities with payments in geometric progression**, we have a formula from which it is easy to calculate the present value and the accumulated value of a **temporary non-deferred immediate** annuity.

Now, as it was done for level annuities, in order to obtain the present value and the accumulated value when having payments in geometric progression of a:

- **temporary deferred immediate annuity**
- **temporary non-deferred annuity-due**
- **temporary deferred annuity-due**

we will use **the already used formula** and the proper **accumulation or discount factor**, depending on the moment in time for which we need the value of the annuity.

Summarizing:

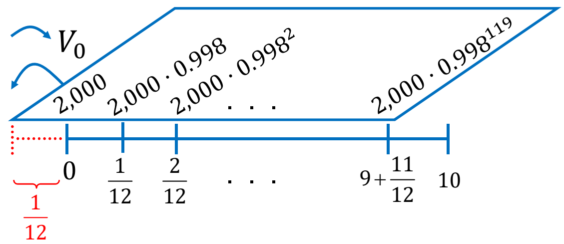
In order to properly calculate the financial value (present, accumulated or at any other moment) of any annuity with payments in geometric progression, the following steps should be followed:

1. Draw the annuity on the timeline.
2. Think of where you want the annuity to be financially valued.
3. Check if the capitalization frequency of the given interest rate is equal to the annuity frequency. If it is not, then obtain the equivalent interest rate.
4. Use the formula of the present value.
5. Reflect on where the financial value obtained with the above formula is placed.
6. If needed, correct this financial value with an accumulation factor or a discount factor, in order to arrive to the financial value at the desired moment.

Example 17. On 1st of January of year *A*, Sarah inherited an annuity-due with monthly payments to be received for 10 years. The first payment was €2,000 and the payments decreased by an accumulative 0.2% each month. Find the value of the bequest at the moment Sarah began to receive it, by using an annual effective interest rate of 3%.

We will draw the annuity on the timeline considering that the payments vary in a geometric progression with a common ratio, $q = 1 - 0.002 = 0.998$ and $C_1 = 2{,}000$.

So, since $n = 120$, $C_r = 2{,}000 \cdot 0.998^{r-1}, \quad \forall r = 1{,}2, \ldots, 120$.



Furthermore, from $I_1 = 0.03$, we can calculate the **<u>monthly effective rate</u>**.

$$I_{12} = (1 + 0.03)^{1/12} - 1 = 0.002466$$

We see that $q = 0.998 \neq 1 + I_{12} = 1.002466$. So, we can write:

$$V_0 = 2{,}000 \cdot \frac{1 - 0.998^{120} \cdot (1 + 0.002466)^{-120}}{1 + 0.002466 - 0.998} \cdot (1 + 0.002466)^1$$

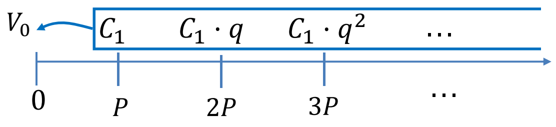
$$V_0 = €186{,}212.98$$

If we wanted to obtain the accumulated value of the inheritance at moment 10, we would have to calculate:

$$V_f = 186{,}212.98 \cdot (1 + 0.002466)^{120} = €169{,}614.21$$

3.3. Financial value of the Non-deferred immediate perpetuity with period P

As for all perpetuities, it makes no sense to calculate the accumulated value. In regards to the present value, graphically:



And, by using the effective interest rate I_M, we have:

$$V_0 = C_1 \cdot (1 + I_M)^{-1} + C_1 \cdot q \cdot (1 + I_M)^{-2} + \cdots$$

$$V_0 = \sum_{r=1}^{\infty} C_1 \cdot q^{r-1} \cdot (1 + I_M)^{-r}$$

Alternatively, it is also possible to write:

$$V_0 = \lim_{n \to \infty} \sum_{r=1}^{n} C_1 \cdot q^{r-1} \cdot (1 + I_M)^{-r}$$

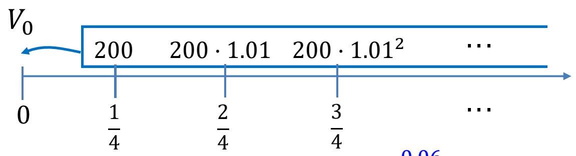
And, in this case:

$$\boxed{V_0 = C_1 \cdot \frac{1}{1 + I_M - q} \quad \text{if} \quad q < 1 + I_M}$$

The present value obtained with this formula is ALWAYS <u>one period before the first payment</u> of the annuity.

Example 18. Let us consider a non-deferred quarterly immediate perpetuity. Its payments increase each quarter by 1% cumulatively, and the first is €200. Find its present value under a nominal interest rate of 6% convertible quarterly.

The payments vary in geometric progression with a common ratio, $q = 1.01$ and $C_1 = 200$. So, $C_r = 200 \cdot 1.01^{r-1}$. Graphically:



$$M = 4 \quad i_4 = 0.06 \rightarrow I_4 = \frac{0.06}{4} = 0.015$$

Since $q = 1.01 < 1 + I_4 = 1.015$:

$$V_0 = 200 \cdot \frac{1}{1 + 0.015 - 1.01} = €40{,}000$$

4. Annuities with payments in arithmetic progression

4.1. Definition

4.2. Financial value of the Non-deferred immediate annuity with n payments and period P

4.3. Financial value of the Non-deferred immediate perpetuity with period P

4.1. Definition

In an annuity with payments varying in arithmetic progression, the following is true:

$$C_r = C_1 + (r-1) \cdot h, \qquad \forall r$$

This means that from a first payment, C_1, any other payment can be obtained by adding a constant, h, called the **common difference**, to the preceding one.

If $h > 0$, the payments of the annuity increase; if $h < 0$, they decrease.

Example 19. On her first birthday, Martha's parents deposit €900 in a bank account. They will keep making deposits on Martha's birthday until she turns 18, with every deposit being €100 greater than the previous one. Find the amount of the last deposit.

The first deposit is:

$$C_1 = 900$$

The second one is:

$$C_2 = 900 + 100$$

The third is:

$$C_3 = (900 + 100) + 100 = 900 + 2 \cdot 100$$

The fourth is:

$$C_4 = (900 + 2 \cdot 100) + 100 = 900 + 3 \cdot 100$$

So, $C_1 = 900$, $h = 100$ and, in general:

$$C_r = 900 + (r-1) \cdot 100, \qquad \forall r = 1,2,\dots,18$$

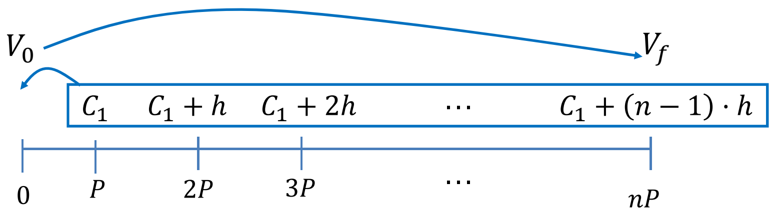
Consequently, the last deposit is:

$$C_{18} = 900 + 17 \cdot 100 = €2{,}600$$

4.2. Financial value of the Non-deferred immediate annuity with n payments and period P

Again, we begin by calculating the **present value**. From it, the final or accumulated value of the annuity will be obtained.

Graphically:



The present value of the annuity, V_0, with the effective interest rate I_M, is:

$$V_0 = C_1 \cdot (1+I_M)^{-1} + C_2 \cdot (1+I_M)^{-2} + \cdots + C_n \cdot (1+I_M)^{-n} =$$

$$= \sum_{r=1}^{n} C_r \cdot (1+I_M)^{-r} = \sum_{r=1}^{n} (C_1 + (r-1)\cdot h) \cdot (1+I_M)^{-r}$$

It can be demonstrated that this expression becomes:

$$\boxed{V_0 = \left(C_1 + n\cdot h + \frac{h}{I_M}\right)\cdot a_{\overline{n}|I_M} - \frac{n\cdot h}{I_M}}$$

And for the accumulated (or final) value of the annuity:

$$\boxed{V_f = V_0 \cdot (1+I_M)^{M\cdot t}}$$

Explanation of the **formula** of the present value of a non-deferred immediate annuity with n payments in arithmetic progression and period P (frequency M)

Common difference of the arithmetic progression

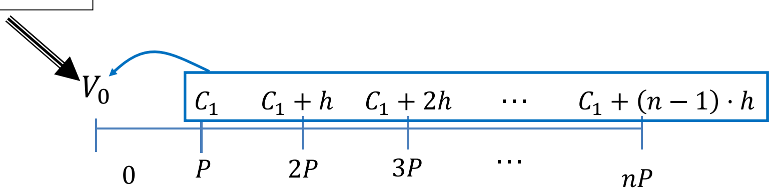
Effective interest rate to determine the present value, whose frequency is the same as that of the annuity, *M*

$$V_0 = \left(C_1 + n\cdot h + \frac{h}{I_M}\right)\cdot \frac{1-(1+I_M)^{-n}}{I_M} - \frac{n\cdot h}{I_M}$$

The present value obtained with this formula is ALWAYS <u>one period before the first payment</u> of the annuity

Amount of the first payment

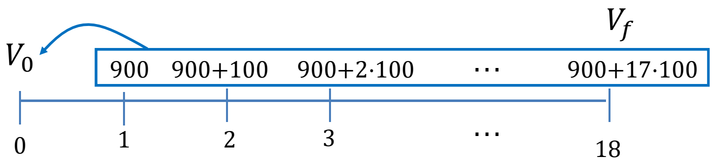
Number of payments (this must be a natural number, that is, <u>it cannot have decimals</u>)



Example 20. For the annuity contained in Example 19, and for an annual effective interest rate equal to 2.5%, find:

- Its financial value when Martha was born.

The graph of the annuity on the timeline is:

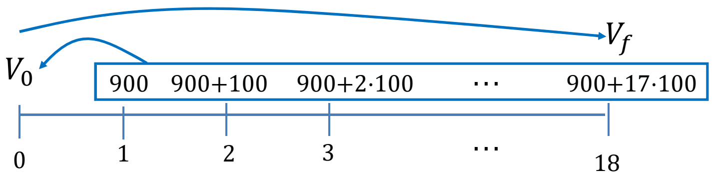


$M = 1$
$n = 18$

The given interest rate is $I_1 = 0.025$. Since it has the same frequency as that of the annuity, we can write:

$$V_0 = \left(900 + 18 \cdot 100 + \frac{100}{0.025}\right) \cdot a_{\overline{18}|0.025} - \frac{18 \cdot 100}{0.025} = €24{,}167.54$$

- Its accumulated value when Martha is 18 years old.



$$V_f = 24{,}167.54 \cdot (1 + 0.025)^{18} = €37{,}693.11$$

NOTE:

For **annuities with payments in arithmetic progression**, we have a formula from which it is easy to calculate the present value and the accumulated value of a **temporary non-deferred immediate** annuity.

As it was done for level annuities and for annuities with payments in geometric progression, to obtain the present value and the accumulated value when having payments in arithmetic progression of a:

- **temporary deferred immediate annuity**
- **temporary non-deferred annuity-due**
- **temporary deferred annuity-due**

we will use **the formula we just saw** and the proper **accumulation or discount factor**, depending on the moment in time for which we need the value of the annuity.

Unit 2. Annuities
4. Annuities with payments in arithmetic progression

Summarizing:

In order to properly calculate the financial value (present, accumulated or at any other moment) of any annuity with payments in arithmetic progression, the following steps should be followed:

1. Draw the annuity on the timeline.
2. Think of where you want the annuity to be financially valued.
3. Check if the capitalization frequency of the given interest rate is equal to the annuity frequency. If it is not, then obtain the equivalent interest rate.
4. Use the formula of the present value.
5. Reflect on where the financial value obtained with the above formula is placed.
6. If needed, correct this financial value with an accumulation factor or a discount factor, in order to arrive at the financial value at the desired moment.

Example 21. Given the following set of payments:

$$\left\{\left(300+(r-1)\cdot 25\,,3+\frac{r}{2}\right)\right\}_{r=1,\dots,11}$$

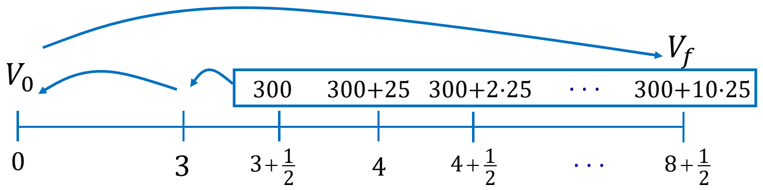
- Draw it on the timeline.

  The first payment is: $\left(300,\ 3+\frac{1}{2}\right)$

  The second payment is: $(300+25,\ \ 4)$

  The third one is: $\left(300+2\cdot 25,\ 4+\frac{1}{2}\right)$

  And so on. Therefore, the set constitutes a temporary annuity with payments in arithmetic progression made every 6-month period, and $C_1=300$, $h=25$, $M=2$, $n=11$.



- For a 3% nominal interest rate payable semi-annually, find its present value (at $T=0$) and the accumulated value at the moment of the last payment.

  From $i_2=0.03$, the **semi-annual effective interest rate** is:

$$I_2=\frac{0.03}{2}=0.015$$

  So:

$$V_0=\left(\left(300+11\cdot 25+\frac{25}{0.015}\right)\cdot a_{\overline{11}|0.015}-\frac{11\cdot 25}{0.015}\right)\cdot(1+0.015)^{-6}$$

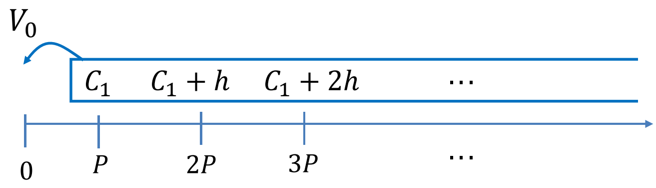
$$V_0=€3{,}880.18$$

  For the accumulated value, we simply use the proper accumulation factor. Thus:

$$V_f=3{,}880.18\cdot(1+0.015)^{17}=€4{,}997.75$$

4.3. Financial value of the Non-deferred immediate perpetuity with period P

Graphically:



By using the effective interest rate I_M, we have:

$$V_0 = \sum_{r=1}^{\infty} (C_1 + (r-1) \cdot h) \cdot (1 + I_M)^{-r}$$

Alternatively, it is also possible to write:

$$V_0 = \lim_{n \to \infty} \sum_{r=1}^{n} (C_1 + (r-1) \cdot h) \cdot (1 + I_M)^{-r}$$

And, in this case:

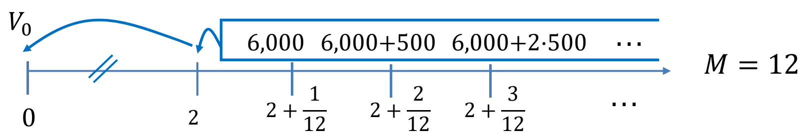
$$\boxed{V_0 = \frac{C_1}{I_M} + \frac{h}{I_M^2}}$$

The present value obtained with this formula is ALWAYS <u>one period before the first payment</u> of the annuity.

As for all perpetuities, it makes no sense to calculate the accumulated value.

Example 22. Let us consider a monthly immediate perpetuity. It is paid after a 2-year deferment and its payments increase by €500 a month with the first one being €6,000. Find its present value if an annual effective interest rate equal to 8% is applied.

The payments vary in arithmetic progression with a common difference, $h = 500$ and $C_1 = 6{,}000$. So, $C_r = 6{,}000 + (r-1)\cdot 500$. Graphically:



<u>Monthly effective interest rate</u>:

$$I_{12} = (1+0.08)^{1/12} - 1 = 0.006434$$

Therefore, the present value of the perpetuity will be:

$$V_0 = \left(\frac{6{,}000}{0.006434} + \frac{500}{0.006434^2}\right)\cdot(1+0.006434)^{-24}$$

$$V_0 = €11{,}154{,}652.14$$

UNIT 2 PROBLEMS

1. For the following annuities, obtain the present value and the accumulated value (if possible), using the compound interest regime:

a) An annual non-deferred immediate annuity, comprising level payments over the next 10 years that are equal to €250. Interest rate: 5% annual.
b) An annuity consisting of monthly immediate level payments of €500 to be made for 5 years after a 6-month deferment. Interest rate: monthly effective interest equal to 1%.
c) A non-deferred sixty-payment quarterly annuity-due. Payments of €200. Interest rate: 8% annual compounded quarterly.
d) A constant annuity-due, to be paid semi-annually over 5 years, after a deferment of 2 years, with payments of €350. Interest rate: 2% effective semi-annual rate for the first 4 years of the financial transaction and 4% annual for the rest of the term.
e) A two-year-deferred immediate perpetuity that pays €70 a year. Interest rate: 3% annual.

2. Pau receives a twenty-payment semi-annual annuity. The first eight payments are level, the amount being €1,000, and the last twelve payments are €1,100 each. The nominal interest rate compounded monthly is 1.5%.

a) If it is assumed that the annuity is immediate, find its present value and its accumulated value.
b) If the first payment is made in exactly 1 year, find its present value.
c) If the payments are made at the beginning of each half year, find its present value.

3. Stuart wishes to have €24,500, 4 years from now, to buy a car. He plans to accomplish this through a savings account with an annual effective rate of 3.40%, by depositing €3,500 today and €800 every two months for the rest of the 4 years. Will Stuart have at least the desired amount in 4 years?

4. Sarah wishes to purchase a flat. She has saved up €13,200 for a down payment. To pay for the flat she has been given a thirty-year mortgage with monthly level payments of €820 with a nominal interest rate of 5.85% convertible monthly. Her payments are due at the end of each month. How much does the flat cost?

5. An investor wishes to accumulate exactly €50,000 within 18 years from now. In order to reach his goal, he opens a bank account today, with an initial deposit of €3,000. He will then make constant deposits at the end of each month for the next 18 years. If the account is governed by an annual interest rate payable monthly, which equals 2%, find the constant monthly amount to be deposited for him to achieve his goal.

6. You have been awarded a prize and have the choice of receiving either a payment of €120,000 immediately or a deferred perpetuity of €1,000 annual payments, the first payment occurring in exactly 4 years. If the current interest rate is an annual 1%, which alternative would you choose? Write the equation that allows you to calculate the annual interest rate for which both alternatives are equivalent.

7. A worker aged 40 wishes to accumulate a retirement fund by depositing €1,000 in a pension scheme at the beginning of each year for the next 25 years.

a) If the pension scheme pays an annual interest of 4%, find the accumulated value at the moment the worker turns 65 years.

b) At the age of 65, the worker decides to give the accumulated amount to the pension scheme in exchange for constant repayments at the end of the following 15 years. Find the amount of the constant repayments assuming that the same interest rate is applied (as in the previous 25 years).

c) If the pension scheme had paid an annual interest rate of 4% for the first 10 years and then it had decreased to an annual 3.5%, what would have been the accumulated value at the moment the worker was 65 years old?

8. A loan with a principal of €150,000 will be repaid over the next 30 years with end-of-month level repayments. If its interest rate is compounded monthly and is equal to 4.80%, find the amount of the level repayments.

9. All of the following problems are related to annuities with payments in geometric progression:

a) A person purchased an immediate annuity with annual payments for 10 years. The first payment was €500 and the rest of the payments increased cumulatively by 8% a year. The purchase price was based on an annual interest rate of 3% compounded quarterly. Find that price.

b) The first of 40 payments of a monthly annuity occurs in exactly 1 year and is equal to €300. The payments increase so that each payment is 0.35% greater than the preceding payment.
   - b.1) Find the present value of this annuity with an annual effective rate of interest of 4.20%.
   - b.2) Find its present value if a nominal interest rate of 4.20% compounded monthly is considered.

c) A person is entitled to receive 12 semi-annual payments at the beginning of each 6-month period. The payments decrease by an accumulative semi-annual 2%, with the third one equal to €7,000. The nominal interest rate is 2.5% convertible monthly. Find the present value of all these payments and their accumulated value at the moment of the last one.

d) Quarterly perpetuity-due, deferred 1 year. The first payment is €1,000 and subsequent ones increase by 1% cumulatively every quarter. Find its present value with a quarterly effective interest rate equal to 1.20%.

10. For the purchase of a luxury good today, 10 annual payments have been agreed. The first payment will be €2,500 and then the payments will rise by 12% annually and cumulatively. If an annual interest rate equal to 6% governs the transaction, find the value of the good today if:

a) The first payment is made in exactly 1 year.
b) The first payment is due today.
c) The first payment matures in 4 years.

11. A person has been making deposits in a savings account at the beginning of each year for 30 years. The first deposit was €2,000, subsequently they increased cumulatively by 3% per year until the twenty-fourth payment, and after that they stayed level at €4,000. Find the accumulated amount in the account at the end of 30 years if the annual effective interest rate has been 3.2% for the entire term.

12. An investor wants to have €200,000 in 20 years. In order to obtain this amount, the investor will deposit the necessary amounts in a savings fund from now on, at the beginning of each month. The amounts will increase in such a way that each amount is 0.25% greater than the previous one. The fund will pay an annual effective interest rate equal to 2.5% for the first 3 years and a 3% nominal interest rate accumulated monthly for the rest of the term. Find the value of the first and the last amounts deposited by the investor in the fund.

13. A loan with a principal of €150,000 will be repaid over the next 30 years with end-of-month repayments. They will increase monthly by an accumulative 0.30%. If an interest rate compounded monthly and equal to 4.80% has been agreed, find the amount of the first and last repayments.

14. All of the following problems are related to annuities with payments in arithmetic progression:

a) The following 20 end-of-month deposits have been made in an account earning a semi-annual effective interest rate of 6%: the first one was €300 while the subsequent deposits increased by €50 each month. What is the balance in the account at the end of the twentieth month?

b) Consider an immediate annuity with annual payments for 20 years. The payments are level throughout the first 10 years, the amount being €1,000. Then, the eleventh payment is €100 greater than the preceding one, the twelfth payment is €100 greater than the eleventh, and so on. Find the present value of this annuity if the annual effective interest rate is 3%.

c) The following semi-annually payments are to be made: €2,750 today, €2,850 6 months from now, €2,950 in a year, and so on, until a final payment of €4,150. Using an annual effective interest rate equal to 10%, find the financial value of these payments today, and at the end of the 6-month period when the last payment will be made.

d) Fran purchases a perpetuity. The perpetuity pays €100 at the end of each quarter for the first 11 years and then the payments increase by €5 each quarter. Find the purchase price if it is determined using a nominal interest rate quarterly compounded of 5%.

15. A loan with a principal of €150,000 will be repaid over the next 30 years with end-of-month repayments, which will increase by €2 monthly. If an interest rate compounded monthly and equal to 4.80% has been agreed, find the amount of the first and the last repayments.

Formulas

$C_r = C$	$C \cdot a_{\overline{n}\rvert I_M} = C \cdot \dfrac{1-(1+I_M)^{-n}}{I_M}$	$C \cdot \dfrac{1}{I_M}$
$C_r = C_1 \cdot q^{r-1}$	$\begin{cases} C_1 \cdot \dfrac{1-q^n \cdot (1+I_M)^{-n}}{1+I_M-q} & 1+I_M \neq q \\ C_1 \cdot n \cdot q^{-1} & 1+I_M = q \end{cases}$	$\dfrac{C_1}{1+I_M-q} \quad q < 1+I_M$
$C_r = C_1 + (r-1) \cdot h$	$\left(C_1 + n \cdot h + \dfrac{h}{I_M}\right) \cdot a_{\overline{n}\rvert I_M} - \dfrac{n \cdot h}{I_M}$	$\dfrac{C_1}{I_M} + \dfrac{h}{I_M^2}$

Answers

1. a) $V_0 = €1{,}930.43 \quad V_f = €3{,}144.47$.
b) $V_0 = €21{,}174.84 \quad V_f = €40{,}834.84$.
c) $V_0 = €7{,}091.22 \quad V_f = €23{,}266.51$.
d) $V_0 = €2{,}963.37 \quad V_f = €3{,}905.60$.
e) $V_0 = €2{,}199.39$.

2. a) $V_0 = €19{,}580.38 \quad V_f = €22{,}747.02$.
b) $V_0 = €19{,}434.16$.
c) $V_0 = €19{,}727.69$.

3. The savings account balance will be €24,486.74. So, Stuart will not have the desired amount.

4. €152,196.97.

5. €175.95.

6. For the given interest of 1% annual, the best alternative is to choose the single payment of €120,000 because the present value of the perpetuity is €97,059.01.
The equation is $120{,}000 = 1{,}000 \cdot \frac{1}{I_1} \cdot (1 + I_1)^{-3}$.

7. a) €43,311.74.
b) €3,895.51.
c) €40,890.02.

8. €787.

9. a) €6,052.68.
b) b.1) $V_0 = €11{,}531.01$.
b.2) $V_0 = €11{,}507.28$.
c) $V_0 = €73{,}536.94 \quad V_f = €84{,}364.34$.
d) $V_0 = €482{,}423.51$.

10. a) €30,595.39.
b) €32,431.12.
c) €25,688.48.

11. €147,612.66.

12. $C_1 = €458.24 \quad C_{240} = €832.26$.

13. $C_1 = €497.60 \quad C_{360} = €1{,}458.53$.

14. a) €16,670.28.
b) $V_0 = €18{,}213.91$.
c) $V_0 = €36{,}893.89 \quad V_f = €75{,}404.90$.
d) $V_0 = €26{,}757.01$.

15. $C_1 = €511.40 \quad C_{360} = €1{,}229.40$.

UNIT 3.

LOANS

1. Definition and classification

1.1. Definition

1.2. Classification

1.1. Definition

A loan is a financial transaction by which a person, called the **lender** or active party, gives another person, called the **borrower** or passive party, a monetary amount, C, called the **principal of the loan**. In exchange for the loan, the borrower undertakes to repay the borrowed amount in a specific term, either through a single repayment or through a set of repayments, plus the payment of interest.

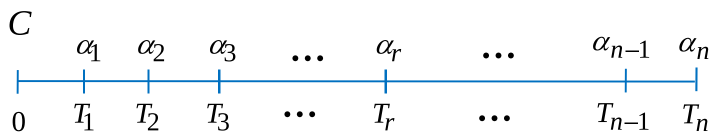
Therefore, at the time of arranging the loan, there exists a financial equivalence between the financial capital that the lender lends (**payment**) and the financial capital that the borrower repays (**repayments**), as a result of the deals that have been agreed by both parties. This financial equivalence is ruled by the compound interest financial regime with the effective interest rate agreed on the loan, I_m.

Thus, a loan can be represented by:

Payment: $(C, 0)$

Repayments: $\{(\alpha_r, T_r)\}_{r=1,2,\dots,n}$

Graphically:



If the effective interest rate of the loan is I_m, then the financial equivalence between the payment and repayments can be expressed as:

$$(C, 0) \sim \{(\alpha_r, T_r)\}_{r=1,2,\dots,n}$$
$$\downarrow$$
$$I_m$$

1.2. Classification

Loans can be classified by considering different aspects.

Regarding how the principal of the loan is repaid:

1) **Loan with a single repayment of the principal**.
   The principal C is paid off by a single repayment at the end of the financial transaction**.**
2) **Loan with periodic repayments (amortizations) of the principal**.
   The principal C is paid off by periodic repayments made over the agreed term.

Regarding how the interest on the loan is paid:

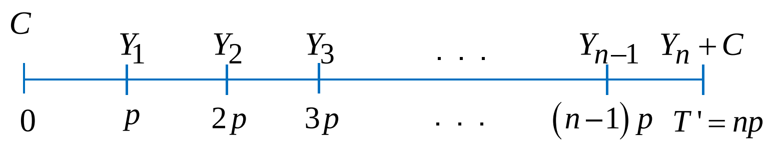
1) **Loan with a single payment of interest**.
   There is only a single payment of interest, usually paid at the end of the financial transaction.
2) **Loan with periodic payments of interest**.
   Payments of interest are made periodically during the agreed term.

More particularly, we will study the following three types of loans:

1) Loan with single repayment of the principal C and single payment of interest Y. In short: **loan repaid by a single repayment**.
Graphically:

being the financial equivalence: $(C,0)\sim(C',T')$ with $C'=C+Y$

2) Loan with a single repayment of the principal C and periodic payments of interest Y_r. For short: **interest-only loan**.
Graphically:



with: $(C,0)\sim\{(Y_r,rp),(C,np)\}_{r=1,2,\dots,n}$

3) Loan with periodic repayments of the principal A_r and periodic payments of interest Y_r. In short: amortizing loan. Specifically, we will see loans repaid with level repayments that are comprised of payment of interest and repayment of principal: **amortizing loan with level repayments.**

with: $(C,0)\sim\{(\alpha,rp)\}_{r=1,2,\dots,n}$ and $\alpha=A_r+Y_r$

In this case, the sum of all the periodic repayments of the principal is equal to the principal of the loan, that is:

$$C=\sum_{r=1}^{n}A_r$$

2.1. Outstanding loan balance

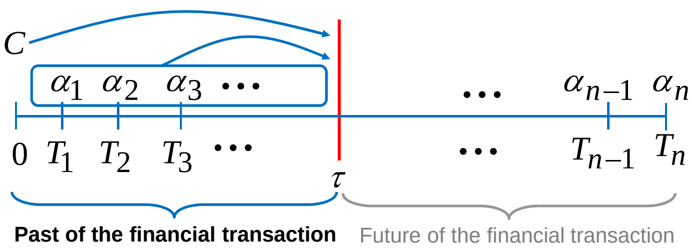
The outstanding balance at any moment τ of the loan is the total amount that the borrower must repay to the lender in order to totally cancel the loan at that moment, by considering only the payment and repayments involved in the financial equivalence (disregarding any fees or costs):

$$\underset{I_m}{\underset{\downarrow}{(C,0)\sim\{(\alpha_r,T_r)\}_{r=1,2,\ldots,n}}}$$

We consider two approaches to computing the outstanding loan balance at moment τ, OLB_τ: the **prospective method** and the **retrospective method**. It will later be shown that the two methods are equivalent.

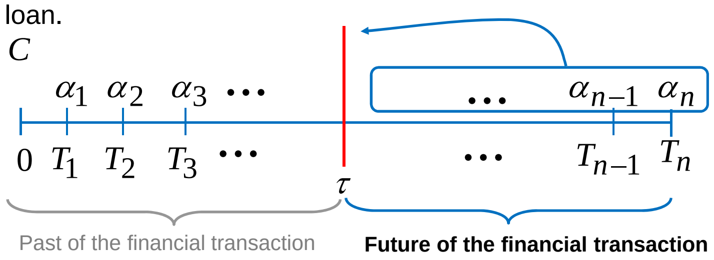
$$\underset{I_m}{\underset{\downarrow}{(C,0)\sim\{(\alpha_r,T_r)\}_{r=1,2,\ldots,n}}}$$

Retrospective method: the retrospective outstanding loan balance at τ is the financial value of the principal of the loan at τ, minus the financial value at τ of all the repayments made up to and including this moment, at the same interest rate, I_m, agreed on the loan.



$$OLB_\tau^{ret} = C\cdot(1+I_m)^{m\cdot\tau} - \sum_{\forall T_r\le\tau} \alpha_r\cdot(1+I_m)^{m\cdot(\tau-T_r)}$$

Prospective method: The prospective outstanding loan balance at τ is the financial value at τ of all the remaining repayments from time τ (not included) until the end of the loan, at the same interest rate, I_m, agreed on the loan.



$$OLB_\tau^{pro} = \sum_{\forall T_r>\tau} \alpha_r \cdot (1+I_m)^{-m\cdot(T_r-\tau)}$$

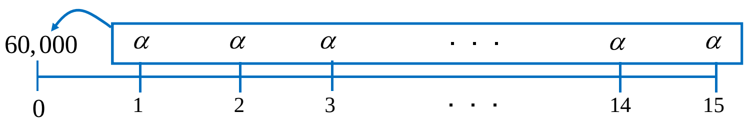
PROPERTY

The retrospective and prospective outstanding loan balances at any moment τ are equal.

$$OLB_\tau^{ret} = OLB_\tau^{pro}$$

Example 1. A loan with a principal of €60,000, paid today, will be repaid through annual level repayments at the end of the following 15 years. It is ruled by an annual effective interest rate equal to 6%.

- Find the amount of the level repayments.



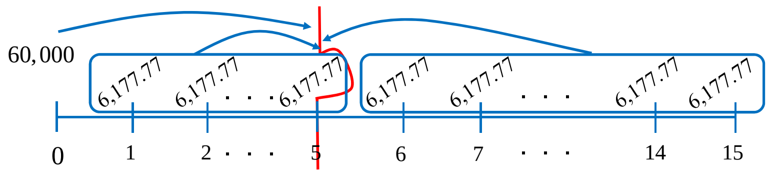
The annual effective rate is: $I_1 = 0.06$

And the amount of the level repayments can be obtained from:

$$60{,}000 = \alpha \cdot a_{\overline{15}|0.06} \Rightarrow \alpha = €6{,}177.77$$

- Find the retrospective and the prospective outstanding loan balance **<u>5 years after</u>** being given.

  By considering the definition of the retrospective outstanding loan balance, we have:



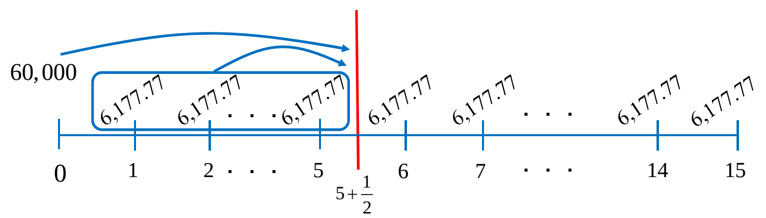
$$OLB_5^{ret} = 60{,}000 \cdot (1+0.06)^5 - 6{,}177.77 \cdot a_{\overline{5}|0.06} \cdot (1+0.06)^5 = €45{,}468.89$$

And taking into account its definition, the retrospective outstanding loan balance after 5 years of being paid is:

$$OLB_5^{pro} = 6{,}177.77 \cdot a_{\overline{10}|0.06} = €45{,}468.89$$

- Find the retrospective and the prospective outstanding loan balance **<u>5 and a half years after</u>** being paid out.

  Following similar reasoning to that explained above, the retrospective outstanding loan balance at 5.5 years is:

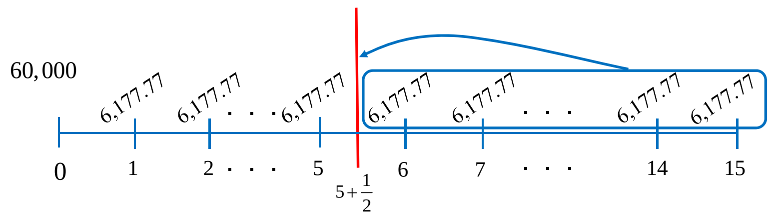


$$OLB_{5.5}^{ret} = 60{,}000 \cdot (1+0.06)^{5.5} - 6.177{,}77 \cdot a_{\overline{5}|0.06} \cdot (1+0.06)^{5.5} = €46{,}813.09$$

It is worth pointing out that $OLB_{5.5}$ can also be calculated from the outstanding loan balance at year 5:

$$OLB_{5.5}^{ret} = OLB_5^{ret} \cdot (1+0.06)^{0.5} = 45{,}468.9 \cdot (1+0.06)^{0.5} = €46.813.09$$

Finally, the prospective outstanding loan balance at 5.5 years is:



$$OLB_{5.5}^{pro} = 6{,}177.77 \cdot a_{\overline{10}|0.06} \cdot (1+0.06)^{0.5} = €46{,}813.09$$

2.2. Other concepts

For all the loans that will be studied in this unit, not only will we analyze financial equivalence and the outstanding loan balance, but also the following concepts:

1) Repayment(s).
2) Outstanding principal.
3) Value of the loan.
4) Lender effective interest rate.
5) Borrower effective interest rate.
6) TAE of the loan.

Repayment(s)

Amount(s) repaid by the borrower included in the financial equivalence. They may be comprised of both the payment of interest and the repayment (amortization) of the principal, depending on the kind of loan considered.

Outstanding principal (or non-repaid principal of the loan), OP_τ

The outstanding principal at moment τ is that part of the principal of the loan that has not been repaid yet at moment τ. The principal of the loan minus the outstanding principal at τ is equal to the repaid principal of the loan until moment τ.

Value of the loan, V_τ

This is the financial value at τ of the outstanding repayments of the loan, using the current market interest rate.

PROPERTY

If the current market interest rate is equal to the interest rate agreed on the loan, I_m, the value of the loan and the outstanding balance of the loan are both equal. Otherwise, they are not.

In addition to interest, loans may have some associated expenses and fees, such as, among others:

- Application or booking fee.
- Arrangement fee (also known as Initial, Underwriting, Administrative, Processing Product, Completion, etc. fee).
- Management expenses.
- Loan broker fee.
- Related taxes.
- Maintenance expenses of associated products (credit cards, bank accounts, ...).
- Life insurance premiums.
- Early repayment fee.
- Land Registry and Public Notary expenses (for mortgages)
- Appraisal expenses, Home insurance premiums and Mortgage default insurance premiums (for mortgages).

Depending on the expenses taken into account, there are 3 different effective interest rates.

Lender effective interest rate

Effective interest rate for the lender, taking into consideration not only the principal of the loan and all the repayments agreed in the financial equivalence but also the fees paid by the borrower in benefit of the lender and all the expenses the lender has over the lifespan of the loan.

Borrower effective interest rate

Effective interest rate for the borrower, taking into account not only the principal of the loan and all the repayments agreed in the financial equivalence, but also all the fees and expenses that are paid both in order to obtain the loan and during the term of the loan.

Given that, over the term of the loan, unexpected repayments, fees or expenses, which are not known at the moment of awarding the loan, may appear, these two effective interest rates can change throughout the life of the loan. Furthermore, for a specific loan, the borrower effective interest rate may not be the same for every borrower.

***TAE* (*Tasa anual equivalente*) of the loan**

As we saw in Unit 1, Spanish financial institutions must inform about the TAE of any financial transaction. The TAE is the annual effective interest rate associated with a financial transaction once the costs and fees indicated by Spanish laws in force have been considered. These costs and fees are known (or can be estimated) at the beginning of the financial transaction and banks have to inform their clients on the *TAE* when this transaction is offered.

We also said that most developed countries have laws in force that contain similar concepts. For example, in the case of credit agreements for consumers, the European Directive 2008/48/EC introduces the concept of annual percentage rate (APR) of charge.

Since we focus on the Spanish financial market, we define the TAE of a loan as the annual effective interest rate associated with this loan when considering the costs and fees known at the moment of agreeing on the loan. But only costs and fees indicated by Spanish laws in force have to be taken into account.

The *TAE* of a loan is calculated when the loan is awarded and it does not change over the term of the loan.

The costs and fees that have to be considered in order to calculate the TAE of a financial transaction have changed over time. At present, Spanish laws in force describe all expenses and fees that must be included in the calculation of the TAE:

> *En el cálculo de la TAE se incluirán los intereses, comisiones y demás gastos que el cliente esté obligado a pagar a la entidad como contraprestación por el crédito o préstamo recibido o los servicios inherentes al mismo. También se incluirán las primas de los seguros que tengan por objeto garantizar a la entidad el reembolso del crédito en caso de fallecimiento, invalidez o desempleo de la persona física que haya recibido el crédito, siempre y cuando la entidad imponga la contratación de dicho seguro como condición para conceder el préstamo o crédito.*

Although other costs and fees may exist, in order to obtain the *TAE* in our examples and problems, we will usually just consider the following ones, that are paid by the borrower in the moment of receiving the loan:

- Application and arrangement fees.
- Other mandatory expenses.

3. Loan repaid by a single repayment

1) The principal of the loan is repaid through a single repayment at the end of the financial transaction.
2) There is only one single payment of interest, which is paid at the end of the financial transaction according to the agreed effective interest rate of the loan, I_m.

Financial equivalence

$$(C,0) \sim (C',T')$$
$$\downarrow$$
$$I_m$$

Repayment

$$C' = C \cdot (1+I_m)^{m \cdot t}$$

with $t = T' - 0$ being the term of the loan.

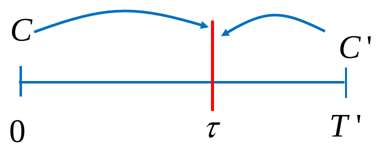
This amount comprises both the **payment of interest**, Y, and the **repayment of the principal**, A.

$$Y = C' - C = C \cdot (1+I_m)^{m \cdot t} - C \qquad A = C$$

Now we obtain the outstanding loan balance and the outstanding principal at 3 different moments in time:

- At the beginning of the loan, 0.
- At any moment between its beginning and its end, $\tau \in (0, T')$.
- At the end of the loan, T' .

Outstanding loan balance

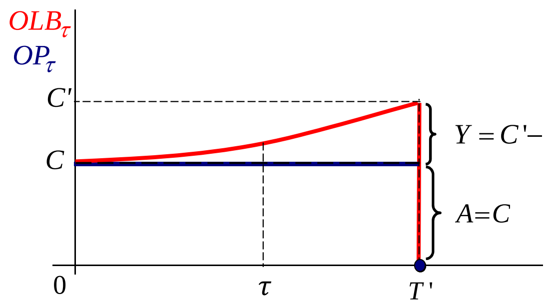


- At 0: $OLB_0 = C$
- At τ: $OLB_\tau = \begin{cases} \text{retrosp.} & C \cdot (1+I_m)^{m \cdot \tau} \\ \text{prosp.} & C' \cdot (1+I_m)^{-m \cdot (T'-\tau)} \end{cases}$
- At T': $OLB_{T'} = 0$

Outstanding principal

- At 0: $OP_0 = C$
- At τ: $OP_\tau = C$
- At T': $OP_{T'} = 0$

Outstanding loan balance vs. outstanding principal over the term of the loan



The outstanding loan balance and the outstanding principal are equal at 0 and at T'

The two are different at $\tau \in (0, T')$. This is because the outstanding loan balance takes into consideration the accrued interest until moment τ and the outstanding principal does not.

Summarizing:

Moment	Outstanding loan balance		Outstanding principal
0	C		C
τ	**Retrospective**	$C \cdot (1+I_m)^{m\cdot\tau}$	C
	Prospective	$C' \cdot (1+I_m)^{-m\cdot(T'-\tau)}$	
T'	0		0

Value of the loan, V_τ

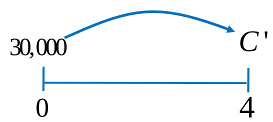
If the current market interest rate is denoted by I_m^{Market}, the financial value at τ of the outstanding repayments of the loan is:

$$V_\tau = C' \cdot \left(1 + I_m^{Market}\right)^{-m\cdot(T'-\tau)}$$

The **lender effective interest rate**, the **borrower effective interest rate** and the ***TAE*** of the loan can be obtained as described in subsection 2.2.

Example 2. A loan with a principal of €30,000 will be repaid through a single repayment, which includes interest and the principal, in 4 years. If its annual effective rate is 7.5%, it has a €300 initial fee and the borrower pays €600 of mandatory expenses at the beginning of the loan.

- Find the **amount of the single repayment** and split it into interest and principal.



$I_1 = 0.075$

$$C' = 30,000 \cdot (1+0.075)^4 = €40,064.07$$

So:

$$Y = C' - C = 40,064.07 - 30,000 = €10,064.07$$

$$A = €30,000$$

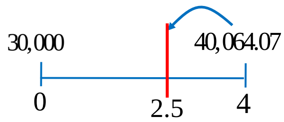
- Obtain the **outstanding loan balance** 2 and a half years after being lent.

$$OLB_{2.5} = \begin{cases} \xrightarrow{\text{retrosp.}} = 30,000 \cdot (1+0.075)^{2.5} = €35,945.33 \\ \xrightarrow{\text{prosp.}} = 40,064.07 \cdot (1+0.075)^{-1.5} = €35,945.33 \end{cases}$$

- Find the **outstanding principal** after 2 and a half years of being lent.

$$OP_{2.5} = €30,000$$

- If today, 2 and a half years after the principal was borrowed, the current market interest rate is an annual 6%, what is the **financial value** of the loan?



$$V_{2.5} = 40,064.07 \cdot (1+0.06)^{-1.5} = €36,711$$

Unit 3. Loans
3. Loan repaid by a single repayment

- Determine the **borrower effective interest rate** and the ***TAE*** of the loan.

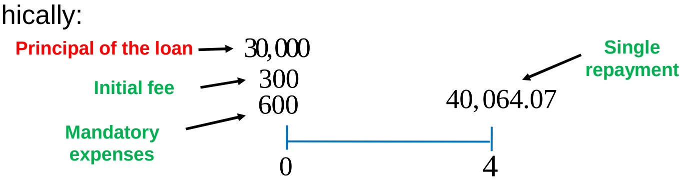
  To determine the borrower effective interest rate, we not only consider the single repayment but also all other fees and expenses.

$$\underbrace{(30,000\,,0)}_{\text{Principal of the loan}} \sim \{\underbrace{(300,0)}_{\text{Initial fee}}, \underbrace{(600,0)}_{\text{Mandatory expenses}}, \underbrace{(40,064.07,4)}_{\text{Single repayment}}\}$$

$$\downarrow$$

Borrower effective interest rate, I_1

Graphically:



If we calculate the value at moment 0, we will have:

$$30,000 = 900 + 40,064.07 \cdot (1+I_1)^{-4}$$

$$I_1 = 0.083217 \equiv 8.3217\%$$

As for the TAE, since Spanish current laws say that both the initial fee and the mandatory expenses have to be considered, we have the same equation:

$$30,000 = 900 + 40,064.07 \cdot (1 + I_1)^{-4}$$

So:

$$TAE = 0.083217 \equiv 8.3217\%$$

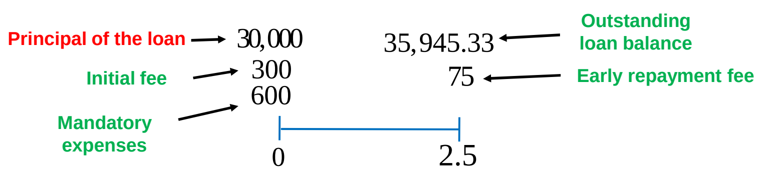
- 2 and a half years after the loan is granted, the borrower decides to completely cancel it. Determine the **borrower effective interest rate** if at that time he pays an early repayment fee of 0.25% on the outstanding principal.

In order to totally cancel the loan the borrower pays its outstanding loan balance, $OLB_{2.5} = €35,945.33$, and the early repayment fee:

$$30,000 \cdot 0.0025 = €75$$

To obtain the borrower effective interest rate, we consider all the payments and repayments involved in the financial transaction.

Graphically:



If we value at moment 0, we have:

$$30,000 = 900 + (35,945.33 + 75) \cdot (1 + I_1)^{-2.5}$$

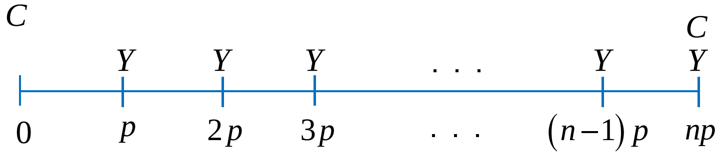
$$I_1 = 0.089085 \equiv 8.9085\%$$

4. Interest-only loan

1) The principal of the loan is repaid through a single repayment at the end of the financial transaction.

2) The term of the loan is split into n periods of length p, $p = \frac{1}{m}$, and payments of interest are made at the end of each period according to the agreed effective interest rate, I_m.

Financial equivalence



$$(C,0) \underset{I_m}{\sim} \{(Y,rp),(C,np)\}_{r=1,2,\ldots,n}$$

$$C = Y \cdot a_{\overline{n|}I_m} + C \cdot (1+I_m)^{-n}$$

The periodic **payments of interest** are level (constant):

$$Y = C \cdot I_m$$

The single **repayment of the principal**, A, that is paid at the end of the loan, is:

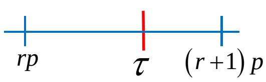
$$A = C$$

Now we obtain the outstanding loan balance and the outstanding principal at 4 different moments in time:

- At the beginning of the loan, 0.
- At the end of each period of the loan: rp.

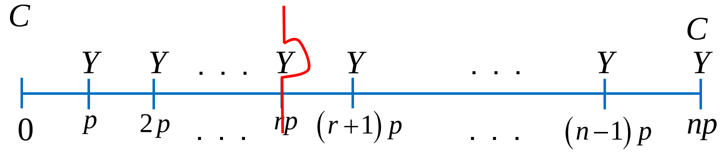
  NOTE: We will always consider, unless otherwise stated, that the payment of interest Y has just been made.

- At a specific moment within a period of the loan: $\tau \in (rp, (r+1)p)$.
- At the end of the loan, np .



Outstanding loan balance

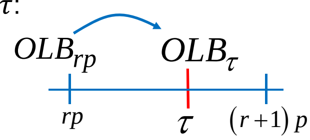
- At 0: $OLB_0 = C$



- At rp: $OLB_{rp} =$
  - retrosp.: $C\cdot(1+I_m)^r - Y\cdot a_{\overline{r|}I_m}\cdot(1+I_m)^r = C$
  - prosp.: $Y\cdot a_{\overline{n-r|}I_m} + C\cdot(1+I_m)^{-(n-r)} = C$

Unit 3. Loans
4. Interest-only loan

- At τ:



$$OLB_\tau = OLB_{rp}\cdot(1+I_m)^{m\cdot(\tau-rp)}$$

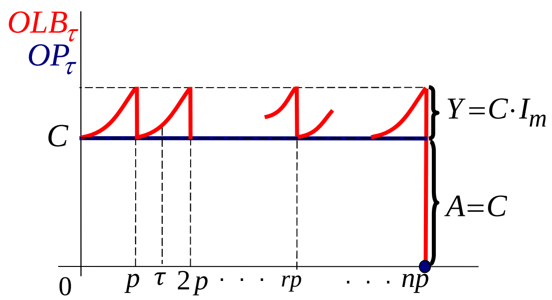
- At np: $OLB_{np} = 0$

Outstanding principal

- At 0: $OP_0 = C$
- At rp: $OP_{rp} = C$
- At τ: $OP_\tau = C$
- At np: $OP_{np} = 0$

The outstanding loan balance and the outstanding principal are equal at 0, at np and at the end of each period once the payment of interest has been made.

Outstanding loan balance vs. outstanding principal over the term of the loan



The outstanding loan balance and the outstanding principal are different at a specific moment τ within a period of the loan. This is because the outstanding loan balance takes into consideration the accrued interest until moment τ and the outstanding principal does not.

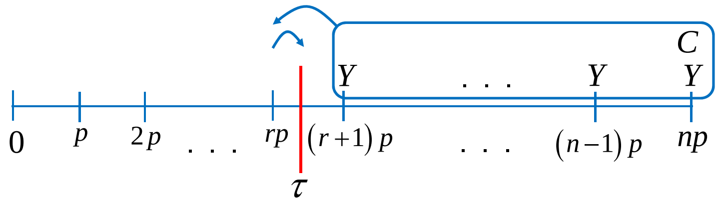
Unit 3. Loans
4. Interest-only loan

Summarizing:

<table>
<tr><th>Moment</th><th colspan="2">Outstanding loan balance</th><th>Outstanding principal</th></tr>
<tr><td>0</td><td colspan="2">C</td><td>C</td></tr>
<tr><td rowspan="2">rp</td><td>Retrospective</td><td>C</td><td rowspan="2">C</td></tr>
<tr><td>Prospective</td><td>C</td></tr>
<tr><td>τ</td><td colspan="2">$OLB_{rp} \to OLB_\tau$
rp τ $(r+1)\,p$
$OLB_\tau = OLB_{rp} \cdot (1+I_m)^{m\cdot(\tau - rp)}$</td><td>$C$</td></tr>
<tr><td>np</td><td colspan="2">0</td><td>0</td></tr>
</table>

<u>Value of the loan</u>, V_τ

If the current market interest rate is denoted by I_m^{Market}, the financial value at τ of the outstanding repayments of the loan is:



$$V_\tau = \left[Y \cdot a_{\overline{n-r}|I_m^{Market}} + C \cdot \left(1 + I_m^{Market}\right)^{-(n-r)} \right] \cdot \left(1 + I_m^{Market}\right)^{m \cdot (\tau - rp)}$$

The **<u>lender effective interest rate</u>**, the **<u>borrower effective interest rate</u>** and the ***<u>TAE</u>*** of the loan can be obtained as described in subsection 2.2.

Example 3. A loan with a principal of €5,000 will be repaid with quarterly payments of interest and a lump sum payment of the principal in 2 years. The nominal rate of interest payable quarterly is 12%, there is an initial fee of 1.5% of the principal and the borrower pays mandatory expenses of €135 when the loan is awarded. Find:

- The amount of the quarterly **payments of interest**.

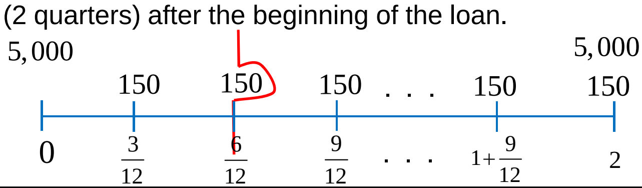
  The quarterly effective interest rate of the loan is:

  $$i_4 = 0.12 \Rightarrow I_4 = \frac{0.12}{4} = 0.03$$

  So, the amount of the quarterly payments of interest is:

  $$Y = C \cdot I_4 = 5{,}000 \cdot 0.03 = €150$$

- The **outstanding loan balance** and the **outstanding principal** 6 months (2 quarters) after the beginning of the loan.

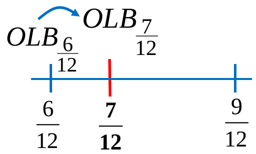


Immediately 6 months after the beginning of the loan, a payment of interest of €150 is made. Therefore, at that moment, the outstanding loan balance and the outstanding principal are equal. That is to say:

$$OLB_{\frac{6}{12}} = OP_2 = €5{,}000$$

- The **outstanding loan balance** and the **outstanding principal** 7 months after the beginning of the loan.

To obtain the outstanding loan balance after 7 months, we can consider the outstanding loan balance after 6 months and the accumulation factor for 1 month, i.e.:

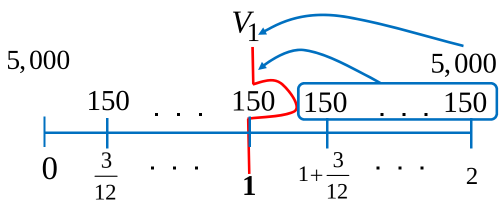


$$OLB_{\frac{7}{12}} = 5{,}000 \cdot (1+0.03)^{4\cdot\frac{1}{12}} = €5{,}049.51$$

And the outstanding principal after 7 months is equal to the principal of the loan: $€5{,}000$

Unit 3. Loans
4. Interest-only loan

- The **value** of the loan 1 year after being awarded, if the market interest rate at that moment is 10% annual effective.



The quarterly effective interest rate equivalent to the market interest rate of a 10% annual effective rate is:

$$I_1 = 0.10 \Rightarrow I_4 = (1+0.10)^{1/4} - 1 = 0.024114$$

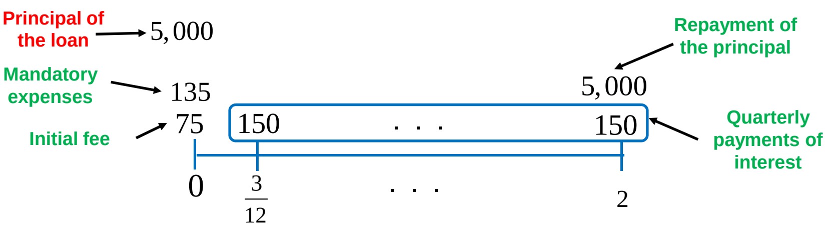
So:

$$V_1 = 150 \cdot a_{\overline{4}|0.024114} + 5{,}000 \cdot (1+0.024114)^{-4} = €5{,}110.96$$

- The equation that allows us to obtain the **borrower effective interest rate**.

  To determine the borrower effective interest rate, we consider all the quarterly payments of interest, $Y = €150$, the lump sum repayment of the principal, $A = C = €5{,}000$, initial fee, $5{,}000 \cdot 0.015 = €75$, and mandatory expenses, $€135$.

  Drawing all this financial capital on the timeline:



  Since the payments of interest are made quarterly, a quarterly effective interest rate has to be used. If we value at 0:

$$5{,}000 = 135 + 75 + 150 \cdot a_{\overline{8}|I_4} + 5{,}000 \cdot (1 + I_4)^{-8}$$

Using a <u>suitable software</u>, I_4 can be obtained.

$$I_4 = 0.03614$$

- The equation that allows us to calculate the ***TAE*** of the loan.

  In this case, since Spanish current laws say that both the initial fee and compulsory expenses have to be included in the calculation of the TAE, we have the same equation and interest rate as in the previous question.

  The *TAE* will be the effective interest rate equivalent to I_4, i.e.:

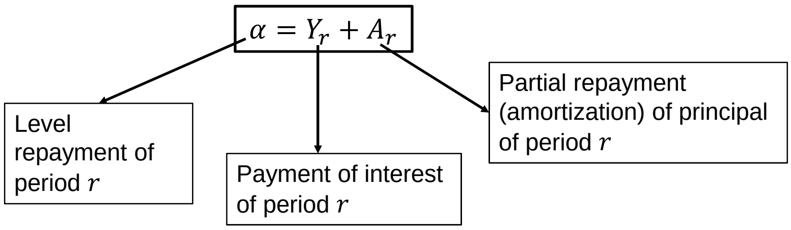
$$TAE = I_1 = (1 + 0.03614)^4 - 1 = 0.15258 \equiv 15.258\%$$

5. Amortizing loan with level repayments

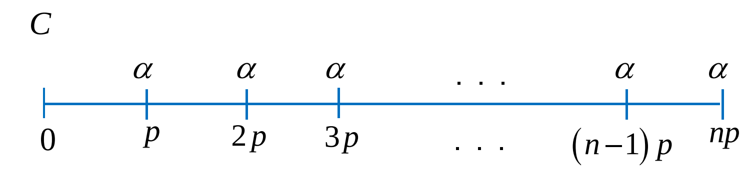
1) The term of the loan is split into n periods of length p, $p = \frac{1}{m}$.

2) The loan is repaid through level repayments, α, at the end of each period according to the agreed effective interest rate, I_m.

3) These level repayments include the payment of interest corresponding to each period and a partial repayment (amortization) of the principal of the loan.



Financial equivalence



$$(C,0) \underset{I_m}{\sim} \{(\alpha, rp)\}_{r=1,2,\dots,n} \qquad C = \alpha \cdot a_{\overline{n}|I_m} \Rightarrow \alpha = \frac{C}{a_{\overline{n}|I_m}}$$

As we previously saw: $\alpha = Y_r + A_r$

- The **payment of interest** made at the end of the period r is:

$$Y_r = OP_{r-1} \cdot I_m$$

- The partial **repayment (amortization) of principal** made at the end of period r is:

$$A_r = \alpha - Y_r$$

In addition to these relationships, there are also other obvious ones. We illustrate them with an example.

Example 4. Let us consider an amortizing loan with 5 level repayments.

C

$\alpha = A_1 + Y_1$ $\quad \alpha = A_2 + Y_2$ $\quad \alpha = A_3 + Y_3$ $\quad \alpha = A_4 + Y_4$ $\quad \alpha = A_5 + Y_5$

0 1 2 3 4 5

$$\boxed{C = \sum_{r=1}^{n} A_r} = A_1 + A_2 + A_3 + A_4 + A_5$$

Just after the moment of making the 3rd repayment (at the end of the 3rd period), the outstanding principal is equal to the sum of all the pending repayments of principal.

Alternatively, it is equal to the principal of the loan minus all the repayments of principal made since that moment.

$$\boxed{OP_r = \sum_{s=r+1}^{n} A_s} \qquad OP_3 = A_4 + A_5$$

$$\boxed{OP_r = C - \sum_{s=1}^{r} A_s} \qquad OP_3 = C - \underbrace{\left(A_1 + A_2 + A_3\right)}_{\text{Part of the principal repaid at the end of the 3rd period}}$$

$$\boxed{OP_{r+1} = OP_r - A_{r+1}} \qquad OP_4 = OP_3 - A_4$$

All the concepts related to an amortizing loan with level repayments can be grouped in its **amortization table** or **amortization schedule**.

r	α_r	Y_r	A_r	OP_r

r	Level repayment α	Payment of interest Y_r	Repayment of principal A_r	Outstanding principal OP_r
0				$OP_0 = C$
1	α	$Y_1 = C \cdot I_m$	$A_1 = \alpha - Y_1$	$OP_1 = OP_0 - A_1$
2	α	$Y_2 = OP_1 \cdot I_m$	$A_2 = \alpha - Y_2$	$OP_2 = OP_1 - A_2$
3	α	$Y_3 = OP_2 \cdot I_m$	$A_3 = \alpha - Y_3$	$OP_3 = OP_2 - A_3$
…	…	…	…	…
n	α	$Y_n = OP_{n-1} \cdot I_m$	$A_n = \alpha - Y_n$	$OP_n = 0$

Example 5. Construct the amortization table for a loan with a principal of €1,000 to be repaid with constant semiannual installments over 3 years, if the loan is governed by 8% annual interest compounded semiannually.

First, we calculate the amount of the level repayments (installments):

$$i_2 = 0.08 \Rightarrow I_2 = \frac{0.08}{2} = 0.04 \qquad 1{,}000 = \alpha \cdot \frac{1-(1+0.04)^{-6}}{0.04} \Rightarrow \alpha = €190.76$$

So, the amortization table of the loan is:

r	α	Y_r	A_r	OP_r
0				1,000.00
1	190.76	40.00	150.76	849.24
2	190.76	33.97	156.79	692.45
3	190.76	27.69	163.07	529.38
4	190.76	21.17	169.59	359.79
5	190.76	14.39	176.37	183.42
6	190.76	7.34	183.42	0

In the previous amortization table, it can be seen that even though the amount of the repayment α is constant, as time goes by the payment of interest, Y_r, decreases and the repayment (amortization) of principal, A_r, increases. This is because the payment of interest made at the end of a period is calculated by applying the effective interest rate to the outstanding principal at the beginning of the period, and this outstanding principal gradually decreases from period to period.

PROPERTY

In amortizing loans with level repayments, **repayments (amortizations) of principal vary in geometric progression**, i.e.:

$$A_r = A_1 \cdot q^{r-1}$$

with the common ratio and the first element given by:

$$q = 1 + I_m \qquad A_1 = \alpha - Y_1 = \alpha - C \cdot I_m$$

$$A_r = A_1 \cdot (1+I_m)^{r-1}$$

Example 6. In the amortization table contained in example 5 we obtained:

$$A_1 = 150.76 \qquad A_2 = 156.79 \qquad A_3 = 163.07$$

$$A_4 = 169.59 \qquad A_5 = 176.37 \qquad A_6 = 183.42$$

The interest rate for this loan was $I_2 = 0.04$.

In this case, the following is satisfied:

$$\boxed{A_r = A_1 \cdot (1 + I_m)^{r-1} = 150.76 \cdot 1.04^{r-1}}$$

Thus, for instance:

$$A_3 = 150.76 \cdot 1.04^{3-1} = 163.07$$

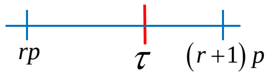
$$A_6 = 150.76 \cdot 1.04^{6-1} = 183.42$$

Now we obtain the outstanding loan balance and the outstanding principal at 4 different moments in time:

- At the beginning of the loan, 0.
- At the end of each period of the loan: rp

NOTE: We will always consider, unless otherwise stated, that the level repayment α has just been made.

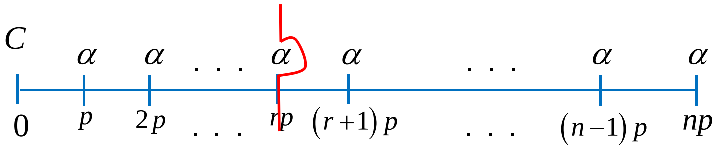
- At a specific moment within a period of the loan: $\tau \in (rp, (r+1)p)$



- At the end of the loan, np .

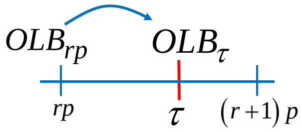
Outstanding loan balance

- At 0: $OLB_0 = C$



- At rp: $OLB_{rp} =$ retrosp. $C \cdot (1+I_m)^r - \alpha \cdot a_{\overline{r}|I_m} \cdot (1+I_m)^r$; prosp. $\alpha \cdot a_{\overline{n-r}|I_m}$

- At τ:



$$OLB_\tau = OLB_{rp} \cdot (1+I_m)^{m \cdot (\tau - rp)}$$

- At np: $OLB_{np} = 0$

Outstanding principal

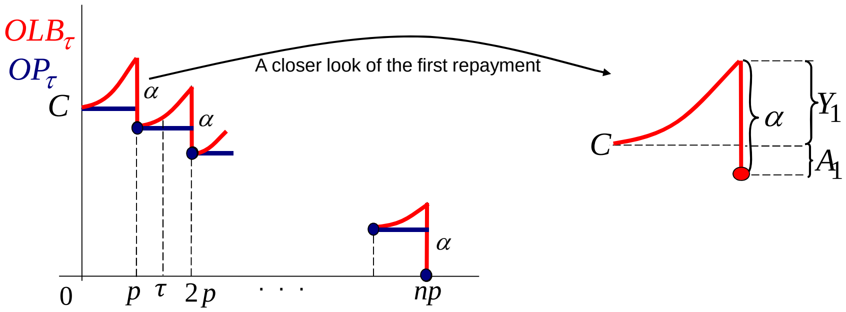
- At 0: $OP_0 = C$
- At rp: $OP_r = OLB_{rp}$

Right after paying the level repayment corresponding to period *r*, the outstanding loan balance does not contain accrued interest, i.e. it is only composed of principal. This is the reason why the outstanding principal and the outstanding loan balance are equal.

- At τ: $OP_\tau = OP_r$
- At np: $OP_n = 0$

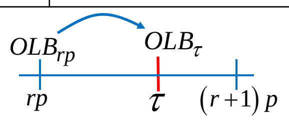
The outstanding loan balance and the outstanding principal are equal at 0, at np and at the end of each period once the level repayment has been made.

Outstanding loan balance vs. outstanding principal over the term of the loan



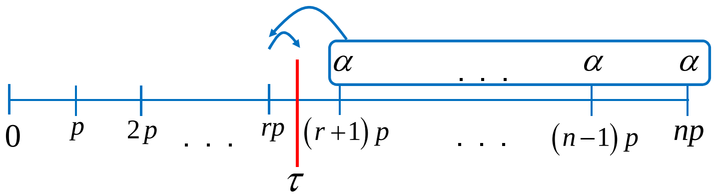
The outstanding loan balance and the outstanding principal are different at a specific moment τ within a period of the loan. This is because the outstanding loan balance takes into consideration the accrued interest until moment τ and the outstanding principal does not.

Summarizing:

Moment	Outstanding loan balance		Outstanding principal
0	C		C
rp	**Retrospective**	$C\cdot(1+I_m)^r-\alpha\cdot a_{\overline{r}\vert I_m}\cdot(1+I_m)^r$	OLB_{rp}
	Prospective	$\alpha\cdot a_{\overline{n-r}\vert I_m}$	
τ	 OLB_{rp} OLB_τ rp τ $(r+1)p$ $OLB_\tau=OLB_{rp}\cdot(1+I_m)^{m\cdot(\tau-rp)}$		OLB_{rp}
np	0		0

Value of the loan, V_τ

If the current market interest rate is denoted by I_m^{Market}, the financial value at τ of the outstanding repayments of the loan is:

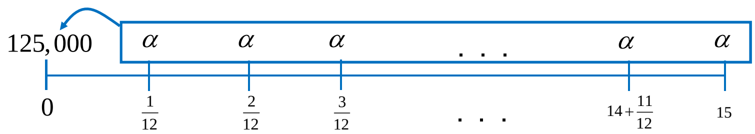


$$V_\tau = \left[\alpha \cdot a_{\overline{n-r|}I_m^{Market}}\right] \cdot \left(1+I_m^{Market}\right)^{m\cdot(\tau - rp)}$$

The **lender effective interest rate**, the **borrower effective interest rate** and the ***TAE*** of the loan can be obtained as described in subsection 2.2.

Example 7. A mortgage of €125,000 is being amortized with level repayments at the end of each month for 15 years, at a 6% interest rate convertible monthly. The borrower pays the bank an initial fee equal to 2% of the principal of the loan and initial obligatory expenses of €550. Your are asked to:

- Obtain the amount of the level monthly **repayments**.



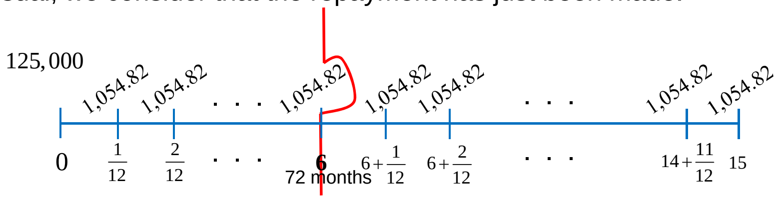
The loan interest rate is: $i_{12} = 0.06 \Rightarrow I_{12} = \dfrac{0.06}{12} = 0.005$

Therefore:

$$125{,}000 = \alpha \cdot \frac{1-(1+0.005)^{-180}}{0.005} \Rightarrow \alpha = €1{,}054.82$$

- Calculate the **outstanding loan balance** and the **outstanding principal** 6 years after the loan was lent.

Six years after securing the loan, 72 months have gone by and, as usual, we consider that the repayment has just been made.



For the outstanding loan balance:

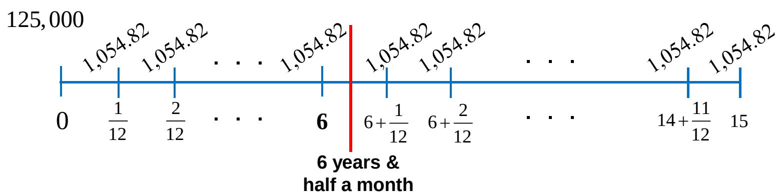
$$OLB_6 = \begin{cases} \text{retrosp.} & 125{,}000\cdot(1+0.005)^{72} - 1{,}054.82\cdot a_{\overline{72}|0.005}\cdot(1+0.005)^{72} = \\ & = €87{,}859.66 \\ \text{prosp.} & 1{,}054.82\cdot a_{\overline{108}|0.005} = €87{,}859.66 \end{cases}$$

This moment is the end of a period, so the outstanding principal and the outstanding loan balance are equal.

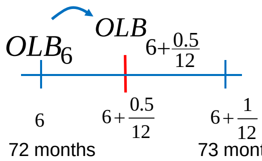
$$OP_{72} = OLB_6 = €87{,}859.66$$

Unit 3. Loans
5. Amortizing loan with level repayments

- Find the **outstanding loan balance** and the **outstanding principal** 6 years and half a month (or 72.5 months) after the loan was lent.



In this case, since we are considering a moment in time between 6 and $6+\frac{1}{12}$, to obtain the outstanding loan balance after 6 years and half a month, we can take the outstanding loan balance after 6 years (72 months) and the accumulation factor for half a month, i.e.:



$$OLB_{6+\frac{0.5}{12}} = 87{,}859.66\cdot(1+0.005)^{0.5} = €88{,}079.04$$

The outstanding principal after 6 years and half a month is the same as the outstanding principal after 6 years.

$$OP_{72.5} = OP_{72} = €87{,}859.66$$

- **Split the level repayment** made 2 years after the loan was given into payment of interest and repayment (amortization) of principal.

  We are going to see two different ways of splitting the repayments into payment of interest and repayment of principal.

  First way

  The repayment made after 2 years is the 24th, so:

$$\alpha = 1{,}054.82 = Y_{24} + A_{24}$$

We first calculate the repayment (amortization) of principal corresponding to the 24th month, A_{24}, by considering that in amortizing loans with level repayments, repayments (amortizations) of principal vary in a geometric progression:

$$A_r = A_1 \cdot (1+I_m)^{r-1} \quad \text{with} \quad A_1 = \alpha - Y_1 = \alpha - C \cdot I_m$$

In our example:

$$A_1 = 1{,}054.82 - \underbrace{125{,}000 \cdot 0.005}_{Y_1=625} = €429.82$$

Therefore:

$$A_r = 429.82 \cdot 1{,}005^{r-1}$$
$$A_{24} = 429.82 \cdot 1.005^{24-1} = €482.07$$
$$Y_{24} = \alpha - A_{24} = 1{,}054.82 - 482.07 = €572.75$$

Second way

In this case, from:

$$\alpha = 1{,}054.82 = Y_{24} + A_{24}$$

we first calculate the payment of interest corresponding to the 24th month, Y_{24}, by considering that in amortizing loans with level repayments:

$$Y_r = OP_{r-1} \cdot I_m$$

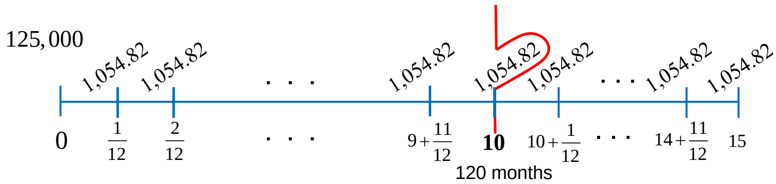
And for $r = 24$, $Y_{24} = OP_{23} \cdot 0.005$

$$OP_{23} = OLB_{23} = 1{,}054.82 \cdot a_{\overline{157}|0.005} = €114{,}550.76$$

It follows that: $Y_{24} = 114{,}550.76 \cdot 0.005 = €572.75$

And then: $A_{24} = 1{,}054.82 - 572.75 = €482.07$

- Obtain the **repaid principal** 10 years after the loan was secured, just after the repayment of the corresponding α.



First, we calculate the outstanding principal after 10 years. Once it has been calculated, the repaid principal at that moment will be the difference between the loan principal and the outstanding principal.

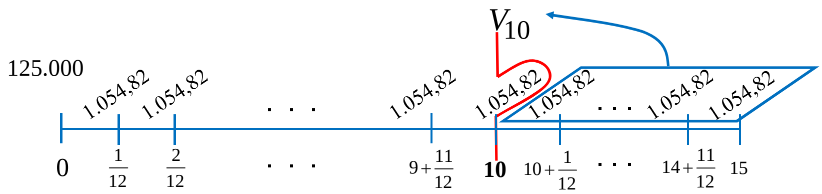
Unit 3. Loans
5. Amortizing loan with level repayments

$$OP_{120} = OLB_{10} = 1{,}054.82 \cdot a_{\overline{60}|0.005} = €54{,}561.16$$

So, the repaid principal after 10 years is:

$$125{,}000 - 54{,}561.16 = €70{,}438.84$$

- Obtain the **value** of the loan 10 years after being awarded, if the market interest rate is 10% annual effective.



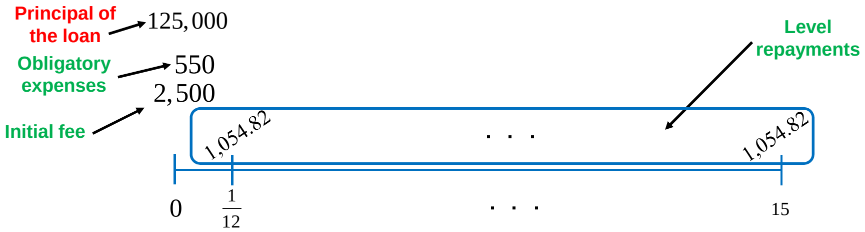
The monthly effective interest rate equivalent to 10% annual effective is:

$$I_1 = 0.10 \;\Rightarrow I_{12} = (1+0.10)^{1/12} - 1 = 0.007974$$

$$V_{10} = 1{,}054.82 \cdot a_{\overline{60}|0.007974} = €50{,}144.56$$

- Write the equation that allows us to obtain the ***TAE*** of the loan.

  To determine the *TAE*, we consider all the monthly repayments and costs and fees known at the moment of awarding the loan. The initial fee is $125{,}000 \cdot 0.02 =$ €$2{,}500$, and the obligatory expenses, €550.



Since the level repayments are made monthly, a monthly effective interest rate has to be used. If we value at 0:

$$125{,}000 = 550 + 2{,}500 + 1{,}054.82 \cdot a_{\overline{180|}I_{12}}$$

or equivalently:

$$125{,}000 = 550 + 2{,}500 + 1{,}054.82 \cdot \frac{1-(1+I_{12})^{-180}}{I_{12}}$$

Using a <u>suitable software</u>, the monthly effective interest rate, I_{12} can be obtained.

The *TAE* would be the annual effective interest rate equivalent to I_{12}, i.e.:

$$I_{12} = 0.005323$$

$$TAE = (1+I_{12})^{12} - 1 = 0.06578 \equiv 6.578\%$$

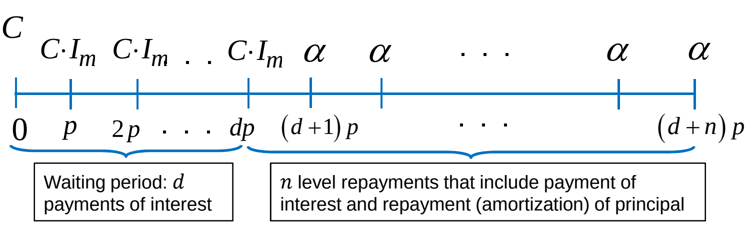
Amortizing loans with a waiting term

Amortizing loans with level repayments can be agreed with a **waiting term**. This means that there is a term before the repayments α begin, during which no amount of the principal is repaid. There may be two different situations:

- **Partial waiting term**: there is an initial waiting term in which only payments of interest, $Y = C \cdot I_m$, are made at the end of each period. When the waiting period finishes, n level repayments, α, that include payment of interest and repayment (amortization) of principal, are paid.
- **Total waiting term**: there is an initial waiting term in which no amount is paid (neither payment of interest nor repayment of principal). When the waiting period finishes, n level repayments, α, that include payment of interest and repayment (amortization) of principal, are paid.

- **Partial waiting term**

  Supposing that the waiting term lasts d periods:



$$C = C \cdot I_m \cdot a_{\overline{d}|I_m} + \alpha \cdot a_{\overline{n}|I_m} \cdot (1+I_m)^{-d}$$

$$C - C \cdot I_m \cdot \frac{1-(1+I_m)^{-d}}{I_m} = \alpha \cdot a_{\overline{n}|I_m} \cdot (1+I_m)^{-d} \qquad \boxed{C = \alpha \cdot a_{\overline{n}|I_m}}$$

This expression is the same as the one we obtained for the case of a loan without a waiting term. This is because, once the partial waiting term ends, the outstanding principal is C, which has to be repaid through n level repayments of amount α.

Example 8. A loan with a principal of €10,000 will be repaid with level repayments at the end of every 6-month period. The term of the loan is 10 years, with the first 2 being a partial waiting term. If the interest rate is 5% semiannual. Find:

- The **amount of the semiannual payments of interest** made in the waiting period.

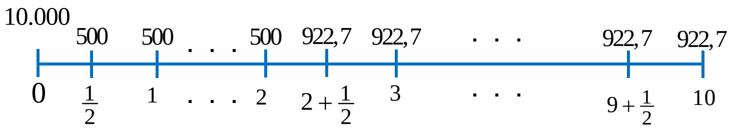
$$I_2 = 0.05 \qquad Y = 10,000 \cdot 0.05 = €500$$

- The **amount of the level repayments** to be made once the waiting period finishes.

  The outstanding principal at the end of the waiting term is $C =$ €10,000. So:

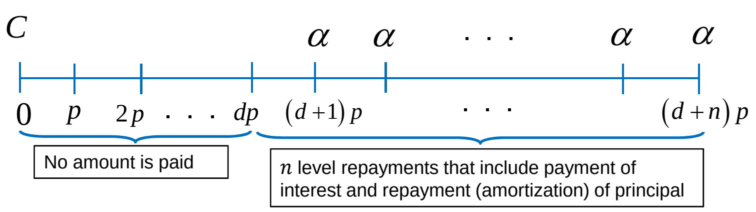
$$10,000 = \alpha \cdot a_{\overline{16}|0.05} \Rightarrow \alpha = €922.70$$

- Graph the loan on the timeline.



- **Total waiting term**

  Supposing that the waiting term lasts d periods:

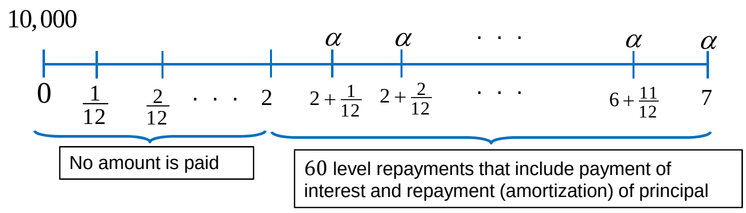


$$C = \alpha \cdot a_{\overline{n}|I_m} \cdot (1+I_m)^{-d}$$

or equivalently**:**

$$C \cdot (1+I_m)^{d} = \alpha \cdot a_{\overline{n}|I_m}$$

Example 9. A person borrows €10,000. The loan is governed by an annual rate of 8% and is being amortized through 60 constant repayments at the end of each month after a total waiting period of 2 years in which no payments are made. Find the amount of the **constant repayment**.



The interest rate for the loan is:

$$I_1 = 0.08 \Rightarrow I_{12} = (1+0.08)^{1/12} - 1 = 0.006434$$

So, the amount of the level repayments is:

$$10,000 = \alpha \cdot a_{\overline{60}|0.006434} \cdot (1+0.006434)^{-24} \Rightarrow \alpha = €234.95$$

6. Changes during the lifetime of the loan

During the lifetime of a loan, some changes with respect to the conditions initially agreed may occur. These changes can be, among others:

- Variation of the repayment.
- Variation of the term of the loan.
- Variation of the interest rate.
- Partial repayment of the outstanding principal.
- Total repayment of the outstanding principal.

In any of these situations the following steps will be followed:

1. Obtain the outstanding principal of the loan at the time of the change of the condition.
2. Apply the condition that has to be modified.
3. Recalculate the new corresponding variable (repayment, term, etc.).

Example 10. 5 years ago a loan of principal €60,000 was given. The loan was agreed with periodic payments of interest every 2 months and a single repayment of the principal 15 years after its grant. The rest of the conditions were:

- Annual interest rate: 2.4% payable every 2-month periods.
- Arrangement fee: 1% of the principal.
- Initial mandatory mortgage default insurance: €1,000.
- Early repayment fee: 0.5% of the principal that is amortized at that time.

Today, after the corresponding interest payment, an early repayment of €9,000 is made. Determine:

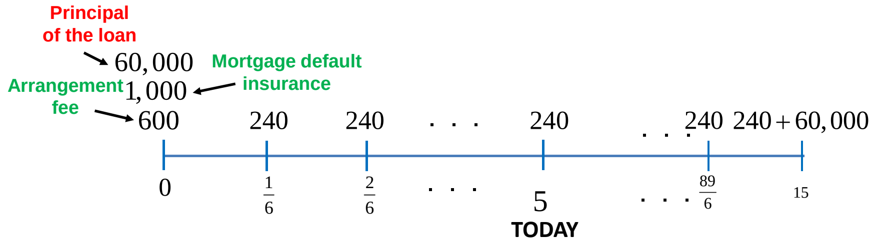
a) The amount of the **payments of interest** over the first 5 years.

$$i_6 = 0.024 \Rightarrow I_6 = \frac{0.024}{6} = 0.004 \Rightarrow Y = 0.004 \cdot 60,000 = €240$$

b) Equation that allows obtaining the ***TAE*** of the loan.

In order to obtain the *TAE*, only the initial conditions which were agreed for the loan have to be taken into account.

Since the payments of interest are made every 2-month period, an effective interest rate for a 2-month period has to be used. If we evaluate at 0:



$$60{,}000 = 1{,}000 + 600 + 240 \cdot \frac{1-(1+I_6)^{-90}}{I_6} + 60{,}000 \cdot (1+I_6)^{-90}$$

From this equation, and using a <u>suitable software</u>, the effective interest rate for a 2-month period, I_6, can be obtained.

The *TAE* would be the annual effective interest rate equivalent to I_6, i.e.:

$$TAE = (1+I_6)^6 - 1$$

c) The amount of the **payments of interest** to be made from today onwards.

First, we calculate the outstanding principal today:

$$OP_5 = OLB_5 = €60,000$$

After the partial repayment of the principal, the new outstanding principal is:

$$OP_{\text{NEW}} = 60,000 - 9,000 = €51,000$$

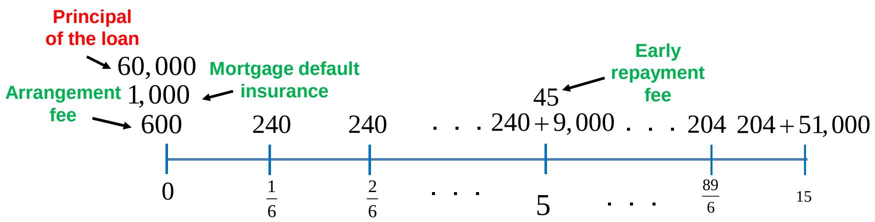
So, the amount of the new payments of interest is:

$$Y_{\text{NEW}} = 0.004 \cdot 51,000 = €204$$

d) Equation that allows us to obtain the **borrower effective** interest rate by considering the partial early repayment made today.

We have to consider the principal of the loan and all the repayments made by the borrower.

Graphically:



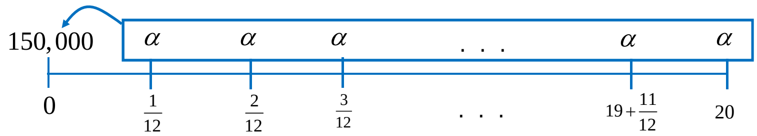
Valuing at 0:

$$60.000 = 600 + 1.000 + 240 \cdot \frac{1-(1+I_6)^{-30}}{I_6} + 9.045 \cdot (1+I_6)^{-30} +$$

$$+ 204 \cdot \frac{1-(1+I_6)^{-60}}{I_6} \cdot (1+I_6)^{-30} + 51.000 \cdot (1+I_6)^{-90}$$

Example 11. Let us consider a loan with a principal of €150,000. It was agreed that it would be repaid over 20 years with level repayments at the end of each month, by applying a 5.64% compounded monthly interest rate.

- Find the amount of the **level repayments**.



The interest rate for the loan is:

$$i_{12} = 0.0564 \Rightarrow I_{12} = \frac{0.0564}{12} = 0.0047$$

So, the amount of the level repayments is:

$$150{,}000 = \alpha \cdot \frac{1-(1+0.0047)^{-240}}{0.0047} \quad \Rightarrow \alpha = €1{,}043.73$$

- Today, 5 years after the loan was awarded, once the corresponding level repayment has been made, the lender and the borrower agree that from now on the level repayment will change to €1,500. If the rest of the initial conditions remain the same, find the **new term of the loan**.

First, we calculate the outstanding principal 5 years (60 months) after the loan was given:

$$OP_{60} = OLB_{60} = 1{,}043.73 \cdot a_{\overline{180}|0.0047} = €126{,}584.61$$

If from now on the level repayments will be €1,500, we have to determine how long this amount has to be paid to be able to repay the current outstanding principal, OP_{60}, that is to say:

$$126{,}584.61 = 1{,}500 \cdot a_{\overline{n}|0.0047} \quad \Rightarrow n = 107.75$$

This means that 107 repayments of €1,500 and an additional repayment of the necessary amount to totally repay the loan will be made. The new term is therefore 108 months.

Example 12. Let us consider the loan included in example 11 with its initial conditions.

Today, 5 years after the loan was given, once the corresponding level repayment has been made, lender and borrower agree to reduce the term of the loan by 6 years. If the rest of the initial conditions remain the same, find the amount of the **new repayments**.

We know, from example 11, that the outstanding principal today is:

$$OP_{\text{TODAY}} = €126{,}584.61$$

The initial term of the loan was 20 years and therefore today there are 15 years left. However, since lender and borrower agree to reduce the term by 6 years, it will now be just 9 more years (108 months).

The amount of the new repayments has to be calculated by considering the current outstanding principal and the number of future repayments to be made:

$$126{,}584.61 = \alpha \cdot \frac{1-(1+0.0047)^{-108}}{0.0047} \Rightarrow \alpha = €1{,}497.31$$

Example 13. Let us consider again the loan included in example 11 with its initial conditions.

Today, 5 years after the loan was given, once the corresponding level repayment has been made, a change in the interest rate of the loan is agreed. It will be the last published Euribor plus a differential of 2.25%.

If the rest of the initial conditions remain the same, and the last published Euribor is 0.55%, find the amount of the **new repayments**.

We know, from example 11, that the outstanding principal today is:

$$OP_{\text{TODAY}} = €126{,}584.61$$

The new interest rate to be applied from today on is:

$$i_{12} = \underbrace{0.0055}_{\text{Last Euribor published}} + \underbrace{0.0225}_{\text{Differential}} = 0.028 \Rightarrow I_{12} = \frac{0.028}{12} = 0.002\dot{3}$$

The new level repayment that must be made to repay the outstanding principal with the new interest rate is:

$$126{,}584.61 = \alpha \cdot \frac{1-(1+0.002\dot{3})^{-180}}{0.002\dot{3}} \Rightarrow \alpha = €862.05$$

Example 14. Let us consider once again the loan included in example 11 with its initial conditions.

Five years after the loan was awarded, besides the corresponding level repayment, the borrower paid an extra €25,000 to partially repay the outstanding principal.

- If the rest of the initial conditions remain the same, find the amount of the **new repayments**.

  We know, from example 11, that the outstanding principal 5 years after the loan was secured is:

$$OP_{60} = €126{,}584.61$$

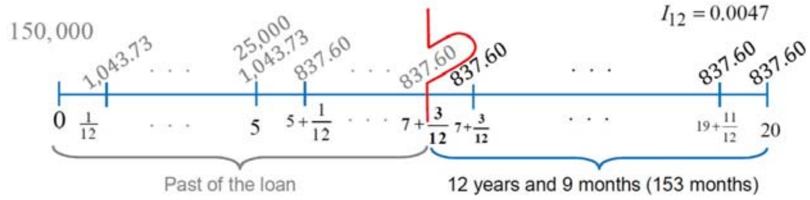
  After the partial repayment of the principal, the new outstanding principal is:

$$OP_{NEW} = 126{,}584.61 - 25{,}000 = €101{,}584.61$$

  The amount of the new level repayments, α', will be that which, paid monthly over 15 years, will repay €101,584.61:

$$101{,}584.61 = \alpha' \cdot a_{\overline{180}|0.0047} \Rightarrow \alpha' = €837.60$$

- Two years and 3 months after making the partial payment of the principal (see the previous section), once the corresponding level repayment has been made, the borrower decides to totally repay the loan. If there is no early repayment fee, obtain the **amount that the borrower must pay**.



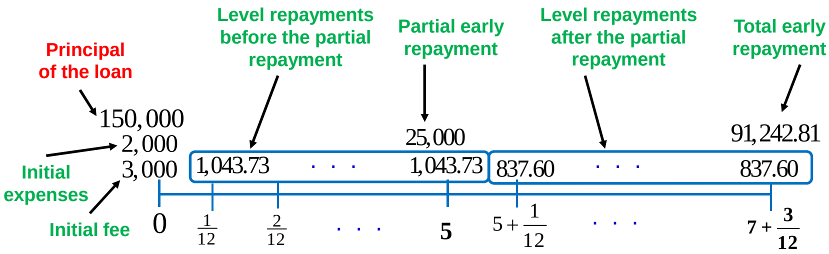
The amount that the borrower must pay is the outstanding principal 7 years and 3 months (87 months) after the loan was awarded. Since the loan was initially agreed for 240 months, there are 153 months left. Therefore, the outstanding principal at that moment is:

$$OP_{87} = OLB_{7+\frac{3}{12}} = 837.60 \cdot a_{\overline{153}|0.0047} = €91{,}242.81$$

- Write the equation that allows us to obtain the **borrower effective interest rate** by considering the previous sections, that there was an initial fee equal to 2% of the principal and that the borrower paid mandatory initial expenses of €2,000.

  To determine the borrower effective interest rate, we consider the principal of the loan and all the repayments made.

  Drawing the financial capital on the timeline:



  Using a monthly effective interest rate:

$$150{,}000 = 3{,}000 + 2{,}000 + 1{,}043.73 \cdot a_{\overline{60}|I_{12}} + 25{,}000 \cdot (1+I_{12})^{-60} +$$
$$+ 837.60 \cdot a_{\overline{27}|I_{12}} \cdot (1+I_{12})^{-60} + 91{,}242.81 \cdot (1+I_{12})^{-87}$$

  By using a suitable software, the borrower effective interest rate, I_{12}, can be obtained.

- With all the information given above, write the equation that allows us to obtain the ***TAE*** of the loan.

  In order to obtain the *TAE*, only the initial conditions which were agreed for the loan have to be taken into account.

  Since the level repayments are made monthly, a monthly effective interest rate has to be used. If we evaluate at 0:

$$150{,}000 = 3{,}000 + 2{,}000 + 1{,}043.73 \cdot a_{\overline{240}|I_{12}}$$

  From this equation, and using a suitable software, the monthly effective interest rate, I_{12}, can be obtained.

  The *TAE* would be the annual effective interest rate equivalent to I_{12}, i.e.:

$$TAE = (1+I_{12})^{12} - 1$$

UNIT 3 PROBLEMS

1. A loan with a principal of €18,000 will be repaid through a single repayment, 3 years after it has been given. This single repayment includes the payment of interest accrued during the term and the repayment of the principal. The interest of the loan is an annual effective rate of 5.5%, it has an initial fee equal to 1.5% on the principal and the borrower pays mandatory initial expenses of €150. Obtain:

a) The amount of the single repayment. Split it into payment of interest and repayment of principal.
b) The annual effective interest rate of the loan, the borrower annual effective interest rate and the *TAE* of the loan.
c) The outstanding principal and the repaid principal 1 year and 2 months after the loan has been lent.
d) The amount to be paid by the borrower, in order to totally cancel the loan, 1 year and 2 months after it has been given.
e) The value of the loan, 1 year and 2 months after it has been awarded, if the current interest on the market is:
   e.1) 5.5% annual effective rate.
   e.2) 6.25% annual effective rate.

2. A person borrowed €50,000 2 years ago. It was agreed that the loan would be repaid by a single repayment, including the payment of interest and the repayment of principal, to be made 4 years after it was given. The interest on the loan is 1% monthly effective. Furthermore, there was an arrangement fee of 2% on the principal. Determine:

a) The repayment amount to be made by the borrower 4 years after the loan was awarded.
b) The payment of interest and repayment of principal included in the above repayment.
c) The amount to be paid by the borrower today to cancel the loan.
d) The annual effective interest rate of the loan and the *TAE* of the loan.
e) Today the borrower makes an extra repayment of €15,000. Calculate the amount that will cancel the loan at its maturity taking into account this extra repayment.
f) The equation that allows us to obtain the borrower annual effective interest rate considering the extra repayment described above.
g) Does the value of the *TAE* change as a result of considering the extra repayment of €15,000?

3. A loan of a principal of €90,000 will be repaid with quarterly payments of interest and a lump-sum repayment of the principal in 5 years. The nominal rate of interest payable quarterly is 6%. Find:

a) The amount of the quarterly payments of interest.
b) The repayment that will be made by the borrower at the end of the last quarter.
c) The repayment that totally cancels the loan 3 years after it has been awarded, once the corresponding payment of interest has been made.
d) The amount to be paid by the borrower, in order to totally cancel the loan, 3 years and 1 month after it has been given.
e) 4 years after the principal has been lent, once the corresponding payment of interest has been made, the interest rate of the loan changes to a nominal interest rate of 6.80% payable quarterly. What is the new amount of payments of interest?

4. 6 years ago a loan with a principal of €48,000, at a nominal interest rate of 4.5% compounded monthly, was agreed. The loan is being repaid through payments of interest at the end of each month and a single repayment of principal 10 years after it was awarded. Today, once the corresponding payment of interest has been made, the borrower makes a partial repayment of the principal of €5,000. Obtain:

a) The amount of the monthly payments of interest over the first 6 years.
b) The outstanding principal and the outstanding loan balance 1 month ago, just after making the corresponding payment of interest.
c) The outstanding principal and the outstanding loan balance 5 years and 11.5 months after the loan was agreed.
d) The amount of the monthly payments of interest to be made from now on.
e) The equation that allows us to obtain the *TAE* of the loan if there was an initial fee of 1% of the principal paid by the borrower.
f) The equation that allows us to obtain the borrower effective interest rate by considering the initial fee, the partial repayment of the principal made today and taking into account that at the moment of partial repayment the borrower pays an early repayment fee of 0.25% on the €5,000.

5. A loan of €170,000 will be amortized with level repayments at the end of each month for 20 years, at an annual interest rate payable monthly equal to 3.60%. Obtain:

a) The amount of the level repayments.
b) The first 2 rows and the last 2 rows of the amortization table.
c) The outstanding principal, the outstanding loan balance and the repaid principal 1 year after the loan is awarded, right after paying the corresponding level repayment. What is the outstanding loan balance 1 year after the loan is awarded, just before paying the corresponding level repayment?
d) The outstanding principal and the outstanding loan balance 1 year and half a month after it is given.
e) The payment of interest and the repayment (amortization) of principal included in the 15^{th} level repayment.
f) The value of the loan 1 year after it is awarded, if the current market interest rate is equal to an annual 3.84% payable monthly.
g) The equation that allows us to calculate the *TAE* of the loan if there exists an initial fee of 3% on the principal of the loan.

6. A person is awarded a loan with the following characteristics:

- Principal: 60,000€.
- Interest rate: 4.8% annual convertible every two months.
- It will be amortized in level repayments at the end of every 2 months.
- Term: 12 years, with the first 2 being a partial waiting term in which only payments of interest, at the end of every two months, are to be made.
- Initial fee: 1% of the principal.
- Early repayment fee: 0.25% of the principal that is amortized at that time.

Find:

a) The amount of the payments of interest to be made during the first 2 years and the amount of the level repayments over the rest of the term.
b) The equation that allows us to obtain the *TAE* of the loan.
c) The outstanding principal of the loan 6 years after it was agreed, once the corresponding repayment has been made.
d) The repaid principal of the loan 6 years and 1 month after it was agreed.
e) If the borrower decides to totally cancel the loan 6 years and a month after it was agreed:
   e.1) the outstanding loan balance at that moment.
   e.2) the equation that allows us to obtain the borrower effective interest rate.

7. Let us consider a loan with a principal of €180,000, agreed at an annual effective rate of 6%, to be repaid with level repayments at the end of each year. The total term of the loan is 15 years, but no amount is paid during the two first years (total waiting term). Find:

a) The amount of the level annual repayments.
b) The equations that allow calculating the *TAE* of the loan and the borrower effective interest rate if, at the moment of borrowing the money, the borrower pays an initial fee of 1% on the principal.
c) The amount to be paid by the borrower 7 years after the loan was given, once the corresponding repayment has been made, in order to totally cancel it.
d) The amount to be paid by the borrower 7 years and 2 months after the loan was agreed to totally cancel it.

8. 3 years ago, a loan with a principal of €150,000 was secured. It is being amortized through 48 equal repayments at the end of each quarter, at an annual effective rate equal to 5.20%. The loan had an initial fee of 1.5% on the principal and an early repayment fee equal to 0.5% of the principal that is amortized at that time. Find:

a) The amount of the repayments made at the end of each quarter.
b) The split of the 20^{th} repayment into payment of interest and repayment (amortization) of principal.
c) The equation that allows us to calculate the *TAE* of the loan.
d) Today, just after making the corresponding repayment, the borrower decides to make a partial repayment of the principal of €12,000.
   d.1) If the rest of the loan conditions remain the same, what is the amount of the new quarterly repayments?
   d.2) Obtain the equation that allows us to calculate the borrower effective interest rate, considering the partial repayment of principal and that the new repayments remain equal over the rest of the term of the loan.
   d.3) What would have been the new term of the loan if, when making the partial repayment of principal, the borrower had decided to continue paying the same initial quarterly repayment?

9. A mortgage with a principal of €220,000 was secured a year ago. It was agreed that it would be repaid over 30 years, with constant instalments at the end of every 6-month period. The interest rate on the loan is 4% nominal compounded semi-annually. Determine:

a) The amount of the semi-annual repayments.
b) The payment of interest and the repayment (amortization) of principal included in the first and in the last repayment.
c) Today, the lender and the borrower decide to change the interest rate, which becomes 4.337% nominal, compounded semi-annually. Obtain the new amount of the semi-annual repayments if the rest of the conditions of the loan remain the same.

<u>Formula</u>

$C_r = C$	$C \cdot a_{\overline{n}\mid I_M} = C \cdot \dfrac{1-(1+I_M)^{-n}}{I_M}$

Answers

1. a) €21,136.34. Payment of interest: €3,136.34. Repayment of principal: €18,000.

b) Annual effective interest rate of the loan: 5.5%.

Borrower annual effective interest rate: 6.3335%.

TAE: 6.3335%.

c) Outstanding principal: €18,000.

Repaid principal: €0.

d) €19,160.21.

e)

e.1) €19,160.21.

e.2) €18,912.98.

2. a) €80,611.30.

b) Payment of interest: €30,611.30.

Repayment of principal: €50,000.

c) €63,486.73.

d) Annual effective interest rate of the loan: 12.6825%.

TAE: 13.2531%.

e) €61,565.28.

f) $50.000 = 1.000 + 15.000 \cdot (1+I_1)^{-2} + 61.565,28 \cdot (1+I_1)^{-4}$.

g) The *TAE* is the same, since the partial repayment of €15,000 does not have to be included in its calculation.

3. a) €1,350.

b) €91,350.

c) €90,000.

d) €90,447.77.

e) €1,530

4. a) €180.

b) Outstanding principal: €48,000.

Outstanding loan balance: €48,000.

c) Outstanding principal: €48,000.

Outstanding loan balance: €48,089.92.

d) €161.25.

e) $48{,}000 = 480 + 180 \cdot a_{\overline{120}|I_{12}} + 48{,}000 \cdot (1+I_{12})^{-120}$ and $TAE = (1+I_{12})^{12} - 1$.

f) $48{,}000 = 480 + 180 \cdot a_{\overline{72}|I_{12}} + 5{,}012.50 \cdot (1+I_{12})^{-72} +$

$$+161.25 \cdot a_{\overline{48}|I_{12}} \cdot (1+I_{12})^{-72} + 43{,}000 \cdot (1+I_{12})^{-120}$$

.

5. a) €994.69.

b)

r	Level repayment	Payment of interest	Repayment of principal	Outstanding principal
0				170,000.00
1	994.69	510.00	484.69	169,515.31
2	994.69	508.55	486.14	169,029.17
...	...	...	...	...
239	994.69	5.94	988.75	991.71
240	994.69	2.98	991.71	0.00

c) Outstanding principal: €164,086.87.
Outstanding loan balance: €164,086.87.
Repaid principal: €5,913.13.
Outstanding loan balance just before paying the corresponding level repayment: €165,081.56.

d) Outstanding principal: €164,086.87.
Outstanding loan balance: €164,332.82.

e) Repayment (amortization) of principal: €505.45.
Payment of interest: €489.24.

f) €160,809.14.

g) $170{,}000 = 5{,}100 + 994.69 \cdot a_{\overline{240}|I_{12}}$ and $TAE = (1+I_{12})^{12} - 1$.

6. a) Amount of interest payments: €480.
Amount of level repayments: €1,263.05.

b) $60{,}000 = 600 + 480 \cdot a_{\overline{12}|I_6} + 1{,}263.05 \cdot a_{\overline{60}|I_6} \cdot (1+I_6)^{-12}$ and $TAE = (1+I_6)^6 - 1$.

c) €39,372.23.

d) €20,627.77.

e)

e.1) €39,529.41.

e.2) $60{,}000 = 600 + 480 \cdot a_{\overline{12}|I_6} + 1{,}263.05 \cdot a_{\overline{24}|I_6} \cdot (1+I_6)^{-12} + 39{,}627.84 \cdot (1+I_6)^{-36{,}5}$.

7. a) €22,845.96.

b) Equation for the *TAE* and for the borrower effective interest rate:

$$180{,}000 = 1{,}800 + 22{,}845.96 \cdot a_{\overline{13}|I_1} \cdot (1+I_1)^{-2}$$

c) €141,868.70.

d) €143,253.17.

8. a) €4,197.82.

b) Repayment (amortization) of principal: €2,906.75.
Payment of interest: €1,291.07.

c) $150{,}000 = 2{,}250 + 4{,}197.82 \cdot a_{\overline{48}|I_4}$ and $TAE = (1+I_4)^4 - 1$.

d)

d.1) €3,780.04.

d.2) $150{,}000 = 2{,}250 + 4{,}197.82 \cdot a_{\overline{12}|I_4} + 12{,}060 \cdot (1+I_4)^{-12} + 3{,}780.04 \cdot a_{\overline{36}|I_4} \cdot (1+I_4)^{-12}$.

d.3) The new term of the loan would have been 32 quarters.

9. a) €6,328.95

b) Repayment (amortization) of principal in the first level repayment: €4,400.00.

Payment of interest in the first level repayment: €1,928.95.

Repayment (amortization) of principal in the last level repayment: €6,204.85.

Payment of interest in the last level repayment: €124.10.

c) €6,583.10.

UNIT 4.

BONDS

1. **Definition and related concepts. Classification**
2. **Zero-coupon bonds**
3. **Coupon bonds**

1. Definition and related concepts. Classification

1.1. Definition and related concepts

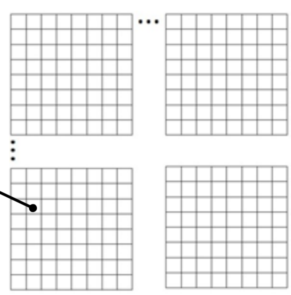
1.2. Classification

1.1. Definition and related concepts

There are two main ways for a company or a government entity to obtain financing: ask a bank for a loan or **issue bonds** and sell them to the public.

Graphically:

A company issues 200,000 bonds with a unit value equal to €10,000



Each of these parts represents €10,000 (**face value** of the bond).

If there are enough buyers willing to buy the bonds and the company sells each bond at a unit price equal to 10,000€, it will receive €2,000,000,000.

A **bond** is a financial product (security) issued by a company or a government entity (**bond issuer**) that promises certain payment(s) at future date(s) to the owner of the bond (**bondholder)**.

The company or government entity that issues the bonds outlines how much money it wants and specifies a length of time, along with the interest it is willing to pay. Investors who buy bonds are then lending money to the issuer and become its creditors through the bonds that they hold.

$$\boxed{\text{Number of bonds issued}} \times \boxed{\text{Issue price}} = \boxed{\textbf{Amount of money the bond issuer wants}}$$

<u>NOTE</u>: As we will see later, the issue price of bonds is not, in general, equal to their face value.

So, bonds are nothing more than loans, in which there is **only one borrower** (the bond issuer) but there are **many lenders** (the bondholders).

The **term** of the bond is the length of time from the **issue date** until the date of the last payment. The date of this last payment is called the **maturity date** or **redemption date**.

The payment(s) made by the bond issuer in exchange for the money borrowed can include both interest and the **redemption amount**.

The **redemption amount** of a bond is the amount that the bondholder receives on the bond maturity date, excluding interest (if there is any) .

All bonds issued by a company or a government entity at the same moment in time have the same **face value**. The face value of a bond is printed on it and represents the debt the bond issuer owes to the bondholder. As for bonds with periodic payments of interest (**coupons**), these are calculated by considering the face value and the effective **interest rate of the bond**. Since, from the issue date, it is known how the coupons will be calculated, bonds are also called **fixed-income securities**.

It is worth highlighting that, in general, the face value of a bond is not equal to neither its issue price nor its redemption value.

After their issue (**primary market**), bonds can be bought and sold on organized markets (**secondary market**). In order to be able to identify bonds from each issue unequivocally on any market, they are assigned an alphanumeric reference of 12 characters, called an **ISIN** (International Securities Identification Number) code.

<u>NOTE</u>: In the Spanish financial market, which is the only one that we will consider when studying bonds, bonds have different names such as, for example, *bonos, obligaciones, pagarés de empresa, Letras del Tesoro*, etc.

Some of the concepts we have just introduced have a specific notation. Let us now see what that is.

0: Issue date

T': Maturity or redemption date

C: Face value of a bond. It is also called the par value

C_e: Issue price of a bond

- If $C_e > C$ the bond is **issued at a premium** (or **above par**)
  The difference $C_e - C$ is called the **original issue premium**
- If $C_e = C$ the bond is **issued at par**
- If $C_e < C$ the bond is **issued at a discount** (or **below par**)
  The difference $C - C_e$ is called the **original issue discount**

The original issue premium and the original issue discount can also be expressed as a percentage of the face value.

C_a: Redemption amount

- If $C_a > C$ the bond is **redeemable at a premium** (or **above par**)
  The difference $C_a - C$ is called **the redemption premium**
- If $C_a = C$ the bond is **redeemable at par**
- The case $C_a < C$ makes no sense

The redemption premium can also be expressed as a percentage of the face value.

Example 1. Obtain the issue price and the redemption amount of 1 bond of an issuance in which bonds have a face value of €3,000. Consider that they are issued at a discount, with an original issue discount of €30 and they are redeemable at a premium, with a redemption premium equal to 2% of the face value.

$$C_e = 3,000 - 30 = €2,970$$

This is equivalent to saying that the bonds have an original issue discount of 1% or that they have been issued at a price of 99%.

$$C_a = 3,000 + 2\% \cdot 3,000 = €3,060$$

This is equivalent to saying that the bonds have a redemption premium of €60 or that they are redeemable at a price of 102%.

Example 2. Let us suppose that in the previous issuance, 500,000 bonds were issued.

Obtain the face value issued, the amount obtained through the issuance of bonds and the total redemption amount to be paid by the bond issuer.

We determined:

$$C_e = €2,970 \quad C_a = €3,060$$

So:

Face value issued: $3,000 \cdot 500,000 = €1,500,000,000$

Amount obtained through the issuance of bonds:

$$2,970 \cdot 500,000 = €1,485,000,000$$

Total redemption amount: $3,060 \cdot 500,000 = €1,530,000,000$

P_τ : Price paid for a bond when it is purchased at any moment τ after its issue

Since not all bonds have the same face value, expressing the price P_τ in monetary units is not informative enough. This is why, in order to avoid misunderstandings, the price of a bond on a financial market is expressed as a percentage of its face value.

Example 3. Let us suppose that the prices of two different bonds, A and B, at moment τ are:

$$P_\tau^A = €2{,}100 \qquad P_\tau^B = €2{,}700$$

If their respective face values are:

$$C^A = €2{,}000 \qquad C^B = €3{,}000$$

Bond A is currently being sold/purchased on the market at a price greater than its face value. More specifically, its current price expressed as a percentage of its face value is:

$$P_\tau^A = \frac{2.100}{2.000} = 1.05 = 105\%$$

Similarly, the current price of bond B expressed as a percentage of its face value is:

$$P_\tau^B = \frac{2.700}{3.000} = 0.9 = 90\%$$

I_m : Effective interest rate for bonds in the same issue (or coupon interest rate).

Moreover, there are other interest rates related to bonds:

<u>Bond issuer interest rate</u>

This is the interest rate for the bond issuer, taking into account the total amount received when issuing the bonds, payments made to the bondholders over the term of the bonds and all the expenses associated with the issue of the bonds.

There is a single **bond issuer interest rate** for each issuance of bonds and it can usually be calculated when bonds are issued.

Bondholder interest rate

This is the interest rate for a specific bondholder, taking into account the amounts paid when purchasing the bonds, the payments received from the bond and all the expenses associated with the bonds over the term they have been held.

Bondholders can either buy bonds at the moment of their issue or at any other moment after being issued.

Furthermore, bondholders can either hold the bonds until their maturity date or sell them at an earlier moment.

This is why the bondholder interest rate may not be the same for every bondholder.

Internal rate of return (IRR) or yield to maturity (YTM) of a bond at moment τ

After their issue, bonds can be bought and sold on organized markets.

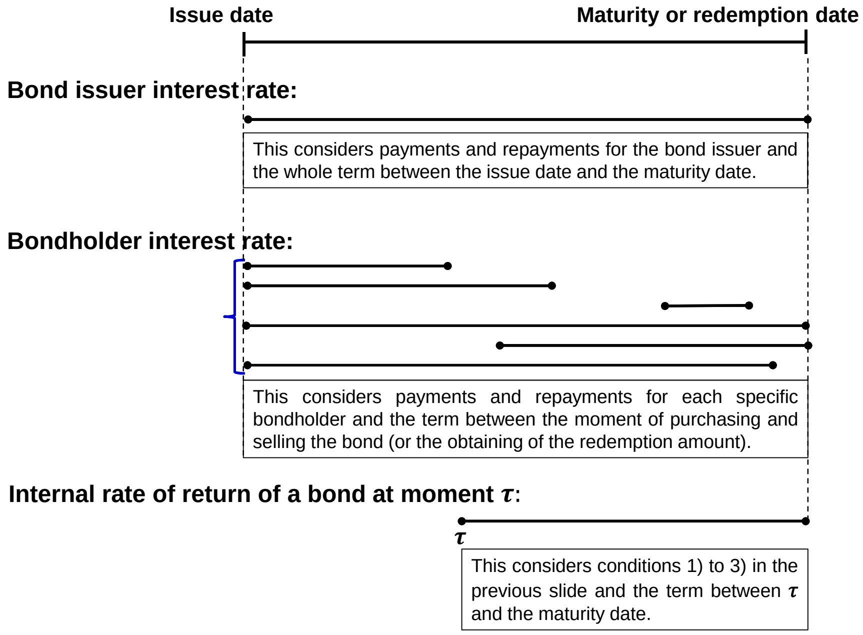
The internal rate of return of a bond at moment τ is the interest rate associated to the purchase of this bond by considering:

1) that it is purchased at a price P_τ,
2) that the bondholder who buys the bond at that moment will hold it until its maturity (thus receiving all the associated payments),
3) and without taking into account any kind of expenses.

On a given day, different bondholders can obtain different IRRs on bonds of the same issuance. So, in order to have information for this issuance on the market, an average of all those IRRs is calculated and published.

This published IRR is known as the **average internal rate of return** or **average interest rate** and it may change every day.

Unit 4. Bonds
1. Definition and related concepts. Classification



Unit 4. Bonds
1. Definition and related concepts. Classification

V_τ : **Value of a bond at time** τ

This is the financial value, at τ, of all outstanding payments promised by the bond from τ until its maturity, using **the current market interest rate**.

In the particular case in which the current market interest rate is equal to the IRR of the bond at moment τ, its financial value, V_τ is equal to the market price of the bond at that moment, P_τ.

PROPERTY

When the financial value of a bond is calculated with its IRR, its market price and its financial value are equal.

1.2. Classification

Bonds can be classified by considering different aspects.

Regarding the issuer:

1) Bonds issued by a government entity

These bonds are usually called public debt**.**

In Spain the central agency which issues public debt is the *Tesoro Público* but other regional or local governments may also issue bonds.

2) Bonds issued by companies

This kind of bonds are issued by large private companies with an important position on the market, such as Banks, electric supply companies, etc. These bonds are called private debt.

As we said earlier, we will only be focusing on the Spanish financial market. Furthermore, we will only study the public debt and, more particularly, the public debt issued by the *Tesoro Público*.

Taking into account the number of times that the bond issuer pays interest to the bondholders.

1) Zero-coupon bonds or **(pure) discount bonds** or **implicit yield bonds**

These are bonds that only promise a single payment on a fixed maturity date. In the Spanish financial market, this single payment is equal to the face value of the bond. Thus, the interest for the bondholder is the difference between the face value received at maturity and the issue price.

Zero-coupon bonds issued in Spain by the *Tesoro Público* are named *Letras del Tesoro*, often translated as Treasury Bills or T-Bills.

2) **Coupon bonds**

These are bonds that promise periodic payments of interest made by the bond issuer prior to and on the maturity or redemption date, as well as an additional redemption payment. In this case, coupon bonds issued by the *Tesoro Público* are called *Bonos del Estado* or *Obligaciones del Estado*.

2. Zero-coupon bonds

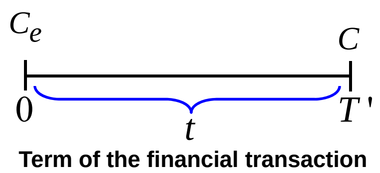
2.1. General characteristics

2.2. *Letras del Tesoro*

2.1. General characteristics

1) Zero-coupon bonds are issued at a discount, i.e., the issue price is lower* than the face value. These are (pure) discount bonds.
2) They are redeemable at par, that is to say, the redemption amount is equal to the face value.
3) The difference between the issue price and the redemption amount on the maturity date is the implicit interest earned by the bondholder*. They are implicit yield bonds.

Graphically:



* In an environment of positive interest rates.

In order to calculate:

- the bond issuer interest rate,
- the bondholder interest rate and
- the internal rate of return of a bond

the term of the financial transaction, t, has to be taken into account:

- If t is lower than or equal to 1 year, the simple interest regime will have to be used.
- If t is greater than 1 year, the compound interest regime will have to be used.

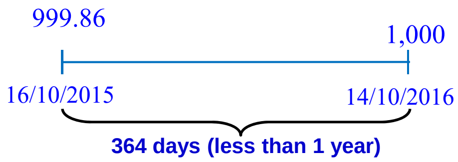
2.2. *Letras del Tesoro*

- These are zero-coupon bonds issued by the *Tesoro Público*.
- Their face value is €1,000.
- Their issue price is established by auction.
- The *Tesoro* currently issues 4 types of *Letras del Tesoro*, each with a different maturity: 3, 6, 9 and 12 months; but years ago there were 18-month *Letras del Tesoro*.
- The term of the financial transaction is always calculated using the criterion Act/360.

Conditions and results of issuances of *Letras del Tesoro* are shown on the website of the *Tesoro Público* (www.tesoro.es).

Once *Letras del Tesoro* are on the secondary market (*BME Renta Fija*), the information for a particular day can be found on the website of BME (www.bmemarketdata.es).

Example 4. Obtain the average interest rate for 1-year *Letras del Tesoro* that were sold/bought on the 16th of October 2015, by considering that they matured on 14th of October 2016 and that their price was €999.86.



By using the simple interest regime:

$$99.986 \cdot \left(1 + i \cdot \frac{364}{360}\right) = 100 \qquad i = \frac{100 - 99.986}{99.986 \cdot \frac{364}{360}} \qquad i = 0.00014 \equiv 0.014\%$$

* *Letras* began to be issued in 1987 and up to this issue in October 2015 had always had a positive interest rate. However, the economic environment changed and the very next issue, which took place on November 17th, 2015, had a negative interest rate. Ever since then, the same has been happening.

Example 5. Find the price today, on the secondary market, of a 1-year *Letra del Tesoro*, with maturity in 330 days, supposing that the average interest rate published today is 0.132%:

P_0 ——————— 100

TODAY — Maturity date

0 — $\frac{330}{360}$

$$P_0 \cdot \left(1 + 0.00132 \cdot \frac{330}{360}\right) = 100 \qquad P_0 = 99.879\%$$

Or alternatively:

$$P_0 \cdot \left(1 + 0.00132 \cdot \frac{330}{360}\right) = 1{,}000 \qquad P_0 = €998.79$$

Example 6. Check the average interest rate of the 9-month *Letras del Tesoro* that were issued on the 15th of December 2020, by considering the following information:

Remember: The face value of *Letras del Tesoro* is €1,000.

LETRAS DEL TESORO

www.tesoro.es
Retrieved on 18th December 2020

(Amounts in millions of euros)

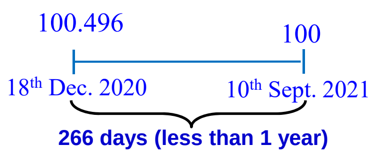
Term		9 MONTHS
Auction dat	MM/DD/YYYY	12/15/2020
Maturity date		09/10/2021
Settlement date		12/18/2020
Nominal bid amount		4,484.73
Nominal allotted amount		950.04
Weighted average price		100.496
Weighted average rate		-0.668

Auction date → Issue date

Settlement date → Moment at which the transaction takes place. On the primary market the settlement date is 3 working days after the trade date (auction date).

Nominal bid amount → It means that 4,484,730 *Letras* were bid (requested) at the auction.

Nominal allotted amount → It means that only 950,040 *Letras* were sold/bought.



$$100.496\cdot\left(1+i\cdot\frac{266}{360}\right)=100 \qquad i=-0.006680\equiv-0.668\%$$

Letras del Tesoro from this issue will be sold and bought on the secondary market until its maturity. In order to see how they were traded on a specific day, it is necessary to check the Public Debt and Private Fixed Income Daily Bulletin for that day at www.bmemarketdata.es, and look for their ISIN.

Example 7. 143 days ago, an investor bought 28 *Letras del Tesoro* at a price of 99.125%. Today, he sells them on the secondary market at a price of 99.829%.

a) Find the total amount obtained by the investor when selling the *Letras del Tesoro*.

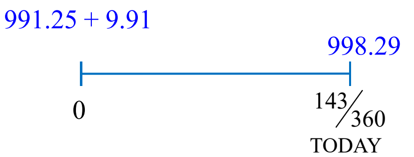
The total amount obtained today is: $1{,}000\cdot 0.99829\cdot 28=€27{,}952.12$

b) Find the annual interest rate for this bondholder.

$$991.25\cdot\left(1+i\cdot\frac{143}{360}\right)=998.29$$

$$i=0.01788\equiv1.788\%$$

c) If the investor had paid a fee of 1% of the purchase price when he bought the *Letras del Tesoro*, what would his annual interest rate have been?



$$(991.25+9.91)\cdot\left(1+i\cdot\frac{143}{360}\right)=998.29$$

$$i=-0.00722\equiv-0.722\%$$

d) Find the annual interest rate today for these *Letras* if they mature in 37 days.

$$99.829\cdot\left(1+i\cdot\frac{37}{360}\right)=100$$

$$i=0.01667\equiv1.667\%$$

3. Coupon bonds

3.1. General characteristics

3.2. *Bonos* and *Obligaciones* del Estado

3.1. General characteristics

1) The issue price of coupon bonds may be equal to or different from their face value.

2) The redemption amount is paid at maturity of the bonds and it can be equal to or different from their face value.

3) The bond issuer pays the bondholder, at the end of each period $p=\frac{1}{m}$, the amount Y, called a **coupon**, which is calculated by applying the effective interest rate of the bonds, I_m, to their face value, C.

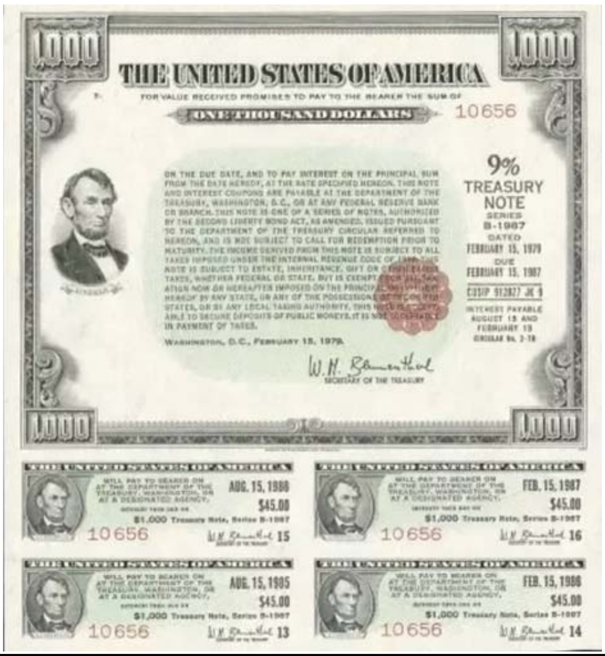
Graphically:

$$
\begin{array}{ccccccc}
C_e & & & & & & C_a \\
 & Y & Y & Y & \cdots & Y & Y \\
\hline
0 & p & 2p & 3p & \cdots & (n-1)\,p & np = T'
\end{array}
$$

where the coupon (or interest) for each period is:

$$Y = C \cdot I_m$$

NOTE: The name "coupon" is due to the fact that, years ago, bonds literally had coupons attached to them. Bondholders received interest by stripping off the coupons and giving them to the bond issuer. This is not the case nowadays since bonds are kept electronically.



www.meridianfinancialpartners.com/
Retrieved on 29th July 2020

To obtain:

- the bond issuer effective interest rate,
- the bondholder effective interest rate and
- the internal rate of return of a bond

the **compound interest regime** will always be used.

Once the coupons bonds have been issued, they can be exchanged on organized markets (secondary markets). If an investor wants to buy or sell one (or more) bond(s) at moment τ, they will have to pay or receive the market price of this bond at moment τ, P_τ.

This price can be determined by summing the amount of the accrued interest and the "clean" price of the bond

Accrued interest: Proportional part of the coupon corresponding to the bond that has been generated from the moment when the last coupon was paid until the moment τ.

Clean price (or ex-coupon price) of the bond: Price of the bond set as a result of the interaction of supply and demand in the market.

$$\boxed{\text{Market price at } \tau,\ P_\tau} = \boxed{\text{Accrued interest}} + \boxed{\text{Clean (ex-coupon) price}}$$

NOTE: The market price is often called the "dirty" price.

On the other hand, recall we said that the financial value of a bond at τ is the financial value of all outstanding payments promised by the bond from τ to its maturity, using the current market interest rate. So, when a coupon bond is considered:

$$\boxed{\text{Financial value at } \tau,\ V_\tau} = \boxed{\text{Financial value at } \tau \text{ of remaining coupons}} + \boxed{\text{Financial value at } \tau \text{ of redemption amount}}$$

3.2. *Bonos* and *Obligaciones del Estado*

- These are both securities issued by the *Tesoro Público*.
- Their face value is €1,000 which is paid on the maturity date, i.e. they are redeemable at par.
- Their issue price is established by auction.
- The *Tesoro* currently issues *Bonos* with maturity terms of 3 and 5 years (approximately) and *Obligaciones* with maturity terms of 10, 15, 30 and 50 years (approximately).
- Coupons are paid in arrears annually, once the bonds have been issued, with the moment of the last coupon being the maturity date.
- The term of the financial transaction is always calculated by using the Act/Act criterion.

Conditions and results of issuances of *Bonos* and *Obligaciones del Estado* are shown on the website of the *Tesoro Público* (www.tesoro.es).

Once these securities are on the secondary market (*BME Renta Fija*) the information for a particular day can be found on the website of BME (www.bmemarketdata.es).

Unit 4. Bonds
3. Coupon bonds

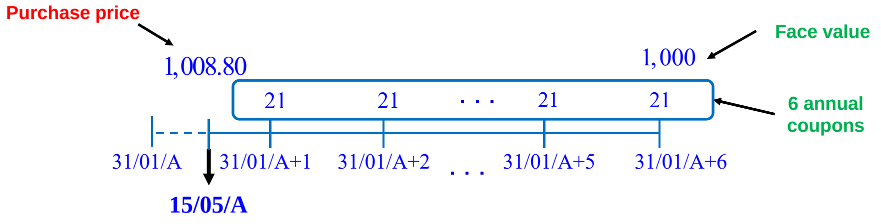
Example 8. An investor buys an *Obligación del Estado* for €1,008.80 today (15th May A). The *Obligación* pays an annual coupon of 2.10% and matures on 31st January A+6.

a) Find the amount of the annual coupon of the *Obligación.*

$$Y = 1{,}000 \cdot 0.0210 = €21$$

b) Depict all the payments and repayments associated with the purchase of the *Obligación*.

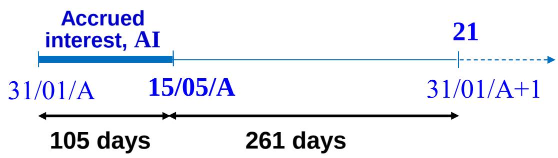
Since the last coupon is received by the bondholder on the maturity date, we locate all the remaining coupons going backwards from the last one.



c) Obtain the amount of the accrued interest of the *Obligación* right after receiving the coupon corresponding to 31st January A+1.

Right after receiving the coupon corresponding to 31st January A+1, the accrued interest of the *Obligación* is, obviously, 0.

d) Obtain the amount of accrued interest on the *Obligación* today, knowing that year A is a leap year.



$$AI = 21 \cdot \frac{105}{366} = €6.03$$

NOTE: If year A was not a leap year, the denominator would be 365.

Example 9. Let us suppose that the information for an issue of *Bonos del Estado* today, 22nd August A, on the secondary market is that contained in the following table.

Coupon interest	Maturity date	Ex-coupon price	IRR
1.01%	31/10/A+4	98.723%	1.325%

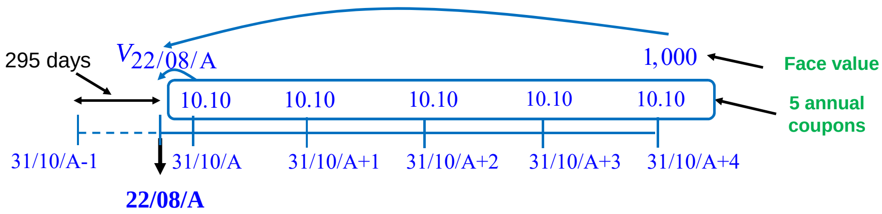
a) Find the financial value today of a *Bono* from this issue by using its IRR.

$\boxed{\text{Financial value at } \tau, V_\tau}$ =

$\boxed{\text{Financial value at } \tau \text{ of remaining coupons}}$ + $\boxed{\text{Financial value at } \tau \text{ of redemption amount}}$

So, we first draw all the remaining coupons from today to the maturity date, as well as the face value.

Since the last coupon is received by the bondholder on the maturity date, we locate all the remaining coupons going backwards from the last one.



So:

$$V_{22/08/A} = \left(10.10 \cdot a_{\overline{5}|0.01325} + 1{,}000 \cdot (1+0.01325)^{-5}\right) \cdot (1+0.01325)^{\frac{295}{365}}$$

$$V_{22/08/A} = €995.39$$

b) Determine the market price today of a *Bono* from this issue.

Market price at τ, P_τ = Accrued interest + Clean (ex-coupon) price

$$\text{AI} = 10.10 \cdot \frac{295}{365} = €8.16$$

$$P^{ex} = €987.23$$

$$P_{22/08/A} = 8.16 + 987.23 = €995.39$$

It is important to highlight that since the financial value of the bond at τ has been calculated with its IRR, this financial value and the market price of the bond at τ are equal. This was already said in a previous property.

As stated earlier, real information on prices and interest rates for *Bonos* and *Obligaciones del Estado* on the secondary market can be found at www.bmemarketdata.es, in the Public Debt and Private Fixed Income Daily Bulletin.

We already know the reason why bond prices are expressed as a percentage of bond face values. But, why are prices agreed (and later published) without including the accrued interest? Why are they agreed (and published) as ex-coupon prices? Why not publish the total price that the buyer paid when the bond was purchased?

To answer this question, first we have to make it clear that when coupons are paid by the issuer, the payment is made to the current owner of the bond. However, it may be that the current owner has not had the bond for the whole period over which the coupon has been generated because the bond was purchased on the secondary market after payment of the previous coupon. So, the new owner would not be entitled to receive the whole coupon and a part of it (the accrued interest until the moment at which the bond was sold/purchased) should be paid to the former owner.

In order to avoid either the issuer or the current owner paying the accrued interest to the former owner, and to make things as easy as possible, it is paid by the buyer to the seller when the bond is purchased/sold.

Including the coupon in published prices would cause them to be affected by the number of days since the date of the last coupon, resulting in non-comparable information.

Example 10. Let us consider two *Bonos del Estado*: A and B, from different issuances but with the same ex-coupon price today, which is equal to 90% (€900). Furthermore, both bonds pay annual coupons at an annual effective interest rate equal to 3%, i.e.

$$Y = €30$$

Moreover, suppose that whereas bond A paid its previous coupon 15 days ago, bond B paid it 360 days ago.

For both bonds, obtain the market price that a buyer has to pay today.

NOTE: Consider that we are not in a leap year.

$$\boxed{\text{Market price at } \tau,\ P_\tau} = \boxed{\text{Accrued interest}} + \boxed{\text{Clean (ex-coupon) price}}$$

Bond A:

$$AI = 30 \cdot \frac{15}{365} = €1.23 \qquad P^{ex} = €900 \qquad P_{TODAY} = 1.23 + 900 = €901.23$$

Bond B:

$$AI = 30 \cdot \frac{360}{365} = €29.59 \qquad P^{ex} = €900 \qquad P_{TODAY} = 29.59 + 900 = €929.59$$

It may seem that the price of bond A is greater than that of bond B. However, since the new owner will recover the accrued interest paid to the seller when receiving the coupon paid by the issuer, both bonds have the same price, as is shown by the ex-coupon price.

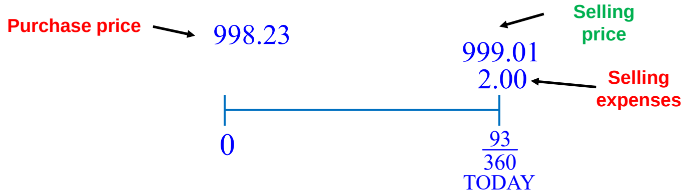
Example 11. An SME has the following securities:

- 6 *Letras del Tesoro*. The SME bought them on the secondary market 93 days ago at a unit price of €998.23.
- 8 *Bonos del Estado*. They were bought at the moment of their issue, 3 years and 240 days ago, they will be redeemed 5 years after their issue and their annual interest rate is equal to 3%. Their unit issue price was €1,014 and the SME paid purchasing expenses of 0.5% of this price.

Obtain:

a) The bondholder annual interest rate for the *Letras*, taking into account that they are sold today at a unit price of €999.01 with selling expenses of 0.2% of the face value.

First, we consider all the payments and repayments associated with the purchase of 1 *Letra* (it would be equivalent to considering all the *Letras*).



The term is less than 1 year, so we use the simple interest regime.

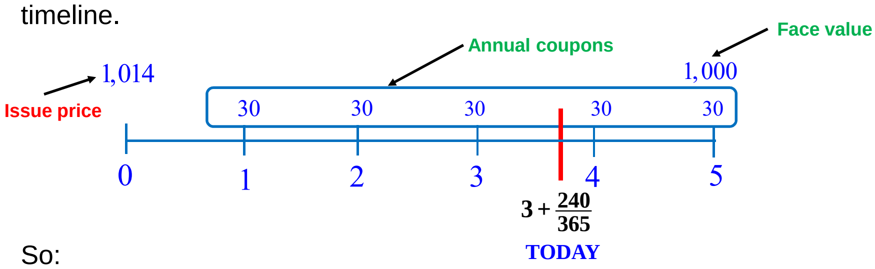
$$998.23\cdot\left(1+i\cdot\frac{93}{360}\right)=999.01-2.00 \qquad i=-0.00473\equiv-0.473\%$$

b) Total amount obtained by the SME if it sells the *Bonos del Estado* today, on the secondary market, at an ex-coupon price of 98.9%.

To calculate the market price today per *Bono* we consider:

$$\boxed{\text{Market price at } \tau,\ P_\tau} = \boxed{\text{Accrued interest}} + \boxed{\text{Clean (ex-coupon) price}}$$

In order to better understand how the accrued interest has to be obtained, let us draw all the payments and repayments related to 1 *Bono* since the moment it was issued until its maturity date on the timeline.



So:

$$AI=30\cdot\frac{240}{365}=€19.73$$

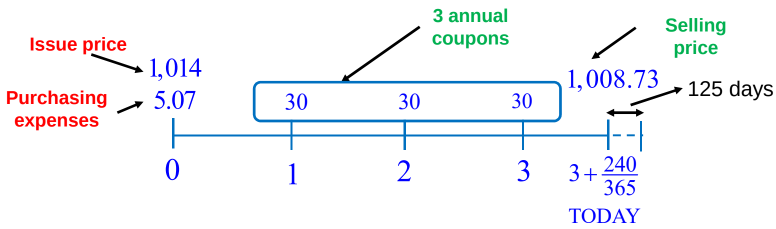
and $P_{\text{TODAY}}=19.73+989=€1,008.73$

The total amount received today when selling the *Bonos del Estado* is:

$$1,008.73\cdot 8=€8,069.84$$

c) The equation that allows us to calculate the annual interest rate for the SME, I_1, associated with the purchase of 1 *Bono del Estado*.

By considering the following timeline:

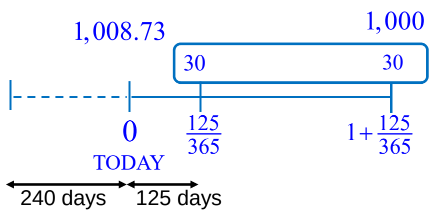


$$1{,}014+5.07=30\cdot a_{\overline{3}|I_1}+1{,}008.73\cdot(1+I_1)^{-\left(3+\frac{240}{365}\right)}$$

d) The equation that allows us to calculate the annual interest today on the secondary market for this *Bono del Estado*.

We have to consider all the remaining coupons from today until the maturity of the *Bono* (in 1 year and 125 days) and its face value at that moment.

Furthermore, we have to bear in mind that the market price today is €1,008.73



Let us denote the annual interest rate of these *Bonos* by I_1. The equation that allows us to obtain this is:

$$1{,}008.73=\left(30\cdot a_{\overline{2}|I_1}+1{,}000\cdot(1+I_1)^{-2}\right)\cdot(1+I_1)^{\frac{240}{365}}$$

UNIT 4 PROBLEMS

NOTE: Unless stated otherwise, all the years that appear are not leap years.

1. For a 1-year *Letra del Tesoro* with a settlement date of 12th May of year A:

a) Obtain the average rate when it was issued by considering that it matures on 11th May of year A+1 and the average price resulting from the auction was 99.835%.

b) Find its price on the secondary market today (9th January A+1), supposing that its internal rate of return (IRR) today is 0.142%.

2. 92 days ago, a small enterprise bought 10 *Letras del Tesoro* at a price of 99.223%. Today, it sells them on the secondary market at a price of 99.742%.

a) Find the total amount obtained by the small enterprise when selling these *Letras del Tesoro.*

b) If the small company paid a fee of 0.5% of the purchase price when it bought the *Letras del Tesoro* and today it has to pay a selling fee equal to 0.5% of their selling price, what is the annual interest rate for the company (bondholder)?

c) If these *Letras del Tesoro* mature in 178 days, find their internal rate of return (IRR) today.

3. An investor buys 25 *Obligaciones del Estado* today that mature in 4 years and 85 days. They pay an annual coupon of 2.35%.

a) For 1 *Obligación*, find the amount of the annual coupons.
b) For 1 *Obligación*, on the timeline draw all the payments and repayments that the investor will make or receive if the *Obligación* is kept until its maturity and a purchasing fee equal to 0.3% of its face value is paid today.
c) If the IRR today for 1 of these *Obligaciones* is equal to 2.578%, obtain its financial value.
d) Determine the amount of the accrued interest of 1 of these *Obligaciones* today.
e) What is its ex-coupon price today?
NOTE: Remember that when the financial value of a bond is calculated with its IRR, its market price and its financial value are equal.
f) What is the total amount paid by the investor today when buying the *Obligaciones*?
g) Write the equation that allows obtaining the annual effective interest rate for the investor (bondholder), associated with the purchase of 1 *Obligación,* if he keeps it until the maturity.

4. A company has 84 *Obligaciones del Estado* paying annual coupons of 3.25% which mature in 6 years and 195 days.

a) Obtain the accrued interest of 1 *Obligación* today if the last coupon was received 170 days ago.

b) If the ex-coupon price of 1 *Obligación* today is 102.45%, find the total amount that the company would receive today if it sold all its *Obligaciones*.

c) Write the equation that allows us to obtain the internal rate of return (IRR) today for one of these *Obligaciones*.

5. An SME has 10 *Bonos del Estado*. They were bought at the moment of their issue, 2 years and 125 days ago, and they will be redeemed 5 years after their issue. Their annual interest (coupon) rate is equal to 2.90% and their unit issue price was €1,002.57. Obtain:

a) The total amount obtained by the SME if it sells the *Bonos del Estado,* on the secondary market today, at an ex-coupon price of 98.9%.

b) The equation that allows us to calculate the annual interest rate for the SME, associated with the purchase of 1 *Bono del Estado* (bondholder effective interest rate).
c) The equation that allows us to calculate the internal rate of return (IRR) today for these *Bonos del Estado.*

Formula

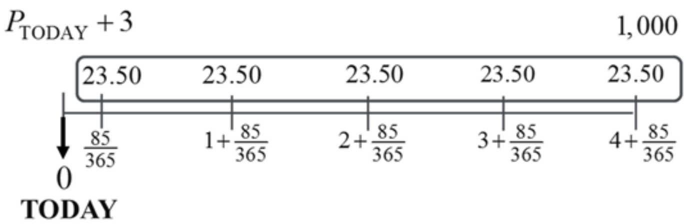
$C_r = C$	$C \cdot a_{\overline{n}\vert I_M} = C \cdot \dfrac{1-(1+I_M)^{-n}}{I_M}$

Answers

1. a) 0.163%.
b) 99.952%.

2. a) €9,974.20
b) -1.867%.
c) 0.523%.

3. a) €23.50
b)



c) €1,008.94.
d) €18.03
e) 99.091%
f) €25.298,50
g) $1{,}008.94 + 3 = \left(23.50 \cdot a_{\overline{5}\vert I_1} + 1{,}000 \cdot (1+I_1)^{-5}\right) \cdot (1+I_1)^{\frac{280}{365}}$

4. a) €15,14.
b) €87,329.76.
c) $1{,}039.64 = \left(32.50 \cdot a_{\overline{7}\vert I_1} + 1{,}000 \cdot (1+I_1)^{-7}\right) \cdot (1+I_1)^{\frac{170}{365}}$

5. a) €9,989.30
b) $1{,}002.57 = 29 \cdot a_{\overline{2}\vert I_1} + 998{,}93 \cdot (1+I_1)^{-\left(2+\frac{125}{365}\right)}$
c) $998.93 = \left(29 \cdot a_{\overline{3}\vert I_1} + 1{,}000 \cdot (1+I_1)^{-3}\right) \cdot (1+I_1)^{\frac{125}{365}}$

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