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Abstract *In this paper we determine the asymptotic behavior of the Non-dictatorship Index (NDI) introduced in Bednay, Moskalenko and Tasnádi (2019). We show that if m denotes the number of alternatives, then as the number of voters tends to infinity the NDI of any anonymous voting rule tends to $(m-1)/m$, which equals the NDI of the constant rule.*

Keywords: Voting rules, dictatorship, non-dictatorship index.
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1 Introduction

Aggregating preferences of individuals to a collective decision (i.e. alternative), is an open problem ever since. Besides the axiomatic approach pioneered by Arrow (1951) there is a fairly large literature initiated by Farkas and Nitzan (1979) and extensively developed by Elkind et al. (2015) employing optimization techniques in order to determine an 'optimal' social choice function. The latter approach takes a distance function and picks for each profile the alternative chosen by the 'closest' profile with a winning alternative determined by some desirable properties.

In Bednay, Moskalenko and Tasnádi (2017) we considered the distances of social choice functions from the dictatorial rules. We derived the plurality rule and the reverse-plurality rule as the solutions of respective optimization problems. By employing the same distance function we have introduced in Bednay, Moskalenko and Tasnádi (2019) a non-dictatorship index (NDI). Concerning the celebrated Gibbard–Satterthwaite theorem (1973/75), the non-dictatorship index focuses on the part of being non-dictatorial, while the well-known Nitzan–Kelly-index (1985/88) on non-manipulability. The degree of manipulability of several social choice rules have been determined by Kelly (1993) and Aleskerov and Kurbanov (1999) via computational experiments among others.

In this paper we determine for any given number of alternatives and for any anonymous social choice function the limit of the NDI as the number of voters tends to infinity. Interestingly, for anonymous social choice functions the limit of the NDI equals the NDI of the constant social choice functions. An analogous convergence result has been established for the Nitzan-Kelly-index (NKI) for a large class of so-called 'classical' social choice functions by Slinko (2002) stating that for these functions the NKI tends to zero as the number of voters tends to infinity; however, this is not true for any social choice function (see Peleg, 1971). Thus, concerning the NKI, the classical social choice functions come close to any dictatorial rule. Regarding a condition and the implication appearing in the Gibbard–Satterthwaite theorem, the NDI of anonymous social choice functions 'approaches' the NDI of a rule extremely violating the condition imposed on the range of social choice functions (i.e. their surjectivity), while the NKI of 'classical' social choice functions 'approaches' the NKI of the dictatorial rules.

2 The framework

Let $A = \{1, \ldots, m\}$ be the set of alternatives, where $m \geq 2$, and $N = \{1, \ldots, n\}$ be the set of voters. We shall denote by $\mathcal{P}$ the set of all linear orderings (irreflexive, transitive and total binary relations) on A and by $\mathcal{P}^n$ the set of all preference profiles. If $\succ \in \mathcal{P}^n$ and $i \in N$, then $\succ_i$ is the preference ordering of voter i over A.

Definition 1. A mapping $f : \mathcal{P}^n \to A$ that selects the winning alternative is called a *social choice function* (or *voting rule*), henceforth, SCF.

An SCF f is called *anonymous* if any reordering of voters' preferences of an arbitrarily given preference profile, does not change the alternative selected by f. As our definition of an SCF does not allow for possible ties, in this event a fixed anonymous[1] tie-breaking rule will be employed. A tie-breaking rule $\tau : \mathcal{P}^n \to \mathcal{P}$ maps preference profiles to linear orderings on A, which will be only employed when a formula does not determine a unique winner. If there are more alternatives chosen by a formula 'almost' specifying an SCF, then the highest ranked alternative is selected, based on the given tie-breaking rule among tied alternatives.

Let $rk[a, \succ]$ denote the *rank* of alternative a in the ordering $\succ \in \mathcal{P}$ (i.e. $rk[a, \succ] = 1$ if a is the top alternative in the ranking $\succ$, $rk[a, \succ] = 2$ if a is second-best, and so on). A voting rule PL is the *plurality rule* if for all $(\succ_i)_{i=1}^n \in \mathcal{P}^n$

$$PL\left((\succ_i)_{i=1}^n\right) = \arg_\tau \max_{a \in A} \; \#\left\{i \in N \mid rk[a, \succ_i] = 1\right\},$$

where the index τ of arg indicates that ties will be resolved based on the tie-breaking rule τ, and PL_SC is the respective *plurality score*

$$PL\_SC\left((\succ_i)_{i=1}^n\right) = \max_{a \in A} \; \#\left\{i \in N \mid rk[a, \succ_i] = 1\right\}.$$

PL chooses an alternative that is ranked first by the maximum number of voters.

A voting rule REV_PL is the *reverse-plurality rule* if for all $(\succ_i)_{i=1}^n \in \mathcal{P}^n$

$$REV\_PL\left((\succ_i)_{i=1}^n\right) = \arg_\tau \min_{a \in A} \; \#\left\{i \in N \mid rk[a, \succ_i] = 1\right\},$$

where the index τ of arg indicates that ties will be resolved based on the tie-breaking rule τ, and REV_PL_SC is the respective *reverse-plurality score*

$$REV\_PL\_SC\left((\succ_i)_{i=1}^n\right) = \min_{a \in A} \; \#\left\{i \in N \mid rk[a, \succ_i] = 1\right\}.$$

REV_PL chooses an alternative that is ranked first by the minimum number of voters.

Let $\mathcal{F} = A^{\mathcal{P}^n}$ be the set of SCFs (i.e. the set of all mappings from $\mathcal{P}^n$ to A) and $\mathcal{F}^{an} \subset \mathcal{F}$ be the set of anonymous voting rules. The subset of $\mathcal{F}$ consisting of the dictatorial rules will

[1] The linear ordering selected by an anonymous tie-breaking is invariant to the ordering of voters' preferences.

be denoted by $\mathcal{D} = \{d_1, \ldots, d_n\}$, where d_i is the dictatorial rule with voter i as the dictator, that is the SCF selects always the top alternative of voter i. By counting the number of profiles, on which f and g choose different alternatives we define a metric:

$$\varrho(f, g) = \#\{\succ \in \mathcal{P}^n | f(\succ) \neq g(\succ)\} \leq |\mathcal{P}^n| = (m!)^n \tag{2.1}$$

on $\mathcal{F} = A^{\mathcal{P}^n}$.

We define our non-dictatorship index (NDI) by taking the distance to the closest dictatorial rule.

Definition 2. The *non-dictatorship index* (NDI) is given by

$$NDI(f) = \frac{\min_{i \in N} \varrho(f, d_i)}{(m!)^n}$$

Assuming that $\mathcal{P}^n$ is a discrete probability space with the uniform distribution, $NDI(f)$ equals the smallest probability that an alternative equals the top ranked alternative of a voter.

3 The limit of the NDI

First we start with bounding the NDI based on our results in Bednay, Moskalenko and Tasnádi (2017) from which it follows that

$$0 \leq NDI(PL) \leq NDI(f) \leq NDI(REV\_PL) \leq 1$$

for any anonymous SCF f. Hence, restricting ourselves to anonymous SCFs, it is sufficient to show that $NDI(PL)$ and $NDI(REV\_PL)$ tend to $(m-1)/m$ when n tends to infinity to derive the limiting result for any anonymous f.

Note that for any anonymous SCF f we have $\varrho(f, d_i) = \varrho(f, d_j)$ for any $i, j \in N$, and it follows that for any anonymous voting rule f

$$\min_{i \in N} \varrho(f, d_i) = \frac{1}{n} \sum_{i \in N} \varrho(f, d_i)$$

holds true. Therefore, for any anonymous f we have

$$\begin{aligned}
NDI(f) &= \frac{\frac{1}{n}\sum_{i \in N} \varrho(f, d_i)}{(m!)^n} = \frac{\frac{1}{n}\sum_{i \in N} \#\{\succ \in \mathcal{P}^n \mid f(\succ) \neq d_i(\succ)\}}{(m!)^n} \\
&= \frac{\frac{1}{n}\sum_{i \in N} \left((m!)^n - \#\{\succ \in \mathcal{P}^n \mid f(\succ) = d_i(\succ)\}\right)}{(m!)^n} \\
&= 1 - \frac{\frac{1}{n}\sum_{i \in N} \#\{\succ \in \mathcal{P}^n \mid f(\succ) = d_i(\succ)\}}{(m!)^n} \\
&= 1 - \frac{\sum_{\succ \in \mathcal{P}^n} \frac{1}{n} \sum_{i \in N} \mathbf{1}_{f(\succ) = d_i(\succ)}}{(m!)^n},
\end{aligned} \tag{3.2}$$

where $\mathbf{1}$ denotes the characteristic function, i.e. $\mathbf{1}_{f(\succ)=d_i(\succ)} = 1$ if $f(\succ) = d_i(\succ)$, and $\mathbf{1}_{f(\succ)=d_i(\succ)} = 0$ otherwise. In particular,

$$NDI(PL) = 1 - \frac{\sum_{\succ \in \mathcal{P}^n} \frac{1}{n} PL\_SC(\succ)}{(m!)^n}. \tag{3.3}$$

We continue with bounding the values of $\frac{1}{n}PL\_SC\left((\succ_i)_{i=1}^n\right)$, where we assume in line with our simulations in Bednay, Moskalenko and Tasnádi (2019) that the preference relations of the individuals are generated independently and distributed uniformly above the set of preference relations $\mathcal{P}$. Since in case of the plurality rule as well as in case of the reverse-plurality rule only the top-ranked alternatives matter the problem reduces to considering for each voter the uniform distribution above the set of alternatives A. We shall denote by $X_i^{(n)} : \mathcal{P}^n \to \{0, 1, \ldots, n\}$ the number of top positions of alternative $i \in A$ in case of n voters. Then $X_1^{(n)}, \ldots, X_m^{(n)}$ are non-independently binomially distributed with parameter values n and $1/m$. Let $Y^{(n)} = \max\left\{X_1^{(n)}, \ldots, X_m^{(n)}\right\}$.

We shall denote by P the uniform distribution above the discrete probability space $\mathcal{P}^n$. Since $X_i^{(n)}$ is the sum of n independent Bernoulli distributions for any $i \in A$, the Chebyshev's inequality, or more precisely the inequality resulting the weak law of large numbers, implies for any $\varepsilon > 0$ that

$$P\left(\left|\frac{X_i^{(n)}}{n} - \frac{1}{m}\right| \geq \varepsilon\right) \leq \frac{1}{4n\varepsilon^2}.$$

Let

$$A_i = \left\{\succ \in \mathcal{P}^n \mid \left|\frac{X_i^{(n)}}{n} - \frac{1}{m}\right| \geq \varepsilon\right\}.$$

Then

$$P\left(\left|\frac{Y^{(n)}}{n} - \frac{1}{m}\right| \geq \varepsilon\right) \leq P\left(\cup_{i=1}^m A_i\right) \leq \sum_{i=1}^m P\left(A_i\right) \leq \frac{m}{4n\varepsilon^2} \tag{3.4}$$

from which it follows that $Y^{(n)}/n$ converges in probability to $1/m$.

Since $PL\_SC(\succ) = Y^{(n)}(\succ)$ for any $\succ \in \mathcal{P}^n$, we get by employing (3.3) that $NDI(PL)$ converges in probability to the common mean $(m-1)/m$ as n tends to infinity. In analogous way we can derive that $NDI(REV\_PL)$ also converges in probability to $(m-1)/m$ as n tends to infinity. Therefore, the limits of all graphs in Bednay, Moskalenko and Tasnádi (2019) associated with the NDI values of well-known SCFs approach $(m-1)/m$ as n tends to infinity. Hence, we have proven the following theorem.

Theorem 1. *Let the number of alternatives m be fixed. Then for any sequence $(f_n)_{n=1}^{\infty}$ of anonymous SCFs, where the index n equals the number of voters, we have*

$$\lim_{n\to\infty} NDI(f_n) = \frac{m-1}{m}.$$

4 Concluding remarks

We shall denote by f^i one of the m *constant* SCFs, which assigns to each profile $\succ \in \mathcal{P}$ alternative $i \in A$, that is $f^i(\succ) = i$ for all $\succ \in \mathcal{P}$. Clearly, f^i is anonymous and it can be easily verified that $NDI(f^i) = (m-1)/m$. Furthermore, we have $NDI(d_i) = 0$ for any dictatorial rule $d_i \in \mathcal{D}$. However, the dictatorial rules are non-anonymous.

As we have already mentioned in the introduction the NKI of classical SCFs tend to zero. Taking into consideration that the NKIs of dictatorial rules and constant rules are all zero, we conclude that the NDI can distinguish between these two types of elementary SCFs (both playing special roles in the statement of the Gibbard–Satterthwaite theorem either implicitly or explicitly), while the NKI and several related indexes on manipulability cannot, this observation might be considered as a fact in favor of the NDI.

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